

Research Article

Radius Problems for Starlike Functions Associated with the Tan Hyperbolic Function

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The aim of this particular article is at studying a holomorphic function f defined on the open-unit disc $\mathfrak{D} = \{z \in \mathbb{C} : |z| < 1\}$ for which the below subordination relation holds $zf'(z)/f(z) < q_0(z) = 1 + \tan h(z)$. The class of such functions is denoted by $\mathfrak{S}_{\tan h}^*$. The radius constants of such functions are estimated to conform to the classes of starlike and convex functions of order β and Janowski starlike functions, as well as the classes of starlike functions associated with some familiar functions.

1. Introduction

To completely comprehend the mathematical concepts used throughout our key observations, some of the essential literature of the geometric function theory must be described and analyzed here. Let us begin with the symbol \mathfrak{A}_n which describes the family of holomorphic (or analytic) functions in a subset \mathfrak{D} of the complex plan \mathbb{C} having the following series expansion

$$f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \dots \quad (1)$$

Also, let the family of all univalent functions be denoted by \mathfrak{S} and is a subset of the class $\mathfrak{A}_1 = \mathfrak{A}$. Next, we define that the subordination between the function belongs to the class \mathfrak{A} . Let $g_1, g_2 \in \mathfrak{A}$. Then, $g_1 < g_2$ or $g_1(z) < g_2(z)$, the mathematical form of the subordination between g_1 and g_2 , if a holomorphic function w occurs in \mathfrak{D} with the restriction $w(0) = 0$ and $|w(z)| < 1$ in such a way that $f(z)$

$= g(w(z))$ hold. Further, if $g_2 \in \mathfrak{S}$ in \mathfrak{D} , then, the following relation holds:

$$g_1(z) < g_2(z), \Leftrightarrow g_1(0) = g_2(0) \text{ and } g_1(\mathfrak{D}) \subset g_2(\mathfrak{D}). \quad (2)$$

Three significant subfamilies of \mathfrak{S} , which are well studied and have nice geometric interpretations, are the families of starlike $\mathfrak{S}^*(\xi)$, convex $\mathfrak{K}(\xi)$, and strongly starlike $\mathfrak{S}^*(\zeta)$ functions of order $\xi (0 \leq \xi < 1)$ and $\zeta (0 < \zeta \leq 1)$, respectively. These families are defined as follows:

$$\begin{aligned} \mathfrak{S}^*(\zeta) &:= \left\{ f \in \mathfrak{S} : \frac{zf'(z)}{f(z)} < \left(\frac{1+z}{1-z} \right)^\zeta, (z \in \mathfrak{D}) \right\}, \\ \mathfrak{S}^*(\xi) &:= \left\{ f \in \mathfrak{S} : \frac{zf'(z)}{f(z)} < \frac{1+(1-2\xi)z}{1-z}, (z \in \mathfrak{D}) \right\}, \\ \mathfrak{K}(\xi) &:= \left\{ f \in \mathfrak{S} : \left(\frac{zf'(z)}{f(z)} \right)' < \frac{1+(1-2\xi)z}{1-z}, (z \in \mathfrak{D}) \right\}. \end{aligned} \quad (3)$$

Particularly, the notations $\mathfrak{S}\mathfrak{S}^*(1) = \mathfrak{S}^*(0) = \mathfrak{S}^*$ and $\mathcal{H}(0) = \mathcal{H}$ represent familiar families of starlike and convex functions, respectively. These subfamilies of \mathfrak{S} satisfy the following relationship

$$\mathcal{H} \subset \mathfrak{S}^* \subset \mathfrak{S}. \tag{4}$$

The reverse of the above relation hold only under certain restriction of the domain. That is; if $f \in \mathfrak{S}$ in \mathfrak{D} , then, it was given in [1], Corollary, p. 98, that f maps the disc $|\mathfrak{z}| < r$ onto a region which is star shaped about the origin for every $r \leq r_0 = \tan h(\pi/4)$. The constant r_0 is known as the radius of starlikeness for the family \mathfrak{S} . Also, given in [1], Corollary, p. 44, the radius of convexity for the families \mathfrak{S}^* and \mathfrak{S} is $2 - \sqrt{3}$.

To make a radius statement for other things than starlikeness and convexity, we choose two subfamilies \mathcal{L} and \mathcal{H} of the set \mathfrak{A} . The \mathcal{L} radius for the family \mathcal{H} , represented by $R_{\mathcal{L}}(\mathcal{H})$, is the largest number R such that $r^{-1}f(r\mathfrak{z}) \in \mathcal{L}$ for every $0 < r \leq R$ and $f \in \mathcal{H}$. Consequently, an alternative formulation of the radius of starlikeness for \mathfrak{S} is that the \mathfrak{S}^* radius for the family \mathfrak{S} is $R_{\mathfrak{S}^*}(\mathfrak{S}) = \tan h(\pi/4)$.

In 1992, Ma and Minda [2] considered the general form of the families as

$$\begin{aligned} \mathfrak{S}^*(\varphi) &:= \left\{ f \in \mathfrak{A} : \frac{\mathfrak{z}f'(\mathfrak{z})}{f(\mathfrak{z})} < \varphi(\mathfrak{z}) \right\}, \\ \mathcal{H}(\varphi) &:= \left\{ f \in \mathfrak{A} : 1 + \frac{\mathfrak{z}f''(\mathfrak{z})}{f(\mathfrak{z})} < \varphi(\mathfrak{z}) \right\}, \end{aligned} \tag{5}$$

where φ is a holomorphic function with $\varphi'(0) > 0$ and has positive real part. Also, the function φ maps \mathfrak{D} onto a star-shaped region with respect to $\varphi(0) = 1$ and is symmetric about the real axis. They addressed some specific results such as distortion, growth, and covering theorems. In recent years, several subfamilies of the set \mathfrak{A} were studied as a special case of the class $\mathfrak{S}^*(\varphi)$. For example,

- (i) If we take $\varphi(\mathfrak{z}) = (1 + L\mathfrak{z})/(1 + M\mathfrak{z})$ with $-1 \leq M < L \leq 1$, then, we achieved the class $\mathfrak{S}^*[L, M] \equiv \mathfrak{S}^*((1 + L\mathfrak{z})/(1 + M\mathfrak{z}))$ which is described by the functions of the Janowski starlike class investigated in [3]. Furthermore, $\mathfrak{S}^*(\xi) := \mathfrak{S}^*[1 - 2\xi, -1]$ is the familiar starlike function family of order ξ with $0 \leq \xi < 1$
- (ii) The family $\mathfrak{S}_{\mathcal{L}}^* := \mathfrak{S}^*(\varphi(\mathfrak{z}))$ with $\varphi(\mathfrak{z}) = \sqrt{1 + \mathfrak{z}}$ was developed in [4] by Sokol and Stankiewicz. The function $\varphi(\mathfrak{z}) = \sqrt{1 + \mathfrak{z}}$ maps the region \mathfrak{D} onto the the image domain which is bounded by $|\omega^2 - 1| < 1$
- (iii) The class $\mathfrak{S}_{\text{car}}^* := \mathfrak{S}^*(\varphi(\mathfrak{z}))$ with $\varphi(\mathfrak{z}) = 1 + (4/3)\mathfrak{z} + (2/3)\mathfrak{z}^2$ was examined by Sharma and his coauthors [5] which consists of function $f \in \mathfrak{A}$ in such

a manner that $(\mathfrak{z}f'(\mathfrak{z}))/f(\mathfrak{z})$ is located in the region bounded by the cardioid given by

$$(9x^2 + 9y^2 - 18x + 5)^2 - 16(9x^2 + 9y^2 - 6x + 1) = 0 \tag{6}$$

- (iv) The family $\mathfrak{S}_R^* := \mathfrak{S}^*(\varphi(\mathfrak{z}))$ with $\varphi(\mathfrak{z}) = 1 + (\mathfrak{z}/J)(J + \mathfrak{z}/J - \mathfrak{z})$, $J = \sqrt{2} + 1$ is studied in [6] while $\mathfrak{S}_{\text{cos}}^* := \mathfrak{S}^*(\cos(\mathfrak{z}))$ and $\mathfrak{S}_{\text{cos } h}^* := \mathfrak{S}^*(\cosh(\mathfrak{z}))$ were contributed by Raza and Bano [7] and Alo-taibi et.al [8], respectively
- (v) By choosing $\varphi(\mathfrak{z}) = 1 + \sin \mathfrak{z}$, we obtain the class $\mathfrak{S}_{\text{sin}}^* := \mathfrak{S}^*(\varphi(\mathfrak{z}))$ which was established in [9]. The authors determined radius problems in this article for the defined class $\mathfrak{S}_{\text{sin}}^*$
- (vi) The class $\mathfrak{S}_e^* := \mathfrak{S}^*(e^{\mathfrak{z}})$ was explored recently in [10]. For such a class \mathfrak{S}_e^* , the authors calculated Hankel determinant bounds of order three in [11]. Also, the class $\mathfrak{S}_{\mathcal{L}}^* := \mathfrak{S}^*(h_{\mathcal{L}}(\mathfrak{z}))$ with

$$h_{\mathcal{L}}(\mathfrak{z}) = \sqrt{2} - (\sqrt{2} - 1) \sqrt{\frac{1 - \mathfrak{z}}{1 + 2(\sqrt{2} - 1)\mathfrak{z}}} \tag{7}$$

was contributed by Mendiratta et al. [12] in which they investigated the radius problems

- (vii) The family $\mathfrak{S}_{\mathcal{C}}^* := \mathfrak{S}^*(\varphi(\mathfrak{z}))$ with $\varphi(\mathfrak{z}) = \mathfrak{z} + \sqrt{1 + \mathfrak{z}^2}$ was introduced and studied by Raina and Sokół [13]
- (viii) By considering the function $\varphi(\mathfrak{z}) = 1 + \sin h^{-1}\mathfrak{z}$, we get the recently examined family $\mathfrak{S}_p^* := \mathfrak{S}^*(1 + \sin h^{-1}\mathfrak{z})$ introduced by Kumar and Arora [14]. They discussed relationships of this class with the already known classes. For more particular classes, see the articles [15–20]

In the present paper, we consider a trigonometric function $\varphi_1(\mathfrak{z}) = 1 + \tan h\mathfrak{z}$ with $\varphi_1(0) = 1$. Also, one can easily obtain that $\Re\varphi_1(\mathfrak{z}) > 0$. By using this function, we define the below family of functions as

$$\mathfrak{S}_{\tan h}^* := \left\{ f \in \mathfrak{S} : \frac{\mathfrak{z}f'(\mathfrak{z})}{f(\mathfrak{z})} < 1 + \tan h\mathfrak{z}, (\mathfrak{z} \in \mathfrak{D}) \right\}. \tag{8}$$

In other words, a function $f \in \mathfrak{S}_{\tan h}^*$ if and only if there exists a holomorphic function q , fulfilling $q(\mathfrak{z}) < q_0(\mathfrak{z}) = 1 + \tan h\mathfrak{z}$, such that

$$f(\mathfrak{z}) = \mathfrak{z} \exp \left(\int_0^{\mathfrak{z}} \frac{q(t) - 1}{t} dt \right). \tag{9}$$

Now, we construct some examples of our newly described family $\mathfrak{S}_{\tan h}^*$. For this, consider the following functions

$$\begin{aligned}
 q_1(z) &= 1 + \frac{z}{3}, \\
 q_2(z) &= \frac{4 + 2z}{4 + z}, \\
 q_3(z) &= \frac{7 + 3z}{7 + z}, \\
 q_4(z) &= 1 + (\tan h)z.
 \end{aligned}
 \tag{10}$$

Since $q_0(z) = 1 + \tanh z$ is univalent in \mathfrak{D} , $q_i(0) = 1 = q_0(0)$, $(i = 1, 2, 3, 4)$, and $q_i(\mathfrak{D}) \subseteq q_0(\mathfrak{D})$, this implies that for each $i = 1, 2, 3, 4$, the relation $q_i \prec q_0$ holds. Thus, from (8), the functions

$$\begin{aligned}
 f_1(z) &= ze^{z/3}, \\
 f_2(z) &= z + \frac{z^2}{4}, \\
 f_3(z) &= z\left(1 + \frac{z}{7}\right)^2, \\
 f_4(z) &= ze^{(\tan h)z},
 \end{aligned}
 \tag{11}$$

corresponding to the functions $q_1(z), q_2(z), q_3(z)$, and $q_4(z)$, respectively, belong to the family \mathfrak{S}_{\tanh}^* . By taking $q(z) = q_0(z) = 1 + \tanh z$ in (8), we get the below function that plays a role of the extremal in many problems of the class $\mathfrak{S}_{\tan h}^*$.

$$\begin{aligned}
 f_0(z) &= z \exp\left(\int_0^z \frac{\tan ht}{t} dt\right) \\
 &= z + z^2 + \frac{1}{2} + \frac{1}{18}z^4 - \frac{5}{72}z^5 + \dots
 \end{aligned}
 \tag{12}$$

In this article, we work on determining the radius of starlikeness and convexity and \mathfrak{S}_{\tanh}^* radius for some subfamilies of starlike functions, mentioned above in which mostly have simple geometric interpretation. Besides these subfamilies, we also discuss the \mathfrak{S}_{\tanh}^* radius for some families of \mathfrak{A} , whose functions have been expressed as a ratio between two functions.

2. Radii of Starlikeness and Convexity

In this portion, we examined the radius of starlikeness and convexity for the family \mathfrak{S}_{\tanh}^* .

Theorem 1. *The $\mathfrak{S}^*(\xi)$ radius for the family $\mathfrak{S}_{\tan h}^*$ is $r_0 = \tan^{-1}(1 - \xi)$ with $0 \leq \xi < 1$.*

Proof. If $f \in \mathfrak{S}_{\tan h}^*$, then, by virtue of (7), a Schwarz function w exists with $|w(z)| \leq |z|$ such as

$$\frac{zf'(z)}{f(z)} = 1 + \tan h(w(z)).
 \tag{13}$$

Now, let $w(z) = Re^{iv}$ with $R \leq |z| = r, -\pi \leq v \leq \pi$. After easy computation, we get

$$\left| \tan h(Re^{iv}) \right|^2 = \frac{\cos h^2(R \cos v) - \cos^2(R \sin v)}{\cos h^2(R \cos v) + \cos^2(R \sin v) - 1} = \Psi(v).
 \tag{14}$$

The equation $\Psi'(v) = 0$ has five roots in $[-\pi, \pi]$, namely, $0, \pm\pi$ and $\pm(\pi/2)$. Since $\Psi(v) = \Psi(-v)$, it is sufficient to show that $v \in [0, \pi]$. Furthermore, we can see that $\Psi(0) = \tan h^2 R = \Psi(\pi)$, $\Psi(\pi/2) = \tan^2 R$, and

$$\max \left\{ \Psi(0), \Psi(\pi), \Psi\left(\frac{\pi}{2}\right) \right\} = \Psi\left(\frac{\pi}{2}\right) = \tan^2 R \leq \tan^2 r.
 \tag{15}$$

Thus, we have

$$\Re e \left(\frac{zf'(z)}{f(z)} \right) \geq 1 - |\tanh(w(z))|, \geq 1 - \tan r \geq \xi,
 \tag{16}$$

whenever $1 - \tan r - \xi \geq 0$. The radius of starlikeness of order ξ , for the family $\mathfrak{S}_{\tan h}^*$, is the smallest positive root $r_0 \in (0, 1)$ of $1 - \tan r - \xi = 0$. \square

Taking $\xi = 0$ in the above Theorem 1, we obtain the following corollary.

Corollary 1. *The \mathfrak{S}^* radius, for the family $\mathfrak{S}_{\tan h}^*$, is $r_0 = \tan^{-1}(1) \approx 0.78$.*

Theorem 2. *The $\mathcal{H}(\xi)$ radius r_0 , for the family $f \in \mathfrak{S}_{\tan h}^*$, is $r_0 = \min \{r_1, r_2\}$, where r_1 is the smallest root of the equation*

$$(1 - \xi - \tan hr \sec^2 r)(1 - r^2)(1 - \tan r) - r \sec^2 r = 0,
 \tag{17}$$

and r_2 is such that $1 - \tan r_2 > 0$.

Proof. If $f \in \mathfrak{S}_{\tan h}^*$, then, a holomorphic function w occurs with $w(0) = 0$ and $|w(z)| \leq |z|$, such that

$$\frac{zf'(z)}{f(z)} = 1 + \tan hw(z).
 \tag{18}$$

By simple computation, it gives

$$1 + \frac{zf''(z)}{f'(z)} = 1 + \tan hw(z) + \frac{zw'(z) \sec h^2 w(z)}{1 + \tan hw(z)}.
 \tag{19}$$

From (18), we get

$$\begin{aligned}
 \Re e \left(1 + \frac{zf''(z)}{f'(z)} \right) &\geq 1 + \Re e(\tan hw(z)) \\
 &\quad - \frac{|z||w'(z)||\sec h^2 w(z)|}{1 - |\tan h(w(z))|}.
 \end{aligned}
 \tag{20}$$

Assume that $w(z) = Re^{iv}$, with $R \leq |z| = r, -\pi \leq v \leq \pi$ for calculating the minimum value of the right side of the last inequality. A simple calculation reveals that

$$\Re \tan h(Re^{iv}) = \frac{\tan h(Rx) \sec^2(Ry)}{1 + \tan h^2(Rx) \tan^2(Ry)}, \quad (21)$$

where $y = \sin v, x = \cos v$, and $x, y \in [-1, 1]$. It is easy to observe that

$$|\sec h^2(e^{iv}R)|^2 = \frac{1}{(\cos h^4(R \cos v) + 2 \cos h^2(R \cos v) \cos^2(R \sin v) + \cos^4(R \sin v) - 2 \cosh^2(R \cos v) - 2 \cos^2(R \sin v) + 1)} =: \phi(v). \quad (24)$$

The equation $\phi'(v) = 0$ attained has five roots in $[-\pi, \pi]$, namely, $0, \pm\pi$ and $\pm(\pi/2)$. Also, $\phi(v) = \phi(-v)$; it is enough to consider only those roots which lie in $[0, \pi]$. Furthermore, we seen that $\phi(0) = \sec h^4 R = \phi(\pi)$, and $\phi(\pi/2) = \sec^4 R$; therefore

$$\max \left\{ \phi(0), \phi(\pi), \phi\left(\frac{\pi}{2}\right) \right\} = \phi\left(\frac{\pi}{2}\right) = \sec^4 R \leq \sec^4 r. \quad (25)$$

Hence,

$$|\sec h^2(e^{iv}R)| \leq \sec^2 R \leq \sec^2 r. \quad (26)$$

Also,

$$|\tan hw(z)| \leq \tan R \leq \tan r. \quad (27)$$

Using the above facts along with the well-known inequality of Schwarz functions w (seeradii19), we have

$$|w'(z)| \leq \frac{1 - |w(z)|^2}{1 - |z|^2} = \frac{1 - R^2}{1 - |z|^2} \leq \frac{1}{1 - |z|^2} \leq \frac{1}{1 - r^2}. \quad (28)$$

Using (19), we obtain

$$\begin{aligned} \Re \left(1 + \frac{zf''(z)}{f'(z)} \right) &\geq 1 + \Re(\tan hw(z)) \\ &\quad - \frac{|\sec h^2 w(z)| |zw'(z)|}{1 - |\tan hw(z)|}, \\ &\geq 1 - \tan hr \sec^2 r \\ &\quad - \frac{(\sec^2 r)r}{(1 - \tan r)(1 - r^2)} \geq \xi. \end{aligned} \quad (29)$$

The last inequality is true if $(1 - \xi - \tan hr \sec^2 r)(1 - r^2) > 0$ with $\tan r < 1$ holds.

$$\begin{aligned} \tan h(Rx) &\geq -\tan hR \geq -\tan hr, \\ 1 &\leq \sec^2(Ry) \leq \sec^2 R \leq \sec^2 r. \end{aligned} \quad (22)$$

Consequently, we have

$$\Re(1 + \tan hw(z)) \geq 1 - \tan hR \sec^2 R \geq 1 - \tan hr \sec^2 r. \quad (23)$$

Now, consider that

Hence, $\mathcal{K}(\xi)$ radius r_0 for the family $\mathfrak{S}_{\tan h}^*$ is the minimum of r_1 and r_2 , where r_1 is the smallest positive root of the equation

$$(1 - \xi - \tan hr \sec^2 r)(1 - r^2)(1 - \tan r) - r \sec^2 r = 0, \quad (30)$$

and r_2 is such that $\tan r_2 < 1$. □

Corollary 2. The \mathcal{K} radius, for the family $\mathfrak{S}_{\tan h}^*$ is $r_0 \approx 0.33286$.

Remark 1. The result in the last Theorem is not the best one. Considering the function f_0 described by (11) provides a sharp result. For the function f_0 , we have

$$\phi(z) = \Re e \frac{(zf'_0(z))'}{f'_0(z)} = \Re e \left(1 + \tan hz + \frac{z \sec h^2 z}{1 + \tan hz} \right), \quad (31)$$

and $\phi(r) = 0$.

3. Radius Problems

To address our main results in this portion, first, we consider a few well-known families as follows.

$$\begin{aligned} \mathcal{P}_n[L, M] &:= \left\{ p(z) = 1 + \sum_{k=n}^{\infty} c_k z^k : p(z) \right. \\ &\quad \left. < \frac{1 + Lz}{1 + Mz}, -1 \leq M < L \leq 1 \right\}. \end{aligned} \quad (32)$$

Also, for $n \in \mathbb{N}$,

$$\begin{aligned} \mathcal{P}_n(\xi) &:= \mathcal{P}_n[1 - 2\xi, -1], \\ \mathcal{P}_n &:= \mathcal{P}_n(0). \end{aligned} \quad (33)$$

If we put $p(z) = (zf'(z))/(f(z))$, for $f \in \mathfrak{A}_n$, then, the family $\mathcal{P}_n[L, M]$ is reduced to $\mathfrak{S}_n^*[L, M]$ and $\mathfrak{S}_n^*(\xi) =: \mathfrak{S}_n^*$ $[1 - 2\xi, -1]$. Let the family $\mathcal{M}(\beta)$ contains the functions $f \in \mathfrak{A}_n$ satisfying that $\text{Re}((zf'(z))/(f(z))) < \beta$, for $\beta > 1$. Furthermore, let

$$\begin{aligned} \mathfrak{S}_{\tan h, n}^* &:= \mathfrak{A}_n \cap \mathfrak{S}_{\tan h}^*, \mathfrak{S}_n^*(\xi) = \mathfrak{A}_n \cap \mathfrak{S}^*(\xi), \mathfrak{S}_{\mathcal{L}, n}^* \\ &= \mathfrak{A}_n \cap \mathfrak{S}_{\mathcal{L}}^*, \\ \mathcal{M}_n(\beta) &= \mathfrak{A}_n \cap \mathcal{M}(\beta). \end{aligned} \tag{34}$$

Ali et al. [21] recently studied the below families

$$\begin{aligned} \mathfrak{S}_n &:= \left\{ f \in \mathfrak{A}_n : \frac{f(z)}{z} \in \mathcal{P}_n \right\}, \\ \mathfrak{G}\mathfrak{S}_n(\xi) &:= \left\{ f \in \mathfrak{A}_n : \frac{f(z)}{g(z)} \in \mathcal{P}_n, g \in \mathfrak{S}_n^*(\xi) \right\} \end{aligned} \tag{35}$$

and calculated $\mathfrak{S}_{\mathcal{L}, n}^*$ radii for certain families. Further, they achieved the conditions on L and M such that $\mathfrak{S}_n^*[L, M] \subset \mathfrak{S}_{\mathcal{L}, n}^*$. In this portion, $\mathfrak{S}_{\tan h, n}^*$ radii for the family of Janowski starlike function and some other geometrically described families are explored. To get our results, we employ the following lemmas.

Lemma 1 [22]. *If $p \in \mathcal{P}_n(\xi)$, then, for $|z| = r$,*

$$\left| \frac{zp'(z)}{p(z)} \right| \leq \frac{2nr^n(1-\xi)}{(1+(1-2\xi)r^n)(1-r^n)}. \tag{36}$$

Lemma 2 [23]. *If $p \in \mathcal{P}_n[L, M]$, then, for $|z| = r$,*

$$\left| p(z) - \frac{1-LMr^{2n}}{1-M^2r^{2n}} \right| \leq \frac{(L-M)r^n}{1-M^2r^{2n}}. \tag{37}$$

In particular, if $p \in \mathcal{P}_n(\xi)$, then, for $|z| = r$,

$$\left| p(z) - \frac{(1+(1-2\xi)r^{2n})}{1-r^{2n}} \right| \leq \frac{2r^n(1-\xi)}{1-r^{2n}}. \tag{38}$$

The aim of the following lemma is at finding the largest and the smallest disks centered at $(a, 0)$ and $(1, 0)$, respectively, such that the domain $\Omega_{\tan h} := q_0(\mathfrak{D})$, where $q_0(z) := 1 + \tan hz$, is contained in the smallest disk and contains the largest disk.

Lemma 3. *Let*

$$1 - \tan h1 \leq a \leq 1 + \tan h1. \tag{39}$$

And $r_a = \tan h1 - |a - 1|$. Then, the following inclusions holds

$$\{w \in \mathbb{C} : |w - a| < r_a\} \subset \Omega_{\tan h} \subset \{w \in \mathbb{C} : |w - 1| < \tan 1\}. \tag{40}$$

Proof. Since $w(z) = Re^{iv}$ with $R \leq |z| = r$, we have

$$1 + \tan hw(z) = \sigma(v) - i\rho(v), \tag{41}$$

with

$$\begin{aligned} \sigma(v) &= 1 + \frac{\tan h(R \cos(v)) \sec^2(R \sin(v))}{1 + \tan h^2(R \cos(v)) \tan^2(R \sin(v))}, \\ \rho(v) &= \frac{\tan h(R \sin(v)) \sec h^2(R \cos(v))}{1 + \tan h^2(R \cos(v)) \tan^2(R \sin(v))}. \end{aligned} \tag{42}$$

First, we consider the square of the distance from $(a, 0)$ to a point on the boundary of $\Omega_{\tan h}$, which is given by

$$\begin{aligned} h(v) = d^2(v) &= \left[a - 1 - \frac{\tan h(\cos(v)) \sec^2(\sin(v))}{1 + \tan h^2(\cos(v)) \tan^2(\sin(v))} \right]^2 \\ &+ \left[\frac{\tan h(\sin(v)) \sec h^2(\cos(v))}{1 + \tan h^2(\cos(v)) \tan^2(\sin(v))} \right]^2. \end{aligned} \tag{43}$$

To show that $|w - a| < r_a$ is the largest disk contained in $\Omega_{\tan h}$, it is sufficient to show that $\min_{-\pi \leq v \leq \pi} d(v) = r_a$. But since $h(v) = h(-v)$, therefore, we consider the range $0 \leq v \leq \pi$. Now, it can easily be obtained that $h'(v) = 0$ has three roots $0, \pi$, and $v_0 \in (0, \pi)$. The root v_0 is dependent on a . The graph of $h(v)$ shows that it is decreasing in $[v_0, \pi]$ and increasing in the interval $[0, v_0]$. Hence, the minimum of $h(v)$ is calculated on either π or 0 . A computation provides

$$\begin{aligned} h(\pi) &:= (a - 1 + \tan h1)^2, \\ h(0) &:= (a - 1 - \tan h1)^2. \end{aligned} \tag{44}$$

Thus, we get

$$\begin{aligned} \min_{-\pi \leq v \leq \pi} h(v) &= \min(h(\pi), h(0)) \\ &= \begin{cases} h(\pi), & \text{if } -\tan h1 \leq a - 1 \leq 0, \\ h(0), & \text{if } 0 \leq a - 1 \leq \tan h1. \end{cases} \end{aligned} \tag{45}$$

Therefore, we can write that

$$\min_{-\pi \leq v \leq \pi} d(v) = \begin{cases} \tan h1 + (a - 1), & \text{if } -\tanh 1 \leq a - 1 \leq 0, \\ \tan h1 - (a - 1), & \text{if } 0 \leq a - 1 \leq \tanh 1, \end{cases} \tag{46}$$

or equivalently

$$\min_{-\pi \leq \nu \leq \lambda} d(\nu) = \tan h1 - |a - 1|. \quad (47)$$

For the circle of the minimum radius centered at $(1, 0)$, which contains $f(\mathfrak{D}) = 1 + \tanh \mathfrak{z}$, we find the maximum distance from $(1, 0)$ to a point on the boundary of $f(\mathfrak{D}) = \Omega_{\tan h}$ and the square of this distance function is given by

$$\phi(\nu) = \frac{\cosh^2(\cos(\nu) - \cos^2(\sin(\nu)))}{\cosh^2(\cos(\nu)) + \cos^2(\cos(\nu)) - 1}. \quad (48)$$

It is easy to verify that $\phi(\nu)$ achieves its maximum value at $\pi/2$, which is $\phi(\pi/2) = \tan^2 1$. Hence, the radius of the smallest disk which contains $\Omega_{\tan h}$ is $\tan 1$. \square

In the following examples, we apply Lemma 3, to find the necessary and sufficient conditions for two specific functions that belong to the family $\mathfrak{S}_{\tan h}^*$.

Example 1.

(a) The function

$$f(\mathfrak{z}) = \mathfrak{z} + d_2 \mathfrak{z}^2 \in \mathfrak{S}_{\tan h}^*, \quad (49)$$

if and only if

$$|d_2| \leq \frac{\tan h1}{1 + \tan h1} \approx 0.43233 \quad (50)$$

(b) The function

$$f(\mathfrak{z}) = \frac{\mathfrak{z}}{(1 - b\mathfrak{z})^2} \in \mathfrak{S}_{\tan h}^*, \quad (51)$$

if and only if

$$|b| \leq \frac{\tan h1}{2 + \tan h1} \approx 0.27578 \quad (52)$$

Proof.

(a) We know that $f(\mathfrak{z}) = \mathfrak{z} + d_2 \mathfrak{z}^2 \in \mathfrak{S}^*$, if and only if $|d_2| \leq (1/2)$. Since $\mathfrak{S}_{\tan h}^* \subset \mathfrak{S}^*$, we get $|d_2| \leq (1/2)$, whenever $f \in \mathfrak{S}_{\tan h}^*$. The function

$$w(\mathfrak{z}) = \frac{\mathfrak{z}f'(\mathfrak{z})}{f(\mathfrak{z})} = \frac{1 + 2d_2\mathfrak{z}}{1 + d_2\mathfrak{z}}, \quad (53)$$

maps \mathfrak{D} onto the disk

$$\left| w - \frac{1 - 2|d_2|^2}{1 - |d_2|^2} \right| < \frac{|d_2|}{1 - |d_2|^2}. \quad (54)$$

Since

$$\frac{1 - 2|d_2|^2}{1 - |d_2|^2} \leq 1, \quad (55)$$

then, from Lemma 3, the above disk will be contained in $\Omega_{\tan h}$ if

$$1 - \tan h1 \leq \frac{1 - 2|d_2|^2}{1 - |d_2|^2}, \quad (56)$$

$$\frac{|d_2|}{1 - |d_2|^2} \leq \frac{1 - 2|d_2|^2}{1 - |d_2|^2} - 1 + \tan h1.$$

The above two inequalities give

$$|d_2| \leq \sqrt{\frac{\tan h1}{1 + \tan h1}}, \quad (57)$$

$$|d_2| \leq \frac{\tan h1}{1 + \tan h1},$$

respectively. Thus, we have

$$|d_2| \leq \min \left\{ \frac{\tan h1}{1 + \tan h1}; \sqrt{\frac{\tan h1}{1 + \tan h1}} \right\} = \frac{\tan h1}{1 + \tan h1} \quad (58)$$

(b) Logarithmic differentiation of the function

$$f(\mathfrak{z}) = \frac{\mathfrak{z}}{(1 - b\mathfrak{z})^2}, \quad (59)$$

yields that

$$w(\mathfrak{z}) = \frac{\mathfrak{z}f'(\mathfrak{z})}{f(\mathfrak{z})} = \frac{(1 + b\mathfrak{z})}{(1 - b\mathfrak{z})}, \quad (60)$$

maps \mathfrak{D} onto the disk

$$\left| w(\mathfrak{z}) - \frac{1 + |b|^2}{1 - |b|^2} \right| \leq \frac{2|b|}{1 - |b|^2}, \quad (61)$$

since

$$1 \leq \frac{1 + |b|^2}{1 - |b|^2}. \quad (62)$$

The disk above is contained in $\Omega_{\tan h}$, in Lemma 3, whenever

$$\frac{1 + |b|^2}{1 - |b|^2} \leq 1 + \tan h, \tag{63}$$

$$\frac{2|b|}{1 - |b|^2} \leq 1 + \tan h - \frac{1 + |b|^2}{1 - |b|^2}.$$

The above two inequalities give

$$|b| \leq \sqrt{\frac{\tan h}{2 + \tan h}}, \tag{64}$$

$$|b| \leq \frac{\tan h}{2 + \tan h},$$

respectively. Thus, we have

$$|b| \leq \min \left\{ \sqrt{\frac{\tan h}{2 + \tan h}}; \frac{\tan h}{2 + \tan h} \right\} = \frac{\tan h}{2 + \tan h}. \tag{65}$$

This completes the required proof. \square

Theorem 3. The sharp $\mathfrak{S}_{\tan h, n}^*$ radius for the family \mathfrak{S}_n is given by

$$\mathcal{R}_{\mathfrak{S}_{\tan h, n}^*}(\mathfrak{S}_n) = \left(\frac{\tan h}{\sqrt{n^2 + \tan h^2} + n} \right)^{1/n}. \tag{66}$$

Proof. Suppose that $f \in \mathfrak{S}_n$. Consider the function $h : \mathfrak{D} \rightarrow \mathbb{C}$ described by

$$h(z) = \frac{f(z)}{z}. \tag{67}$$

Using logarithmic differentiation, we get

$$\frac{zf'(z)}{f(z)} - 1 = \frac{zh'(z)}{h(z)}. \tag{68}$$

Implementing Lemma 1, we have

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| = \left| \frac{zh'(z)}{h(z)} \right| \leq \frac{2nr^n}{1 - r^{2n}}. \tag{69}$$

According to Lemma 3, if the following inequality holds, the image of $|z| \leq r$ under the function $(zf'(z))/(f(z))$ lies on disk $\Omega_{\tan h}$:

$$\frac{2nr^n}{1 - r^{2n}} \leq \tan h, \tag{70}$$

or equivalently

$$(\tan h)r^{2n} + 2nr^n - \tan h \leq 0. \tag{71}$$

Thus, $\mathfrak{S}_{\tan h, n}^*$ radius of \mathfrak{S}_n is the smallest positive root of

$$(\tan h)r^{2n} + 2nr^n - \tan h = 0, \tag{72}$$

in (0,1). Assume the function

$$f_0(z) = \frac{z(1 + z^n)}{(1 - z^n)}. \tag{73}$$

Then, it is clear to see that $\Re((f_0(z))/z) > 0$ in the unit disk \mathfrak{D} . Hence, $f_0 \in \mathfrak{S}_n$ and

$$\frac{zf_0'(z)}{f_0(z)} = 1 + \frac{2nz^n}{1 - z^{2n}}. \tag{74}$$

Further, f_0 assures the sharpness of the results since at $z = \mathcal{R}_{\mathfrak{S}_{\tan h, n}^*}(\mathfrak{S}_n)$, we obtain

$$\frac{zf_0'(z)}{f_0(z)} - 1 = \frac{2nz^n}{1 - z^{2n}} = \tan h. \tag{75}$$

This completes the proof. \square

Theorem 4. The sharp $\mathfrak{CS}_{\tan h, n}^*$ radius for the family $\mathfrak{CS}_n(\xi)$ is given by

$$\mathcal{R}_{\mathfrak{CS}_{\tan h, n}^*}(\mathfrak{CS}_n(\xi)) = \left(\frac{\tan h}{(n - \xi + 1) + \sqrt{[\tan h + 2(1 - \xi)] \tan h + (1 + n - \xi)^2}} \right)^{1/n}. \tag{76}$$

Proof. Let $f \in \mathfrak{CS}_n(\xi)$ and describe a function

$$h(z) = \frac{f(z)}{g(z)}, \tag{77}$$

where $g \in \mathfrak{S}_n^*(\xi)$. Then, $h \in \mathfrak{P}_n$. According to the definition of h , we get

$$\frac{zf'(z)}{f(z)} = \frac{zh'(z)}{h(z)} + \frac{zg'(z)}{g(z)}. \tag{78}$$

Utilizing Lemma 1 and Lemma 2, we conclude that

$$\left| \frac{\mathfrak{z}f'(\mathfrak{z})}{f(\mathfrak{z})} - \frac{1 + (1 - 2\xi)r^{2n}}{1 - r^{2n}} \right| \leq \frac{2(1 + n - \xi)r^n}{1 - r^{2n}}. \quad (79)$$

Since

$$\frac{1 + (1 - 2\xi)r^{2n}}{1 - r^{2n}} \geq 1, \quad (80)$$

it follows from Lemma 3 and (78) that the function $f \in \mathfrak{C}_{\tan h, n}^*$ if the following holds:

$$\frac{1 + (1 - 2\xi)r^{2n} + 2(1 + n - \xi)r^n}{1 - r^{2n}} \leq 1 + \tan h1, \quad (81)$$

or equivalently, the inequality

$$(2 - 2\xi + \tan h1)r^{2n} + 2(1 + n - \xi)r^n - \tan h1 \leq 0 \quad (82)$$

holds. Thus, the $\mathfrak{C}_{\tan h, n}^*$ radius for the class $\mathfrak{C}\mathfrak{S}_n(\xi)$ is the smallest positive root of

$$2(1 + n - \xi)r^n + (2 - 2\xi + \tan h1)r^{2n} - \tan h1 = 0. \quad (83)$$

Now, assume the functions described by

$$\begin{aligned} f_0(\mathfrak{z}) &= \frac{\mathfrak{z}(1 + \mathfrak{z}^n)}{(1 - \mathfrak{z}^n)^{(n+2-2\xi)/n}}, \\ g_0(\mathfrak{z}) &= \frac{\mathfrak{z}}{(1 - \mathfrak{z}^n)^{(2(1-\xi))/n}}. \end{aligned} \quad (84)$$

Then, we get

$$\frac{f_0(\mathfrak{z})}{g_0(\mathfrak{z})} = \frac{(1 + \mathfrak{z}^n)}{(1 - \mathfrak{z}^n)} \text{ and } \frac{zg'_0(\mathfrak{z})}{g_0(\mathfrak{z})} = \frac{(1 + (1 - 2\xi)z^n)}{(1 - \mathfrak{z}^n)}. \quad (85)$$

Furthermore, it is obvious that

$$\begin{aligned} \Re\left(\frac{f_0(\mathfrak{z})}{g_0(\mathfrak{z})}\right) &> 0, \\ \Re\left(\frac{\mathfrak{z}g'_0(\mathfrak{z})}{g_0(\mathfrak{z})}\right) &> \xi, \end{aligned} \quad (86)$$

in the unit disk \mathfrak{D} . Therefore, $f_0 \in \mathfrak{C}\mathfrak{S}_n(\xi)$. The function f_0 described in (83), at $\mathfrak{z} = \mathcal{R}_{\mathfrak{C}_{\tan h, n}^*}(\mathfrak{C}\mathfrak{S}_n(\xi))$ satisfies that

$$\frac{\mathfrak{z}f'_0(\mathfrak{z})}{f_0(\mathfrak{z})} = \frac{1 + (1 - 2\xi)\mathfrak{z}^{2n} + 2(1 + n - \xi)\mathfrak{z}^n}{1 - \mathfrak{z}^{2n}} = 1 + \tan h(1). \quad (87)$$

Hence, the verified result is sharp. \square

Theorem 5. The $\mathfrak{C}_{\tan h, n}^*$ radius for the family $\mathfrak{C}_n^*[L, M]$ is given by

$$\mathcal{R}_{\mathfrak{C}_{\tan h, n}^*}(\mathfrak{C}_n^*[L, M]) = \left\{ \begin{array}{l} \min(1 : r_1), -1 \leq M \leq 0 < L \leq 1 \\ \min(1 : r_2), 0 < M < L \leq 1 \end{array} \right\}, \quad (88)$$

where

$$\begin{aligned} r_1 &= \left(\frac{2 \tan h1}{(L - M) + \sqrt{(L - M)^2 + 4[M^2(1 + \tan h1) - LM] \tan h1}} \right)^{1/n}, \\ r_2 &= \left(\frac{2 \tanh 1}{(L - M) + \sqrt{(L - M)^2 + 4[M^2(\tanh 1 - 1) + LM] \tanh 1}} \right)^{1/n}. \end{aligned} \quad (89)$$

Proof. Let $f \in \mathfrak{C}_n^*[L, M]$. Then, by Lemma 2, we get

$$\left| \frac{\mathfrak{z}f'(\mathfrak{z})}{f(\mathfrak{z})} - b \right| \leq \frac{(L - M)r^n}{1 - M^2r^{2n}}, \quad (90)$$

where center of the disk is $b = (1 - LMr^{2n})/(1 - M^2r^{2n})$, $|z| = r$. Applying Lemma 3, it is easy to see that $b \geq 1$ for $M < 0$ and we achieved

$$\frac{1 - LMr^{2n} + (L - M)r^n}{1 - M^2r^{2n}} \leq 1 + \tan h1. \quad (91)$$

After some simple calculation, we have

$$r \leq \left(\frac{2 \tan h1}{(L - M) + \sqrt{(L - M)^2 + 4[M^2(1 + \tan h1) - LM] \tan h1}} \right)^{1/n} = r_1. \quad (92)$$

In addition, if $M = 0$ for $b = 1$ and from (89), we get

$$\left| \frac{\mathfrak{z}f'(\mathfrak{z})}{f(\mathfrak{z})} - 1 \right| \leq Lr^n \quad (0 < L \leq 1). \quad (93)$$

Implementing Lemma 3 with $a = 1$ leads to $f \in \mathfrak{C}_{\tan h, n}^*$, if

$$r \leq \left(\frac{\tanh 1}{L} \right)^{1/n}. \quad (94)$$

For $0 < M < L \leq 1$, we get $b < 1$. Thus, from (89) and Lemma 3, we see that $f \in \mathfrak{C}_{\tan h, n}^*$, if the following holds:

$$\frac{(L - M)r^n + LMr^{2n} - 1}{1 - M^2r^{2n}} \leq b + \tan h1 - 1, \quad (95)$$

or equivalently, if

$$r \leq \left(\frac{2 \tan h1}{(L - M) + \sqrt{(L - M)^2 + 4[M^2(\tan h1 - 1) + LM] \tan h1}} \right)^{1/n} = r_2. \tag{96}$$

This completes the proof. \square

Theorem 6. Let $-1 < M < L \leq 1$. If either

(a) $L - 1 \leq (1 - M)(\tanh 1 - 1)$ and $(1 - \tanh 1)(1 - M^2) \leq L - M \leq 1 - M^2$ or

(b) $L + 1 \leq (1 + M)(\tanh 1 + 1)$ and $1 - M^2 \leq L - M \leq 1 + \tanh 1$

holds, then, $\mathfrak{C}_n^*[L, M] \subset \mathfrak{C}_{\tan h, n}^*$.

Proof. Let $p(z) = (zf'(z))/(f(z))$. Since $f \in \mathfrak{C}_n^*[L, M]$, using Lemma 2, we get

$$\left| p(z) - \frac{1 - LM}{1 - M^2} \right| \leq \frac{L - M}{1 - M^2}. \tag{97}$$

Therefore, either $1 - LM/1 - M^2 \leq 1$ or $1 - LM/1 - M^2 \geq 1$.

For $(1 - LM)/(1 - M^2) \leq 1$, using Lemma 3, we see that $f \in \mathfrak{C}_{\tan h, n}^*$, if the following holds:

$$\begin{aligned} \frac{L - M}{1 - M^2} &\leq \frac{1 - LM}{1 - M^2} - (1 - \tan h1), \\ 1 - \tan h1 &\leq \frac{L - M}{1 - M^2} \leq 1, \end{aligned} \tag{98}$$

which, upon simplification, reduces to the condition stated in (a).

For $(1 - LM)/(1 - M^2) \geq 1$, again, applying Lemma 3, we see that $f \in \mathfrak{C}_{\tan h, n}^*$, if the following holds:

$$\begin{aligned} \frac{L - M}{1 - M^2} &\leq (1 + \tan h1) - \frac{1 - LM}{1 - M^2}, \\ 1 &\leq \frac{L - M}{1 - M^2} \leq (1 + \tan h1), \end{aligned} \tag{99}$$

which, upon simplification, reduces to the condition stated in (b). \square

Theorem 7. The sharp $\mathfrak{C}_{\tan h}^*$ radii for the families $\mathfrak{C}_{\mathcal{L}}^*$, $\mathfrak{C}_{\mathcal{RL}}^*$, \mathfrak{C}_{car}^* , $\mathfrak{C}_{\mathcal{E}}^*$, $\mathcal{B}\mathfrak{C}^*(\xi)$, $\mathcal{M}(\beta)$, $\mathfrak{C}_{\mathcal{E}\mathcal{F}}^*$, $\mathfrak{C}\mathfrak{C}^*(\xi)$, $\mathfrak{C}_{\mathcal{B}}^*$, and $\mathfrak{C}_{\mathcal{F}}^*(u)$ are

$$\mathcal{R}_{\mathfrak{C}_{\mathcal{L}}^*} = (2 - \tan h1) \tan h1 \approx 0.944,$$

$$\mathcal{R}_{\mathfrak{C}_{\mathcal{RL}}^*} = \frac{(2 + (1 + \sqrt{2}) \tan h1) \tan h1}{6 - 3\sqrt{2} + 4(\sqrt{2} - 1) \tan h1 - \sec^2 h1} \approx 0.992,$$

$$\mathcal{R}_{\mathfrak{C}_{car}^*} = \frac{1}{2} (\sqrt{2(2 + 3 \tan h1)} - 2) \approx 0.463,$$

$$\mathcal{R}_{\mathfrak{C}_{\mathcal{E}}^*} = \frac{\tan h1(2 + \tan h1)}{2(1 + \tan h1)} \approx 0.217,$$

$$\mathcal{R}_{\mathcal{B}\mathfrak{C}^*(\xi)} = \frac{2 \tan h1}{1 + \sqrt{1 + 4\xi \tan h1}}, \quad \text{for } 0 \leq \xi < 1,$$

$$\mathcal{R}_{\mathcal{M}(\beta)} = \frac{\tan h1}{2(\beta - 1) + \tan h1}, \quad \beta > 1,$$

$$\mathcal{R}_{\mathfrak{C}_{\mathcal{E}\mathcal{F}}^*} = \sqrt{1 + 2 \tan h1} - 1 \approx 0.589,$$

$$\mathcal{R}_{\mathfrak{C}\mathfrak{C}^*(\xi)} = \frac{(1 + \tan h1)^{1/\xi} - 1}{(1 + \tan h1)^{1/\xi} + 1}, \quad (0 \leq \xi \leq 1),$$

$$\mathcal{R}_{\mathfrak{C}_{\mathcal{B}}^*} = \ln [1 + \ln (1 + \tan h1)] \approx 0.449,$$

$$\mathcal{R}_{\mathfrak{C}_{\mathcal{F}}^*(u)} = \frac{\sqrt{1 + 2u \ln (1 + \tan h1)} - 1}{u}. \tag{100}$$

Proof.

(1) Let $f \in \mathfrak{C}_{\mathcal{L}}^*$, then,

$$\frac{zf'(z)}{f(z)} < \sqrt{1 + z}. \tag{101}$$

Thus, for $|z| = r$, we have

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq 1 - \sqrt{1 - r} \leq \tan h1. \tag{102}$$

For

$$r \leq (2 - \tan h1) \tan h1 = \mathcal{R}_{\mathfrak{C}_{\mathcal{L}}^*}. \tag{103}$$

For checking the sharpness of the result, we assume the function f_0 described by

$$f_0(z) = \frac{4z \exp \{2(\sqrt{z+1} - 1)\}}{(1 + \sqrt{1+z})^2}. \tag{104}$$

Since

$$\frac{zf_0'(z)}{f_0(z)} = \sqrt{1 + z}, \tag{105}$$

it follows that $f_0 \in \mathfrak{C}_{\mathcal{L}}^*$ and at $\mathfrak{z} = -\mathcal{R}_{\mathfrak{E}_{\tan h}^*}(\mathfrak{C}_{\mathcal{L}}^*)$ and we see that

$$\frac{\mathfrak{z}f'_0(\mathfrak{z})}{f_0(\mathfrak{z})} - 1 = -\tan h1, \quad (106)$$

and hence, the result is sharp

(2) For function $f \in \mathfrak{C}_{\mathcal{R}\mathcal{L}}^*$, then,

$$\frac{\mathfrak{z}f'(\mathfrak{z})}{f(\mathfrak{z})} < \sqrt{2} - (\sqrt{2} - 1) \sqrt{\frac{1 - \mathfrak{z}}{1 + 2(\sqrt{2} - 1)\mathfrak{z}}}. \quad (107)$$

Thus, for $|\mathfrak{z}| = r$, we get

$$\begin{aligned} \left| \frac{\mathfrak{z}f'(\mathfrak{z})}{f(\mathfrak{z})} - 1 \right| &\leq \left| \sqrt{2} - (\sqrt{2} - 1) \sqrt{\frac{1 - \mathfrak{z}}{1 + 2(\sqrt{2} - 1)\mathfrak{z}}} - 1 \right| \\ &\leq 1 - \sqrt{2} - (\sqrt{2} - 1) \sqrt{\frac{1 + r}{1 - 2(\sqrt{2} - 1)r}} \\ &\leq \tan h1. \end{aligned} \quad (108)$$

For

$$r \leq \frac{(2 + (1 + \sqrt{2}) \tan h1) \tan h1}{6 - 3\sqrt{2} + 4(\sqrt{2} - 1) \tan h1 - \sec h^2} = \mathcal{R}_{\mathfrak{E}_{\tan h}^*}(\mathfrak{C}_{\mathcal{R}\mathcal{L}}^*). \quad (109)$$

For checking the sharpness, assume the function f_0 described by

$$f_0(\mathfrak{z}) = \mathfrak{z} \exp \left(\int_0^{\mathfrak{z}} \frac{q_0(t) - 1}{t} dt \right), \quad (110)$$

where

$$q_0(\mathfrak{z}) = \sqrt{2} - (\sqrt{2} - 1) \sqrt{\frac{1 - \mathfrak{z}}{(1 + 2(\sqrt{2} - 1)\mathfrak{z})}}. \quad (111)$$

Since $q_0(\mathfrak{z}) = (\mathfrak{z}f'_0(\mathfrak{z}))/f_0(\mathfrak{z})$ and from the definition of f_0 at $\mathfrak{z} = -\mathcal{R}_{\mathfrak{E}_{\tan h}^*}(\mathfrak{C}_{\mathcal{R}\mathcal{L}}^*)$, we have

$$\frac{\mathfrak{z}f'_0(\mathfrak{z})}{f_0(\mathfrak{z})} = \sqrt{2} - (\sqrt{2} - 1) \sqrt{\frac{1 - \mathfrak{z}}{(1 + 2(\sqrt{2} - 1)\mathfrak{z})}} = -\tan h1, \quad (112)$$

and hence, the sharpness of the result is verified

(3) Let $f \in \mathfrak{C}_{\text{car}}^*$, then,

$$\frac{\mathfrak{z}f'(\mathfrak{z})}{f(\mathfrak{z})} < 1 + \frac{4\mathfrak{z}}{3} + \frac{2\mathfrak{z}^2}{3}. \quad (113)$$

Therefore, for $|\mathfrak{z}| = r$, we get

$$\left| \frac{\mathfrak{z}f'(\mathfrak{z})}{f(\mathfrak{z})} - 1 \right| = \left| 1 + \frac{4\mathfrak{z}}{3} + \frac{2\mathfrak{z}^2}{3} - 1 \right| \leq \frac{4r}{3} + \frac{2r^2}{3} \leq \tan h1. \quad (114)$$

For

$$r \leq \frac{\sqrt{2(2 + 3 \tan h1)} - 2}{2}. \quad (115)$$

For checking the sharpness, assume the function f_0 described by

$$f_0(\mathfrak{z}) = \mathfrak{z} \exp \left\{ \frac{4\mathfrak{z} + \mathfrak{z}^2}{3} \right\}. \quad (116)$$

Since

$$\frac{\mathfrak{z}f'_0(\mathfrak{z})}{f_0(\mathfrak{z})} = 1 + \frac{4\mathfrak{z}}{3} + \frac{2\mathfrak{z}^2}{3}, \quad (117)$$

it follows that $f_0 \in \mathfrak{C}_{\text{car}}^*$ and at $\mathfrak{z} = \mathcal{R}_{\mathfrak{E}_{\tan h}^*}(\mathfrak{C}_{\text{car}}^*)$ and we get

$$\frac{\mathfrak{z}f'_0(\mathfrak{z})}{f_0(\mathfrak{z})} - 1 = \tan h1. \quad (118)$$

Hence, the result is sharp

(4) Let $f \in \mathfrak{C}_{\mathfrak{E}}^*$. Then,

$$\frac{\mathfrak{z}f'(\mathfrak{z})}{f(\mathfrak{z})} < \mathfrak{z} + \sqrt{1 + \mathfrak{z}^2}. \quad (119)$$

Thus, for $|\mathfrak{z}| = r$, we get

$$\left| \frac{\mathfrak{z}f'(\mathfrak{z})}{f(\mathfrak{z})} - 1 \right| \leq r + \sqrt{1 + r^2} - 1 \leq \tan h1, \quad (120)$$

for

$$r \leq \frac{\tan h1(2 + \tan h1)}{2(1 + \tan h1)}. \quad (121)$$

For checking the sharpness of the result, consider the function f_0 defined by

$$f_0(z) = z \exp \left(\int_0^z \frac{q_0(t) - 1}{t} dt \right). \tag{122}$$

Since

$$q(z_0) = \frac{zf'_0(z)}{f_0(z)} = z + \sqrt{1 + z^2}, \tag{123}$$

it follows that $f_0 \in \mathfrak{C}_{\mathcal{E}}^*$ and at $z = \mathcal{R}_{\mathfrak{E}_{\tanh}^*}(\mathfrak{C}_{\mathcal{E}}^*)$, we have

$$\frac{zf'_0(z)}{f_0(z)} - 1 = \tan h1. \tag{124}$$

Hence, the result is sharp

(5) For function $f \in \mathcal{B}\mathfrak{C}^*(\xi)$, we have

$$\frac{zf'(z)}{f(z)} < 1 + \frac{z}{(1 - \xi z^2)}, \quad (0 \leq \xi < 1). \tag{125}$$

Therefore, for $|z| = r$, we have

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \frac{r}{(1 - \xi r^2)} \leq \tan h1. \tag{126}$$

For $0 \leq \xi < 1$, we obtain

$$r \leq \frac{2 \tan h1}{1 + \sqrt{1 + 4\xi \tan h^2 1}} = \mathcal{R}_{\mathfrak{E}_{\tanh}^*}(\mathcal{B}\mathfrak{C}^*(\xi)). \tag{127}$$

For checking the sharpness of the result, we assume the function

$$f_0(z) = z \exp \left(\int_0^z \frac{q_0(t) - 1}{t} dt \right), \tag{128}$$

where

$$q_0(z) = 1 + \frac{z}{(1 - \xi z^2)}. \tag{129}$$

Since $q_0(z) = (zf'_0(z))/(f_0(z))$, it follows that $f_0 \in \mathcal{B}\mathfrak{C}^*(\xi)$ and at $z = \mathcal{R}_{\mathfrak{E}_{\tanh}^*}(\mathcal{B}\mathfrak{C}^*(\xi))$ and we have

$$\frac{zf'_0(z)}{f_0(z)} - 1 = \tan h1. \tag{130}$$

Hence, the verified result is sharp

(6) Let $f \in \mathcal{M}(\beta)$. Then, by Lemma 2, for $n = 1$, we have

$$\left| \frac{zf'(z)}{f(z)} - \frac{1 + (1 - 2\beta)^2}{1 - r^2} \right| \leq \frac{2(\beta - 1)r}{1 - r^2}. \tag{131}$$

Obviously,

$$\left| \frac{1 + (1 - 2\beta)r^2}{1 - r^2} \right| \leq 1. \tag{132}$$

Hence, by Lemma 3, the above disk contains in $\Omega_{\tan h}$, so

$$\begin{aligned} 1 - \tan h1 &\leq \frac{1 + (1 - 2\beta)r^2}{1 - r^2}, \\ \frac{2(\beta - 1)r}{1 - r^2} &\leq \frac{1 + (1 - 2\beta)r^2}{1 - r^2} - 1 + \tan h1. \end{aligned} \tag{133}$$

Simple calculation gives

$$r \leq \frac{\tan h1}{2(\beta - 1) + \tan h1}. \tag{134}$$

For checking the sharpness, assume the function defined as

$$f_0(z) = z \exp \left(\int_0^z \frac{q_0(t) - 1}{t} dt \right). \tag{135}$$

Since

$$\frac{zf'_0(z)}{f_0(z)} = 1 + 2(\beta - 1)z + [2(\beta - 1) + \tanh 1]z^2, \tag{136}$$

it follows that $f_0 \in \mathcal{M}(\beta)$, at $z = \mathcal{R}_{\mathfrak{E}_{\tanh}^*}(\mathcal{M}(\beta))$, we get

$$\frac{zf'_0(z)}{f_0(z)} - 1 = \tanh 1. \tag{137}$$

Hence, this verified that the result is sharp.

(7) Let $f \in \mathfrak{C}_{\mathcal{E}\mathcal{F}}^*$. Then

$$\frac{zf'(z)}{f(z)} < 1 + z + \frac{z^2}{2}. \tag{138}$$

Therefore, for $|z| = r$, it gives

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq r + \frac{r^2}{2} \leq \tanh 1. \tag{139}$$

For

$$r \leq \sqrt{1 + 2 \tan h1} - 1 = \mathcal{R}_{\mathfrak{E}_{\tanh}^*}(\mathfrak{C}_{\mathcal{E}\mathcal{F}}^*). \tag{140}$$

For checking the sharpness, assume the function defined as

$$f_0(\mathfrak{z}) = \mathfrak{z}e^{(\mathfrak{z}+(\mathfrak{z}^2/4))}. \quad (141)$$

Since

$$\frac{\mathfrak{z}f'_0(\mathfrak{z})}{f_0(\mathfrak{z})} = 1 + \mathfrak{z} + \frac{\mathfrak{z}^2}{2}, \quad (142)$$

it follows that $f_0 \in \mathfrak{E}_{\mathfrak{E}\mathcal{F}}^*$ and at $\mathfrak{z} = \mathcal{R}_{\mathfrak{E}_{\tan h}}^*(\mathfrak{E}_{\mathfrak{E}\mathcal{F}}^*)$ and we have

$$\frac{\mathfrak{z}f'_0(\mathfrak{z})}{f_0(\mathfrak{z})} - 1 = \tanh 1. \quad (143)$$

This shows that the result is sharp

(8) Supposing that $f \in \mathfrak{E}\mathfrak{E}^*(\zeta)$, we have

$$\frac{\mathfrak{z}f'(\mathfrak{z})}{f(\mathfrak{z})} < \left(\frac{(1+\mathfrak{z})}{(1-\mathfrak{z})}\right)^\zeta, \quad (0 < \zeta \leq 1). \quad (144)$$

Thus, for $|\mathfrak{z}| = r$, we get

$$\left|\frac{\mathfrak{z}f'(\mathfrak{z})}{f(\mathfrak{z})} - 1\right| \leq \left(\frac{(1+r)}{(1-r)}\right)^\zeta - 1 \leq \tan h1. \quad (145)$$

For

$$r \leq \frac{(1 + \tan h1)^{1/\zeta} - 1}{(1 + \tan h1)^{1/\zeta} + 1} = \mathcal{R}_{\mathfrak{E}_{\tan h}}^*(\mathfrak{E}\mathfrak{E}^*(\zeta)). \quad (146)$$

For checking the sharpness, assume the function described as

$$f_0(\mathfrak{z}) := \mathfrak{z} \exp\left(\int_0^{\mathfrak{z}} \frac{(q_0(t) - 1)}{t} dt\right), \quad (147)$$

where

$$q_0(\mathfrak{z}) = \left(\frac{(1+\mathfrak{z})}{(1-\mathfrak{z})}\right)^\zeta. \quad (148)$$

Since $q_0(\mathfrak{z}) = (\mathfrak{z}f'_0(\mathfrak{z}))/f_0(\mathfrak{z})$, it follows that $f_0 \in \mathfrak{E}\mathfrak{E}^*(\zeta)$ and at $\mathfrak{z} = \mathcal{R}_{\mathfrak{E}_{\tan h}}^*(\mathfrak{E}\mathfrak{E}^*(\zeta))$ and we have

$$\frac{\mathfrak{z}f'_0(\mathfrak{z})}{f_0(\mathfrak{z})} - 1 = \tan h1. \quad (149)$$

Hence, this showed that the result is sharp

(9) Supposing that $f \in \mathfrak{E}_{\mathfrak{E}}^*$, then,

$$\frac{\mathfrak{z}f'(\mathfrak{z})}{f(\mathfrak{z})} < e^{e^{\mathfrak{z}}-1}. \quad (150)$$

Thus, for $|\mathfrak{z}| = r$, we have

$$\left|\frac{\mathfrak{z}f'(\mathfrak{z})}{f(\mathfrak{z})} - 1\right| \leq e^{e^r-1} - 1 \leq \tan h1. \quad (151)$$

For

$$r \leq \ln [1 + \ln (1 + \tan h1)]. \quad (152)$$

To show the sharpness of the result, we assume the function described by

$$f_0(\mathfrak{z}) = \mathfrak{z} \exp\left(\int_0^{\mathfrak{z}} \frac{q_0(t) - 1}{t} dt\right), \quad (153)$$

where

$$q_0(\mathfrak{z}) = e^{\mathfrak{z}} - 1 = \frac{\mathfrak{z}f'_0(\mathfrak{z})}{f_0(\mathfrak{z})}. \quad (154)$$

Since $(\mathfrak{z}f'_0(\mathfrak{z}))/f_0(\mathfrak{z}) = q_0(\mathfrak{z})$, it follows that $f_0 \in \mathfrak{E}_{\mathfrak{E}}^*$ and $\mathfrak{z} = \mathcal{R}_{\mathfrak{E}_{\tan h}}^*(\mathfrak{E}_{\mathfrak{E}}^*)$ and we have

$$\frac{\mathfrak{z}f'_0(\mathfrak{z})}{f_0(\mathfrak{z})} - 1 = \tan h1, \quad (155)$$

and hence, the sharpness of the result is verified.

(10) Let $f \in \mathfrak{E}_{\mathfrak{E}}^*(u)$. Then,

$$\frac{\mathfrak{z}f'(\mathfrak{z})}{f(\mathfrak{z})} < e^{\mathfrak{z}+u\mathfrak{z}^2/2}, \quad (u \geq 1). \quad (156)$$

Thus, for $|\mathfrak{z}| = r$, we easily get

$$\left|\frac{\mathfrak{z}f'(\mathfrak{z})}{f(\mathfrak{z})} - 1\right| \leq e^{r+ur^2/2} \leq \tan h1. \quad (157)$$

For

$$r \leq \frac{\sqrt{1 + 2u \ln (1 + \tan h1)} - 1}{u} = \mathcal{R}_{\mathfrak{E}_{\tan h}}^*(\mathfrak{E}_{\mathfrak{E}}^*(u)). \quad (158)$$

Now, we choose the following function to confirm its sharpness

$$f_0(\mathfrak{z}) = \mathfrak{z} \exp\left(\int_0^{\mathfrak{z}} \frac{q_0(t) - 1}{t} dt\right). \quad (159)$$

Since

$$\frac{\mathfrak{z}f'_0(\mathfrak{z})}{f_0(\mathfrak{z})} = q_0(\mathfrak{z}) = e^{\mathfrak{z}+u\mathfrak{z}^2/2}, \tag{160}$$

it follows that $f_0 \in \mathfrak{E}^*_\mathcal{F}(u)$ and at $\mathfrak{z} = \mathcal{R}_{\mathfrak{E}^*_{\tan h}}(\mathfrak{E}^*_\mathcal{F}(u))$ and we have

$$\frac{\mathfrak{z}f'_0(\mathfrak{z})}{f_0(\mathfrak{z})} - 1 = \tan h1. \tag{161}$$

This result is sharp. □

4. Functions Defined in terms of the Ratio of Functions

Now, for the following families, we will talk about the radius problem. For brevity, we shall denote them by

$$\begin{aligned} \mathcal{F}_1 &= \left\{ f \in \mathfrak{A}_n : \Re\left(\frac{f(\mathfrak{z})}{g(\mathfrak{z})}\right) > 0 \text{ and } \Re\left(\frac{g(\mathfrak{z})}{\mathfrak{z}}\right) > 0, g \in \mathfrak{A}_n \right\}, \\ \mathcal{F}_2 &= \left\{ f \in \mathfrak{A}_n : \Re\left(\frac{f(\mathfrak{z})}{g(\mathfrak{z})}\right) > 0 \text{ and } \Re\left(\frac{g(\mathfrak{z})}{\mathfrak{z}}\right) > \frac{1}{2}, g \in \mathfrak{A}_n \right\}, \\ \mathcal{F}_3 &= \left\{ f \in \mathfrak{A}_n : \left| \frac{f(\mathfrak{z})}{g(\mathfrak{z})} - 1 \right| < 1 \text{ and } \Re\left(\frac{g(\mathfrak{z})}{\mathfrak{z}}\right) > 0, g \in \mathfrak{A}_n \right\}. \end{aligned} \tag{162}$$

Theorem 8. *The sharp $\mathfrak{E}^*_{\tan h,n}$ radii for function in the families $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$, respectively, are*

$$\begin{aligned} \mathcal{R}_{\mathfrak{E}^*_{\tan h,n}}(\mathcal{F}_1) &= \left(\sqrt{1 + 4n^2 \cot h^2 1 - 2n \cot h1} \right)^{1/n}, \\ \mathcal{R}_{\mathfrak{E}^*_{\tan h,n}}(\mathcal{F}_2) &= \left(\frac{\sqrt{9n^2 + 4n \tan h1 + 4 \tan h^2 1 - 3n}}{2(n + \tan h1)} \right)^{1/n}, \\ \mathcal{R}_{\mathfrak{E}^*_{\tan h,n}}(\mathcal{F}_3) &= \left(\frac{\sqrt{9n^2 + 4n \tan h1 + 4 \tan h^2 1 - 3n}}{2(n + \tan h1)} \right)^{1/n}. \end{aligned} \tag{163}$$

Proof.

(1) Let $f \in \mathcal{F}_1$ and describe the function $p, h : \mathfrak{D} \rightarrow \mathbb{C}$ by

$$\begin{aligned} p(\mathfrak{z}) &= \frac{g(\mathfrak{z})}{\mathfrak{z}}, \\ h(\mathfrak{z}) &= \frac{f(\mathfrak{z})}{g(\mathfrak{z})}. \end{aligned} \tag{164}$$

Then, obviously $p, h \in \mathcal{P}_n$. Since

$$f(\mathfrak{z}) = \mathfrak{z}p(\mathfrak{z})h(\mathfrak{z}), \tag{165}$$

it follows from Lemma 1 that

$$\left| \frac{\mathfrak{z}f'(\mathfrak{z})}{f(\mathfrak{z})} - 1 \right| \leq \frac{4nr^n}{1 - r^{2n}} \leq \tan h1, \tag{166}$$

for

$$r \leq \left(\sqrt{4n^2 \cot h^2 1 + 1 - 2n \cot h1} \right)^{1/n} = \mathcal{R}_{\mathfrak{E}^*_{\tan h,n}}(\mathcal{F}_1). \tag{167}$$

For checking the sharpness of the result, we assume the functions

$$\begin{aligned} f_0(\mathfrak{z}) &= \mathfrak{z} \left(\frac{1 + \mathfrak{z}^n}{1 - \mathfrak{z}^n} \right)^2, \\ g_0(\mathfrak{z}) &= \mathfrak{z} \left(\frac{1 + \mathfrak{z}^n}{1 - \mathfrak{z}^n} \right). \end{aligned} \tag{168}$$

Thus, obviously,

$$\begin{aligned} \Re\left(\frac{f_0(\mathfrak{z})}{g_0(\mathfrak{z})}\right) &> 0, \\ \Re\left(\frac{g_0(\mathfrak{z})}{\mathfrak{z}}\right) &> 0, \end{aligned} \tag{169}$$

and hence, $f \in \mathcal{F}_1$. A computation shows that at $\mathfrak{z} = \mathcal{R}_{\mathfrak{E}^*_{\tan h,n}}(\mathcal{F}_1)e^{i(\pi/n)}$

$$\frac{\mathfrak{z}f'_0(\mathfrak{z})}{f_0(\mathfrak{z})} = 1 + \frac{4n\mathfrak{z}^n}{1 - \mathfrak{z}^{2n}} = 1 - \tan h1. \tag{170}$$

Hence, the result is sharp

(2) Let $f \in \mathcal{F}_2$. Describe the function $p, h : \mathfrak{D} \rightarrow \mathbb{C}$ by

$$\begin{aligned} p(\mathfrak{z}) &= \frac{g(\mathfrak{z})}{\mathfrak{z}}, \\ h(\mathfrak{z}) &= \frac{f(\mathfrak{z})}{g(\mathfrak{z})}. \end{aligned} \tag{171}$$

Then, $p \in \mathcal{P}_n$ and $h \in \mathcal{P}_n(1/2)$. Since

$$f(\mathfrak{z}) = \mathfrak{z}p(\mathfrak{z})h(\mathfrak{z}), \tag{172}$$

it follows from Lemma 1 that

$$\left| \frac{\mathfrak{z}f'(\mathfrak{z})}{f(\mathfrak{z})} - 1 \right| \leq \frac{nr^n}{1 - r^2} + \frac{4nr^n}{1 - r^{2n}} = \frac{nr^{2n} + 3nr^n}{1 - r^{2n}} \leq \tan h1. \tag{173}$$

For

$$r \leq \left(\frac{\sqrt{9n^2 + 4n \tan h1 + 4 \tan h^2 1 - 3n}}{2(n + \tan h1)} \right)^{1/n} = \mathcal{R}_{\mathfrak{E}_{\tan h,n}^*}(\mathcal{F}_2). \tag{174}$$

Thus, $f \in \mathfrak{E}_{\tan h,n}^*$ for $r \leq \mathcal{R}_{\mathfrak{E}_{\tan h,n}^*}(\mathcal{F}_2)$. For checking the sharpness of the result, assume the functions

$$\begin{aligned} f_0(z) &= \frac{z(1+z^n)}{(1-z^n)^2}, \\ g_0(z) &= \frac{z}{1-z^n}. \end{aligned} \tag{175}$$

Then, obviously,

$$\begin{aligned} \Re\left(\frac{f_0(z)}{g_0(z)}\right) &\geq 0, \\ \Re\left(\frac{g_0(z)}{z}\right) &\geq \frac{1}{2}, \end{aligned} \tag{176}$$

and hence, $f \in \mathcal{F}_2$. The sharpness is obvious, since at $z = \mathcal{R}_{\mathfrak{E}_{\tan h,n}^*}(\mathcal{F}_2)$, we get

$$\frac{zf'_0(z)}{f_0(z)} - 1 = \frac{3nz^n + nz^{2n}}{1-z^{2n}} = \tan h1 \tag{177}$$

(3) Let $f \in \mathcal{F}_3$. Describe the functions $p, h : \mathfrak{D} \rightarrow \mathbb{C}$ by

$$\begin{aligned} p(z) &= \frac{g(z)}{z}, \\ h(z) &= \frac{g(z)}{f(z)}. \end{aligned} \tag{178}$$

Then, $p \in \mathcal{P}_n$. We know that $|(1/h(z)) - 1| < 1$ if and only if $\Re(h(z)) > (1/2)$, and therefore, $h \in \mathcal{P}_n(1/2)$. Using Lemma 1, we have

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \frac{nr^{2n} + 3nr^n}{1-r^{2n}}. \tag{179}$$

Applying Lemma 3, we obtain

$$\frac{nr^{2n} + 3nr^n}{1-r^{2n}} \leq \tan h1. \tag{180}$$

For checking the sharpness, consider the functions

$$\begin{aligned} f_0(z) &= \frac{(1+z^n)^2 z}{(1-z^n)}, \\ g_0(z) &= \frac{(1+z^n)z}{1-z^n}. \end{aligned} \tag{181}$$

From the definition of f_0 and g_0 , we get

$$\begin{aligned} \Re\left(\frac{g_0(z)}{f_0(z)}\right) &= \Re\left(\frac{1}{1+z^n}\right) > \frac{1}{2}, \\ \Re\left(\frac{g_0(z)}{z}\right) &= \Re\left(\frac{1+z^n}{1-z^n}\right) > 0. \end{aligned} \tag{182}$$

Hence, $f_0 \in \mathcal{F}_3$. Now, at $z = \mathcal{R}_{\mathfrak{E}_{\tan h,n}^*}(\mathcal{F}_3)e^{in/n}$, we get

$$\frac{zf'_0(z)}{f_0(z)} - 1 = \frac{3nz^n - nz^{2n}}{1-z^{2n}} = -\tan h1. \tag{183}$$

This result is sharp. □

Data Availability

No data are used.

Conflicts of Interest

The authors declare no conflict of interest.

Authors' Contributions

All authors have equally contributed to complete this manuscript.

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