

Research Article

On Behavior Laplace Integral Operators with Generalized Bessel Matrix Polynomials and Related Functions

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Recently, the applications and importance of integral transforms (or operators) with special functions and polynomials have received more attention in various fields like fractional analysis, survival analysis, physics, statistics, and engendering. In this article, we aim to introduce a number of Laplace and inverse Laplace integral transforms of functions involving the generalized and reverse generalized Bessel matrix polynomials. In addition, the current outcomes are yielded to more outcomes in the modern theory of integral transforms.

1. Introduction

Recently, the integral transforms (or operators) have been extensively used tools in solving certain boundary value problems and certain integral equations. They are also useful in evaluating infinite integrals involving special functions or in solving differential equations of mathematical physics (see, e.g., [1–6] and the references cited therein). Laplace transform is a type of the integral transforms that is the most popular and widely used in several branches of astronomy, engineering, applied statistics, probability distributions, and applied mathematics (see, for instance, [7–13]).

A number of studies on the generalizations of Laplace transform associated with special polynomials have been contributed by Ortigueira and Machado [14], Jarad and Abdeljawad [15], Ganie and Jain [16], and Saifa et al. [17].

In 1949, Krall and Frink [18] introduced and discussed several properties of the generalized Bessel polynomials (GBPs), which are given by

$$\mathfrak{B}_n(\alpha, \beta; \xi) = \sum_{s=0}^n \binom{n}{s} (n + \alpha - 1)_s \left(\frac{\xi}{\beta}\right)^s. \quad (1)$$

These polynomials, which seem to have been considered first by Bochner [19], are also mentioned in Romanovsky [20] and Krall [21].

Recently, these polynomials have been investigated in diverse ways and turned out to be applicable in a number of research fields (see, to exemplify, [22–25]).

Additionally, various extensions of the classical orthogonal polynomials to matrix setting were investigated. The matrix generalization of the generalized Bessel polynomials $\mathcal{B}_n^{\theta, \phi}(z)$, $z \in \mathbb{C}$, for parameters (square) matrices θ and ϕ , was also introduced in diverse ways ([26]; see also [27]). Various studies of the generalized Bessel matrix polynomials have been presented and discussed (see [27, 28]).

Recently, many works established Laplace integral transforms of special functions like Gauss's and Kummer's

functions [29], generalized hypergeometric functions [30, 31], Aleph-Functions [32], and Bessel functions [33]. Whereas, some formulas corresponding to integral transforms of orthogonal matrix polynomials are little known and traceless in the literature. This motivates us to discuss Laplace integral transforms for functions involving generalized Bessel matrix polynomials. In particular, we obtain a number of useful Laplace and inverse Laplace type integrals of the generalized Bessel matrix polynomials together with certain elementary matrix functions, exponential function, logarithmic function, generalized hypergeometric matrix functions, and Bessel functions and products of generalized Bessel matrix polynomials. We also discuss some interesting and special cases of our main results.

2. Preliminaries

Here, we state some basic definitions and preliminaries which will be used in the article (see, for details, [34–36]).

Here and in the following sections, C and N denote the sets of complex numbers and positive integers, respectively, and $N_0 = N \cup \{0\}$. We denote by $M_r(\mathbb{C})$ the space of $r \times r$ complex matrices endowed with classical norm defined by

$$\|\theta\| = \sup_{y \neq 0} \left\{ \frac{\|\theta y\|}{\|y\|} \right\} = \sup \{ \|\theta y\|, \|y\| = 1 \}. \quad (2)$$

This norm satisfies the inequality $\|\theta\phi\| \leq \|\theta\| \|\phi\|$, where θ and ϕ are in $M_r(\mathbb{C})$.

Definition 1. For any matrix θ in $M_r(\mathbb{C})$, the spectrum $\sigma(\theta)$ is the set of all eigenvalues of θ for which we denote

$$\alpha(\theta) = \max \{ \Re(\eta) : \eta \in \sigma(\theta) \} \text{ and } \beta(\theta) = \min \{ \Re(\eta) : \eta \in \sigma(\theta) \}, \quad (3)$$

where $\alpha(\theta)$ refers to the spectral abscissa of θ and for which $\beta(\theta) = -\alpha(-\theta)$. A matrix $\theta \in M_r(\mathbb{C})$ is said to be positive stable if and only if $\beta(\theta) > 0$.

Definition 2 (see [35, 36]). If $\theta \in M_r(\mathbb{C})$, and $w \in C$, then the matrix exponential $e^{\theta w}$ is given to be

$$e^{\theta w} = I + \theta w + \dots + \frac{\theta^n}{n!} w^n + \dots, \quad (4)$$

where I is the identity matrix in $M_r(\mathbb{C})$.

Definition 3 (see [37]). Let θ be a positive stable matrix in $M_r(\mathbb{C})$ with $\theta + nI$ is invertible for all integers $n \in N_0$, the Gamma matrix function $\Gamma(\theta)$ and the Digamma matrix function $\psi(\theta)$ are defined, respectively, as follows:

$$\Gamma(\theta) = \int_0^\infty e^{-u} u^{\theta-1} du; \quad u^{\theta-1} = \exp((\theta - I) \ln u). \quad (5)$$

$$\psi(\theta) = \Gamma^{-1}(\theta) \Gamma'(\theta), \quad (6)$$

where $\Gamma^{-1}(\theta)$ and $\Gamma'(\theta)$ are reciprocal and derivative of the Gamma matrix function.

Note that the scalar Gamma and Digamma functions are easily found when $r = 1$ in (5) and (6), respectively (see, e.g., [38, Section 1.1]).

Definition 4 (see [?]). For all θ in $M_r(\mathbb{C})$, we assume

$$\theta + kI \text{ is invertible for all } k \in \mathbb{N}_0, \quad (7)$$

and the Pochhammer symbol (the shifted factorial) is defined by

$$(\theta)_r = \begin{cases} \theta(\theta + I) \cdots (\theta + (r - 1)I) = \Gamma^{-1}(\theta) \Gamma(\theta + rI), & r \in \mathbb{N}, \\ I, & r = 0. \end{cases} \quad (8)$$

Lemma 5 (see [34]). Let θ be a matrix in $M_r(\mathbb{C})$ such that $\|\theta\| < 1$ and $\|I\| = 1$. Then, $(I + \theta)^{-1}$ exists, and we have

$$(I + \theta)^{-1} = I - \theta + \theta^2 - \theta^3 + \theta^4 - \theta^5 + \dots \quad (9)$$

Definition 6 (see [39]). Let m and n be finite positive integers, the generalized hypergeometric matrix function is given by

$${}_m F_n(\theta; \phi; z) = \sum_{k=0}^\infty \prod_{i=1}^m (\theta_i)_k \prod_{j=1}^n [(\phi_j)_k]^{-1} \frac{z^k}{k!}, \quad (10)$$

where $\theta_i, 1 \leq i \leq m$ and $\phi_j, 1 \leq j \leq n$ are commutative matrices in $M_r(\mathbb{C})$ with $\phi_j + kI$ are invertible for all integers $k \in N_0$ and $1 \leq i \leq m$. In [39], Abdalla discussed regions of convergence of (2.6).

Note that for $m = 1, n = 0$ in (10), we have the Binomial type matrix function ${}_1 F_0(\theta; -; z)$ [39] as follows:

$${}_1 F_0(\theta; -; z) = (1 - z)^{-\theta} = I + \theta z + \frac{\theta(\theta + I)z^2}{2!} + \dots + \frac{(\theta)_n z^n}{n!} + \dots, |z| < 1. \quad (11)$$

Also, for $m = 2, n = 1$ in (10), we get the hypergeometric matrix function ${}_2 F_1$ (cf. [40]).

Further, the substitution $r = 1$ in (10) leads to the classical generalized hypergeometric functions [38, Section 1.5], see also, [41].

Definition 7 (see [26]). Let θ and ϕ be commuting matrices in $M_r(\mathbb{C})$ such that ϕ is an invertible matrix. For any natural number $n \geq 0$, the n^{th} generalized Bessel matrix polynomial $\mathcal{B}_n^{\theta, \phi}(z)$ is defined as

$$\begin{aligned} \mathcal{B}_n^{\theta, \phi}(z) &= \sum_{r=0}^n \binom{n}{r} (\theta + (n-1)I)_r (z\phi^{-1})^r \\ &= \sum_{r=0}^n \frac{(-1)^r}{r!} (-nI)_r (\theta + (n-1)I)_r (z\phi^{-1})^r \\ &= {}_2F_0(-nI, \theta + (n-1)I; -; -z\phi^{-1}). \end{aligned} \tag{12}$$

In addition, the n^{th} reverse generalized Bessel matrix polynomial $\Theta_n^{(\theta, \phi)}(z)$ is given by (see [27])

$$\begin{aligned} \Theta_n^{(\theta, \phi)}(z) &= z^n \mathcal{B}_n^{\theta, \phi}(z^{-1}) = (-1)^n \Gamma^{-1}(-\theta - (2n-2)I) \Gamma \\ &\quad \cdot (-\theta + (n-2)I) \times {}_1F_1(-nI; -\theta - (2n-2)I; \phi z) \cdot \Theta_n^{(\theta, \phi)}(z) \\ &= z^n \mathcal{B}_n^{\theta, \phi}(z^{-1}) = (-1)^n \Gamma^{-1}(-\theta - (2n-2)I) \Gamma \\ &\quad \cdot (-\theta + (n-2)I) \times {}_1F_1(-nI; -\theta - (2n-2)I; \phi z). \end{aligned} \tag{13}$$

Obviously, the n^{th} generalized Bessel matrix polynomial $\mathcal{B}_n^{(\theta, \phi)}(z)$ when $r = 1$ is easily found to be the scalar generalized Bessel polynomials (1.1).

Definition 8. Let $g(\tau)$ be a function of τ specified for $\tau > 0$. Then, the Laplace transform of $g(\tau)$ is defined by

$$\mathcal{G}(\lambda) = \mathcal{L}\{g(\tau): \lambda\} = \int_0^\infty e^{-\lambda\tau} g(\tau) d\tau, \quad \Re(\lambda) > 0, \tag{14}$$

provided that the improper integral exists, $e^{-\lambda u}$ is the kernel of the transformation and the function $g(\tau)$ is called the inverse Laplace transform of $\mathcal{G}(\lambda)$ (see [1, Chapter 3]; see also [7]).

The following Lemma, which may be easily derivable from (14), will be desired in the sequel.

Lemma 9. Let θ be a positive stable and invertible matrix in $M_r(\mathbb{C})$ and $\Re(\lambda) > 0$. Then, we have

$$\mathcal{L}\{\tau^\theta : \lambda\} = \int_0^\infty e^{-\lambda\tau} \tau^\theta d\tau = \lambda^{-(\theta+I)} \Gamma(\theta+I), \tag{15}$$

$$\mathcal{L}\{\tau^\theta (\tau+I)^{-1} : \lambda\} = \Gamma(\theta+I) e^\lambda \Gamma(-\theta, \lambda), \tag{16}$$

where $\Gamma(\theta, \lambda)$ is the incomplete Gamma matrix function [42].

$$\begin{aligned} \mathcal{L}\{g(\tau)e^{\theta\tau} : \lambda\} &= \mathcal{G}(\lambda I - \theta), \\ \mathcal{L}^{-1}\{\lambda^{-\theta} : \tau\} &= \tau^{(\theta-I)} \Gamma^{-1}(\theta). \end{aligned} \tag{17}$$

3. Laplace Type Integrals of Functions Involving $\mathcal{B}_n^{\theta, \phi}(z)$ and $\Theta_n^{\theta, \phi}(z)$

In this section, we investigate several Laplace-type transforms of functions involving generalized and reverse general-

ized Bessel matrix polynomials asserted in the following theorems:

Theorem 10. Let $z, \lambda \in \mathbb{C}, \Re(\lambda) > 0, n \in \mathbb{N}_0$, and $r \in \mathbb{N}$. Also, let θ, ϕ and A be matrices in $M_r(\mathbb{C})$ such that $\beta(A) > 0$ and $\phi + kI$ are invertible for all $k \in \mathbb{N}_0$. For the function

$$g_1(z) = z^{A-I} \mathcal{B}_n^{\theta, \phi}(z), \tag{18}$$

we have

$$\mathcal{G}_1(\lambda) = \mathcal{L}\{g_1(z): \lambda\} = \lambda^{-A} \Gamma(A) {}_3F_0 \left[\begin{matrix} -nI, \theta + (n-1)I, A \\ \\ \end{matrix} ; -(\lambda\phi)^{-1} \right]. \tag{19}$$

Proof. From the expansion series of the $\mathcal{B}_n^{\theta, \phi}(z)$ in (12) and upon using (15) in Lemma 9, we obtain

$$\begin{aligned} \mathcal{G}_1(\lambda) &= \sum_{s=0}^n \frac{(-nI)_s (\theta + (n-1)I)_s (-\phi^{-1})^s}{s!} \mathcal{L}\{z^{A+(s-1)I}\} \\ &= \sum_{s=0}^n \frac{(-nI)_s (\theta + (n-1)I)_s (-\phi^{-1})^s}{s!} \lambda^{-(A+sI)} \Gamma(A+sI) \\ &= \lambda^{-A} \Gamma(A) \sum_{s=0}^n \frac{(-nI)_s (\theta + (n-1)I)_s (A)_s (-\lambda\phi)^{-1})^s}{s!}. \end{aligned} \tag{20}$$

Thus, we get the required result (19).

Theorem 11. Let $z, \lambda \in \mathbb{C}, \Re(\lambda) > 0, n \in \mathbb{N}_0$, and $r \in \mathbb{N}$. Also, let θ, ϕ and A be matrices in $M_r(\mathbb{C})$ such that $\beta(A) > 0, \phi + kI$ are invertible for all $k \in \mathbb{N}_0$ and $I - A$ satisfies the spectral condition (7). Further, let

$$g_2(z) = z^{A-(n+1)I} \Theta_n(\theta, \phi; z). \tag{21}$$

Then,

$$\mathcal{G}_2(\lambda) = \mathcal{L}\{g_2(z): \lambda\} = \lambda^{-A} \Gamma(A) {}_2F_1 \left[\begin{matrix} -nI, \theta + (n-1)I \\ \\ I - A \end{matrix} ; \lambda\theta^{-1} \right]. \tag{22}$$

Proof. Starting from Definition 7, and applying the relation (15), it follows that

$$\begin{aligned}
\mathcal{G}_2(\lambda) &= \sum_{s=0}^n \frac{(-nI)_s (\theta + (n-1)I)_s (-\phi^{-1})^s}{s!} \mathcal{L} \left\{ z^{A-(s+1)I} \right\} \\
&= \sum_{s=0}^n \frac{(-nI)_s (\theta + (n-1)I)_s (-\phi^{-1})^s}{s!} \lambda^{-(A-sI)} \Gamma(A-sI) \\
&= \lambda^{-A} \Gamma(A) \sum_{s=0}^n \frac{(-nI)_s (\theta + (n-1)I)_s [(I-A)_s]^{-1} (\lambda\phi^{-1})^s}{s!}.
\end{aligned} \tag{23}$$

Thus, the result (22) is established.

Theorem 12. Let $z, \mu, \lambda \in \mathbb{C}$, $\Re(\lambda - \mu) > 0$, $n \in \mathbb{N}_0$, and $r \in \mathbb{N}$. Also, let θ, ϕ and A be matrices in $M_r(\mathbb{C})$ such that $\beta(A) > 0$, $\phi + kI$ are invertible for all $k \in \mathbb{N}_0$ and $I - A$ satisfies the spectral condition (7). If

$$g_3(z) = z^{A-I} e^{\mu z} \mathcal{B}_n^{\theta, \phi}(z^{-1}). \tag{24}$$

Then,

$$\mathcal{G}_3(\lambda) = \mathcal{L}\{g_3(z); \lambda\} = (\lambda - \mu)^{-A} \Gamma(A)_2 F_1 \left[\begin{matrix} -nI, \theta + (n-1)I \\ I - A \end{matrix}; (\lambda - \mu)\phi^{-1} \right]. \tag{25}$$

Proof. For convenience, let the left-hand side of (25) be denoted by S and by invoking the series expression of (12) to S , we obtain

$$\begin{aligned}
S &= \sum_{k=0}^n \frac{(-nI)_s (\theta + (n-1)I)_k (-\phi^{-1})^s}{s!} \int_0^\infty z^{A-(s-1)I} e^{-(\mu+\lambda)z} dz \\
&= \sum_{s=0}^n \frac{(-nI)_s (\theta + (n-1)I)_s (-\phi^{-1})^s}{s!} (-\mu + \lambda)^{-(A-sI)} \Gamma(A-sI) \\
&= \Gamma(A) (\lambda - \mu)^{-A} \sum_{s=0}^n \frac{(-nI)_s (\theta + (n-1)I)_s}{s!} \\
&\quad \cdot [(I-A)_s]^{-1} \frac{((\lambda - \mu)\phi^{-1})^s}{s!},
\end{aligned} \tag{26}$$

therefore, (25) as desired.

Theorem 13. Let $z, w, \lambda \in \mathbb{C}$, $\Re(\lambda) > 0$, $n \in \mathbb{N}_0$, and $r \in \mathbb{N}$. Also, let θ, ϕ and A be matrices in $M_r(\mathbb{C})$ such that $\beta(A) > 0$ and $\phi + kI$ are invertible for all $k \in \mathbb{N}_0$. For the function

$$g_4(z) = z^{A-I} (z+w)^{-1} \mathcal{B}_n^{\theta, \phi}(z), \tag{27}$$

we have

$$\begin{aligned}
\mathcal{G}_4(\lambda) &= \mathcal{L}\{g_4(z); \lambda\} = w^{A-I} \Gamma(A) e^{\lambda w} \\
&\quad \times \sum_{k=0}^n \frac{(-nI)_k (\theta + (n-1)I)_k (A)_k}{k!} \Gamma \\
&\quad \cdot (I - A - kI; \lambda w) (-w\phi^{-1})^k,
\end{aligned} \tag{28}$$

where $\Gamma(A, z)$ is the incomplete Gamma matrix function defined in [42].

Proof. To prove (28), we consider

$$\begin{aligned}
\mathcal{G}_4(\lambda) &= \int_0^\infty z^{A-I} (z+w)^{-1} \mathcal{B}_n^{\theta, \phi}(z) e^{-\lambda z} dz \\
&= \sum_{k=0}^n \frac{(-nI)_k (\theta + (n-1)I)_k}{k!} (-\phi^{-1})^k \\
&\quad \times \int_0^\infty z^{A+(k-1)I} (z+w)^{-1} e^{-\lambda z} dz.
\end{aligned} \tag{29}$$

According to (16) in Lemma 9, we get

$$\begin{aligned}
\mathcal{G}_4(\lambda) &= \sum_{k=0}^n \frac{(-nI)_k (\theta + (n-1)I)_k}{k!} \Gamma(A+kI) \\
&\quad \times w^{A+(k-1)I} e^{\lambda w} \Gamma((1-k)I - A, w\lambda) (-\phi^{-1})^k \\
&= \Gamma(A) w^{A-I} e^{w\lambda} \sum_{k=0}^n \frac{(-nI)_k (\theta + (n-1)I)_k (A)_k}{k!} \\
&\quad \times \Gamma((1-k)I - A, w\lambda) (-w\phi^{-1})^k.
\end{aligned} \tag{30}$$

This completes the proof of Theorem 13.

Theorem 14. Let $z, \lambda, \nu \in \mathbb{C}$, $\Re(\lambda) > 0$, $\Re(\nu) > 0$, $n, m \in \mathbb{N}_0$, and $r \in \mathbb{N}$. Also let θ, ϕ and A be matrices in $M_r(\mathbb{C})$ such that $\beta(A) > 0$, $\phi + kI$ are invertible for all $k \in \mathbb{N}_0$, $(I+n)I - A$ and $(2-n)I - A - \theta$ satisfies the spectral condition (7). Further, let

$$g_5(z) = z^{A-I} \mathcal{B}_n^{\theta, \lambda I}(z^{-1}) \mathcal{B}_m^{\nu I, \phi}(z^{-1}). \tag{31}$$

Then,

$$\begin{aligned}
\mathcal{G}_5(\lambda) &= \{g_5(z); \lambda\} = \lambda^{-A} \Gamma(A) \Gamma(I-A) \Gamma(2I-A-\theta) \times \Gamma^{-1} \\
&\quad \cdot ((I+n)I - A) \Gamma^{-1} ((2-n)I - A - \theta) \times {}_3F_2 \\
&\quad \cdot \left[\begin{matrix} -mI, (\nu+n-1)I, 2I-A-\theta \\ (I+n)I-A, (2-n)I-A-\theta \end{matrix}; \lambda\phi^{-1} \right].
\end{aligned} \tag{32}$$

Proof. To prove (32), we require the relation (15) and Definition 7, thus we arrive at

$$\begin{aligned}
 \mathcal{E}_5(\lambda) &= \sum_{s=0}^n \sum_{j=0}^m \frac{(-nI)_s(\theta + (n-1)I)_s (-\lambda^{-1})^s}{s!} \\
 &\quad \times \frac{(-mI)_j(\nu I + (m-1)I)_j (-\phi^{-1})^j}{j!} \mathcal{L}\left\{z^{A-(s+j+1)I}\right\} \\
 &= \sum_{s=0}^n \sum_{j=0}^m \frac{(-nI)_s(\phi + (n-1)I)_s (-\lambda^{-1})^s}{s!} \\
 &\quad \times \frac{(-mI)_j(\nu I + (m-1)I)_j (-\phi^{-1})^j}{r!} \Gamma(A - (s+j)I) \lambda^{-(A-(s+j)I)} \\
 &= \lambda^{-A} \Gamma(A) \sum_{j=0}^m \frac{(-mI)_j(\nu I + (m-1)I)_j (-\phi^{-1})^j}{j!} [(I-A)_j]^{-1} \\
 &\quad \times \sum_{s=0}^n \frac{(-nI)_s(\phi + (n-1)I)_s}{s!} [((1-j)I - A)_s]^{-1} \\
 &= \lambda^{-A} \Gamma(A) \Gamma(I-A) \Gamma(2I-A-\theta) \Gamma^{-1}(I-\theta+nI) \Gamma^{-1} \\
 &\quad \cdot (2I-A-\theta-nI) \times \sum_{j=0}^m \frac{(-mI)_j(\nu I + (m-1)I)_j (-\lambda\phi^{-1})^j}{j!} \\
 &\quad \cdot (2I-A-\theta)_j \times [((1+n)I-A)_j]^{-1} [((2-n)I-A-\theta)_j]^{-1}. \tag{33}
 \end{aligned}$$

This completes the proof of Theorem 14.

Theorem 15. Let $z, \lambda \in \mathbb{C}$, $\Re(\lambda) > 0$, $n, m \in \mathbb{N}_0$, and $r \in \mathbb{N}$. Also, let θ, ϑ, ϕ and A be matrices in $M_r(\mathbb{C})$ such that $\beta(A) > 0, \phi + kI$, are invertible for all $k \in \mathbb{N}_0, \vartheta, (\theta + A)$ and $\theta + A - I$ satisfies the spectral condition (7). Further, let

$$g_6(z) = z^{A-I} \mathcal{B}_n^{\theta, \lambda z I}(1) \mathcal{B}_m^{\vartheta, \phi}(z) \mathcal{B}_m^{\vartheta, \phi}(-z). \tag{34}$$

Then,

$$\begin{aligned}
 \mathcal{E}_6(\lambda) = \mathcal{L}\{g_6(z); \lambda\} &= \frac{2^{A-I}}{\sqrt{\pi}} (\theta + A - I)_n \Gamma(A) \lambda^{-A} [(I-A)_n]^{-1} \\
 &\quad \times {}_8F_3 \left[\begin{matrix} -mI, \vartheta + (m-1)I, \frac{1}{2}(\theta - I), \frac{1}{2}\vartheta, \frac{1}{2}(A + (1-n)I), \\ \frac{1}{2}(A - nI), \frac{1}{2}(\theta + A + nI), \frac{1}{2}(\theta + A + (n-1)I) \\ \vartheta I, \frac{1}{2}(\theta + A), \frac{1}{2}(\theta + A - I) \end{matrix} ; 16(\lambda\phi)^{-2} \right]. \tag{35}
 \end{aligned}$$

Proof. Applying the following formula (see [39])

$$\mathcal{B}_m^{\vartheta, \phi}(z) \mathcal{B}_m^{\vartheta, \phi}(-z) = {}_4F_1 \left[\begin{matrix} -mI, \vartheta + (m-1)I, \frac{1}{2}(\vartheta - I), \frac{1}{2}\vartheta \\ \vartheta - I \end{matrix} ; 4z^2 \phi^{-2} \right]. \tag{36}$$

We thus find that

$$\begin{aligned}
 \mathcal{E}_6(\lambda) &= \mathcal{L} \left\{ z^{A-I} \mathcal{B}_n^{\theta, \lambda z I}(1) {}_4F_1 \left[\begin{matrix} -mI, \vartheta + (m-1)I, \frac{1}{2}(\vartheta - I), \frac{1}{2}\vartheta \\ \vartheta - I \end{matrix} ; 4z^2 \phi^{-2} \right] \right\} \\
 &= \sum_{s=0}^n \frac{(-nI)_s(\theta + (n-1)I)_s (-\lambda^{-1})^s}{s!} \times \sum_{j=0}^m \frac{(-mI)_j(\vartheta + (m-1)I)_j}{j!} \\
 &\quad \cdot \left(\frac{1}{2}(\vartheta - I)\right)_j \left(\frac{1}{2}\vartheta\right)_j [(\vartheta - I)_j]^{-1} (4\phi^{-2})^j \times \mathcal{L}\left\{z^{A-(s+1+2j)I}\right\}. \tag{37}
 \end{aligned}$$

Making use of (15), we observe that

$$\begin{aligned}
 \mathcal{E}_6(\lambda) &= \sum_{s=0}^n \frac{(-nI)_s(\theta + (n-1)I)_s (-\lambda^{-1})^s}{s!} \\
 &\quad \times \sum_{j=0}^m \frac{(-mI)_j(\vartheta + (m-1)I)_j}{j!} \left(\frac{1}{2}(\vartheta - I)\right)_j \left(\frac{1}{2}\vartheta\right)_j \\
 &\quad \cdot [(\vartheta - I)_j]^{-1} \times (4\phi^{-2})^j \lambda^{-(A+(s-2j)I)} \Gamma(A - (s-2j)I) \\
 &= \lambda^{-A} \Gamma(A) \sum_{j=0}^m \frac{(-mI)_j(\vartheta + (m-1)I)_j}{j!} \left(\frac{1}{2}(\vartheta - I)\right)_j \\
 &\quad \cdot \left(\frac{1}{2}\vartheta\right)_j \times [(\vartheta - I)_j]^{-1} (A)_{2j} (4(\lambda\phi)^{-2})^j \\
 &= \sum_{s=0}^n \frac{(-nI)_s(\theta + (n-1)I)_s}{s!} [(I-A-2jI)_s]^{-1} \\
 &= \lambda^{-A} \frac{2^{A-I}}{\sqrt{\pi}} \Gamma(A) \Gamma(I-A) \Gamma(2I-A-\theta) \Gamma^{-1} \\
 &\quad \cdot (I-A+nI) \Gamma^{-1} (2I-A-\theta-nI) \\
 &\quad \times \sum_{j=0}^m \frac{(-mI)_j(\vartheta + (m-1)I)_j}{j!} \left(\frac{1}{2}(\vartheta - I)\right)_r \left(\frac{1}{2}\vartheta\right)_j \\
 &\quad \cdot \left(\frac{1}{2}A\right)_j [(\vartheta - I)_j]^{-1} \times \left(\frac{1}{2}(A+I)\right)_j \\
 &\quad \cdot \left(\frac{1}{2}(A+(1-n)I)\right)_j \left(\frac{1}{2}(A-nI)\right)_j \\
 &\quad \times \left(\frac{1}{2}(A+\theta+nI)\right)_j \left(\frac{1}{2}(A+\theta+(n-1)I)\right)_j \\
 &\quad \cdot \left[\left(\frac{1}{2}(A+I)\right)_j\right]^{-1} \times \left[\left(\frac{1}{2}A\right)_j\right]^{-1} \left[\left(\frac{1}{2}(A+\theta)\right)_j\right]^{-1} \\
 &\quad \cdot \left[\left(\frac{1}{2}(A+\theta-I)\right)_j\right]^{-1} \cdot (16(\lambda\phi)^{-2})^j. \tag{38}
 \end{aligned}$$

Thus, after a simplification, we get the required result (35).

Theorem 16. Let $z, \lambda \in \mathbb{C}$, $\Re(\lambda) > 0$, $n \in \mathbb{N}_0$, and $r \in \mathbb{N}$. Also, let θ, ϕ and A be matrices in $M_r(\mathbb{C})$ such that $\beta(A) > 0$ and $\phi + kI$ are invertible for all $k \in \mathbb{N}_0$. For the function

$$g_7(z) = z^{A-I} \log z \mathcal{B}_n^{\theta, \phi}(z), \quad (39)$$

then, we have

$$\begin{aligned} \mathcal{G}_7(\lambda) &= \mathcal{L}\{g_7(z); \lambda\} = \lambda^{-A} \Gamma(A) \sum_{s=0}^n (-nI)_s (\theta + (n-1)I)_s (A)_s \\ &\quad \times \frac{(-(\lambda\phi)^{-1})^s}{s!} (\psi(A+sI) - \log \lambda), \end{aligned} \quad (40)$$

where $\psi(A)$ is the Digamma matrix function defined in (6).

Proof. The proof of this Theorem is quite straight forward as

$$\begin{aligned} \mathcal{G}_7(\lambda) &= \int_0^\infty z^{A-I} \log z \mathcal{B}_n^{\theta, \phi}(z) e^{-\lambda z} dz \\ &= \sum_{s=0}^n \frac{(-nI)_s (\theta + (n-1)I)_k}{s!} (-\phi^{-1})^s \\ &\quad \times \int_0^\infty z^{A+(s-1)I} \log z e^{-\lambda z} dz. \end{aligned} \quad (41)$$

Upon using (2,2), we have

$$\Gamma(A+sI) = \int_0^\infty z^{A+(s-1)I} e^{-z} dz. \quad (42)$$

Hence,

$$\Gamma'(A+sI) = \int_0^\infty z^{A+(s-1)I} e^{-z} \log z dz. \quad (43)$$

We thus arrive at

$$\begin{aligned} \Psi(A+sI) &= \Gamma'(A+sI) \Gamma^{-1}(A+sI) \\ &= \Gamma^{-1}(A+sI) \int_0^\infty z^{A+(s-1)I} e^{-z} \log z dz. \end{aligned} \quad (44)$$

Therefore, we get

$$\begin{aligned} \Psi(A+sI) &= \lambda^{A+sI} \Gamma^{-1}(A+sI) \int_0^\infty z^{A+(s-1)I} e^{-\lambda z} \log(\lambda z) dz \\ &= \lambda^{A+sI} \Gamma^{-1}(A+sI) \int_0^\infty z^{A+(s-1)I} e^{-\lambda z} [\log(\lambda) + \log(z)] dz \\ &= \lambda^{A+sI} \Gamma^{-1}(A+sI) \int_0^\infty z^{A+(s-1)I} e^{-\lambda z} \log(\lambda) dz \\ &\quad + \lambda^{A+sI} \Gamma^{-1}(A+sI) \int_0^\infty z^{A+(s-1)I} e^{-\lambda z} \log(z) dz \\ &= \log(\lambda) + \lambda^{A+sI} \Gamma^{-1}(A+sI) \int_0^\infty z^{A+(s-1)I} e^{-\lambda z} \log(z) dz. \end{aligned} \quad (45)$$

We thus have

$$\int_0^\infty z^{A+(s-1)I} e^{-\lambda z} \log(z) dz = \lambda^{-(A+sI)} \Gamma(A+sI) [\Psi(A+sI) - \log \lambda]. \quad (46)$$

From the above equations, we get the required result as follows:

$$\begin{aligned} \mathcal{G}_7(\lambda) &= \sum_{s=0}^n \frac{(-nI)_s (\theta + (n-1)I)_s (A)_s}{s!} \\ &\quad \cdot (-(\lambda\phi)^{-1})^s \times \lambda^{-A} \Gamma(A) [\Psi(A+sI) - \log \lambda] \\ &= \lambda^{-A} \Gamma(A) \sum_{s=0}^n \frac{(-nI)_s (\theta + (n-1)I)_s (A)_s}{s!} \\ &\quad \times (-(\lambda\phi)^{-1})^s [\Psi(A+sI) - \log \lambda]. \end{aligned} \quad (47)$$

Theorem 17. Let $z, \lambda \in \mathbb{C}$, $\Re(\lambda) > 0$, $n, m, q \in \mathbb{N}_0$, and $r \in \mathbb{N}$. Also, let θ, ϕ, E, D and A be matrices in $M_r(\mathbb{C})$ such that $\beta(A) > 0$, and $\phi + kI$ are invertible for all $k \in \mathbb{N}_0$. Further, let

$$g_8(z) = z^{2A-I} {}_mF_q(E; D; z^2) \mathcal{B}_n^{\theta, \phi}(z^2). \quad (48)$$

Then,

$$\begin{aligned} \mathcal{G}_8(\lambda) &= \mathcal{L}\{g_8; \lambda\} = \frac{2^{2A-I}}{\sqrt{\pi}} \Gamma(A) \Gamma\left(A + \frac{1}{2}\right) \lambda^{-2A} \\ &\quad \times \sum_{k=0}^n \frac{1}{k!} (-nI)_k (\theta + (n-1)I)_k (A)_k \left(A + \frac{1}{2}\right)_k \left(-4(\lambda^2\phi)^{-1}\right)^k \\ &\quad \times {}_{m+2}F_q\left(E, A+kI, A + \left(k + \frac{1}{2}\right)I; D; 4(\lambda)^{-2}\right), \end{aligned} \quad (49)$$

where ${}_mF_q(E; D; z)$ is the generalized hypergeometric type matrix functions defined in (10) such that $\Re(\lambda) > 0$ if $m < q - 1$ and $\Re(\lambda) > |\beta(A)|$ if $m = q - 1$.

Proof. Using Definitions (10) and (12) and upon using (15), we obtain

$$\begin{aligned} \mathcal{G}_8(\lambda) &= \sum_{k=0}^n \frac{1}{k!} (-nI)_k (\theta + (n-1)I)_k (4(\phi)^{-1})^k \\ &\quad \times \sum_{r=0}^\infty \prod_{i=1}^m (E_i)_r \prod_{j=1}^q [(D_j)_r]^{-1} \frac{1}{k!} \mathcal{L}\left\{z^{2A-(1-2k-2r)I}\right\} \\ &= \sum_{k=0}^n \frac{1}{k!} (-nI)_k (\theta + (n-1)I)_k (4(\phi)^{-1})^k \\ &\quad \times \sum_{r=0}^\infty \prod_{i=1}^m (E_i)_r \prod_{j=1}^q [(D_j)_r]^{-1} \frac{1}{k!} \times \lambda^{-2A-(2k+2r)I} \Gamma \\ &\quad \cdot (2A + (2k + 2r)I). \end{aligned} \quad (50)$$

Thus, after a simplification, we obtain the result (49) in Theorem 3.11.

Theorem 18. Let $z, v, \sigma, \lambda \in \mathbb{C}$, $\Re(\lambda) > 0$, $\Re(v) > -1$, $\Re(\sigma) > 0$, $n, m \in \mathbb{N}_0$, and $r \in \mathbb{N}$. Also, let θ be matrix in $M_r(\mathbb{C})$ such that $\beta(A) > 0$ and $\phi + kI$ are invertible for all $k \in \mathbb{N}_0$. For the function

$$g_9(z) = z^{v/2} J_v(2(\sigma z)^{1/2}) \mathcal{B}_n^{\theta, \lambda z}(1). \quad (51)$$

Then, we have

$$\begin{aligned} \mathcal{G}_9(\lambda) &= \mathcal{L}\{g_9(z): \lambda\} = \sigma^{v/2} (\theta + vI)_n \left(\frac{I}{(-v)_n} \right) \lambda^{-(v+1)} \\ &\times {}_2F_2 \left[\begin{matrix} 1 + v - m, \theta + (n + v)I \\ 1 + v, \theta + vI \end{matrix} ; -\frac{\sigma}{\lambda} \right], \end{aligned} \quad (52)$$

where $J_v(z)$ is the Bessel function of the first kind of order v defined by (see, e.g., [38, 41, 43])

$$J_v(z) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s! \Gamma(1 + v + s)} \left(\frac{z}{2} \right)^{v+2s}. \quad (53)$$

Proof. According to (12) and (53) and upon using (15), it follows that

$$\begin{aligned} \mathcal{G}_9(\lambda) &= \mathcal{L}\left\{ z^{v/2} J_v(2(\sigma z)^{1/2}) \mathcal{B}_n^{\theta, \lambda z}(1) \right\} = \sum_{m=0}^{\infty} \frac{(-1)^m (\sigma)^{m+(v/2)}}{m! \Gamma(1 + v + m)} \\ &\times \sum_{k=0}^n \frac{(-nI)_k (\theta + (n-1)I)_k}{k!} (-\lambda^{-1})^k \mathcal{L}\left\{ z^{\frac{v}{2} + \frac{v}{2} - k + m} \right\} \\ &= (\sigma)^{v/2} \sum_{m=0}^{\infty} \frac{(-\sigma)^m}{m! \Gamma(1 + v + m)} \times \sum_{k=0}^n \frac{(-nI)_k (\theta + (n-1)I)_k}{k!} \\ &\cdot (-\lambda^{-1})^k \Gamma(1 + v + m - k) \lambda^{v-m+k-1} \\ &= (\sigma)^{v/2} \lambda^{v-1} \sum_{m=0}^{\infty} \frac{(-\sigma)^m \lambda^{-m} \Gamma(1 + v + m)}{m! \Gamma(1 + v + m)} \\ &\times \sum_{k=0}^n \frac{(-nI)_k (\theta + (n-1)I)_k}{(-v+m)_k k!} = (\sigma)^{v/2} \lambda^{v-1} \frac{(\theta + vI)_n}{(-v)_n} \\ &\cdot \sum_{m=0}^{\infty} \frac{(1 + v - n)_m ((v+n)I + \theta)_m [(\theta + vI)_m]^{-1}}{m! (1 + v)_m} \left(\frac{-\sigma}{\lambda} \right)^m. \end{aligned} \quad (54)$$

This completes the proof of Theorem 18.

4. Inverse Laplace Type Integrals of Functions Involving $\mathcal{B}_n^{P, Q}(z)$

Here, we obtain the following inverse Laplace type transforms of generalized Bessel matrix polynomials with products of some functions in the following theorem:

Theorem 19. Let $z, \lambda, \sigma \in \mathbb{C}$, $\Re(\lambda) > 1/2 |\Re(\sigma)|$, $n \in \mathbb{N}_0$, and $r \in \mathbb{N}$. Also, let A be matrix in $M_r(\mathbb{C})$ such that $\beta(A) > 0$. If

$$\mathcal{G}_{10}(\lambda) = \Gamma(A) \left(\lambda + \frac{1}{2} \sigma \right)^{-A} \mathcal{B}_n^{A-(n+1)I, \frac{1}{\lambda+1/2\sigma}}(-\sigma). \quad (55)$$

Then,

$$g_{10}(z) = z^{A-I} \exp\left(\frac{-1}{2}\sigma z\right) (1 - \sigma z)^n. \quad (56)$$

Proof. It is sufficient to find Laplace transform of $g_{10}(z)$

$$\begin{aligned} \mathcal{G}_{10}(\lambda) &= \mathcal{L}\left\{ z^{A-I} \exp\left(\frac{-1}{2}\sigma z\right) (1 - \sigma z)^n \right\} \\ &= \mathcal{L}\left\{ z^{A-I} \exp\left(\frac{-1}{2}\sigma z\right) {}_1F_0\left(\begin{matrix} -n \\ - \end{matrix}; \sigma z\right) \right\} \\ &= \sum_{k=0}^n \frac{(-nI)_k \sigma^k}{k!} \mathcal{L}\left\{ z^{A-(1-k)I} \exp\left(\frac{-1}{2}\sigma z\right) \right\} \\ &= \sum_{k=0}^n \frac{(-nI)_k \sigma^k}{k!} \Gamma(A + kI) \left(\lambda + \frac{1}{2} \sigma \right)^{-(A+kI)} \\ &= \Gamma(A) \left(\lambda + \frac{1}{2} \sigma \right)^{-A} \sum_{k=0}^n \frac{(-nI)_k (A)_k}{k!} \left(\frac{\sigma}{(\lambda + 1/2\sigma)} \right)^k, \end{aligned} \quad (57)$$

This finalizes the proof of Theorem 19.

Theorem 20. Let $z, \lambda, \sigma \in \mathbb{C}$, $\Re(\lambda) > 0, \Re(\sigma) > 0$, $n \in \mathbb{N}_0$, and $r \in \mathbb{N}$. Also, let A be matrix in $M_r(\mathbb{C})$ such that $\beta(A + nI) > 0$. Further, let

$$\mathcal{G}_{11}(\lambda) = (-1)^n \sigma^{\frac{1}{2}A+nI} \lambda^{-(A+(2n+1)I)} \exp\left(\frac{-\sigma}{\lambda} z\right) \mathcal{B}_n^{I-A-2nI, \sigma}(\lambda). \quad (58)$$

Then,

$$g_{11}(z) = z^{\frac{1}{2}A+nI} J_v(2(\sigma z)^{1/2}). \quad (59)$$

Proof. By invoking to (15) and (53), we consider

$$\begin{aligned}
 \mathcal{E}_{11}(\lambda) &= \mathcal{L} \left\{ z^{\frac{A}{2}+nI} J_\nu(2(\sigma z)^{1/2}) \right\} \\
 &= \sum_{r=0}^{\infty} \frac{\Gamma^{-1}(A+(1+r)I) (-\sigma)^r \sigma^{A/2}}{r!} \mathcal{L} \left\{ z^{A+(n+r)I} \right\} \\
 &= \sigma^{A/2} \Gamma^{-1}(A+I) \sum_{r=0}^{\infty} \frac{(-)^r [(A+I)_r]^{-1}}{r!} \Gamma \\
 &\quad \cdot (A+(r+n+1)I) \lambda^{-(A+(r+n+1)I)} \\
 &= \sigma^{A/2} (A+I)_n \lambda^{-(A+(n+1)I)} \exp\left(\frac{-\sigma}{\lambda}\right) \\
 &\quad \cdot \sum_{r=0}^n \frac{(-nI)_r [(A+I)_r]^{-1}}{r!} \left(\frac{\sigma}{\lambda}\right)^r \\
 &= \sigma^{\frac{A}{2}+nI} (A+I)_n \lambda^{-(A+(2n+1)I)} \exp\left(\frac{-\sigma}{\lambda}\right) \\
 &\quad \cdot \sum_{r=0}^n \frac{(-nI)_r [(A+I)_r]^{-1}}{r!} \left(\frac{\sigma}{\lambda}\right)^{r-n}.
 \end{aligned} \tag{60}$$

Putting $n - r = k$, we obtain

$$\begin{aligned}
 \mathcal{E}_{11}(\lambda) &= (-1)^n \sigma^{\frac{A}{2}+nI} \lambda^{-(A+(2n+1)I)} \exp\left(\frac{-\sigma}{\lambda}\right) \\
 &\quad \times \sum_{k=0}^n \frac{(-nI)_k (-(A+nI))_k}{k!} \left(\frac{-\lambda}{\sigma}\right)^k \\
 &= (-1)^n \sigma^{\frac{A}{2}+nI} \lambda^{-(A+(2n+1)I)} \exp\left(\frac{-\sigma}{\lambda} z\right) \mathcal{B}_n^{I-A-2nI, \sigma}(\lambda).
 \end{aligned} \tag{61}$$

This finalizes the proof of Theorem 20.

The remaining results, which are given in the following theorems, can also be proven in a similar way. So we prefer to omit the details.

Theorem 21. Let $z, \lambda \in \mathbb{C}$, $\Re(\lambda) > 0$, $n \in \mathbb{N}_0$, and $r \in \mathbb{N}$. Also, let θ and ϕ be matrices in $M_r(\mathbb{C})$ such that $\phi + kI$ are invertible for all $k \in \mathbb{N}_0$. Further, let

$$\mathcal{E}_{12}(\lambda) = (-\phi)^n \lambda^{\theta+(2n-2)I} \Gamma(2I-\theta) \mathcal{B}_n^{2I-\theta-2nI, \frac{\phi-\lambda I}{\lambda}}(-n). \tag{62}$$

Then,

$$g_{12}(z) = z^{-(\theta+(n-1)I)} \mathcal{B}_n^{\theta, \phi}(z^{-1}). \tag{63}$$

Theorem 22. Let $z, \lambda \in \mathbb{C}$, $\Re(\lambda) > 0$, $n \in \mathbb{N}_0$, and $r \in \mathbb{N}$. Also let θ and ϕ be matrices in $M_r(\mathbb{C})$ such that $\phi + kI$ are invertible for all $k \in \mathbb{N}_0$. Further, let

$$\mathcal{E}_{13}(\lambda) = \frac{1}{\lambda_2} F_0 \left[\begin{matrix} -n, \theta - (n+1)I \\ - \end{matrix} ; \lambda \phi^{-1} \right]. \tag{64}$$

Then,

$$g_{13}(z) = \mathcal{B}_n^{\theta, \phi}(z^{-1}). \tag{65}$$

Theorem 23. Let $z, \lambda, \mu \in \mathbb{C}$, $\Re(\lambda) > \Re(\mu) > 0$, $n \in \mathbb{N}_0$, and $r \in \mathbb{N}$. Also let θ and ϕ be matrices in $M_r(\mathbb{C})$ such that $\phi + kI$ are invertible for all $k \in \mathbb{N}_0$. Further, let

$$\mathcal{E}_{14}(\lambda) = (\lambda - \mu)^{-1} {}_2F_0 \left[\begin{matrix} -n, \theta - (1-n)I \\ - \end{matrix} ; (\lambda - \mu)\phi^{-1} \right]. \tag{66}$$

Then,

$$g_{14}(z) = \exp(\mu z) \mathcal{B}_n^{\theta, \phi}(z^{-1}). \tag{67}$$

5. Conclusion

In fact, this work is a continuation of the recent paper by Abdalla [44]. In the current manuscript, the authors introduced various Laplace integral formulas of generalized Bessel matrix polynomials with certain elementary matrix functions, Binomial matrix functions exponential function, logarithmic function, generalized hypergeometric matrix functions, and Bessel function of the first kind. We also presented inverse Laplace transforms of generalized Bessel matrix polynomials with some functions. It is obvious that the results presented here which are involved in certain matrices in $M_r(\mathbb{C})$ may reduce to yield the corresponding scalar ones when $r = 1$. Furthermore, the results derived in this article yields to many special cases; the interested reader may be referred to (see, e.g., [1, 7, 45]).

A remarkably large number of Laplace transforms and inverse Laplace transforms involving a variety of functions and polynomials have been presented (see, e.g., [45, pp. 129–299]). In this connection, we tried to give matrix versions of those outcomes for Laplace transforms and inverse Laplace formulas involving a variety of functions and polynomials (see, [45, pp. 129–299]).

Data Availability

No data were used to support the study.

Conflicts of Interest

This work does not have any conflicts of interest.

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