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Research Article

Well-Posedness and Stability Result of the Nonlinear Thermodiffusion Full von Kármán Beam with Thermal Effect and Time-Varying Delay

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In this work, we consider a new full von Kármán beam model with thermal and mass diffusion effects according to the Gurtin-Pinkin model combined with time-varying delay. Heat and mass exchange with the environment during thermodiffusion in the von Kármán beam. We establish the well-posedness and the exponential stability of the system by the energy method under suitable conditions.

1. Introduction and Preliminaries

In this paper, we are concerned with the following problem:

$$\begin{cases} w_{tt} - d_1\left[\left(u_x + \frac{1}{2}(w_x)^2\right)w_x\right]_x + d_2 w_{xxxx} + \mu_1 w_t + \mu_2 w_t(x, t-\tau(t)) = 0, \\ u_{tt} - d_1\left[u_x + \frac{1}{2}(w_x)^2\right]_x - \delta_1\theta_x - \delta_2 P_x = 0, \\ c\theta_t + dP_t - \int_0^\infty \beta_1(\sigma)\theta_{xx}(t-\sigma)d\sigma - \delta_1 u_{tx} = 0, \\ d\theta_t + rP_t - \int_0^\infty \beta_2(\sigma)P_{xx}(t-\sigma)d\sigma - \delta_2 u_{tx} = 0, \end{cases} \tag{1}$$

where

$$(x, \sigma, t) \in (0, L) \times \mathbb{R}_+ \times (0,\infty). \tag{2}$$

Here, $\tau(t) > 0$ represents the time-varying delay, and d_1, d_2, δ_1, δ_2, c, d, r, and μ_1 are positive constants; μ_2 is a real number, and β_1 and β_2 are the relaxation functions, with the initial data

$$w(x, 0) = w_0(x),$$

$$w_t(x, 0) = w_1(x),$$

$$u(x, 0) = u_0(x),$$

$$u_t(x,0) = u_1(x),$$
$$\theta(x,0) = \theta_0(x),$$
$$P(x,0) = P_0(x),$$
$$w_t(x, t-\tau(0)) = f_0(x, t-\tau(0)), \quad (3)$$

where

$$(x,t) \in (0,L)\times(0,\tau(0)), \quad (4)$$

and Neumann-Dirichlet boundary conditions

$$\begin{aligned} &w(x,t) = u(x,t) = P(x,t) = 0, \quad x = 0, L, \forall t \geq 0,\\ &w_x(x,t) = \theta(x,t) = 0, \quad x = 0, L, \forall t \geq 0. \end{aligned} \quad (5)$$

The case of time-varying delay in the wave equation has been studied recently by Nicaise et al. [1]; they proved the exponential stability under the condition

$$\mu_2 < \sqrt{1-d}\mu_1, \quad (6)$$

where d is a constant that satisfies

$$\tau'(t) \leq d < 1, \quad \forall t > 0. \quad (7)$$

For the wave equation with a time-varying delay, in [1], the authors consider the system

$$\begin{cases} u_{tt} - \Delta u = 0,\\ u(x,t) = 0,\\ \dfrac{du}{dv}(x,t) = \mu_1 u_t(x,t) + \mu_2 u_t(x, t-\tau(t)), \end{cases} \quad (8)$$

where the time-varying delay $\tau(t) > 0$ satisfies

$$0 \leq \tau(t) \leq \overline{\tau}, \quad \forall t > 0, \quad (9)$$

$$\tau'(t) \leq 1, \quad \forall t > 0, \quad (10)$$

$$\tau(t) \in W^{2,\infty}([0,T]), \quad \forall T > 0. \quad (11)$$

They proved the exponential stability under suitable conditions.

The purpose of this work is to study problem (1)–(5), with a delay term appearing in the control term at the first equation, introducing the time-varying delay term $\beta_2 w_t(x, t-\tau(t))$; thermal and mass diffusion effects make the problem different from those considered in the literature (see [2–30]).

This paper is organized as follows: in the rest of this section, we put the preliminaries necessary for problem (1); in Section 2, we establish the well-posedness. As for Section 3, we prove the exponential stability result by the energy method and Lyapunov function.

In order to prove the existence of a unique solution of problem (1)–(5), we introduce the new variable

$$z(x,\rho,t) = w_t(x, t-\tau(t)\rho). \quad (12)$$

Then, we obtain

$$\begin{cases} \tau(t) z_t(x,\rho,t) + \left(1-\tau'(t)\rho\right) z_\rho(x,\rho,t) = 0,\\ z(x,0,t) = w_t(x,t). \end{cases} \quad (13)$$

And it is more convenient to work in the history space setting by introducing the so-called summed past history of θ and P defined by (see [31–36])

$$\begin{cases} \eta^t(\sigma) = \displaystyle\int_0^\sigma \theta(t-\zeta)\,d\zeta,\\ \nu^t(\sigma) = \displaystyle\int_0^\sigma P(t-\zeta)\,d\zeta, \quad (t,\sigma) \in [0,\infty)\times\mathbb{R}_+. \end{cases} \quad (14)$$

Differentiating $(14)_1$ and $(14)_2$, we get

$$\begin{cases} \eta_t^t(\sigma) + \eta_\sigma^t(\sigma) = \theta(t),\\ \nu_t^t(\sigma) + \nu_\sigma^t(\sigma) = P(t), \end{cases} \quad (15)$$

with the boundary and initial conditions

$$\begin{aligned} &\eta^t(0) = \nu^t(0) = 0, \quad t \geq 0,\\ &\eta^0(\sigma) = \eta_0(\sigma), \nu^0(\sigma) = \nu_0(\sigma), \quad \sigma \geq 0. \end{aligned} \quad (16)$$

We set

$$\begin{cases} \eta_0(\sigma) = \displaystyle\int_0^\sigma \overline{\theta}_0(\tau)\,d\tau,\\ \nu_0(\sigma) = \displaystyle\int_0^\sigma \overline{P}_0(\tau)\,d\tau, \sigma \in \mathbb{R}_+. \end{cases} \quad (17)$$

Concerning the memory kernels β_1 and β_2, we set

$$\begin{aligned} \beta(\sigma) &= -\beta_1{}'(\sigma),\\ \lambda(\sigma) &= -\beta_2{}'(\sigma). \end{aligned} \quad (18)$$

Assuming $\beta_1(\infty) = \beta_2(\infty) = 0$, then from (14), we infer

$$\begin{cases} \displaystyle\int_0^\infty \beta_1(\sigma)\theta(t-\sigma)\,d\sigma = -\int_0^\infty \beta_1{}'(\sigma)\eta^t(\sigma)\,d\sigma,\\ \displaystyle\int_0^\infty \beta_2(\sigma)P(t-\sigma)\,d\sigma = -\int_0^\infty \beta_2{}'(\sigma)\nu^t(\sigma)\,d\sigma, \end{cases} \quad (19)$$

and therefore,

$$\begin{cases} \int_0^\infty \beta_1(\sigma)\theta_{xx}(t-\sigma)d\sigma = \int_0^\infty \beta(\sigma)\eta_{xx}^t(\sigma)d\sigma, \\ \int_0^\infty \beta_2(\sigma)P_{xx}(t-\sigma)d\sigma = \int_0^\infty \lambda(\sigma)\nu^t(\sigma)d\sigma. \end{cases} \quad (20)$$

Consequently, the problem is equivalent to

$$\begin{cases} w_{tt} - d_1\left[\left(u_x + \frac{1}{2}(w_x)^2\right)w_x\right]_x + d_2 w_{xxxx} + \mu_1 w_t + \mu_2 z(x,1,t) = 0, \\ u_{tt} - d_1\left[u_x + \frac{1}{2}(w_x)^2\right]_x - \delta_1\theta_x - \delta_2 P_x = 0, \\ c\theta_t + dP_t - \int_0^\infty \beta(\sigma)\eta_{xx}^t(\sigma)d\sigma - \delta_1 u_{tx} = 0, \\ d\theta_t + rP_t - \int_0^\infty \lambda(\sigma)\nu_{xx}^t(\sigma)d\sigma - \delta_2 u_{tx} = 0, \\ \eta_t^t + \eta_\sigma^t = \theta, \\ \nu_t^t + \nu_\sigma^t = P, \\ \tau(t)z_t(x,\rho,t) + \left(1-\tau'(t)\rho\right)z_\rho(x,\rho,t) = 0, \end{cases} \quad (21)$$

where

$$(x,\sigma,\rho,t) \in (0,L)\times\mathbb{R}_+\times(0,1)\times(0,\infty), \quad (22)$$

with the initial and boundary conditions

$$\begin{cases} w(x,t) = w_x(x,t) = u(x,t) = P(x,t) = \theta(x,t) = 0, \quad x = 0, L, \\ w(x,0) = w_0(x), w_t(x,0) = w_1(x), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), \\ \theta(x,0) = \theta_0(x), P(x,0) = P_0(x), \\ z(x,\rho,0) = f_0(x,-\rho\tau(0)), \\ \eta^0(x,\sigma) = \eta_0(x,\sigma), \nu^0(x,\sigma) = \nu_0(x,\sigma), \quad (x,\sigma)\in(0,1)\times\mathbb{R}_+, \end{cases} \quad (23)$$

$$\forall (x,\rho,\sigma,t) \in (0,L)\times(0,1)\times(0,\infty)\times(0,\infty), \quad (24)$$

where the function $\tau(t)$ satisfies (7), (11), and the condition

$$0 < \tau_0 < \tau(t) < \bar{\tau}, \quad \forall t > 0. \quad (25)$$

In this paper, we establish the well-posedness and prove the exponential stability by using the variable of Kato under some restrictions and assumptions:

(H1).

$$|\mu_2| \le \sqrt{1-d}\mu_1. \quad (26)$$

(H2). The symmetric matrix Λ is positive definite, where

$$\Lambda = \begin{pmatrix} c\, d \\ d\, r \end{pmatrix}. \quad (27)$$

That is, $|\Lambda| = cr - d^2 > 0$ implies that

$$c\int_0^L \theta^2 dx + 2d\int_0^L \theta P dx + r\int_0^L P^2 dx > 0. \quad (28)$$

Condition (28) is needed to stabilize the system when diffusion effects are added to thermal effects (see, e.g., [31–38] for more information on this). By virtue of $cr > d^2$, we deduce that $d/c < r/d$. Let, then, ζ be a number chosen in such a way that

$$\frac{d}{c} < \zeta < \frac{r}{d}. \quad (29)$$

Thus, Young's inequality leads to

$$2d\int_0^L \theta P dx \le \frac{d}{\zeta}\int_0^L \theta^2 dx + d\zeta\int_0^L P^2 dx. \quad (30)$$

(H3). We assume the following set of hypotheses on μ and λ:

$$\begin{gathered} \beta,\lambda \in C^1(\mathbb{R}_+)\cap L^1(\mathbb{R}_+), \\ \beta(\sigma),\lambda(\sigma) \ge 0, \beta'(\sigma),\lambda'(\sigma) \le 0, \quad \forall\sigma\in\mathbb{R}_+, \\ \beta'(\sigma)+\alpha_1\beta(\sigma) \le 0, \lambda'(\sigma)+\alpha_2\lambda(\sigma) \le 0, \quad \text{for some } \alpha_1,\alpha_2 > 0, \forall\sigma\in\mathbb{R}_+, \end{gathered} \quad (31)$$

$$\begin{cases} \beta(0) = \int_0^\infty \beta(\sigma)d\sigma := \beta_0 > 0, \\ \lambda(0) = \int_0^\infty \lambda(\sigma)d\sigma := \lambda_0 > 0. \end{cases} \quad (32)$$

Let f be a memory kernel satisfying the assumptions (31) and (32).

Now, we consider the weighted Hilbert spaces

$$\begin{aligned} \mathscr{M}_f &= L^2\left(\mathbb{R}_+, H_0^1(0,L)\right) \\ &= \left\{\Phi : \mathbb{R}_+ \to H_0^1(0,L) / \int_0^L\int_0^\infty f(\sigma)\Phi_x^2(\sigma)d\sigma dx < \infty\right\}, \end{aligned} \quad (33)$$

equipped with the inner product

$$\langle \Phi,\Psi\rangle_{\mathscr{M}_f} = \int_0^L\int_0^\infty f(\sigma)\Phi_x(\sigma)\Psi_x(\sigma)d\sigma dx, \quad (34)$$

and the norm

$$\|\Phi\|^2_{\mathscr{M}_f} = <\Phi, \Phi>_{\mathscr{M}_f} = \int_0^L \int_0^\infty f(\sigma)\Phi_x^2(\sigma)d\sigma dx. \quad (35)$$

We also introduce the linear operator T on $\mathscr{M}_f$ defined by

$$T_\Phi = -\Phi_\sigma, \quad \Phi \in \mathscr{D}(T), \quad (36)$$

with

$$\mathscr{D}(T) = \{\Phi \in \mathscr{M}_f / \Phi_\sigma \in \mathscr{M}_f, \Phi(0) = 0\}, \quad (37)$$

where Φ_σ is the distributional derivative of Φ with respect to the internal variable σ, and then, the operator T is the infinitesimal generator of a C_0-semigroup of contractions. Following Ref. [39], there holds

$$\begin{aligned} <T\Phi, \Phi>_{\mathscr{M}_f} &= <-\Phi_\sigma, \Phi>_{\mathscr{M}_f} \\ &= -\frac{1}{2}\int_0^\infty f(\sigma)\frac{d}{d\sigma}\int_0^L \Phi_x^2(\sigma)dxd\sigma, \quad \forall \Phi \in \mathscr{D}(T). \end{aligned} \quad (38)$$

Integration by parts yields

$$\begin{aligned} &\int_0^\infty f(\sigma)\frac{d}{d\sigma}\int_0^L \Phi_x^2(\sigma)dxd\sigma \\ &= f(\sigma)\int_0^L \Phi_x^2(\sigma)dx\Big|_0^\infty - \int_0^\infty f'(\sigma)\int_0^L \Phi_x^2(\sigma)dxd\sigma. \end{aligned} \quad (39)$$

Hence, from (31), we obtain

$$<T\Phi, \Phi>_{\mathscr{M}_f} = \frac{1}{2}\int_0^\infty f'(\sigma)\int_0^L \Phi_x^2(\sigma)dxd\sigma \le 0. \quad (40)$$

As a direct consequence, we deduce from (32) and (40) that

$$\begin{aligned} <T\eta, \eta>_{\mathscr{M}_\beta} &= \frac{1}{2}\int_0^\infty \beta'(\sigma)\int_0^L \eta_x^2(\sigma)dxd\sigma \\ &\le -\frac{\alpha_1}{2}\int_0^\infty \beta(\sigma)\int_0^L \eta_x^2(\sigma)dxd\sigma = -\frac{\alpha_1}{2}\|\eta_x\|^2_{\mathscr{M}_\beta}, \\ <T\nu, \nu>_{\mathscr{M}_\lambda} &= \frac{1}{2}\int_0^\infty \lambda'(\sigma)\int_0^L \nu_x^2(\sigma)dxd\sigma \\ &\le -\frac{\alpha_2}{2}\int_0^\infty \lambda(\sigma)\int_0^L \nu_x^2(\sigma)dxd\sigma = -\frac{\alpha_2}{2}\|\nu_x\|^2_{\mathscr{M}_\lambda}, \end{aligned} \quad (41)$$

for all $\eta, \nu \in \mathscr{D}(T)$. Finally, we define the operator $\mathbb{L}_f : \mathscr{D}(\mathbb{L}_f) \to L^2(0, L)$ by

$$\mathbb{L}_f\Phi = \int_0^\infty f(\sigma)\Phi_{xx}(\sigma)d\sigma, \quad (42)$$

with the domain

$$\mathscr{D}(\mathbb{L}_f) = \left\{\Phi \in \mathscr{M}_f / \int_0^\infty f(\sigma)\Phi_{xx}(\sigma)d\sigma \in L^2(0, L), \Phi(0) = 0\right\}. \quad (43)$$

2. Well-Posedness

In this section, we give sufficient conditions that guarantee the well-posedness of this problem. Let

$$U = (w, w_t, u, u_t, \theta, \eta^t, P, \nu^t, z)^T. \quad (44)$$

For the sake of simplicity, we write $\eta = \eta^t(\sigma)$ and $\nu = \nu^t(\sigma)$ and the new dependent variables $\varphi = \omega_t$ and $\psi = u_t$; then, (21)–(23) can be written as

$$\begin{cases} U' = \mathscr{A}(t)U + \mathscr{F}(U), \\ U(0) = (w_0, w_1, u_0, u_1, \theta_0, \eta_0, P_0, \nu_0, f_0(., -\rho\tau(0)))^T, \end{cases} \quad (45)$$

with the linear problem

$$\begin{cases} U' = \mathscr{A}(t)U, \\ U(0) = (w_0, w_1, u_0, u_1, \theta_0, \eta_0, P_0, \nu_0, f_0(., -\rho\tau(0)))^T, \end{cases} \quad (46)$$

where the time-varying operator $\mathscr{A}$ is defined by

$$\mathscr{A}(t)\begin{pmatrix} w \\ \varphi \\ u \\ \psi \\ \theta \\ \eta \\ P \\ \nu \\ z \end{pmatrix} = \begin{pmatrix} \varphi \\ -d_2 w_{xxxx} - \mu_1\varphi - \mu_2 z(x, 1, t) \\ \psi \\ d_1 u_{xx} + \delta_1\theta_x + \delta_2 P_x \\ -\frac{1}{\alpha_1}\left[(d\delta_2 - r\delta_1)\psi_x - r\mathbb{L}_\beta\eta + d\mathbb{L}_\lambda\nu\right] \\ \theta + T\eta \\ -\frac{1}{\alpha_2}\left[(d\delta_1 - c\delta_2)\psi_x + d\mathbb{L}_\beta\eta - c\mathbb{L}_\lambda\nu\right] \\ P + T\nu \\ \frac{(\tau'(t)\rho - 1)}{\tau(t)} z_\rho \end{pmatrix}, \quad (47)$$

$$\mathscr{F}(U)=\begin{pmatrix}0\\ d_1\left[u_x+\frac{1}{2}(w_x)^2\right]_x\\ 0\\ \frac{d_1}{2}(w_x)^2\\ 0\\ 0\\ 0\\ 0\\ 0\end{pmatrix}. \quad (48)$$

The energy space $\mathscr{H}$ is defined as

$$\begin{aligned}\mathscr{H}=&\left[H^4(0,L)\cap H_0^2(0,L)\right]\times H_0^1(0,L)\\ &\times\left[H^2(0,L)\cap H_0^2(0,L)\right]\times H_0^1(0,L)\times L^2(0,L)\\ &\times\mathscr{M}_\beta\times L^2(0,L)\times\mathscr{M}_\lambda\times L^2((0,L),(0,1)),\end{aligned} \quad (49)$$

and the domain of $\mathscr{A}$ is

$$\begin{aligned}\mathscr{D}(\mathscr{A}(t))=\{&U\in\mathscr{H}/\varphi=z(.,0),\theta,P\in H_0^1(0,L),\mathbb{L}_\beta\eta,\mathbb{L}_\lambda\nu\\ &\in L^2(0,L),\eta,\nu\in\mathscr{D}(T)\}.\end{aligned} \quad (50)$$

We equip $\mathscr{H}$ with the inner product

$$\begin{aligned}<U,\bar{U}>_{\mathscr{H}}=&\int_0^L\{\varphi\bar{\varphi}+d_1u_x\bar{u}_x+\psi\bar{\psi}+d_2w_{xx}\bar{w}_{xx}\}dx\\ &+\int_0^L\int_0^1 z(x,\rho,t)\bar{z}(x,\rho,t)d\rho dx\\ &+<\Lambda(\theta,P)^T,(\bar{\theta},\bar{P})^T>+<\eta,\bar{\eta}>_{\mathscr{M}_\beta}+<\nu,\bar{\nu}>_{\mathscr{M}_\lambda},\end{aligned} \quad (51)$$

with the existence and the uniqueness in the following result.

Theorem 1. *Let (7), (11), and (25) be satisfied and assume that (26)–(31) hold. Then, for all $U_0\in\mathscr{D}(\mathscr{A}(0))$, there exists a unique solution U of problem (21)–(23) satisfying*

$$U\in C([0,+\infty),\mathscr{D}(\mathscr{A}(0))\cap C^1([0,+\infty),\mathscr{H}). \quad (52)$$

In order to prove Theorem 1, we will use the variable norm technique developed by Kato in [40]. The following theorem is proved in [40].

Theorem 2. *Assume that*

(1) $\mathscr{D}(\mathscr{A}(0))$ is a dense subset of $\mathscr{H}$

(2) $\mathscr{D}(\mathscr{A}(t))=\mathscr{D}(\mathscr{A}(0)),\forall t>0$

(3) For all $t\in[0,T]$, $\mathscr{A}(t)$ generates a strongly continuous semigroup on $\mathscr{H}$ and the family $\mathscr{A}=\{\mathscr{A}(t): t\in[0,T]\}$ is stable with stability constants C and m independent of t; i.e., the semigroup $(S_t(s))_{s\geq 0}$ generated by $\mathscr{A}(t)$ satisfies

$$\|S_t(s)(u)\|_{\mathscr{H}}\leq Ce^{ms}\|u\|_{\mathscr{H}},\quad \forall u\in\mathscr{H},s\geq 0. \quad (53)$$

(4) $d_t\mathscr{A}(t)\in L_*^\infty([0,T],B(\mathscr{D}(\mathscr{A}(0)),\mathscr{H}))$, where $L_*^\infty([0,T],B(\mathscr{D}(\mathscr{A}(0)),\mathscr{H}))$ is the space of equivalent classes of essentially bounded, strongly measurable functions from $[0,T]$ into the set $B(\mathscr{D}(\mathscr{A}(0)),\mathscr{H})$ of bounded operators from $\mathscr{D}(\mathscr{A}(0))$ into $\mathscr{H}$

Then, problem (46) has a unique solution

$$U\in C([0,T],\mathscr{D}(\mathscr{A}(0)))\cap C^1([0,T],\mathscr{H}), \quad (54)$$

for any initial datum in $\mathscr{D}(\mathscr{A}(0))$.

Proof. To prove Theorem 1, we use the method in [1] with the necessary modification.

(1) First, we show that $\mathscr{D}(\mathscr{A}(0))$ is dense in $\mathscr{H}$

Let $F=(f_1,f_2,f_3,f_4,f_5,f_6,f_7,f_8,f_9)\in\mathscr{H}$ be orthogonal to all elements of $\mathscr{D}(\mathscr{A}(0))$ with respect to the inner product $\langle .,.\rangle_{\mathscr{H}}$:

$$\begin{aligned}0=<U,F>_{\mathscr{H}}=&\int_0^L\{\varphi f_2+\psi f_4+d_2w_{xx}f_{1xx}+d_1u_xf_{3x}\}dx\\ &+\int_0^L\int_0^1 z(x,\rho,t)f_9d\rho dx+<\Lambda(\theta,P)^T,(f_5,f_7)^T>\\ &+<\eta,f_6>_{\mathscr{M}_\beta}+<\nu,f_8>_{\mathscr{M}_\lambda}.\end{aligned} \quad (55)$$

For all $U=(w,\varphi,u,\psi,\theta,\eta,P,\nu,z)^T\in\mathscr{D}(\mathscr{A}(0))$, our goal is to prove that $f_i=0,\forall i=1,\cdots,9$. Let us first take $z\in\mathscr{D}((0,L)\times(0,1))$ and $w=\varphi=\psi=u=\theta=q=\phi=0$, so the vector $U=(0,0,0,0,0,0,0,0,z)^T\in\mathscr{D}(\mathscr{A}(0))$, and therefore, from (55), we deduce that

$$\int_0^L\int_0^1 z(x,\rho)f_7d\rho dx=0. \quad (56)$$

Since $\mathscr{D}((0,L)\times(0,1))$ is dense in $L^2((0,L)\times(0,1))$, it follows then that $f_9=0$.

Similarly, let $\varphi\in H_0^1(0,L)$; then, $U=(0,\varphi,0,0,0,0,0,0,0)^T\in\mathscr{D}(\mathscr{A}(0))$, which implies from (55) that

$$\int_0^L\varphi f_2dx=0. \quad (57)$$

So, as above, $f_2=0$.

And let $U = (w, 0, 0, 0, 0, 0, 0, 0, 0)^T$; then, we obtain from (55) that

$$\int_0^L w_{xx} f_{1xx} dx = 0. \quad (58)$$

It is obvious that $U = (w, 0, 0, 0, 0, 0, 0, 0, 0)^T \in \mathscr{D}(\mathscr{A}(0))$ only if $w \in H^4(0, L) \cap H_0^2(0, L)$ is dense in $H_0^2(0, L)$, with respect to the inner product

$$<g, h>_{H_0^2(0,L)} \int_0^L g_{xx} h_{xx} dx. \quad (59)$$

We get $f_1 = 0$. By the same ideas as above, we can also show that $f_3 = 0$.

For $u \in \mathscr{D}(\mathscr{A}(t))$, we get from (55) that

$$\int_0^L u_x f_{3x} dx = 0, \quad (60)$$

and by the density of $\mathscr{D}(\mathscr{A}(t))$ in $H_0^1(0, L)$, we obtain $f_3 = 0$.

For $\psi \in \mathscr{D}(\mathscr{A}(t))$, we get from (55) that

$$\int_0^L \psi f_4 dx = 0, \quad (61)$$

and by the density of $\mathscr{D}(\mathscr{A}(t))$ in $H^1(0, L)$, we obtain $f_4 = 0$.

Next, let $U = (0, 0, 0, 0, \theta, 0, 0, 0, 0)^T$; then, we obtain from (55) that

$$\int_0^L \theta f_5 dx = 0. \quad (62)$$

It is obvious that $U = (0, 0, 0, 0, \theta, 0, 0, 0, 0)^T \in \mathscr{D}(\mathscr{A}(0))$ only if $\theta \in L^2(0, L)$ is dense in $L^2(0, L)$; we get $f_5 = 0$; for $\eta \in \mathscr{M}_\beta$, we get from (55) that

$$\int_0^L \int_0^\infty \beta(\sigma) \eta_x f_{6x} d\sigma dx = 0, \quad (63)$$

which gives $f_6 = 0$. Similarly, for P and ν. This completes the proof of (1).

(2) With our choice, $\mathscr{D}(\mathscr{A}(t))$ is independent of t; consequently,

$$\mathscr{D}(\mathscr{A}(t)) = \mathscr{D}(\mathscr{A}(0)), \quad \forall t > 0. \quad (64)$$

(3) Now, we show that the operator $\mathscr{A}(t)$ generates a C_0-semigroup in $\mathscr{H}$ for a fixed t. We define the time-dependent inner product on $\mathscr{H}$:

$$\begin{aligned} <U, \bar{U}>_t = & \int_0^L \{\varphi\bar{\varphi} + d_1 u_x \bar{u}_x + \psi\bar{\psi} + d_2 w_{xx} \bar{w}_{xx}\} dx \\ & + \xi\tau(t) \int_0^L \int_0^1 z(x, \rho, t) \bar{z}(x, \rho, t) d\rho dx \\ & + <\Lambda(\theta, P)^T, (\bar{\theta}, \bar{P})^T> + <\eta, \bar{\eta}>_{\mathscr{M}_\beta} + <\nu, \bar{\nu}>_{\mathscr{M}_\lambda}, \end{aligned} \quad (65)$$

where ξ satisfies

$$\frac{|\mu_2|}{\sqrt{1-d}} \le \xi \le \left(2\mu_1 - \frac{|\mu_2|}{\sqrt{1-d}}\right), \quad (66)$$

thanks to hypothesis (26).

Let us set

$$\kappa(t) = \frac{\left(\tau'(t)^2 + 1\right)^{1/2}}{2\tau(t)}. \quad (67)$$

In this step, we prove the dissipativity of the operator $\bar{\mathscr{A}}(t) = \mathscr{A}(t) - \tau(t) I$.

For a fixed t and $U = (w, \varphi, u, \psi, \theta, \eta, P, \nu, z)^T \in \mathscr{D}(\mathscr{A}(t))$, we have

$$\begin{aligned} <\mathscr{A}(t)U, U>_t = & -\mu_1 \int_0^L \varphi^2 dx - \mu_2 \int_0^L \varphi z(x, 1, t) dx \\ & + <T\eta, \eta>_{\mathscr{M}_\beta} + <T\nu, \eta>_{\mathscr{M}_\lambda} \\ & - \xi \int_0^L \int_0^1 \left(1 - \tau'(t)\rho\right) z(x, \rho, t) z_\rho(x, \rho, t) d\rho dx. \end{aligned} \quad (68)$$

Observe that

$$\begin{aligned} & \int_0^L \int_0^1 \left(1 - \tau'(t)\rho\right) z(x, \rho, t) z_\rho(x, \rho, t) d\rho dx \\ & = \frac{1}{2} \int_0^L \int_0^1 \left(1 - \tau'(t)\rho\right) \frac{d}{d\rho} z^2 d\rho dx \\ & = \frac{\tau'(t)}{2} \int_0^L \int_0^1 z^2(x, \rho, t) d\rho dx \\ & \quad + \frac{1}{2} \int_0^L \left\{ z^2(x, 1, t)\left(1 - \tau'(t)\right) - z^2(x, 0, t) \right\} d\rho dx, \\ & <T\eta, \eta>_{\mathscr{M}_\beta} + <T\nu, \eta>_{\mathscr{M}_\lambda} \\ & = + \frac{1}{2} \int_0^\infty \beta'(\sigma) \int_0^L \eta_x^2(\sigma) dx d\sigma + \frac{1}{2} \int_0^\infty \lambda'(\sigma) \int_0^L \nu_x^2(\sigma) dx d\sigma \\ & \le -\frac{\alpha_1}{2} \|\eta(\sigma)\|_{\mathscr{M}_\beta}^2 - \frac{\alpha_2}{2} \|\nu(\sigma)\|_{\mathscr{M}_\lambda}^2, \end{aligned} \quad (69)$$

whereupon

$$<\mathscr{A}(t)U, U>_t = -\mu_1 \int_0^L \varphi^2 dx - \mu_2 \int_0^L \varphi z(x,1,t) dx - \frac{\alpha_1}{2} \|\eta(\sigma)\|^2_{\mathscr{M}_\beta} - \frac{\alpha_2}{2} \|\nu(\sigma)\|^2_{\mathscr{M}_\lambda} - \frac{\xi \tau'(t)}{2} \int_0^L \int_0^1 z^2(x,\rho,t) d\rho dx - \frac{\xi}{2} \int_0^L z^2(x,1,t)\left(1-\tau'(t)\right) dx + \frac{\xi}{2} \int_0^L \varphi^2 dx. \quad (70)$$

By using Young's inequality and (7), we get

$$<\mathscr{A}(t)U, U>_t \le \left(-\mu_1 + \frac{|\mu_2|}{2\sqrt{1-d}} + \frac{\xi}{2}\right) \int_0^L \varphi^2 dx + \left(\frac{|\mu_2|\sqrt{1-d}}{2} - \xi \frac{(1-d)}{2}\right) \int_0^L z^2(x,1,t) dx - \frac{\alpha_1}{2} \|\eta(\sigma)\|^2_{\mathscr{M}_\beta} - \frac{\alpha_2}{2} \|\nu(\sigma)\|^2_{\mathscr{M}_\lambda} + \kappa(t) < U, U>_t, \quad (71)$$

under condition (66) which allows to write

$$\begin{aligned} -\mu_1 + \frac{|\mu_2|}{2\sqrt{1-d}} + \frac{\xi}{2} &\le 0, \\ \frac{|\mu_2|\sqrt{1-d}}{2} - \xi \frac{(1-d)}{2} &\le 0. \end{aligned} \quad (72)$$

Consequently, the operator $\tilde{\mathscr{A}}(t) = \mathscr{A}(t) - \kappa(t) I$ is dissipative.

Now, we prove the subjectivity of the operator $I - \mathscr{A}(t)$ for fixed $t > 0$.

Let $(f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9)^T \in \mathscr{H}$; we seek $U = (w, \varphi, u, \psi, \theta, \eta, P, \nu, z)^T \in \mathscr{D}(\mathscr{A}(t))$ solution of the following system:

$$\begin{cases} w - \varphi = f_1, \\ \varphi + d_2 w_{xxxx} + \mu_1 \varphi + \mu_2 z(.,1,t) = f_2, \\ u - \psi = f_3, \\ \psi - d_1 u_{xx} - \delta_1 \theta_x - \delta_2 P_x = f_4, \\ \alpha_1 \theta + (d\delta_2 - r\delta_1)\psi_x - r\mathbb{L}_\beta \eta + d\mathbb{L}_\lambda \nu = \alpha_1 f_5, \\ \eta - \theta - T\eta = f_6, \\ \alpha_2 P + (d\delta_1 - c\delta_2)\psi_x + d\mathbb{L}_\beta \eta + -c\mathbb{L}_\lambda \nu = \alpha_2 f_7, \\ \nu - P - T\nu = f_8, \\ z - \dfrac{\left(\tau'(t)\rho - 1\right)}{\tau(t)} z_\rho = f_9. \end{cases} \quad (73)$$

Suppose that we have found w and u. Then,

$$\begin{cases} w - \varphi = f_1, \\ u - \psi = f_3. \end{cases} \quad (74)$$

Furthermore, by (73), we can find z as

$$z(x,0) = \varphi(x), \quad x \in (0, L). \quad (75)$$

Following the same approach as in [1], we obtain, by using the last equation in (73),

$$\begin{cases} z(x,\rho) = \varphi(x) e^{-\rho\tau(t)} + \tau(t) e^{-\rho\tau(t)} \displaystyle\int_0^1 f_9(x,y) e^{y\tau(t)} dy, & \text{if } \tau'(t) = 0, \\ z(x,\rho) = \varphi(x) e^{\eta_\rho(t)} + e^{\eta_\rho(t)} \displaystyle\int_0^1 \frac{\tau(t)}{1-\tau'(t)y} f_9(x,y) e^{-\eta_y(t)} dy, & \text{if } \tau'(t) \neq 0, \end{cases} \quad (76)$$

where $\eta_y(t) = (\tau(t)/\tau'(t)) \ln (1 - \tau'(t)\rho)$. Whereupon, from (74), we obtain

$$\begin{cases} z(x,\rho) = \varphi(x) e^{-\rho\tau(t)} - f_1 e^{-\rho\tau(t)} + \tau(t) e^{-\rho\tau(t)} \displaystyle\int_0^1 f_9(x,y) e^{y\tau(t)} dy, & \text{if } \tau'(t) = 0, \\ z(x,\rho) = \varphi(x) e^{\eta_\rho(t)} - f_1 e^{\eta_\rho(t)} + e^{\eta_\rho(t)} \displaystyle\int_0^1 \frac{\tau(t)}{1-\tau'(t)y} f_9(x,y) e^{-\eta_y(t)} dy, & \text{if } \tau'(t) \neq 0. \end{cases} \quad (77)$$

Integrating $(73)_6$ and $(73)_8$ with $\eta(0)=\nu(0)=0$, we have

$$\begin{cases} \eta(\sigma)=(1-e^{-\sigma})\theta+\int_0^\sigma e^{s-\sigma}f_6(s)ds, \\ \nu(\sigma)=(1-e^{-\sigma})P+\int_0^\sigma e^{s-\sigma}f_8(s)ds. \end{cases} \quad (78)$$

Substituting $(73)_{1,3,6,8,9}$ into the others, we obtain the following system. Now, we have to find w, u, θ, and P as solutions of the equations:

$$\begin{cases} w+d_2w_{xxxx}+\mu_1\varphi+\mu_2z(.,1,t)=f_2+f_1+\beta_1f_1, \\ u-d_1u_{xx}-\delta_1\theta_x-\delta_2P_x=f_4+f_3, \\ \alpha_1\theta-rC_\beta\theta_{xx}+dC_\lambda P_{xx}+(d\delta_2-r\delta_1)u_x=h_3, \\ \alpha_2P+dC_\beta\theta_{xx}-cC_\lambda P_{xx}+(d\delta_1-c\delta_1)u_x=h_4. \end{cases} \quad (79)$$

Solving (79), we get

$$\begin{cases} \mu_3w+d_2w_{xxxx}=h_1, \\ u-d_1u_{xx}-\delta_1\theta_x-\delta_2P_x=h_2, \\ \alpha_1\theta-rC_\beta\theta_{xx}+dC_\lambda P_{xx}+(d\delta_2-r\delta_1)u_x=h_3, \\ \alpha_2P+dC_\beta\theta_{xx}-cC_\lambda P_{xx}+(d\delta_1-c\delta_1)u_x=h_4, \end{cases} \quad (80)$$

where

$$\begin{cases} \mu_3=1+\mu_1+e^{-\tau(t)}, \\ h_1=f_2+(1+\mu_1)f_2-\mu_2z_0, \\ h_2=f_4+f_3, \\ h_3=\alpha_1f_5+(d\delta_2-r\delta_1)f_{3x}+r\int_0^\infty\beta(\sigma)\int_0^\sigma e^{s-\sigma}f_{6xx}(s)dsd\sigma-d\int_0^\infty\lambda(\sigma)\int_0^\sigma e^{s-\sigma}f_{8xx}(s)dsd\sigma, \\ h_4=\alpha_2f_7+(d\delta_1-c\delta_2)f_{5x}-d\int_0^\infty\beta(\sigma)\int_0^\sigma e^{s-\sigma}f_{6xx}(s)dsd\sigma+c\int_0^\infty\lambda(\sigma)\int_0^\sigma e^{s-\sigma}f_{8xx}(s)dsd\sigma. \end{cases} \quad (81)$$

From (77), we have

$$z(x,1)=\begin{cases} w(x)e^{-\tau(t)}+z_0(x), & \text{if } \tau'(t)=0, \\ w(x)e^{\eta_\rho(t)}+z_0(x), & \text{if } \tau'(t)\neq 0, \end{cases} \quad (82)$$

where $x\in(0,L)$ and

$$z_0(x)=\begin{cases} -f_1e^{-\rho\tau(t)}+\tau(t)e^{-\rho\tau(t)}\int_0^1 f_9(x,y)e^{y\tau(t)}dy, & \text{if } \tau'(t)=0, \\ -f_1e^{\eta_\rho(t)}+e^{\eta_\rho(t)}\int_0^1\frac{\tau(t)}{1-\tau'(t)y}f_9(x,y)e^{-\eta_y(t)}dy, & \text{if } \tau'(t)\neq 0. \end{cases} \quad (83)$$

It is clear from the above formula that z_0 depends only on f_1, f_9. Consequently, problem (80) is equivalent to

$$\zeta\left((w,u,\theta,P),\left(\widehat{w},\widehat{u},\widehat{\theta},\widehat{P}\right)\right)=\Gamma\left(\widehat{w},\widehat{u},\widehat{\theta},\widehat{P}\right), \quad (84)$$

where the bilinear form $\zeta:[H_0^2(0,L)\times H_0^1(0,L)\times L^2(0,L)\times L^2(0,L)]^2\to\mathbb{R}$ and the linear form $\Gamma:[H_0^2(0,L)\times H_0^1(0,L)\times L^2(0,L)\times L^2(0,L)]\to\mathbb{R}$ are defined by

$$\begin{aligned} &\zeta\left((w,u,\theta,P),\left(\widehat{w},\widehat{u},\widehat{\theta},\widehat{P}\right)\right) \\ &=\int_0^L(\mu_3w\widehat{w}+d_2w_{xx}\widehat{w}_{xx}+u\widehat{u}+d_1u_x\widehat{u}_x)dx+\alpha_1\int_0^L\theta\widehat{\theta}dx \\ &+\alpha_2\int_0^L P\widehat{P}dx+rC_\beta\int_0^L\theta_x\widehat{\theta}_xdx+cC_\lambda\int_0^L P_x\widehat{P}_xdx \\ &-dC_\beta\int_0^L\theta_x\widehat{P}_xdx-dC_\lambda\int_0^L P_x\widehat{\theta}_xdx \\ &+(d\delta_2-r\delta_1)\int_0^L u_x\widehat{\theta}dx+(d\delta_1-c\delta_2)\int_0^L u_x\widehat{P}dx \\ &+\int_0^L(\delta_1\theta+\delta_2P)\widehat{u}_xdx, \end{aligned}$$

$$\Gamma\left(\widehat{w},\widehat{u},\widehat{\theta},\widehat{P}\right)=\int_0^L h_1\widehat{w}dx+\int_0^L h_2\widehat{u}dx+\int_0^L h_3\widehat{\theta}dx+\int_0^L h_4\widehat{P}dx. \quad (85)$$

Now, for $\mathscr{H}_1 = H_0^2(0,L) \times H_0^1(0,L) \times L^2(0,L) \times L^2(0,L)$, equipped with the norm

$$\|(w,u,\theta,P)\|_{\mathscr{H}_1}^2 = \|w\|_2^2 + \|w_{xx}\|_2^2 + \|u\|_2^2 + \|u_x\|_2^2 + \|\theta\|_2^2 + \|\theta_x\|_2^2 + \|P_x\|_2^2 + \|P\|_2^2, \tag{86}$$

then, we have

$$\begin{aligned} B((w,u,\theta,P),(w,u,\theta,P)) &= \mu_3 \int_0^L w^2 dx + d_2 \int_0^L w_{xx}^2 dx + \int_0^L u^2 dx + d_1 \int_0^L u_x^2 dx \\ &+ \alpha_1 \int_0^L \theta^2 dx + \alpha_2 \int_0^L P^2 dx + rC_\beta \int_0^L \theta_x^2 dx + cC_\lambda \int_0^L P_x^2 dx \\ &- \left(dC_\beta + dC_\lambda\right) \int_0^L P_x \theta_x dx + (d\delta_2 - r\delta_1) \int_0^L u_x \theta dx \\ &+ (d\delta_1 - c\delta_2) \int_0^L u_x P dx + \int_0^L (\delta_1 \theta + \delta_2 P) u_x dx. \end{aligned} \tag{87}$$

Then, for some $M_0 > 0$,

$$B((w,u,\theta,P),(w,u,\theta,P)) \geq M_0 \|(w,u,\theta,P)\|_{\mathscr{H}_1}^2. \tag{88}$$

Thus, B is coercive.

By Cauchy-Schwarz's and Poincaré's inequalities, we obtain

$$B\left((w,u,\theta,P),\left(\widehat{w},\widehat{u},\widehat{\theta},\widehat{P}\right)\right) \leq M_1 \|(w,u,\theta,P)\|_{\mathscr{H}_1}^2 \left\|\left(\widehat{w},\widehat{u},\widehat{\theta},\widehat{P}\right)\right\|_{\mathscr{H}_1}^2. \tag{89}$$

Similarly, we get

$$\Gamma\left(\widehat{w},\widehat{u},\widehat{\theta},\widehat{P}\right) \leq M_2 \left\|\left(\widehat{w},\widehat{u},\widehat{\theta},\widehat{P}\right)\right\|_{\mathscr{H}_1}^2. \tag{90}$$

Consequently, applying the Lax-Milgram theorem, problem (84) admits a unique solution $(w,u,\theta,P) \in \mathscr{H}_1$, for all $(\widehat{w},\widehat{u},\widehat{\theta},\widehat{P}) \in \mathscr{H}_1$. Applying the classical elliptic regularity, it follows from (80) that $(w,u,\theta,P) \in \mathscr{H}_1$.

Therefore, the operator $I - \mathscr{A}(t)$ is surjective for any fixed $t > 0$. Since $\kappa(t) > 0$ and

$$I - \bar{\mathscr{A}}(t) = (1 + \kappa(t))I - \mathscr{A}(t), \tag{91}$$

we deduce that the operator $I - \bar{\mathscr{A}}(t)$ is also surjective for any $t > 0$.

To complete the proof of (3), it suffices to show that

$$\frac{\|U\|_t}{\|U\|_s} \leq e^{(c/2\tau_0)|t-s|}, \quad \forall t,s \in [0,T], \tag{92}$$

where $U = (w,\varphi,u,\psi,\theta,\eta,P,\nu,z)^T$ and $\|.\|_t$ is the norm associated with the inner product (56).

For $t,s \in [0,T]$, we have from (56) that

$$\begin{aligned} \|U\|_t^2 - \|U\|_s^2 e^{(c/\tau_0)|t-s|} &= \left(1 - e^{(c/\tau_0)|t-s|}\right) \int_0^L \left\{\varphi^2 + d_2 w_{xx}^2 + d_1 u_x^2 + \psi^2\right\} dx \\ &+ \left(1 - e^{(c/\tau_0)|t-s|}\right) < \Lambda(\theta,P)^T, (\theta,P)^T > \\ &+ \left(1 - e^{(c/\tau_0)|t-s|}\right) \left\{\|\eta\|_{\mathscr{M}_\beta}^2 + \|\nu\|_{\mathscr{M}_\lambda}^2\right\} \\ &+ \xi\left(\tau(t) - \tau(s) e^{(c/\tau_0)|t-s|}\right) \int_0^L \int_0^1 z^2(x,\rho,t) d\rho dx. \end{aligned} \tag{93}$$

It is clear that $(1 - e^{(c/\tau_0)|t-s|}) \leq 0$. Now, we will prove that $(\tau(t) - \tau(s)e^{(c/\tau_0)|t-s|}) \leq 0$ for $c > 0$. To do this, we have

$$\tau(t) = \tau(s) + \tau'(a)(t-s), \tag{94}$$

where $a \in (s,t)$, which implies

$$\frac{\tau(t)}{\tau(s)} \leq 1 + \frac{|\tau(a)|}{\tau(s)} |t-s|. \tag{95}$$

By using (11), we deduce that

$$\frac{\tau(t)}{\tau(s)} \leq 1 + \frac{c}{\tau_0} |t-s| \leq e^{(c/\tau_0)|t-s|}, \tag{96}$$

which proves (92); therefore, this completes the proof of (3).

(4) It is clear that

$$\frac{d}{dt}\mathscr{A}(t)U = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \dfrac{\left(\tau''(t)\tau'(t)\rho - \tau'(t)\left(\tau'(t)\rho - 1\right)\right)}{\tau^2(t)} z_\rho \end{pmatrix}. \tag{97}$$

Then, by (11) and (25), (4) holds exactly as in [1]. Consequently, from the above analysis, we deduce that the problem

$$\begin{cases} \bar{U}_t = \bar{\mathscr{A}}(t)\bar{U}_t, \\ \bar{U}_t(0) = U_0, \end{cases} \tag{98}$$

has a solution $\bar{U} \in C([0,\infty),\mathscr{H})$, and if $U_0 \in \mathscr{D}(\mathscr{A}(0))$, then

$$\bar{U} \in C([0,\infty),\mathscr{D}(\mathscr{A}(0))) \cap C^1([0,\infty),\mathscr{H}). \tag{99}$$

Now, let

$$U(t) = e^{\vartheta(t)}\bar{U}(t), \tag{100}$$

with $\vartheta(t) = \int_0^t \kappa(s)ds$; then, by using (98), we have

$$\begin{aligned} U_t(t) &= \kappa(t)e^{\vartheta(t)}\bar{U}(t) + e^{\vartheta(t)}\bar{U}_t(t) \\ &= \kappa(t)e^{\vartheta(t)}\bar{U}(t) + e^{\vartheta(t)}\bar{\mathscr{A}}(t)\bar{U}(t) \\ &= e^{\vartheta(t)}\left(\kappa(t)\bar{U}(t) + \bar{\mathscr{A}}(t)\bar{U}(t)\right) \\ &= e^{\vartheta(t)}\left(\mathscr{A}(t)\bar{U}(t)\right) = \mathscr{A}(t)U(t)). \end{aligned} \tag{101}$$

Consequently, $U(t)$ is the unique solution of (46).

It remains to prove that the operator $\mathscr{F}$ defined in (48) is locally Lipschitz in $\mathscr{H}$.

Let $U_1 = (w_1,\varphi_1,u_1,\psi_1,\theta_1,\eta_1,P_1,\nu_1,z_1)^T \in \mathscr{H}$ and $U_2 = (w_2,\varphi_2,u_2,\psi_2,\theta_2,\eta_2,P_2,\nu_2,z_2)^T \in \mathscr{H}$. Then, we have

$$\|\mathscr{F}(U_1) - \mathscr{F}(U_2)\| = d_1\left(|R|^2 + |K|^2\right), \tag{102}$$

where

$$\begin{aligned} R &= \left[\left(u_{1x} + \frac{1}{2}w_{1x}^2\right)w_{1x} - \left(u_{2x} + \frac{1}{2}w_{2x}^2\right)w_{2x}\right], \\ K &= \frac{1}{2}\left(w_{1x}^2 - w_{2x}^2\right). \end{aligned} \tag{103}$$

Adding and subtracting the term $(u_{1x} + (1/2)w_{1x}^2)w_{2x}$ inside the norm $|R|$, we find

$$\begin{aligned} |R| \le{}& \|w_{1x} - w_{2x}\|_{L^\infty(0,L)}|u_{1x} + \frac{1}{2}w_{1x}^2| + \|w_{2x}\|_{L^\infty}|u_{1x} - u_{2x}| \\ &+ \frac{1}{2}\|w_{2x}\|_{L^\infty}|w_{1x} + w_{2x}|\|w_{1x} - w_{2x}\|_{L^\infty(0,L)}. \end{aligned} \tag{104}$$

Using the embedding of $H^1(0,L)$ into $L^\infty(0,L)$, from (104), one has

$$|R| \le k_1(\|U_1\|_{\mathscr{H}},\|U_2\|_{\mathscr{H}})\|U_1 - U_2\|. \tag{105}$$

Using once again the embedding of $H^1(0,L)$ into $L^\infty(0,L)$, one also sees that

$$|K| \le k_2(\|U_1\|_{\mathscr{H}},\|U_2\|_{\mathscr{H}})\|U_1 - U_2\|. \tag{106}$$

Combining (102), (105), and (106), consequently, $\mathscr{F}(U)$ is locally Lipschitz continuous in $\mathscr{H}$. This ends the proof of Theorem 1.

3. General Decay

In this section, we state and prove the stability of system (21)–(23) using the multiplier technique under the assumptions (26)–(31).

We define the energy functional E by

$$\begin{aligned} E(t) ={}& \frac{1}{2}\int_0^L \left\{w_t^2 + u_t^2 + d_2 w_{xx}^2 + d_1\left(u_x + \frac{1}{2}w_x^2\right)^2 + c\theta^2 + rP^2\right\}dx \\ &+ d < \theta, P > + \frac{1}{2}\|\eta\|_{\mathscr{M}_\beta}^2 + \frac{1}{2}\|\nu\|_{\mathscr{M}_\lambda}^2 \\ &+ \frac{\xi}{2}\int_0^L\int_0^1 \tau(t)z^2(x,\rho,t)d\rho dx, \end{aligned} \tag{107}$$

where

$$\frac{|\mu_2|}{\sqrt{1-d}} \le \xi \le \left(2\mu_1 - \frac{|\mu_2|}{\sqrt{1-d}}\right). \tag{108}$$

The following lemma shows that the energy is decreasing.

Lemma 3. *Assume that (26)–(31) hold and the hypotheses (7), (11), and (25) are satisfied. Then, for $\forall C \ge 0$,*

$$\begin{aligned} E'(t) \le{}& -C\left(\int_0^L w_t^2 dx + \int_0^L z^2(x,1,t)dx\right) - \frac{\alpha_1}{4}\|\eta\|_{\mathscr{M}_\beta}^2 \\ &+ \frac{1}{4}\int_0^\infty \beta'(\sigma)\|\eta_x(\sigma)\|^2 d\sigma - \frac{\alpha_2}{4}\|\nu\|_{\mathscr{M}_\lambda}^2 \\ &+ \frac{1}{4}\int_0^\infty \lambda'(\sigma)\|\nu_x(\sigma)\|^2 d\sigma \le 0. \end{aligned} \tag{109}$$

Proof. Multiplying the equations of (21) by w_t, u_t, θ, η, P, ν, and ξz, respectively, then by integration by parts, we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_0^L \left\{w_t^2 + u_t^2 + d_2 w_{xx}^2 + d_1\left(u_x + \frac{1}{2}w_x^2\right)^2 + c\theta^2 + rP^2\right\}dx \\ &\quad + \frac{d}{dt}\left\{d < \theta, P > + \frac{1}{2}\|\eta\|_{\mathscr{M}_\beta}^2 + \frac{1}{2}\|\nu\|_{\mathscr{M}_\lambda}^2\right\} \\ &\quad + \frac{\xi}{2}\frac{d}{dt}\int_0^L\int_0^1 \tau(t)z^2(x,\rho,t)d\rho dx \\ &= -\mu_1\int_0^1 \omega_t^2 dx - \mu_2\int_0^L w_t z(x,1,t)dx \\ &\quad + \frac{1}{2}\int_0^\infty \beta'(\sigma)\|\eta_x(\sigma)\|^2 d\sigma + \frac{1}{2}\int_0^\infty \lambda'(\sigma)\|\nu_x(\sigma)\|^2 d\sigma \\ &\quad + \frac{\xi}{2}\int_0^L\int_0^1 \tau'(t)z^2(x,\rho,t)d\rho dx \\ &\quad - \xi\int_0^L\int_0^1 \left(1 - \tau'(t)\rho\right)z(x,\rho,t)z_\rho(x,\rho,t)d\rho dx \\ &\le -\mu_1\int_0^1 \omega_t^2 dx - \mu_2\int_0^L w_t z(x,1,t)dx - \frac{\alpha_1}{4}\|\eta\|_{\mathscr{M}_\beta}^2 - \frac{\alpha_2}{4}\|\nu\|_{\mathscr{M}_\lambda}^2 \end{aligned}$$

$$+\frac{1}{4}\int_0^\infty \beta'(\sigma)\|\eta_x(\sigma)\|^2 d\sigma + \frac{1}{4}\int_0^\infty \lambda'(\sigma)\|\nu_x(\sigma)\|^2 d\sigma$$
$$-\frac{\xi}{2}\int_0^L\int_0^1 \frac{d}{d\rho}\left(\left(1-\tau'(t)\rho\right)z^2(x,\rho,t)\right)d\rho dx$$
$$= -\mu_1\int_0^1 \omega_t^2 dx - \mu_2\int_0^L w_t z(x,1,t)dx - \frac{\alpha_1}{4}\|\eta\|^2_{\mathscr{M}_\beta} - \frac{\alpha_2}{4}\|\nu\|^2_{\mathscr{M}_\lambda}$$
$$+\frac{1}{4}\int_0^\infty \beta'(\sigma)\|\eta_x(\sigma)\|^2 d\sigma + \frac{1}{4}\int_0^\infty \lambda'(\sigma)\|\nu_x(\sigma)\|^2 d\sigma$$
$$+\frac{\xi}{2}\int_0^L\left(z^2(x,0,t)-z^2(x,1,t)\right)dx$$
$$+\frac{\xi\tau'(t)}{2}\int_0^L z^2(x,1,t))dx. \quad (110)$$

From (110), we find

$$E'(t) \le -\left(\mu_1 - \frac{\xi}{2}\right)\int_0^L w_t^2 dx + \left(\frac{\xi\tau'(t)}{2} - \frac{\xi}{2}\right)\int_0^L z^2(x,1,t)dx$$
$$-\mu_2\int_0^L w_t z(x,1,t)dx - \frac{\alpha_1}{4}\|\eta\|^2_{\mathscr{M}_\beta} - \frac{\alpha_2}{4}\|\nu\|^2_{\mathscr{M}_\lambda}$$
$$+\frac{1}{4}\int_0^\infty \beta'(\sigma)\|\eta_x(\sigma)\|^2 d\sigma + \frac{1}{4}\int_0^\infty \lambda'(\sigma)\|\nu_x(\sigma)\|^2 d\sigma. \quad (111)$$

Using Young's inequality, we have

$$-\mu_2\int_0^L w_t z(x,1,t)dx \le \frac{|\mu_2|}{2\sqrt{1-d}}\int_0^L w_t^2 dx + \frac{|\mu_2|\sqrt{1-d}}{2}\int_0^L z^2(x,1,t)dx. \quad (112)$$

Inserting (112) into (111), we get

$$E'(t) \le -\left(\mu_1 - \frac{\xi}{2} - \frac{|\mu_2|}{2\sqrt{1-d}}\right)\int_0^L w_t^2 dx$$
$$+\left(\frac{\xi}{2}\left(\tau'(t)-1\right) + \frac{|\mu_2|\sqrt{1-d}}{2}\right)\int_0^L z^2(x,1,t)dx$$
$$-\frac{\alpha_1}{4}\|\eta\|^2_{\mathscr{M}_\beta} + \frac{1}{4}\int_0^\infty \beta'(\sigma)\|\eta_x(\sigma)\|^2 d\sigma - \frac{\alpha_2}{4}\|\nu\|^2_{\mathscr{M}_\lambda}$$
$$+\frac{1}{4}\int_0^\infty \lambda'(\sigma)\|\nu_x(\sigma)\|^2 d\sigma. \quad (113)$$

Then, by using (7), (28)–(31), and (108), we obtain (109).

In the following, we state and prove our stability result; we introduce and prove several lemmas.

Lemma 4. *The functional*

$$F_1(t) := \int_0^L \left(u_t u + \frac{1}{2}w_t w + \frac{\beta_1}{4}w^2\right)dx, \quad (114)$$

satisfies, for any $\varepsilon_1 > 0$,

$$F_1'(t) \le -d_1\int_0^L \left(u_x + \frac{1}{2}w_x^2\right)^2 dx - \frac{d_2}{4}\int_0^L w_{xx}^2 dx + \int_0^L u_t^2 dx$$
$$+\frac{1}{2}\int_0^L w_t^2 dx + 2\varepsilon_1\int_0^L u_x^2 dx + \frac{\delta_1^2}{4\varepsilon_1}\int_0^L \theta^2 + \frac{\delta_2^2}{4\varepsilon_1}\int_0^L P^2$$
$$+c\int_0^L z^2(x,1,t)dx. \quad (115)$$

Proof. By differentiating F_1, then by integration by parts, we obtain

$$F_1'(t) = \int_0^L u_t^2 dx + \frac{1}{2}\int_0^L w_t^2 dx - \frac{1}{2}d_1\int_0^L \left(u_x + \frac{1}{2}w_x^2\right)w_x^2 dx$$
$$-d_1\int_0^L u_x\left(u_x + \frac{1}{2}w_x^2\right)dx - \frac{\mu_2}{2}\int_0^L wz(x,1,t)dx$$
$$-\frac{d_2}{2}\int_0^L w_{xx}^2 dx + \delta_1\int_0^L \theta u_x dx + \delta_2\int_0^L P u_x dx. \quad (116)$$

In what follows, using Young's and Poincaré's inequalities, we obtain (115).

Then, we have the following lemma.

Lemma 5. *The functional*

$$F_2(t) := \int_0^L u_t \Phi dx, \quad (117)$$

where $-\delta_1\Phi_x = c\theta + dP$, with $\Phi(0) = \Phi(L) = 0$, satisfies

$$F_2'(t) \le -\int_0^L u_t^2 dx + \varepsilon_2\int_0^L \left(u_x + \frac{1}{2}w_x^2\right)^2 dx + c\|\eta\|^2_{\mathscr{M}\mu}$$
$$+c\left(1+\frac{1}{\varepsilon_2}\right)\int_0^L \theta^2 dx + c\left(1+\frac{1}{\varepsilon_2}\right)\int_0^L P^2 dx. \quad (118)$$

Proof. For direct computations, we have

$$F_2'(t) = \underbrace{\int_0^L u_{tt}\Phi dx}_{f}$$
$$1(t) + \underbrace{\int_0^L u_t\Phi_t dx}_{f_2(t)}. \quad (119)$$

Using Young's inequality and integrating by parts, we obtain

$$f_1(t) \le \varepsilon_2 \int_0^L \left(u_x + \frac{1}{2} w_x^2\right)^2 dx + c\left(1 + \frac{1}{\varepsilon_2}\right)\int_0^L \theta^2 dx + c\left(1 + \frac{1}{\varepsilon_2}\right)\int_0^L P^2 dx, \tag{120}$$

$$\begin{aligned} f_2(t) &= -\frac{1}{\delta_1}\int_0^L u_t \partial_x^{-1}\left(\int_0^\infty \beta(\sigma)\eta_{xx}(\sigma)d\sigma + \delta_1 u_{tx}\right)dx \\ &= -\frac{1}{\delta_1}\int_0^L u_t\left(\int_0^\infty \beta(\sigma)\eta_x(\sigma)d\sigma + \delta_1 u_t\right)dx \\ &\le -\int_0^L u_t^2 dx + c\|\eta\|^2_{\mathscr{M}_\beta}. \end{aligned} \tag{121}$$

From (120) and (121), we obtain (118).

Lemma 6. *Assuming that assumptions (31) and (32) hold, the functional*

$$F_3(t) := \underbrace{-\int_0^\infty \beta(\sigma)\int_0^L (c\theta + dP)\eta dx d\sigma}_{\mathscr{G}_1} \underbrace{- \int_0^\infty \lambda(\sigma)\int_0^L (d\theta + rP)\nu dx d\sigma}_{\mathscr{G}_2}, \tag{122}$$

satisfies

$$\begin{aligned} F_3'(t) \le& -\hat{c}\int_0^L \theta^2 dx - \hat{r}\int_0^L P^2 dx + \beta_0\|\eta\|^2_{\mathscr{M}_\beta} + \lambda_0\|\nu\|^2_{\mathscr{M}_\lambda} \\ &+ \frac{c}{\varepsilon_3}\int_0^L u_t^2 dx - C_{\beta_0}\int_0^\infty \beta'(\sigma)\|\eta_x(\sigma)\|^2(\sigma)d\sigma \\ &- C_{\lambda_0}\int_0^\infty \lambda'(\sigma)\|\nu_x(\sigma)\|^2(\sigma)d\sigma, \end{aligned} \tag{123}$$

where

$$\begin{aligned} \hat{c} &= \frac{1}{2}\left(\beta_0 c - (\beta_0 + \lambda_0)\frac{d}{\zeta}\right), \\ \hat{r} &= \frac{1}{2}(\lambda_0 r - (\mu_0 + \lambda_0)d\zeta), \end{aligned} \tag{124}$$

and $\zeta > 0$ satisfies (29).

Proof. We take the derivative of $F_3 = \mathscr{G}_1 + \mathscr{G}_2$, which gives

$$\begin{aligned} \mathscr{G}_1'(t) =& -\int_0^\infty \beta(\sigma)\int_0^L (c\theta + dP)_t \eta dx d\sigma \\ &- \int_0^\infty \beta(\sigma)\int_0^L (c\theta + dP)\eta_t dx d\sigma \\ =& -\int_0^\infty \beta(\sigma)\int_0^L (c\theta_t + dP_t)\eta dx d\sigma + c\int_0^\infty \beta(\sigma)\int_0^L \theta\eta_\sigma dx d\sigma \\ &+ d\int_0^\infty \beta(\sigma)\int_0^L P\eta_\sigma dx d\sigma - c\beta_0\int_0^L \theta^2 dx - d\int_0^\infty \beta(\sigma)\int_0^L P\theta dx d\sigma. \end{aligned} \tag{125}$$

The first term on the right-hand side of (125) is

$$\begin{aligned} &-\int_0^\infty \beta(\sigma)\int_0^L (c\theta + dP)_t \eta dx d\sigma \\ &\quad = -\delta_1\int_0^\infty \beta(\sigma)\int_0^L u_{tx}\eta dx d\sigma \\ &\qquad - \int_0^L \left(\int_0^\infty \beta(\sigma)\eta_{xx} d\sigma\right)\left(\int_0^\infty \beta(\sigma)\eta d\sigma\right)dx, \end{aligned} \tag{126}$$

and can be controlled in the following way:

$$\left| -\delta_1\int_0^\infty \beta(\sigma)\int_0^L u_{tx}\eta dx d\sigma\right| \le C(\varepsilon_3)\|\eta\|^2_{\mathscr{M}_\beta} + \frac{c}{\varepsilon_3}\int_0^L u_t^2 dx, \tag{127}$$

$$-\int_0^L \left(\int_0^\infty \beta(\sigma)\eta_{xx} d\sigma\right)\left(\int_0^\infty \beta(\sigma)\eta d\sigma\right)dx \le \beta_0\|\eta\|^2_{\mathscr{M}_\beta}. \tag{128}$$

Moreover, by integration by parts, we get

$$\begin{aligned} \left| c\int_0^\infty \beta(\sigma)\int_0^L \theta\eta_\sigma dx d\sigma\right| &= c\left| -\int_0^\infty \beta'(\sigma)\int_0^L \theta\eta dx d\sigma\right| \\ &\le \frac{c\mu_0}{4}\int_0^L \theta^2 dx - C_{\beta_0}\int_0^\infty \beta'(\sigma)\|\eta_x(\sigma)\|^2 d\sigma, \end{aligned} \tag{129}$$

where $C_{\beta_0} > 0$. Similarly, we obtain

$$\begin{aligned} \left| d\int_0^\infty \beta(\sigma)\int_0^L P\eta_\sigma dx d\sigma\right| &= c\left| -\int_0^\infty \beta'(\sigma)\int_0^L P\eta dx d\sigma\right| \\ &\le \frac{r\lambda_0}{4}\int_0^L P^2 dx - C'_{\beta_0}\int_0^\infty \beta'(\sigma)\|\eta_x(\sigma)\|^2 d\sigma, \end{aligned} \tag{130}$$

where $C'_{\beta_0} > 0$. Using (29), we get

$$-d\int_0^\infty \beta(\sigma)\left(\int_0^L \theta P dx\right)d\sigma \le \beta_0\frac{d}{2\zeta}\int_0^L \theta^2 dx + \beta_0\frac{d\zeta}{2}\int_0^L P^2 dx. \tag{131}$$

Then, we obtain

$$\begin{aligned} \mathscr{G}_1'(t) \le& \frac{\beta_0}{2}\left(\frac{d}{\zeta} - \frac{3c}{2}\right)\int_0^L \theta^2 dx + \frac{1}{2}\left(\beta_0 d\zeta + \frac{r\lambda_0}{2}\right)\int_0^L P^2 dx \\ &+ \frac{c}{\varepsilon_3}\int_0^L u_t^2 dx - \mathscr{C}_{\beta_0}\int_0^\infty \beta'(\sigma)\|\eta_x(\sigma)\|^2 d\sigma \\ &+ (\beta_0 + C(\varepsilon_3)\|\eta\|^2_{\mathscr{M}_\beta}, \end{aligned} \tag{132}$$

where $\mathscr{C}_{\beta_0} = C_{\beta_0} + C'_{\beta_0}$. Then, using the same arguments, we find

$$\begin{aligned}\mathscr{G}_2'(t) \le& \frac{1}{2}\left(\lambda_0 \frac{d}{\zeta} + \frac{\beta_0 c}{2}\right)\int_0^L \theta^2 dx + \frac{\lambda_0}{2}\left(d\zeta - \frac{3r}{2}\right)\int_0^L P^2 dx \\ &+ \frac{c}{\varepsilon_3}\int_0^L u_t^2 dx - \mathscr{C}_{\lambda_0}\int_0^\infty \lambda'(\sigma)\|v_x(\sigma)\|^2 d\sigma \\ &+ (\lambda_0 + C(\varepsilon_3)\|v\|^2_{\mathscr{M}_\lambda}.\end{aligned} \tag{133}$$

Adding (127) and (133), we obtain (123).

We choose ζ in such a way that

$$\begin{aligned}\hat{c} &= \frac{1}{2}\left(\beta_0 c - (\beta_0 + \lambda_0)\frac{d}{\zeta}\right) > 0, \\ \hat{r} &= \frac{1}{2}(\lambda_0 r - (\beta_0 + \lambda_0)d\zeta) > 0,\end{aligned} \tag{134}$$

which implies

$$\frac{d}{c} < \frac{\beta_0 + \lambda_0}{\beta_0}\frac{d}{c} < \zeta < \frac{\lambda_0}{\beta_0 + \lambda_0}\frac{r}{d} < \frac{r}{d}. \tag{135}$$

Then, ζ satisfies (29).

Now, let us introduce the following functional.

Lemma 7. *The functional*

$$F_4(t) := \xi\tau(t)\int_0^L\int_0^1 e^{-2\tau(t)\rho} z^2(x,\rho,t)\,d\rho\,dx, \tag{136}$$

satisfies

$$F_4'(t) \le -2F_4(t) - \eta_1\int_0^L z^2(x,1,t)dx + \xi\int_0^L w_t^2 dx, \tag{137}$$

where η_1 is a positive constant.

Proof. By differentiating F_4, with respect to t, we have

$$\begin{aligned}F_4'(t) =& \xi\tau'(t)\int_0^L\int_0^1 e^{-2\tau(t)\rho} z^2(x,\rho,t)\,d\rho\,dx \\ &+ \xi\tau(t)\int_0^L\int_0^1 \left\{-2\tau'(t)\rho e^{-2\tau(t)\rho}z^2 + e^{-2\tau(t)\rho}z_t z\right\}d\rho\,dx.\end{aligned} \tag{138}$$

By using the last equation of (21), we have

$$\begin{aligned}&\tau(t)\int_0^L\int_0^1 e^{-2\tau(t)\rho} z_t z\,d\rho\,dx \\ &= \int_0^L\int_0^1 \left(\tau'(t)\rho - 1\right)e^{-2\tau(t)\rho} z_\rho z\,d\rho\,dx \\ &= \frac{1}{2}\int_0^L\int_0^1 \frac{d}{d\rho}\left\{\left(\tau'(t)\rho - 1\right)e^{-2\tau(t)\rho}z^2\right\}d\rho\,dx \\ &\quad + \tau(t)\int_0^L\int_0^1 \left(\tau'(t)\rho - 1\right)e^{-2\tau(t)\rho}z^2\,d\rho\,dx \\ &\quad - \frac{\tau'(t)}{2}\int_0^L\int_0^1 e^{-2\tau(t)\rho}z^2 dx.\end{aligned} \tag{139}$$

Using (137)–(139), we get

$$\begin{aligned}F_4'(t) =& -2\xi\tau(t)\int_0^L\int_0^1 e^{-2\tau(t)\rho} z^2(x,\rho,t)\,d\rho\,dx + \xi\int_0^L z^2(x,0,t)dx \\ &- \xi\left(1-\tau'(t)\right)e^{-2\tau(t)}\int_0^L z^2(x,1,t)dx.\end{aligned} \tag{140}$$

Then, by using (7), (25), and the fact that $z(x,0,t) = w_t(x,t)$ and setting $\eta_1 = \xi(1-d)e^{-2\bar{\tau}}$, we obtain (137).

We are now ready to prove the following result.

Theorem 8. *Assume (26)–(31) hold; there exist positive constants C_1 and C_2 such that the energy functional given by (107) satisfies*

$$E(t) \le C_2 e^{-C_1 t}, \quad \forall t \ge 0. \tag{141}$$

Proof. We define a Lyapunov functional

$$\mathscr{L}(t) := NE(t) + \sum_{i=1}^{i=3} N_i F_i(t) + F_4(t), \tag{142}$$

where N and N_i, $i = 1, 2, 3$, are positive constants to be selected later.

By differentiating (142) and using (109), (115), (118), (123), and (137), including the relation

$$\begin{aligned}\int_0^L u_x^2 dx &= \int_0^L \left(u_x^2 + \frac{1}{2}w_x^2 - \frac{1}{2}w_x^2\right)dx \\ &\le 2\int_0^L \left(u_x + \frac{1}{2}w_x^2\right)^2 dx - \frac{1}{2}\int_0^L w_x^4 dx \\ &\le 2\int_0^L \left(u_x + \frac{1}{2}w_x^2\right)^2 dx - \frac{L}{4}\int_0^L w_{xx}^2 dx,\end{aligned} \tag{143}$$

we get

$$\begin{aligned}\mathscr{L}'(t) \leq &-[(d_1-2\varepsilon_1)N_1-\varepsilon_2N_2]\int_0^L\left(u_x+\frac{1}{2}w_x^2\right)^2dx\\&-\left[N_2-N_1-\frac{c}{\varepsilon_3}N_3\right]\int_0^L u_t^2dx\\&-\left[\left(\frac{d_2}{4}-\frac{L}{2}\varepsilon_1\right)N_1\right]\int_0^L w_{xx}^2dx\\&-\left[CN-\frac{1}{2}N_1-\xi\right]\int_0^L\omega_t^2dx\\&-\left[\widehat{c}N_3-\frac{\delta_1^2}{4\varepsilon_1}N_1-c\left(1+\frac{1}{\varepsilon_2}\right)N_2\right]\int_0^L\theta^2dx\\&-\left[\widehat{r}N_3-\frac{\delta_2^2}{4\varepsilon_1}N_1-c\left(1+\frac{1}{\varepsilon_2}\right)N_2\right]\int_0^L P^2dx\\&-[CN-cN_1+\eta_1]\int_0^L z^2(x,1,t)dx-2F_4(t)\\&-\left[\frac{\alpha_1}{4}N-cN_2-\mu_0N_3\right]\|\eta\|^2_{\mathscr{M}_\beta}\\&-\left[\frac{\alpha_2}{4}N-\lambda_0N_3\right]\|\nu\|^2_{\mathscr{M}_\lambda}\\&+\left[\frac{1}{4}N-C_{\mu_0}N_3\right]\int_0^\infty\beta'(\sigma)\|\eta_x(\sigma)\|^2d\sigma\\&+\left[\frac{1}{4}N-C_{\lambda_0}N_3\right]\int_0^\infty\lambda'(\sigma)\|\nu_x(\sigma)\|^2d\sigma.\end{aligned}\tag{144}$$

First, we choose ε_1 small enough such that

$$\begin{aligned}&d_1-2\varepsilon_1>0,\\&\frac{d_2}{4}-\frac{L}{2}\varepsilon_1>0.\end{aligned}\tag{145}$$

By setting

$$\begin{aligned}\varepsilon_2&=\frac{(d_1-2\varepsilon_1)N_1}{2N_2},\\\varepsilon_3&=\frac{2cN_3}{N_2},\end{aligned}\tag{146}$$

we obtain

$$\begin{aligned}\mathscr{L}'(t) \leq &-\left[\frac{1}{2}(d_1-2\varepsilon_1)N_1\right]\int_0^L\left(u_x+\frac{1}{2}w_x^2\right)^2dx\\&-\left[\frac{1}{2}N_2-N_1\right]\int_0^L u_t^2dx-\left[\left(\frac{d_2}{4}-\frac{L}{2}\varepsilon_1\right)N_1\right]\int_0^L w_{xx}^2dx\\&-\left[CN-\frac{1}{2}N_1-\xi\right]\int_0^L\omega_t^2dx\\&-\left[\widehat{c}N_3-\frac{\delta_1^2}{4\varepsilon_1}N_1-c\left(1+\frac{N_2}{N_1}\right)N_2\right]\int_0^L\theta^2d\\&-\left[\widehat{r}N_3-\frac{\delta_2^2}{4\varepsilon_1}N_1-c\left(1+\frac{N_2}{N_1}\right)N_2\right]\int_0^L P^2dx\\&-[CN-cN_1+\eta_1]\int_0^L z^2(x,1,t)dx-2F_4(t)\\&-\left[\frac{\alpha_1}{4}N-cN_2-\mu_0N_3\right]\|\eta\|^2_{\mathscr{M}_\beta}\\&-\left[\frac{\alpha_2}{4}N-\lambda_0N_3\right]\|\nu\|^2_{\mathscr{M}_\lambda}\\&+\left[\frac{1}{4}N-C_{\mu_0}N_3\right]\int_0^\infty\beta'(\sigma)\|\eta_x(\sigma)\|^2d\sigma\\&+\left[\frac{1}{4}N-C_{\lambda_0}N_3\right]\int_0^\infty\lambda'(\sigma)\|\nu_x(\sigma)\|^2d\sigma.\end{aligned}\tag{147}$$

Next, we carefully choose our constants so that the terms inside the brackets are positive.

We choose N_2 large enough such that

$$k_1=\frac{1}{2}N_2-N_1>0.\tag{148}$$

Then, we choose N_3 large enough such that

$$\begin{aligned}k_2&=\widehat{r}N_3-\frac{\delta_2^2}{4\varepsilon_1}N_1-c\left(1+\frac{N_2}{N_1}\right)N_2>0,\\k_3&=\widehat{r}N_3-\frac{\delta_2^2}{4\varepsilon_1}N_1-c\left(1+\frac{N_2}{N_1}\right)N_2>0.\end{aligned}\tag{149}$$

Thus, we arrive at

$$\begin{aligned}\mathscr{L}'(t)\leq &-k_0\int_0^L\left(u_x+\frac{1}{2}w_x^2\right)^2dx-k_1\int_0^L u_t^2dx-k_4\int_0^L w_{xx}^2dx\\&-(CN-c)\int_0^L\omega_t^2dx-k_2\int_0^L\theta^2dx-k_3\int_0^L P^2dx\\&-(CN-c)\int_0^L z^2(x,1,t)dx-2F_4(t)\\&-\left(\frac{\alpha_1}{4}N-c\right)\|\eta\|^2_{\mathscr{M}_\beta}-\left(\frac{\alpha_2}{4}N-c\right)\|\nu\|^2_{\mathscr{M}_\lambda}\\&+\left(\frac{1}{4}N-c\right)\int_0^\infty\beta'(\sigma)\|\eta_x(\sigma)\|^2d\sigma\\&+\left(\frac{1}{4}N-c\right)\int_0^\infty\lambda'(\sigma)\|\nu_x(\sigma)\|^2d\sigma,\end{aligned}\tag{150}$$

where $k_0=(1/2)(d_1-2\varepsilon_1)N_1$ and $k_4=((d_2/4)-(L/2)\varepsilon_1)N_1$.

On the other hand, we let

$$\mathfrak{T}(t)=\sum_{i=1}^{i=3}N_iF_i(t)+F_4(t).\tag{151}$$

Exploiting Young's, Cauchy-Schwarz's, and Poincaré's inequalities, we get

$$|\mathfrak{T}(t)| \leq c\int_0^L \left(\omega_t^2 + u_t^2 + \left(u_x + \frac{1}{2}\omega_x^2\right)^2 + \omega_{xx}^2 + \theta^2 + P^2\right)dx + c\|\eta\|_{\mathscr{M}_\beta}^2 + c\|\nu\|_{\mathscr{M}_\lambda}^2 + c\int_0^L\int_0^1 z^2(x,\rho,t)d\rho dx. \tag{152}$$

Then,

$$|\mathfrak{T}(t)| \leq cE(t). \tag{153}$$

Consequently, we obtain

$$|\mathfrak{T}(t)| = |\mathscr{L}(t) - NE(t)| \leq cE(t), \tag{154}$$

that is,

$$(N-c)E(t) \leq \mathscr{L}(t) \leq (N+c)E(t). \tag{155}$$

Now, we choose N large enough such that

$$\begin{aligned} N - c &> 0, \\ \frac{\alpha_1}{4}N - c &> 0, \\ \frac{\alpha_2}{4}N - c &> 0, \\ \frac{1}{4}N - c &> 0 > 0, \\ CN - c &> 0. \end{aligned} \tag{156}$$

Exploiting (107), estimates (150) and (155), respectively, give

$$\mathscr{L}'(t) \leq -a_1E(t), \tag{157}$$

for some $a_1 > 0$, and

$$c_1E(t) \leq \mathscr{L}(t) \leq c_2E(t), \quad \forall t \geq 0, \tag{158}$$

for some $c_1, c_2 > 0$; we have

$$\mathscr{L}(t) \sim E(t). \tag{159}$$

A combination with (157) and (158) gives

$$\mathscr{L}'(t) \leq -C_1\mathscr{L}(t), \quad \forall t \geq 0, \tag{160}$$

where $C_1 = a_1/c_2$.

Finally, by simple integration of (159) and (160), we obtain the result (141).

Data Availability

No data were used to support the study.

Conflicts of Interest

This work does not have any conflicts of interest.

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