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Research Article

A Newton Linearized Crank-Nicolson Method for the Nonlinear Space Fractional Sobolev Equation

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In this paper, one class of finite difference scheme is proposed to solve nonlinear space fractional Sobolev equation based on the Crank-Nicolson (CN) method. Firstly, a fractional centered finite difference method in space and the CN method in time are utilized to discretize the original equation. Next, the existence, uniqueness, stability, and convergence of the numerical method are analyzed at length, and the convergence orders are proved to be $O(\tau^2 + h^2)$ in the sense of l^2-norm, $H^{\alpha/2}$-norm, and l^∞-norm. Finally, the extensive numerical examples are carried out to verify our theoretical results and show the effectiveness of our algorithm in simulating spatial fractional Sobolev equation.

1. Introduction

The main propose of this paper is to construct one class of the Newton linearized finite difference method based on CN discretization in temporal direction to efficiently solve the following spatial fractional Sobolev equation:

$$\begin{cases} \partial_t u - \mu \partial_x^\alpha \partial_t u = \kappa \partial_x^\beta u + f(u), & \text{in } \mathbb{R} \times (0, T], \\ u(x, 0) = u_0(x), & \text{in } \mathbb{R} \times \{0\}. \end{cases} \tag{1}$$

where $1 < \alpha, \beta \leqslant 2$, μ and κ are given positive constants, $u_0(x)$ and $f(u)$ are known sufficiently smooth functions. ∂_x^α in (1) denotes the Riesz fractional derivative operator for $1 < \alpha \leqslant 2$ and is defined in [1] as follows:

$$\partial_x^\alpha u(x, t) = -\frac{1}{2\cos(\pi\alpha/2)\Gamma(2-\alpha)} \frac{\partial^2}{\partial x^2} \int_{-\infty}^{\infty} |x - \xi|^{1-\alpha} u(\xi, t) d\xi. \tag{2}$$

This type of equation is widely used as a mathematical model for fluid flow through thermodynamics [2], shear in second-order fluids [3], consolidation of clay [4], and so on. Note that some special forms of equation (1) are frequently encountered in many fields. For example, taking $\alpha, \beta = 2$, (1) reduces to a one-dimensional integral-order Sobolev equation in the bounded domain [5]. When $f(u) = \sum_{i=1}^{p} \gamma_i u^p$ with integer p and given constants $\gamma_i (i = 1, 2, \cdots, p)$, then the equation is called a semiconductor equation [6]. When $f(u) = 0$, it is reduced to a homogeneous space fractional Sobolev equation. When $\mu = 0$, (1) is reduced to the classical nonlinear reaction-diffusion equations. Recently, many scholars are dedicated to the numerical investigation on fractional diffusion equations and Sobolev equations based on finite difference or finite element methods in the literature. For example, Çelik and Duman [7] investigated the CN method to approximate the fractional diffusion equation with the Riesz fractional derivative in a finite domain. Wang et al. [8] studied the finite difference method for the space fractional Schrödinger equations under the framework of the fractional Sobolev space. Ran and He [9] investigated the nonlinear multidelay fractional diffusion equation based on the CN method in time and the fractional centered difference in space. Chen et al. [5] proposed a Newton linearized compact finite difference scheme to numerically solve a class of Sobolev equations based on the CN method and proved the unique solvability, convergence, and stability of the proposed scheme. Wang and Huang [10] constructed a conservative

linearized difference scheme for the nonlinear fractional Schrödinger equation. Zhang et al. [11] established the numerical asymptotic stability result of the compact θ-method for the generalized delay diffusion equation. More researches on delay fractional problems can be referred to [12, 13] and the references therein.

The main work in this paper is to develop an efficient Newton linearized CN method to solve the nonlinear space fractional Sobolev problem (1). The existence, uniqueness, stability, and convergence of the proposed numerical scheme are demonstrated, and the convergent orders are obtained in the sense of l^2-norm, $H^{\alpha/2}$-norm, and l^∞-norm. Besides, we also prove that the convergence orders of the constructed linearized numerical scheme are $O(\tau^2+h^2)$ under three types of norms. The extensive numerical examples are proposed to argue a second-order accuracy in both temporal and spatial dimensions.

The organization of this paper is as follows. In Section 2, we define the fractional Sobolev norm and introduce the second-order centered finite difference approximation for the space Riesz derivative. In Section 3, we construct a CN finite difference scheme for the space fractional Sobolev equation. The existence, uniqueness, stability, and convergence of the proposed scheme in three classes of conventional norms are proved. Finally, the theoretical results are verified by several numerical examples.

2. Preliminaries

Firstly, we present some notations and lemmas which will be used to construct and analyze our numerical scheme.

2.1. Fractional Sobolev Norm. Firstly, we define the fractional Sobolev norm (cf. [14]). Let $h\mathbb{Z}$ be denoted by the infinite grid with grid points $x_j = jh$ $(j \in \mathbb{Z})$. For arbitrary grid functions $u = \{u_j\}, v = \{v_j\}$ on $h\mathbb{Z}$, we define the discrete inner products and the corresponding l^2-norm and l^∞-norm

$$(u,v) = h\sum_{j\in\mathbb{Z}} u_j v_j, \|u\|^2 = (u,u), \|u\|_{l^\infty} = \sup_{j\in\mathbb{Z}} |u_j|. \tag{3}$$

Denote $l^2 := \{u \mid u = \{u_j\}, \|u\|^2 < +\infty\}$. For $u \in l^2$, the semidiscrete Fourier transformation $\hat{u}$ is written as

$$\hat{u}(k) := \frac{1}{\sqrt{2\pi}} h \sum_{j\in\mathbb{Z}} u_j e^{-ikx_j}. \tag{4}$$

It is easy to get $\hat{u} \in L^2[-\pi/h, \pi/h]$ due to $u \in l^2$. The inversion formula is defined by

$$u_j = \frac{1}{\sqrt{2\pi}} \int_{-\pi/h}^{\pi/h} \hat{u}(k) e^{ikx_j} dk, \tag{5}$$

then we can easily check that Parseval's equality

$$(u,v) = \int_{-\pi/h}^{\pi/h} \hat{u}(k)\hat{v}(k) dk, \tag{6}$$

holds. Moreover, For the given constant $0 \leqslant \sigma \leqslant 1$, the fractional Sobolev norm $\|\cdot\|_{H^\sigma}$ and seminorm $|\cdot|_{H^\sigma}$ are defined as follows:

$$\|u\|_{H^\sigma}^2 = \int_{-\pi/h}^{\pi/h} \left(1+|k|^{2\sigma}\right)|u\wedge(k)|^2 dk, |u|_{H^\sigma}^2 = \int_{-\pi/h}^{\pi/h} |k|^{2\sigma}|u\wedge(k)|^2 dk. \tag{7}$$

Obviously, $\|u\|_{H^\sigma}^2 = \|u\| + |u|_{H^\sigma}^2$.

2.2. Second-Order Approximation of Spatial Riesz Fractional Derivative. In this section, we will review a second-order approximation for the Riesz fractional derivative. Introduce

$$\mathscr{C}^{n+\alpha}(R) = \left\{ f \left| \int_{-\infty}^{\infty} (1+|\omega|)^{n+\alpha} |\hat{f}(\omega)| d\omega < \infty, f \in L^1(\mathbb{R}) \right. \right\}, \tag{8}$$

where $\hat{f}(\omega) = \int_{-\infty}^{\infty} e^{i\omega t} f(t) dt$ denotes the Fourier transformation of $f(x)$.

Lemma 1. *(cf. [7]). Suppose the function $f(\cdot) \in \mathscr{C}^{2+\alpha}(\mathbb{R})$ and the fractional central difference is defined as follows:*

$$\delta_x^\alpha f(x) = -h^{-\alpha} \sum_{k=-\infty}^{+\infty} g_k^{(\alpha)} f(x-kh). \tag{9}$$

Then, it holds

$$\delta_x^\alpha f(x) = \partial_x^\alpha f(x) + O(h^2). \tag{10}$$

$g_k^{(\alpha)}$ is defined as

$$g_k^{(\alpha)} = \frac{(-1)^k \Gamma(\alpha+1)}{\Gamma(\alpha/2-k+1)\Gamma(\alpha/2+k+1)}, k \in \mathbb{Z}. \tag{11}$$

This is consistently established for arbitrary $x \in \mathbb{R}$.

Remark 2. (cf. [15, 16]). *If we define f^* by*

$$f^*(x) = \begin{cases} f(x), & x \in [a,b], \\ 0, & x \notin [a,b], \end{cases} \tag{12}$$

such that $f^*(x) \in \mathscr{C}^{2+\alpha}(\mathbb{R})$. We get

$$\delta_x^\alpha f(x) = -h^{-\alpha} \sum_{k=-[(b-x)/h]}^{[(x-a)/h]} g_k^{(\alpha)} f(x-kh) + O(h^2). \tag{13}$$

For any $t \in [0,T]$, we define

$$u^*(x) = \begin{cases} u(x), & x \in [a,b], \\ 0, & x \notin [a,b], \end{cases} \tag{14}$$

and suppose $u^*(x) \in \mathscr{C}^{2+\alpha}(\mathbb{R})$.

3. Second-Order CN Method and Theoretical Analysis

In this section, we are concentrated on the derivation and theoretical analysis of the finite different scheme. In practical computation, it is necessary to truncate the whole space problem onto a finite interval (boundaries are usually chosen sufficient large such that the truncation error is negligible or the exact solution has compact support in the bounded domain [17]). Here, we will truncate (1) on the interval $\Omega = (a, b)$ as follows:

$$\partial_t u - \mu \partial_x^\alpha \partial_t u = \kappa \partial_x^\beta u + f(u), \text{ in } \Omega \times (0, T], \tag{15}$$

$$u(x, 0) = u_0(x), \text{ in } \mathbb{R} \times \{0\}, \tag{16}$$

$$u(x, t) = 0, \text{ on } \mathbb{R} \setminus \Omega \times [0, T]. \tag{17}$$

3.1. The Derivation of the Linearized Numerical Scheme. Take positive integers M, N and let $\tau = T/N$, $h = (b-a)/M$ be the temporal and spatial step sizes, respectively. Denote $x_i = a + ih$, $0 \leqslant i \leqslant M$; $t_k = k\tau$, $0 \leqslant k \leqslant N$; $t_{k+1/2} = (k + 1/2)\tau$, $0 \leqslant k \leqslant N-1$; $\Omega_h = \{x_i \mid 0 \leqslant i \leqslant M\}$, $\Omega_\tau = \{t_k \mid 0 \leqslant k \leqslant N\}$. Define $\bar{\omega} = \{j \mid j = 0, 1, \cdots, M\}$, $\omega = \{j \mid j = 1, 2, \cdots, M-1\}$, $\partial\omega = \bar{\omega} \setminus \omega$. Let $V_h = \{u \mid u = u_i^k \mid 0 \leqslant i \leqslant M, 0 \leqslant k \leqslant N, u_0^k = u_M^k = 0\}$ be grid function space defined on $\Omega_{h\tau} = \Omega_h \times \Omega_\tau$. Then, for a given grid function $u \in V_h$, we introduce the following notations:

$$u_i^{k+(1/2)} = \frac{1}{2}\left(u_i^{k+1} + u_i^k\right), \delta_t u_i^{k+(1/2)} = \frac{1}{\tau}\left(u_i^{k+1} - u_i^k\right). \tag{18}$$

Define the grid function

$$U_i^k = u(x_i, t_k), i \in \bar{\omega}, 0 \leqslant k \leqslant N. \tag{19}$$

Then, we consider (15) at the point $(x_i, t_{k+(1/2)})$ and have

$$\begin{aligned} &\partial_t u\left(x_i, t_{k+(1/2)}\right) - \mu \partial_x^\alpha \partial_t u\left(x_i, t_{k+(1/2)}\right) \\ &= \kappa \partial_x^\beta u\left(x_i, t_{k+(1/2)}\right) + f\left(u\left(x_i, t_{k+(1/2)}\right)\right), i \in \omega, 0 \leqslant k \leqslant N-1. \end{aligned} \tag{20}$$

Utilizing the Taylor expansion, the first term on the left hand side (LHS) in (20) can be estimated as

$$\partial_t u\left(x_i, t_{k+(1/2)}\right) = \delta_t U_i^{k+(1/2)} + O\left(\tau^2\right). \tag{21}$$

Noticing Lemma 1, for the second term on LHS in (20), we have

$$\begin{aligned} \partial_x^\alpha \partial_t u\left(x_i, t_{k+(1/2)}\right) &= -\frac{1}{h^\alpha} \sum_{j=0}^{M} g_{i-j}^{(\alpha)} \delta_t U_j^{k+(1/2)} + O\left(\tau^2 + h^2\right) \\ &= \delta_t \delta_x^\alpha U_i^{k+(1/2)} + O\left(\tau^2 + h^2\right). \end{aligned} \tag{22}$$

For the first term on the right hand side (RHS) in (20), it yields

$$\begin{aligned} \partial_x^\beta u\left(x_i, t_{k+(1/2)}\right) &= -\frac{1}{h^\beta} \sum_{j=0}^{M} g_{i-j}^{(\beta)} U_j^{k+(1/2)} + O\left(\tau^2 + h^2\right) \\ &= \delta_x^\beta U_i^{k+(1/2)} + O\left(\tau^2 + h^2\right). \end{aligned} \tag{23}$$

Moreover, we have

$$\begin{aligned} u\left(x_i, t_{k+(1/2)}\right) &= \frac{1}{2}\left(U_i^{k+1} + U_i^k\right) + O\left(\tau^2\right), \\ u(x_i, t_{k+1}) - u(x_i, t_k) &= O(\tau) \leqslant c_0 \tau, \end{aligned} \tag{24}$$

where c_0 is a positive constant.

Applying the Newton linearized method to the nonlinear term f on RHS in (20) and using Taylor expansion at the point U_i^k, it yields

$$\begin{aligned} f\left(u\left(x_i, t_{k+(1/2)}\right)\right) &= f\left(U_i^k\right) + \left(U_i^{k+(1/2)} - U_i^k\right) f'\left(U_i^k\right) + O\left(\tau^2\right) \\ &= f\left(U_i^k\right) + \frac{1}{2}\left(U_i^{k+1} - U_i^k\right) f'\left(U_i^k\right) + O\left(\tau^2\right), \end{aligned} \tag{25}$$

where $f'(U_i^k) = \partial_U f|_{U=U_i^k}$. Plugging (21)–(23) and substituting (25) into (20), we have

$$\begin{aligned} \delta_t U_i^{k+(1/2)} - \mu \delta_t \delta_x^\alpha U_i^{k+(1/2)} &= \kappa \delta_x^\beta U_i^{k+(1/2)} + f\left(U_i^k\right) \\ &+ \frac{1}{2} f'\left(U_i^k\right)\left(U_i^{k+1} - U_i^k\right) + R_i^k, i \in \omega, 0 \leqslant k \leqslant N-1. \end{aligned} \tag{26}$$

There exists a positive constant $c_1 > 0$ such that

$$\left|R_i^k\right| \leqslant c_1\left(\tau^2 + h^2\right), i \in \omega, 0 \leqslant k \leqslant N-1. \tag{27}$$

Omitting R_i^k in (26), replacing $U_i^{k+(1/2)}$ with $u_i^{k+(1/2)}$ in (26), then the finite difference scheme reads

$$\begin{aligned} \delta_t u_i^{k+(1/2)} - \mu \delta_t \delta_x^\alpha u_i^{k+(1/2)} &= \kappa \delta_x^\beta u_i^{k+(1/2)} + f\left(u_i^k\right) + \frac{1}{2} f'\left(u_i^k\right) \\ &\cdot \left(u_i^{k+1} - u_i^k\right), i \in \omega, 0 \leqslant k \leqslant N-1, \end{aligned} \tag{28}$$

$$u_i^0 = u_0(x_i), i \in \bar{\omega}, \tag{29}$$

$$u_i^k = 0, i \in \partial\omega, 1 \leqslant k \leqslant N. \tag{30}$$

3.2. The Unique Solvability of Finite Difference Scheme. This section is concerned with the solvability of scheme (28)–(30). Now, we give some lemmas which will be used in the demonstration of solvability.

Lemma 3. *(cf. [7]). Let*

$$A^{\alpha}=\begin{pmatrix} g_0^{(\alpha)} & g_{-1}^{(\alpha)} & \cdots & g_{3-M}^{(\alpha)} & g_{2-M}^{(\alpha)} \\ g_1^{(\alpha)} & g_0^{(\alpha)} & g_{-1}^{(\alpha)} & \cdots & g_{3-M}^{(\alpha)} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ g_{M-3}^{(\alpha)} & \cdots & g_1^{(\alpha)} & g_0^{(\alpha)} & g_{-1}^{(\alpha)} \\ g_{M-2}^{(\alpha)} & g_{M-3}^{(\alpha)} & \cdots & g_1^{(\alpha)} & g_0^{(\alpha)} \end{pmatrix}. \tag{31}$$

It holds

$$g_0^{(\alpha)}=\frac{\Gamma(\alpha+1)}{\Gamma^2(\alpha/2+1)}\leqslant 0,\ \sum_{j=-\infty}^{+\infty} g_j^{(\alpha)}=0, g_j^{(\alpha)}=g_{-j}^{(\alpha)}\leqslant 0, \tag{32}$$

where for any $|j|\leqslant 1$, and $0<\lambda_i<2g_0^{(\alpha)}(i\in\omega)$, λ_i is the ith the eigenvalue of matrix A^{α}. A^{β} is given in a similar way. It implies that the matrices A^{α} and A^{β} are real symmetric positive definite matrices.

Lemma 4. *(discrete Sobolev inequality (cf. [14])) For every $1/2<\sigma\leqslant 1$, there exists a constant $C_{\sigma}=C(\sigma)>0$, independent of $h>0$, such that*

$$\|v\|_{l^{\infty}}\leqslant C_{\sigma}\|v\|_{H^{\sigma}}. \tag{33}$$

Lemma 5. *(cf. [8]). For any $1<\alpha\leqslant 2$ and any grid function $v\in V_h$, we have*

$$C_{\alpha}|v|_{H^{\alpha/2}}^2\leqslant(-\delta_x^{\alpha}v,v)\leqslant|v|_{H^{\alpha/2}}^2, \tag{34}$$

where $C_{\alpha}=(\pi/2)^{\alpha}$.

Lemma 6. *(cf. [17]). For any grid function $v\in V_h$, there exists a fractional symmetric positive quotient operator $\delta_x^{\alpha/2}$, such that*

$$(-\delta_x^{\alpha}v,v)=\left(\delta_x^{\alpha/2}v,\delta_x^{\alpha/2}v\right). \tag{35}$$

Lemma 7. *(cf. [18]) (discrete uniform Sobolev inequality). For every $1/2<\sigma\leqslant 1$, there exists a constant $C_{\sigma}=C(\sigma)>0$ independent of $h>0$ such that*

$$\|u\|_{l^{\infty}}\leqslant\|u\|_{H^{\sigma}}. \tag{36}$$

Lemma 8. *(cf. [19]). Suppose $\{F^k\}_{k=0}^{\infty}$ be nonnegative sequence and satisfy*

$$F^k\leqslant c\tau\sum_{l=0}^{k-1}F^l+g,\ k=0,1,2,\cdots, \tag{37}$$

Then, we have

$$F^k\leqslant ge^{ck\tau},\ k=0,1,2,\cdots, \tag{38}$$

where c and g are nonnegative constants.

Theorem 9. *The linearized finite difference scheme (28)–(30) is uniquely solvable.*

Proof. Denote $u^k=(u_1^k,u_2^k,\cdots,u_{M-1}^k)^T$. We will prove the above result by the mathematical induction. Obviously, (29) is true for $k=0$. Now, we suppose $u^l(0\leqslant k\leqslant l\leqslant N-1)$ has been uniquely determined; then, we only need to prove that u^{l+1} is uniquely determined by (28). We can rewrite (28) in the following matrix form

$$\begin{aligned}&\left(I+\frac{\mu}{h^{\alpha}}A^{\alpha}+\frac{\tau\kappa}{2h^{\beta}}A^{\beta}-\frac{\tau}{2}\operatorname{diag}\left(f'\left(u^l\right)\right)\right)u^{l+1}\\&=\left(I+\frac{\mu}{h^{\alpha}}A^{\alpha}-\frac{\tau\kappa}{2h^{\beta}}A^{\beta}-\frac{\tau}{2}\operatorname{diag}\left(f'\left(u^l\right)\right)\right)u^l\\&\quad+\tau f\left(u^l\right)+\tilde{G}^{l+1},\end{aligned} \tag{39}$$

where $\tilde{G}^{l+1}$ is a vector which depends only on the boundary value. By using Lemma 3, when τ is sufficiently small, it is easy to verify that the coefficient matrix of (39) is strictly diagonally dominant, which implies that there exists a unique solution u^{l+1}. This completes the proof.

3.3. The Convergence and Stability of the Finite Difference Scheme. Firstly, we easily have the estimation of the local truncation error, according to (27).

Lemma 10. *Let $u(x,\cdot)\in\mathscr{C}^{(2+\alpha)}(x,\cdot)$ be the solution of the problem (15)–(17). Then, we have*

$$\|R^k\|^2\leqslant(b-a)c_1^2\left(\tau^2+h^2\right)^2,\ 0\leqslant k\leqslant N-1, \tag{40}$$

where c_1 is a positive constant independent of τ and h.

Denote

$$e_i^k=U_i^k-u_i^k,\ i\in\omega,\ 0\leqslant k\leqslant N. \tag{41}$$

We will obtain the main convergence result.

Theorem 11. *Let $u(x,\cdot)\in\mathscr{C}^{(2+\alpha)}(x,\cdot)$ be the solution of the problem (15)–(17). Then, there exist positive constants τ_0 and h_0, when $\tau<\tau_0$ and $h<h_0$, for $0\leqslant k\leqslant N$, we have*

$$\|e^k\|\leqslant C_1\left(\tau^2+h^2\right),\ \left|e^k\right|_{H^{\alpha/2}}\leqslant C_2\left(\tau^2+h^2\right),\|e^k\|_{l^{\infty}}\leqslant C_3\left(\tau^2+h^2\right), \tag{42}$$

where $C_1,C_2,C_3>0$ are positive constants independent of τ and h.

Proof. The mathematical induction will be employed. Firstly, it is obvious (42) is true for $k=0$, via (29). Then, it assumes that (42) is true for $1 \leqslant k \leqslant m \leqslant N-1$. We will discuss that (42) holds for $k=m+1$. According to the hypothesis, we can obtain the following estimation:

$$\|u^k\|_{l^\infty} \leqslant \|e^k\|_{l^\infty} + \|U^k\|_{l^\infty} \leqslant C_3\left(\tau^2+h^2\right) + \tilde{c}_0 \leqslant 1+\tilde{c}_0,\ 1 \leqslant k \leqslant m, \tag{43}$$

where $\tau < \tau_0 = (2C_3)^{-1/2}$, $h < h_0 = (2C_3)^{-1/2}$, and $\tilde{c}_0 = \max_{(x,t)\in\Omega\times[0,T]} |U(x,t)|$.

In the view of Lipschitz condition, we have

$$|f\left(U_i^k\right) - f\left(u_i^k\right)| \leqslant c_2\left|e_i^k\right|,\ i \in \bar{\omega},\ 0 \leqslant k \leqslant N, \tag{44}$$

$$|f'\left(U_i^k\right) - f'\left(u_i^k\right)| \leqslant c_3\left|e_i^k\right|,\ i \in \bar{\omega},\ 0 \leqslant k \leqslant N, \tag{45}$$

$$\left|f'\left(u_i^k\right)\right| \leqslant c_4,\ i \in \bar{\omega},\ 0 \leqslant k \leqslant N, \tag{46}$$

where c_2, c_3, and c_4 are positive constants independent of τ and h.

Now, subtracting (28) from (26), we can obtain the error equation

$$\delta_t e_i^{k+(1/2)} - \mu\delta_t\delta_x^\alpha e_i^{k+(1/2)} = \kappa\delta_x^\beta e_i^{k+(1/2)} + P_i^k + \mathrm{R}_i^k,\ i \in \omega,\ 0 \leqslant k \leqslant N-1, \tag{47}$$

where

$$P_i^k = f\left(U_i^k\right) - f\left(u_i^k\right) + \frac{1}{2}\left(f'\left(U_i^k\right)\left(U_i^{k+1}-U_i^k\right) - f'\left(u_i^k\right)\left(u_i^{k+1}-u_i^k\right)\right). \tag{48}$$

Firstly, we establish l^2-error estimation. Taking the discrete inner product of (47) with $e^{k+(1/2)}$, we have

$$\begin{aligned}&\left(\delta_t e^{k+(1/2)}, e^{k+(1/2)}\right) - \mu\left(\delta_t\delta_x^\alpha e^{k+(1/2)}, e^{k+(1/2)}\right)\\ &\quad = \kappa\left(\delta_x^\beta e^{k+(1/2)}, e^{k+(1/2)}\right) + \left(P^k, e^{k+(1/2)}\right) + \left(R^k, e^{k+(1/2)}\right).\end{aligned} \tag{49}$$

Now, we estimate each term in (49). The first term on LHS in (49) can be estimated as

$$\left(\delta_t e^{k+(1/2)}, e^{k+(1/2)}\right) = \frac{1}{2\tau}\left(\|e^{k+1}\|^2 - \|e^k\|^2\right). \tag{50}$$

Noticing Lemma 6, for the second term on the LHS in (49), we have

$$\begin{aligned}\left(\delta_t\delta_x^\alpha e^{k+(1/2)}, e^{k+(1/2)}\right) &= -\left(\delta_x^{\alpha/2}\left(\frac{e^{k+1}-e^k}{\tau}\right), \delta_x^{\alpha/2}\left(\frac{e^{k+1}+e^k}{2}\right)\right)\\ &= -\frac{1}{2\tau}\left(\|\delta_x^{\alpha/2}e^{k+1}\|^2 - \|\delta_x^{\alpha/2}e^k\|^2\right).\end{aligned} \tag{51}$$

Similarly, the first term on RHS in (49) can be obtained by

$$\begin{aligned}\left(\delta_x^\beta e^{k+(1/2)}, e^{k+(1/2)}\right) &= -\left(\delta_x^{\beta/2}\left(\frac{e^{k+1}+e^k}{2}\right), \delta_x^{\beta/2}\left(\frac{e^{k+1}+e^k}{2}\right)\right)\\ &= -\frac{1}{4}\|\delta_x^{\beta/2}\left(e^{k+1}+e^k\right)\|^2.\end{aligned} \tag{52}$$

According to (44)–(46), we have

$$\begin{aligned}|P_i^k| &= \left|f\left(U_i^k\right) - f\left(u_i^k\right) + \frac{1}{2}\left(\left(f'\left(U_i^k\right) - f'\left(u_i^k\right)\right)\right.\right.\\ &\qquad \left.\left.\cdot\left(U_i^{k+1}-U_i^k\right) + f'\left(u_i^k\right)\left(e_i^{k+1}-e_i^k\right)\right)\right|\\ &\leqslant c_2|e_i^k| + \frac{1}{2}\left(c_3c_0\tau\,|e_i^k| + c_4\,|e_i^{k+1}-e_i^k|\right).\end{aligned} \tag{53}$$

Using the Cauchy-Schwarz inequality and Young inequality, the second term on the RHS in (49) becomes

$$\begin{aligned}\left(P^k, e^{k+(1/2)}\right) &\leqslant \|P^k\|\|e^{k+(1/2)}\| \leqslant \frac{3}{4}\|P^k\|^2 + \frac{1}{3}\|e^{k+(1/2)}\|^2\\ &\leqslant \frac{9}{4}\left(c_2^2\|e^k\|^2 + \frac{1}{4}c_3^2c_0^2\tau^2\|e^k\|^2 + \frac{1}{4}c_4^2\|e^{k+1}-e^k\|^2\right)\\ &\quad + \frac{1}{6}\left(\|e^{k+1}\|^2 + \|e^k\|^2\right)\\ &\leqslant \left(\frac{9c_2^2}{4} + \frac{9c_3^2c_0^2\tau^2}{16} + \frac{9}{8}c_4^2 + \frac{1}{6}\right)\|e^k\|^2\\ &\quad + \left(\frac{9c_4^2}{8} + \frac{1}{6}\right)\|e^{k+1}\|^2.\end{aligned} \tag{54}$$

The last term of RHS in (49) is estimated as

$$\begin{aligned}\left(R^k, e^{k+(1/2)}\right) &\leqslant \|R^k\|\|e^{k+(1/2)}\| \leqslant \frac{3}{4}\|R^k\|^2 + \frac{1}{3}\|e^{k+(1/2)}\|^2\\ &\leqslant \frac{3}{4}(b-a)c_1^2\left(\tau^2+h^2\right)^2 + \frac{1}{6}\left(\|e^{k+1}\|^2 + \|e^k\|^2\right).\end{aligned} \tag{55}$$

Substituting (50)–(55) into (49), we get

$$\begin{aligned}&\frac{\|e^{k+1}\|^2 - \|e^k\|^2}{2\tau} + \mu\frac{\|\delta_x^{\alpha/2}e^{k+1}\|^2 - \|\delta_x^{\alpha/2}e^k\|^2}{2\tau} + \kappa\frac{\|\delta_x^{\beta/2}\left(e^{k+1}+e^k\right)\|^2}{4}\\ &\qquad \leqslant c_5\left(\|e^{k+1}\|^2 + \|e^k\|^2\right) + \frac{3}{4}(b-a)c_1^2\left(\tau^2+h^2\right)^2,\end{aligned} \tag{56}$$

where $c_5 = (9c_2^2/4) + (9c_3^2c_0^2\tau^2/16) + ((9/8)9/8c_4^2) + (1/3)$.

Summing for k from 0 to m, we have

$$\frac{\|e^{m+1}\|^2-\|e^0\|^2}{2\tau}+\mu\frac{\|\delta_x^{\alpha/2}e^{m+1}\|^2-\|\delta_x^{\alpha/2}e^0\|^2}{2\tau}+\frac{\kappa}{4}\sum_{k=0}^{m}\|\delta_x^{\beta/2}\left(e^{k+1}+e^k\right)\|^2 \leqslant c_5\sum_{k=0}^{m}\left(\|e^{k+1}\|^2+\|e^k\|^2\right)+\frac{3}{4}(b-a)c_1^2\sum_{k=0}^{m}\left(\tau^2+h^2\right)^2. \tag{57}$$

Noticing that $e^0=0$ and $\kappa>0$, we have

$$\|e^{m+1}\|^2+\mu\|\delta_x^{\alpha/2}e^{m+1}\|^2 \leqslant 4\tau c_5\sum_{k=0}^{m}\|e^k\|^2+2\tau c_5\|e^{m+1}\|^2 + \frac{3\tau}{2}(b-a)c_1^2\sum_{k=0}^{m}\left(\tau^2+h^2\right)^2. \tag{58}$$

Let $F^m=\|e^m\|^2+\mu\|\delta_x^{\alpha/2}e^m\|^2$, we have

$$F^{m+1}\leqslant 4c_5\tau\sum_{k=0}^{m}F^k+2c_5\tau F^{m+1}+\frac{3\tau}{2}(b-a)c_1^2\sum_{k=0}^{m}\left(\tau^2+h^2\right)^2. \tag{59}$$

It implies when $\tau\leqslant\tau_0=1/3c_5$, we have

$$F^{m+1}\leqslant 12\tau c_5\sum_{k=0}^{m}F^k+\frac{9}{2}\tau(b-a)c_1^2\sum_{k=0}^{m}\left(\tau^2+h^2\right)^2. \tag{60}$$

Using Gronwall Lemma 8, we have

$$F^{m+1}\leqslant\exp\left(12c_5m\tau\right)\left(\frac{9}{2}\tau(b-a)c_1^2\sum_{k=0}^{m}\left(\tau^2+h^2\right)^2\right). \tag{61}$$

Therefore, we have

$$\|e^{m+1}\|\leqslant C_1\left(\tau^2+h^2\right), \tag{62}$$

where $C_1:=\sqrt{(9(b-a)c_1^2T\exp(12c_5T))/2}$.

Similarly, applying Lemma 5 yields

$$\left|e^{m+1}\right|_{H^{\alpha/2}}\leqslant C_2\left(\tau^2+h^2\right), \tag{63}$$

where $C_2:=\sqrt{(9(b-a)c_1^2T\exp(12c_5T))/2C_\alpha\mu}$.

Finally, we can establish l^∞-error estimate by combining (62) with (63). Denoting $C_3:=C_\sigma\sqrt{C_1^2+C_2^2}$, it follows from Lemma 7 that

$$\|e^{m+1}\|_{l^\infty}\leqslant C_3\left(\tau^2+h^2\right). \tag{64}$$

We complete the proof.

Next, we will analyze the stability of the scheme (28)–(30). Let $\{v_i^k\mid 0\leqslant i\leqslant M, 0\leqslant k\leqslant N\}$ be the solution of the fractional Sobolev equation

$$\delta_t v_i^{k+(1/2)}-\mu\delta_t\delta_x^\alpha v_i^{k+(1/2)}=\kappa\delta_x^\beta v_i^{k+(1/2)}+f\left(v_i^k\right)+\frac{1}{2}f'\left(v_i^k\right)\cdot\left(v_i^{k+1}-v_i^k\right),\ i\in\omega,\ 0\leqslant k\leqslant N-1, \tag{65}$$

$$v_i^0=u_0(x_i)+\phi_i^0,\ i\in\bar{\omega}, \tag{66}$$

$$v_i^k=0,\ i\in\partial\omega,\ 1\leqslant k\leqslant N, \tag{67}$$

where ϕ_i^0 is the perturbation of the initial value. Subtracting (65)–(67) from (28)–(30) and denoting $\rho_i^k=v_i^k-u_i^k$, we have

$$\delta_t\rho_i^{k+(1/2)}-\mu\delta_t\delta_x^\alpha\rho_i^{k+(1/2)}=\kappa\delta_x^\beta\rho_i^{k+(1/2)}+f\left(v_i^k\right)-f\left(u_i^k\right)+\frac{1}{2}\cdot\left[f'\left(v_i^k\right)\left(v_i^{k+1}-v_i^k\right)-f'\left(u_i^k\right)\left(u_i^{k+1}-u_i^k\right)\right],$$

$$i\in\omega,\ 0\leqslant k\leqslant N-1,$$

$$\rho_i^0=\phi_i^0,\ i\in\bar{\omega},$$

$$\rho_i^k=0,\ i\in\partial\omega,\ 1\leqslant k\leqslant N. \tag{68}$$

Similar to the proof of Theorem 11, we have the following result.

Theorem 12. *Denote $\rho_i^k=v_i^k-u_i^k, i\in\bar{\omega}, 0\leqslant k\leqslant N$. Then, there exist positive constants τ_0 and h_0, when $\tau<\tau_0$ and $h<h_0$, we have*

$$\|\rho^k\|\leqslant C_4\|\rho^0\|,\left|\rho^k\right|_{H^{\alpha/2}}\leqslant C_5\left|\rho^0\right|_{H^{\alpha/2}},\|\rho^k\|_{l^\infty}\leqslant C_6\|\rho^0\|_{l^\infty}, \tag{69}$$

where $C_4, C_5, C_6>0$ are positive constants independent of τ and h.

4. Numerical Examples

In this section, we will provide extensive numerical examples to testify the theoretical results. we will define the discrete l^2-norm and l^∞-norm separately and the corresponding convergence orders are defined as follows:

$$E(h,\tau)=\sqrt{h\sum_{i=0}^{M}\left(\mathrm{U}_i^N-u_i^N\right)^2},\ E_\infty(h,\tau)=\max_{0\leqslant i\leqslant M,0\leqslant k\leqslant N}\left|U_i^k-u_i^k\right|,$$

$$\mathrm{Ord}_2=\log_2\left(\frac{\|E(h,\tau)\|}{\|E(h/2,\tau/2)\|}\right),$$

$$\mathrm{Ord}_\infty=\log_2\left(\frac{\|E(h,\tau)\|_\infty}{\|E(h/2,\tau/2)\|_\infty}\right). \tag{70}$$

Table 1: l^2- and l^∞-errors and their convergence orders of (28)–(30) for $1 < \alpha < 2$ in the spatial direction for (72) with fixed time step $\tau = 1/2000$ for Example 1.

(α, β)	h	$\|e\|$	Ord_2	$\|e\|_\infty$	Ord_∞
(1.2,1.8)	1/10	$7.5025e-4$	–	$1.0862e-3$	–
	1/20	$1.7359e-4$	2.1117	$2.5686e-4$	2.0802
	1/40	$4.0650e-5$	2.0702	$6.1167e-5$	2.0702
(1.5,1.5)	1/10	$7.7557e-4$	–	$1.1391e-3$	–
	1/20	$1.7445e-4$	2.1524	$2.6326e-4$	2.1133
	1/40	$3.9827e-5$	2.1310	$6.1202e-5$	2.1048
(1.8,1.2)	1/10	$1.1777e-3$	–	$1.6525e-3$	–
	1/20	$2.7908e-4$	2.0773	$3.9673e-4$	2.0584
	1/40	$6.6203e-5$	2.0757	$9.5326e-5$	2.0572

Table 2: l^2- and l^∞-errors and their convergence orders of (28)–(30) for $1 < \alpha < 2$ in the temporal direction for (72) with fixed spatial step $h = 1/2000$ for Example 1.

(α, β)	τ	$\|e\|$	Ord_2	$\|e\|_\infty$	Ord_∞
(1.2,1.8)	1/10	$1.9481e-4$	–	$3.1163e-4$	–
	1/20	$4.8708e-5$	1.9998	$7.7916e-5$	1.9998
	1/40	$1.2169e-5$	2.0010	$1.9465e-5$	2.0010
(1.5,1.5)	1/10	$1.5223e-4$	–	$2.4163e-4$	–
	1/20	$3.8057e-5$	2.0000	$6.0406e-5$	2.0001
	1/40	$9.5069e-6$	2.0011	$1.5088e-5$	2.0013
(1.8,1.2)	1/10	$1.2416e-4$	–	$1.9507e-4$	–
	1/20	$3.1025e-5$	2.0006	$4.8747e-5$	2.0006
	1/40	$7.7412e-6$	2.0028	$1.2163e-5$	2.0028

Table 3: l^2-and l^∞-errors and their convergence orders of (28)–(30) for $1 < \alpha < 2$ in the spatial direction for (73) with $\tau = 1/2000$ for Example 2.

(α, β)	h	$\|e\|$	Ord_2	$\|e\|_\infty$	Ord_∞
(1.1,1.9)	1/100	$1.3736e-4$	–	$2.9683e-4$	–
	1/200	$4.9959e-5$	1.4591	$9.9718e-5$	1.5737
	1/400	$1.6497e-5$	1.5985	$3.1155e-5$	1.6784
(1.5,1.5)	1/100	$4.6568e-4$	–	$6.5750e-4$	–
	1/200	$1.3983e-4$	1.7357	$2.0089e-4$	1.7106
	1/400	$4.0066e-5$	1.8032	$5.8340e-5$	1.7839
(1.9,1.1)	1/100	$1.7483e-4$	–	$2.1547e-4$	–
	1/200	$4.6475e-5$	1.9115	$5.7682e-5$	1.9013
	1/400	$1.2052e-5$	1.9471	$1.5024e-5$	1.9408

Example 1. We firstly consider the following fractional Sobolev equation as

$$\partial_t u - \mu \partial_x^\alpha \partial_t u = \kappa \partial_x^\beta u + \sin(u) + g(x,t), (x,t) \in (0,1) \times (0,1], \tag{71}$$

The exact solution is

$$u(x,t) = t^3 x^2 (1-x)^2. \tag{72}$$

The initial boundary conditions and $g(x,t)$ are determined by (72).

Table 4: l^2-and l^∞-errors and their convergence orders of (28)–(30) for $1 < \alpha < 2$ in the temporal direction for (73) with $h = 1/1000$ for Example 2.

(α, β)	τ	$\|e\|$	Ord_2	$\|e\|_\infty$	Ord_∞
	1/100	$2.0697e-2$	–	$5.6643e-2$	–
(1.1,1.9)	1/200	$8.1358e-3$	1.3471	$1.3366e-2$	2.0833
	1/400	$2.0509e-3$	1.9880	$2.9698e-3$	2.1701
	1/100	$2.0526e-2$	–	$5.6522e-2$	–
(1.5,1.5)	1/200	$8.1446e-3$	1.3336	$1.3303e-2$	2.0871
	1/400	$2.0523e-3$	1.9886	$2.9423e-3$	2.1767
	1/100	$2.0734e-2$	–	$5.6310e-2$	–
(1.9,1.1)	1/200	$8.1558e-3$	1.3461	$1.3202e-2$	2.0927
	1/400	$2.0536e-3$	1.9897	$2.9033e-3$	2.1850

Table 5: l^2-and l^∞-errors and their convergence orders of (28)–(30) for $1 < \alpha < 2$ in the spatial direction for (75) with $\tau = 1/1000$ for Example 3.

(α, β)	h	$\|e\|$	Ord_2	$\|e\|_\infty$	Ord_∞
	1/10	$1.4761e-2$	–	$1.6303e-2$	–
(1.3,1.7)	1/20	$3.3659e-3$	2.1327	$3.8460e-3$	2.0837
	1/40	$8.0373e-4$	2.0662	$9.0995e-4$	2.0795
	1/10	$1.6167e-2$	–	$1.8043e-2$	–
(1.5,1.5)	1/20	$3.6642e-3$	2.1414	$4.2293e-3$	2.0930
	1/40	$8.7053e-4$	2.0735	$9.9262e-4$	2.0911
	1/10	$2.0933e-2$	–	$2.2257e-2$	–
(1.7,1.3)	1/20	$4.8190e-3$	2.1190	$5.3128e-3$	2.0667
	1/40	$1.1227e-3$	2.1018	$1.2648e-3$	2.0705

Table 6: l^2-and l^∞-errors and their convergence orders of (28)–(30) for $1 < \alpha < 2$ in the temporal direction for (75) with $h = 1/1000$ for Example 3.

(α, β)	τ	$\|e\|$	Ord_2	$\|e\|_\infty$	Ord_∞
	1/10	$9.1725e-3$	–	$1.0470e-2$	–
(1.3,1.7)	1/20	$2.5218e-3$	1.8629	$2.8729e-3$	1.8657
	1/40	$6.6380e-4$	1.9256	$7.5586e-4$	1.9263
	1/10	$8.7813e-3$	–	$9.8734e-3$	–
(1.5,1.5)	1/20	$2.4228e-3$	1.8578	$2.7208e-3$	1.8595
	1/40	$6.3844e-4$	1.9240	$7.1689e-4$	1.9242
	1/10	$8.2127e-3$	–	$9.1056e-3$	–
(1.7,1.3)	1/20	$2.2771e-3$	1.8507	$2.5228e-3$	1.8517
	1/40	$6.0134e-4$	1.9209	$6.6620e-4$	1.9210

Taking $\mu = 1$, $\kappa = 1$, the linearized numerical scheme (28)–(30) with $\tau = h$ is applied to solve the above Sobolev equation. The global numerical errors and convergence orders with respect to different α and β are listed in the following tables. Table 1 lists the l^2-norm and l^∞-norm errors and spatial convergence orders with fixed time step $\tau = 1/2000$. Table 2 tests the temporal convergence orders with fixed spatial step $h = 1/2000$. It demonstrates that the convergence orders of the scheme (28)–(30) is second-order accurate in both spatial and temporal directions which is consistent with Theorem 11.

All the data are referred to MATLAB codes in Example 1 in the supplementary files.

Table 7: l^2-and l^∞-errors and their convergence orders of (28)–(30) for $1 < \alpha \leqslant 2$ in the spatial direction with $\tau = 1/1000$ for Example 4.

(α, β)	h	$\|e\|$	Ord_2	$\|e\|_\infty$	Ord_∞
	1/40	$1.7548e-2$	–	$1.1092e-2$	–
(1.2,1.8)	1/80	$3.9216e-3$	2.1618	$2.3542e-3$	2.2362
	1/160	$9.5790e-4$	2.0558	$5.6623e-4$	2.0558
	1/40	$1.2035e-2$	–	$7.0740e-3$	–
(1.5,1.5)	1/80	$2.5833e-3$	2.2199	$1.4349e-3$	2.3016
	1/160	$6.2934e-4$	2.0373	$3.4473e-4$	2.0574
	1/40	$7.8751e-3$	–	$3.5547e-3$	–
(1.8,1.2)	1/80	$1.6022e-3$	2.2972	$8.7242e-4$	2.0266
	1/160	$3.9110e-4$	2.0345	$2.1876e-4$	1.9957
	1/40	$1.4624e-2$	–	$8.1939e-3$	–
(2.0,2.0)	1/80	$3.1088e-3$	2.2339	$1.5922e-3$	2.3635
	1/160	$7.5817e-4$	2.0358	$3.8083e-4$	2.0638

Table 8: l^2-and l^∞-errors and their convergence orders of (28)–(30) for $1 < \alpha \leqslant 2$ in the temporal direction with $h = 1/1000$ for Example 4.

(α, β)	τ	$\|e\|$	Ord_2	$\|e\|_\infty$	Ord_∞
	1/40	$9.2300e-9$	–	$5.5579e-9$	–
(1.2,1.8)	1/80	$2.3078e-9$	1.9998	$1.3897e-9$	1.9998
	1/160	$5.7564e-10$	2.0033	$3.4662e-10$	2.0034
	1/40	$7.5499e-9$	—	$3.2739e-9$	–
(1.5,1.5)	1/80	$1.8868e-9$	2.0005	$8.1803e-10$	2.0008
	1/160	$4.7057e-10$	2.0034	$2.0378e-10$	2.0051
	1/40	$6.7080e-9$	–	$2.7409e-9$	–
(1.8,1.2)	1/80	$1.6769e-9$	2.0001	$6.8522e-10$	2.0000
	1/160	$4.1960e-10$	1.9987	$1.7116e-10$	2.0012
	1/40	$9.2576e-9$	–	$4.4312e-9$	–
(2.0,2.0)	1/80	$2.3143e-9$	2.0001	$1.1077e-9$	2.0001
	1/160	$5.7857e-10$	2.0000	$2.7694e-10$	2.0000

Example 2. Next, we consider the nonlinear fractional Sobolev equation as

The exact solution

$$u(x,t) = \sin\,(t+1)(2+x)^2(2-x)^2 \tag{73}$$

is oscillatory along with the temporal direction, where $\mu = 1$ and $\kappa = 1$. And the initial boundary conditions and $g(x,t)$ are determined by (73).

In this example, we examine the spatial convergence orders with the fixed time step $\tau = 1/2000$ and the temporal convergence orders with the fixed spatial step $h = 1/1000$ in l^2-norm and l^∞-norm errors, respectively. All the numerical results in the example are listed in Tables 3 and 4. Similar results are observed. All the data are referred to MATLAB codes in Example 2 in the supplementary files.

Example 3. Then, we calculate the nonlinear fractional Sobolev equation as

$$\partial_t u - \partial_x^\alpha \partial_t u = \partial_x^\beta u + u^2 - u^4 + g(x,t),\ (x,t) \in (-1,1) \times (0,1], \tag{74}$$

We choose the exact solution

$$u(x,t) = \left(t + t^3\right)(1+x)^2(1-x)^2. \tag{75}$$

The initial boundary conditions and $g(x,t)$ are determined by (75).

Similar to above example, Tables 5 and 6 list the l^2-norm and l^∞-norm errors and corresponding spatial and temporal convergence orders of (28)–(30), respectively. To testify the spatial convergence orders, we fixed the time step $\tau = 1/1000$. Similarly, we take the fixed spatial step $h = 1/1000$ to obtain the temporal convergence orders. The numerical

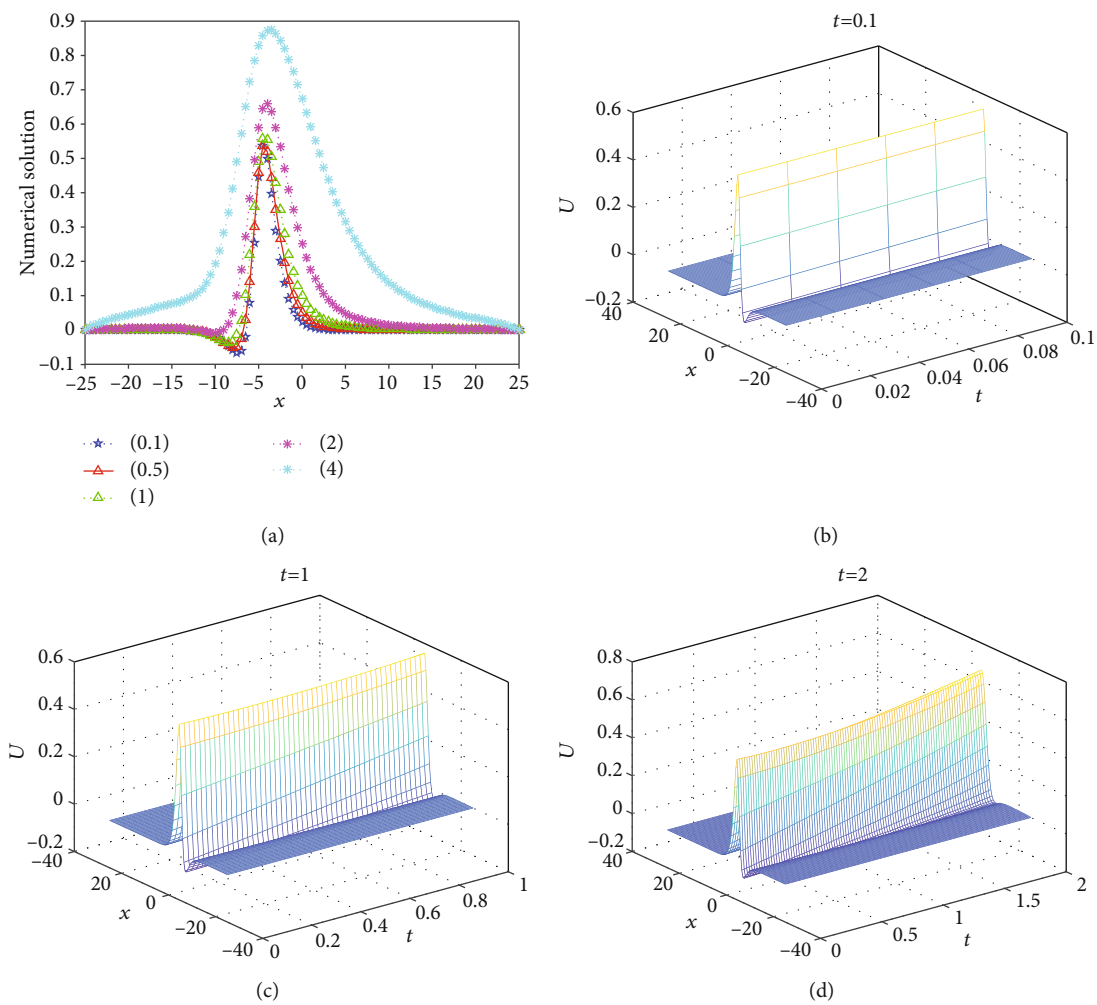

Figure 1: Using scheme (28)–(30), curves of $u(x,t)$ with respect to x at different time with $h=0.5$, $\tau=0.02$ (a) and the evolutionary surfaces of $u(x,t)$ at $T=0.1$ (b), $T=1$ (c), and $T=2$ (d) in Example 4.

results show that (28)–(30) is close to second-order accurate in spatial and temporal directions.

All the data are referred to MATLAB codes in Example 3 in the supplementary files.

In the following model, the exact solution is unknown, we test convergence orders using the posterior error estimation

$$\begin{aligned}\mathrm{Ord}_2 &= \log_2\left(\frac{\|u(h,\tau)-u(h,\tau/2)\|}{\|u(h,\tau/2)-u(h,\tau/2)\|}\right), \mathrm{Ord}_\infty \\ &= \log_2\left(\frac{\|u(h,\tau)-u(h,\tau/2)\|_\infty}{\|u(h,\tau/2)-u(h,\tau/4)\|_\infty}\right).\end{aligned} \tag{76}$$

Example 4. We consider the following equation:

$$\begin{aligned}&\partial_t u - \partial_x^\alpha \partial_t u = \partial_x^\beta u + u - u^2, \quad (x,t)\in(-25,25)\times(0,0.1],\\ &u(x,0)=\sqrt{2}sech(x+5)\cos(4/x), x\in[-25,25],\\ &u(-25,t)=u(25,t)=0, t\in[0,0.1],\end{aligned} \tag{77}$$

with the exact solution is unknown.

In the computation, we take different spatial and temporal step sizes. The l^2-norm, l^∞-norm errors, and their convergence orders of (28)–(30) are listed in Table 7 with the fixed temporal step size $\tau=1/1000$. Similarly, the spatial step size fixed at $h=1/1000$ in Table 8. Tables 7 and 8 show that the numerical results have second-order accurate in spatial and temporal directions. Figure 1 presents curves of $u(x,t)$ with respect to x at different time with the step sizes $h=0.5$ and $\tau=0.02$. All the data are referred to MATLAB codes in Example 4 in the supplementary files.

5. Conclusion

In the article, we establish an efficient finite difference scheme for nonlinear spatial fractional Sobolev equation based on Newton linearized technique. We have proved that the numerical solution of the scheme is unique solvable, stable, and convergent. The pointwise error estimate is proved with the convergence order $O(\tau^2+h^2)$. Extensive numerical examples are carried out to testify the numerical theoretical results. Extending the current work to high dimensional cases is possible, which will leave as our future work.

Data Availability

All the data are available and referred to the supplementary file.

Conflicts of Interest

The authors declare that they have no competing interests.

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