Research Article

Multiple Standing Waves for Nonlinear Schrödinger-Poisson Systems

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In this paper, we consider the following nonlinear Schrödinger-Poisson systems. Under suitable conditions on V , K , g , and h , when $1 < s < 6$, we obtain two nontrivial solutions for the problem and when $g(x, \cdot)$ is odd and $6 < s < \infty$, we obtain infinitely many solutions for the problem.

1. Introduction

In this paper, we consider the following nonlinear Schrödinger-Poisson equations on $\mathbb{R}^3$

$$\begin{cases} -\Delta u + V(x)u + K(x)\phi u = g(x, u) - h(x)|u|^{s-2}u, & \text{in } \mathbb{R}^3, \\ -\Delta \phi = K(x)u^2, & \text{in } \mathbb{R}^3. \end{cases} \quad (1)$$

Such problem arises when one is looking for standing wave solutions $\psi(t, x) = e^{-i\omega t/\hbar}u(x)$ for the nonlinear Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + U(x)\psi + K(x)\phi \psi - f(x, |\psi|)\psi, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^3, \quad (2)$$

coupled with the Poisson equation

$$-\Delta \phi = K(x)u^2, \quad x \in \mathbb{R}^3. \quad (3)$$

It is well known that the Schrödinger-Poisson systems have a strong physical meaning because they appear in quantum mechanics models and in semiconductor theory

(see [1–3]). Problem (1) is a special case of the following Schrödinger-Poisson system:

$$\begin{cases} -\Delta u + V(x)u + K(x)\phi u = f(x, u), & \text{in } \mathbb{R}^3, \\ -\Delta \phi = K(x)u^2, & \text{in } \mathbb{R}^3. \end{cases} \quad (4)$$

Cerami and Vaira in [4] studied the existence of positive solutions for the problem where they considered $V(x) = 1$, $f(x, u) = k(x)|u|^{p-2}u$, and $4 < p < 6$. After that, many researchers have focused on the problem under various conditions (see [5, 6]).

In recent years, the Schrödinger-Poisson system has been studied widely under variant assumptions on V , K , and f (see [7–10]). Because the problem is set on the whole space $\mathbb{R}^3$, it is well known that the main difficulty of this problem is the lack of compactness for Sobolev embedding, and then, it is usually difficult to prove that a minimizing sequence or a (PS) sequence is strongly convergent if we seek solutions by variational methods. In order to overcome this difficulty, most of them dealt with the situation where V is a positive constant or being radially symmetric (see [11–13]). When V is not a constant and not radially symmetric, there have been many works by developing various variational techniques (see [4, 14–17]).

In paper [18], Sun et al. considered the system

$$\begin{cases} -\Delta u + a(x)u + \phi u = k(x)|u|^{q-2}u - h(x)|u|^{p-2}u, & \text{in } \mathbb{R}^3, \\ -\Delta \phi = u^2, \quad \lim_{|x| \rightarrow \infty} \phi(x) = 0, & \text{in } \mathbb{R}^3, \end{cases} \quad (5)$$

where $1 < q < 2 < p < +\infty$, $a(x)$, $k(x)$, and $h(x)$ are measurable functions satisfying suitable assumptions. They obtained infinitely many solutions in $H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$ with negative energy. Since $2 < p < \infty$ is allowed to be supercritical, the usual Sobolev space $H^1(\mathbb{R}^3)$ cannot be used for the study; to overcome this difficulty we introduce a new space which is motivated by [19].

We should also mention another recent paper [20]; Wang et al. considered a similar problem

$$\begin{cases} -\Delta u + V(x)u + \phi(x)u = k(x)|u|^{q-2}u - h(x)|u|^{p-2}u + g(x), & \text{in } \mathbb{R}^3, \\ -\Delta \phi = u^2, & \text{in } \mathbb{R}^3, \end{cases} \quad (6)$$

and a nontrivial solution is obtained (see [20], Theorem 2). Note that there is an inhomogeneous term on the right-hand side of the first equation. Here $1 < q < 2 < p < 4$ and the potential V satisfies a coercive condition so that the working space can be compactly embedded into Lebesgue spaces.

Now, we turn to our problem (1). Since the space dimension is $N = 3$, the critical Sobolev exponent $2^* = 6$. For $p \in (1, 6)$ we denote

$$p_0 = \frac{2N}{2N - p(N - 2)} = \frac{6}{6 - p}, \quad p' = \frac{p}{p - 1}. \quad (7)$$

Note that p' is the conjugate exponent in Hölder inequality. To state our results on the problem (1), we make the following assumptions:

(V) $V \in C(\mathbb{R}^3, \mathbb{R})$ satisfies $0 < \inf_{\mathbb{R}^3} V(x) \leq \sup_{\mathbb{R}^3} V(x) < +\infty$.

(K) $K \in L^\infty(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$, $K(x) \geq 0$ for a.e. $x \in \mathbb{R}^3$.

(g₁) $g \in C(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$; for some $r \in (1, 2)$ and $b \in L^\infty(\mathbb{R}^3) \cap L^{r_0}(\mathbb{R}^3)$, we have

$$|g(x, u)| \leq b(x)|u|^{r-1}, \quad (x, u) \in \mathbb{R}^3 \times \mathbb{R}. \quad (8)$$

(g₂) There exist $\delta > 0$, $\theta \in (1, 2)$, and $a \in L^\infty(\mathbb{R}^3) \cap L^{\theta_0}(\mathbb{R}^3)$; $a(x) \geq 0$ such that

$$G(x, u) = \int_0^u g(x, \tau) d\tau \geq a(x)|u|^\theta, \quad (x, u) \in \mathbb{R}^3 \times (-\delta, \delta). \quad (9)$$

(h₁) $s \in (\theta, 6)$ and $h \in L^\infty(\mathbb{R}^3) \cap L^{s_0}(\mathbb{R}^3)$; $h(x) \geq 0$ for $x \in \mathbb{R}^3$.

(h₂) $s > \theta$, $h \in L^\infty(\mathbb{R}^3)$, and $h(x) \geq 0$ for $x \in \mathbb{R}^3$.

Remark 1. It follows from (g₁) and (g₂) that $r \leq \theta$ for $a(x) > 0$ and δ small enough.

Under these assumptions, it is clear that the zero function $u(x) = 0$ is the trivial solution of problem (1). Our main results on the existence of multiple nontrivial solutions are the following theorems.

Theorem 2. Suppose (V), (K), (g₁), (g₂), and (h₁) are satisfied; then, problem (1) has two nontrivial solutions.

Theorem 3. Suppose (V), (K), (g₁), (g₂), and (h₂) are satisfied. If $g(x, \cdot)$ is odd, then problem (1) admits infinitely many solutions.

The paper is organized as follows. In Section 2, we give some useful notions and set up the variational framework of the problem. In Section 3, we prove Theorem 2, and the proof of Theorem 3 is given in Section 4. For simplicity, throughout this paper, we denote the norm on $L^\theta = L^\theta(\mathbb{R}^3)$ with $1 < \theta < \infty$ by $|u|_\theta = \left(\int_{\mathbb{R}^3} |u|^\theta dx \right)^{1/\theta}$.

2. Preliminary

Thanks to condition (V), the norm

$$\|u\| = \left(\int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2) dx \right)^{1/2} \quad (10)$$

is an equivalent norm on $H^1 = H^1(\mathbb{R}^3)$. Let $q \in [2, 6]$; then, we have a continuous embedding $H^1 \hookrightarrow L^q$. Hence, there is a constant S_q such that

$$|u|_q \leq S_q \|u\|, \quad u \in H^1. \quad (11)$$

For later use, we also denote by S the best Sobolev constant for the continuous embedding $\mathcal{D}^{1,2} \hookrightarrow L^6$.

For $u \in H^1$, it is well known that the Poisson equation

$$-\Delta \phi = K(x)u^2 \quad (12)$$

has a unique solution $\phi = \phi_u$ in $\mathcal{D}^{1,2}(\mathbb{R}^3)$. Note that according to [21], Theorem 2.2.1,

$$\phi_u(x) = \frac{1}{4\pi} \int \frac{K(y)u^2(y)}{|x-y|} dy. \quad (13)$$

Consequently, $\phi_u \geq 0$ in $\mathbb{R}^3$. We also know that there exists $a_1 > 0$ such that

$$0 \leq \frac{1}{4} \int K(x)\phi_u u^2 \leq a_1 \|u\|^4, \quad \text{for all } u \in H^1. \quad (14)$$

See [14], Lemma 1.1.

Define a functional $\Phi : H^1 \longrightarrow \mathbb{R}$,

$$\begin{aligned} \Phi(u) = & \frac{1}{2} \int (|\nabla u|^2 + V(x)u^2) + \frac{1}{4} \int K(x)\phi_u u^2 - \int G(x, u) \\ & + \frac{1}{s} \int h(x)|u|^s. \end{aligned} \quad (15)$$

Under our assumptions, it is easy to see that $\Phi \in C^1(H^1)$. According to Benci and his collaborators [1, 22], it is well known that if u is a critical point of Φ , then (u, ϕ_u) is a (weak) solution of (1).

To find critical points of Φ , some compactness conditions, such as the well-known Palais-Smale condition ((PS) for short), are crucial. To establish the (PS) condition for Φ , we need the following results.

Proposition 4. *Let $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ such that*

$$|f(x, u)| \leq \alpha(x)|u|^{r-1} + \beta(x)|u|^{s-1}, \quad (x, u) \in \mathbb{R}^N \times \mathbb{R}, \quad (16)$$

for $1 \leq r \leq 2 \leq s \leq 2^*$, $\alpha \in L^\infty(\mathbb{R}^N) \cap L^{r_0}(\mathbb{R}^N)$, and $\beta \in L^\infty(\mathbb{R}^N) \cap L^{s_0}(\mathbb{R}^N)$. Then, the functional $\Psi : \mathcal{D}^{1,2}(\mathbb{R}^N) \longrightarrow \mathbb{R}$,

$$\Psi(u) = \int F(x, u), \quad \text{where } F(x, u) = \int_0^u f(x, \tau) d\tau, \quad (17)$$

is of class C^1 and $\nabla \Psi : \mathcal{D}^{1,2}(\mathbb{R}^N) \longrightarrow (\mathcal{D}^{1,2}(\mathbb{R}^N))^*$ is compact.

Remark 5. Proposition 4 is a special case of do Ó ([23], Lemma 1). Note that since the embedding $H^1(\mathbb{R}^N) \hookrightarrow \mathcal{D}^{1,2}(\mathbb{R}^N)$ is continuous, the statement of Proposition 4 remains valid if we replace $\mathcal{D}^{1,2}(\mathbb{R}^N)$ with $H^1(\mathbb{R}^N)$.

Motivated by Ruiz ([13], Lemma 2.1), we also have the following result.

Lemma 6. *Suppose $K \in L^2(\mathbb{R}^N)$ and $u_n \rightharpoonup u$ in H^1 . Then, $\phi_{u_n} \longrightarrow \phi_u$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$.*

Proof. As in the proof of [13], Lemma 2.1, we define linear functionals $T_n, T : \mathcal{D}^{1,2} \longrightarrow \mathbb{R}$,

$$T_n(v) = \int K(x)u_n^2 v, \quad T(v) = \int K(x)u^2 v. \quad (18)$$

It can be shown that T_n and T are continuous. Note that

$$\int \nabla \phi_{u_n} \cdot \nabla v = T_n(v), \quad \int \nabla \phi_u \cdot \nabla v = T(v), \quad (19)$$

by the isometry between $\mathcal{D}^{1,2}$ and $(\mathcal{D}^{1,2})^*$ via the Riesz representation theorem; it suffices j. Let $\varepsilon > 0$; we choose $R > 0$ such that

$$\int_{|x| \geq R} K^2(x) < \varepsilon^2. \quad (20)$$

Now, let $v \in \mathcal{D}^{1,2}$ with $\|v\|_{\mathcal{D}^{1,2}} \leq 1$; using the Hölder inequality, we have

$$\begin{aligned} |T_n(v) - T(v)| & \leq \left(\int_{|x| \geq R} + \int_{|x| < R} \right) K(x) |u_n^2 - u^2| |v| \\ & \leq \left(\int_{|x| \geq R} K^2(x) \right)^{1/2} |u_n^2 - u^2|_3 \|v\|_6 + \sup_{\mathbb{R}^3} K \\ & \quad \cdot \left(\int_{|x| < R} |u_n^2 - u^2|^{6/5} \right)^{5/6} \left(\int_{|x| < R} |v|^6 \right)^{1/6} \\ & \leq \varepsilon \cdot S^3 \left(\sup_n \|u_n\|_{\mathcal{D}^{1,2}}^2 + \|u\|_{\mathcal{D}^{1,2}}^2 \right) \\ & \quad + \sup_{\mathbb{R}^3} K \cdot S \left(\int_{|x| < R} |u_n^2 - u^2|^{6/5} \right)^{5/6}. \end{aligned} \quad (21)$$

Since $\{u_n\}$ is bounded in $\mathcal{D}^{1,2}$ and

$$\int_{|x| < R} |u_n^2 - u^2|^{6/5} \longrightarrow 0 \quad (22)$$

by the compactness of the Sobolev embedding, letting $n \longrightarrow \infty$ in (21), we deduce

$$\lim_n |T_n(v) - T(v)| \leq C\varepsilon \quad (23)$$

uniformly for $\|v\|_{\mathcal{D}^{1,2}} \leq 1$. It follows that $T_n \longrightarrow T$ in $(\mathcal{D}^{1,2})^*$. The proof is completed. $\square$

Remark 7. Even though we have this lemma in hand, we do not know how to deduce

$$\int K(x) (\phi_{u_n} u_n - \phi_u u) (u_n - u) \longrightarrow 0 \quad (24)$$

from $u_n \rightharpoonup u$. However, we can still deduce the (PS) condition for our functional (see the proof of Lemma 9).

3. Proofs of Theorem 2

To prove Theorem 2, we will apply the truncated method, see e.g. [24]. For a function $u : \mathbb{R}^3 \longrightarrow \mathbb{R}$, we set $u^\pm = \max\{\pm u, 0\}$. Then, we define the truncated functional $\Phi_+ : H^1 \longrightarrow \mathbb{R}$,

$$\begin{aligned} \Phi_+(u) = & \frac{1}{2} \int (|\nabla u|^2 + V(x)u^2) + \frac{1}{4} \int K(x)\phi_u u^2 - \int G(x, u^+) \\ & + \frac{1}{s} \int h(x)(u^+)^s. \end{aligned} \quad (25)$$

It is well known that Φ_+ is of class C^1 ; the critical points of Φ_+ are solutions of the truncated problem

$$\begin{cases} -\Delta u + V(x)u + K(x)\phi u = g(x, u^+) - h(x)(u^+)^{s-1}, & \text{in } \mathbb{R}^3, \\ -\Delta \phi = K(x)u^2, & \text{in } \mathbb{R}^3. \end{cases} \quad (26)$$

Moreover, suppose $u \in H^1$ is a critical point of Φ_+ , then $u \geq 0$. Hence, u is a solution of (1).

Lemma 8. *Under the assumptions of Theorem 2, the functional Φ_+ is coercive. As a consequence, Φ_+ is bounded from below.*

Proof. By (g_1) , we have

$$|G(x, u)| \leq \frac{1}{r} b(x) |u|^r, \quad (x, u) \in \mathbb{R}^3 \times \mathbb{R}. \quad (27)$$

Note that $K(x)\phi_u u^2 \geq 0$ and $h(x) \geq 0$ on $\mathbb{R}^3$; using the Hölder inequality and the fact

$$\begin{aligned} r \cdot r'_0 &= 6, \\ \frac{1}{r'_0} &= \frac{r}{6}, \end{aligned} \quad (28)$$

we have

$$\begin{aligned} \Phi_+(u) &= \frac{1}{2} \|u\|^2 + \frac{1}{4} \int K(x)\phi_u u^2 - \int G(x, u^+) + \frac{1}{s} \int h(x)(u^+)^s \\ &\geq \frac{1}{2} \|u\|^2 - \frac{1}{r} \int b(x) |u|^r \\ &\geq \frac{1}{2} \|u\|^2 - \frac{1}{r} \left(\int |b|^{r_0} \right)^{1/r_0} \left(\int |u|^{r \cdot r'_0} \right)^{1/r'_0} \\ &\geq \frac{1}{2} \|u\|^2 - \frac{1}{r} |b|_{r_0} |u|_6^r \\ &\geq \frac{1}{2} \|u\|^2 - \frac{1}{r} |b|_{r_0} \cdot S_6^r \|u\|^r. \end{aligned} \quad (29)$$

Since $r < 2$, we get $\Phi_+(u) \rightarrow +\infty$ as $\|u\| \rightarrow \infty$. Thus, we have proved Lemma 8. $\square$

Lemma 9. *Under the assumptions of Theorem 2, the functional Φ_+ satisfies the (PS) condition.*

Proof. Let $\{u_n\} \subset H^1$ be a (PS) sequence of Φ_+ . By Lemma 8, $\{u_n\}$ is bounded. Hence, up to a subsequence, we have $u_n \rightarrow u$ in H^1 . Firstly, we have

$$\langle \Phi'_+(u_n) - \Phi'_+(u), u_n - u \rangle \rightarrow 0. \quad (30)$$

To apply Proposition 4, let $\Psi_1, \Psi_2 : H^1 \rightarrow \mathbb{R}$,

$$\begin{aligned} \Psi_1(u) &= \int G(x, u^+), \\ \Psi_2(u) &= \frac{1}{s} \int h(x)(u^+)^s. \end{aligned} \quad (31)$$

By our assumptions (g_1) and (h_1) , we can apply Proposition 4 (see also Remark 5) and deduce

$$\begin{aligned} \nabla \Psi_1(u_n) &\rightarrow \nabla \Psi_1(u) \text{ in } H^1, \\ \nabla \Psi_2(u_n) &\rightarrow \nabla \Psi_2(u) \text{ in } H^1. \end{aligned} \quad (32)$$

Therefore, since $\{u_n\}$ is bounded in H^1 , we have

$$\begin{aligned} \int [g(x, u_n^+) - g(x, u^+)](u_n - u) &= \langle \nabla \Psi_1(u_n) - \nabla \Psi_1(u), u_n - u \rangle \rightarrow 0, \\ \int h(x) [(u_n^+)^{s-1} - (u^+)^{s-1}](u_n - u) &= \langle \nabla \Psi_2(u_n) - \nabla \Psi_2(u), u_n - u \rangle \rightarrow 0. \end{aligned} \quad (33)$$

Using Lemma 6 and the continuous embedding $\mathcal{D}^{1,2} \hookrightarrow L^6$, we see that $\phi_{u_n} \rightarrow \phi_u$ in L^6 . Hence, by the boundedness of $\{u_n\}$ in $L^{12/5}$, we deduce

$$\begin{aligned} \left| \int K(x) (\phi_{u_n} - \phi_u) u_n (u_n - u) \right| \\ \leq |K|_\infty \|\phi_{u_n} - \phi_u\|_6 \|u_n\|_{12/5} \|u_n - u\|_{12/5} \rightarrow 0. \end{aligned} \quad (34)$$

Combining (30)–(34), we obtain

$$\begin{aligned} \|u_n - u\|^2 &= \langle \Phi'_+(u_n) - \Phi'_+(u), u_n - u \rangle \\ &\quad - \int K(x) (\phi_{u_n} u_n - \phi_u u) (u_n - u) \\ &\quad + \int [g(x, u_n^+) - g(x, u^+)](u_n - u) \\ &\quad - \int h(x) [(u_n^+)^{s-1} - (u^+)^{s-1}](u_n - u) \\ &= o(1) - \int K(x) (\phi_{u_n} u_n - \phi_u u) (u_n - u) \\ &= o(1) - \int K(x) (\phi_{u_n} - \phi_u) u_n (u_n - u) \\ &\quad - \int K(x) \phi_u (u_n - u)^2 \\ &= o(1) - \int K(x) \phi_u (u_n - u)^2. \end{aligned} \quad (35)$$

Since the integral in the final line is nonnegative, we deduce that $u_n \rightarrow u$ in H^1 ; thus, we have proved Lemma 9. $\square$

Now we are ready to present the proof of Theorem 2.

Proof of Theorem 2. Firstly, we claim that the zero function $\mathbf{0}$ is not a minimizer of Φ_+ . For this purpose, we choose a non-negative function $v \in C_0^\infty \setminus \{\mathbf{0}\}$. Suppose $t \in (0, |v|_\infty^{-1}\delta)$; then, for all $x \in \mathbb{R}^3$, we have $0 \leq tv(x) < \delta$. Thus, using (g_2) and noting that $\phi_{tv} = t^2\phi_v$, we deduce

$$\begin{aligned} \Phi_+(tv) &= \frac{t^2}{2} \int (|\nabla v|^2 + V(x)v^2) + \frac{1}{4} \int K(x)\phi_{tv}(tv)^2 \\ &\quad - \int G(x, tv) + \frac{t^s}{s} \int h(x)v^s \\ &\leq \frac{t^2}{2} \int (|\nabla v|^2 + V(x)v^2) \\ &\quad + \frac{t^4}{4} \int K(x)\phi_v v^2 - \int a(x)(tv)^\theta + \frac{t^s}{s} \int h(x)v^s \\ &= \frac{t^2}{2} \int (|\nabla v|^2 + V(x)v^2) + \frac{t^4}{4} \int K(x)\phi_v v^2 \\ &\quad - t^\theta \int a(x)v^\theta + \frac{t^s}{s} \int h(x)v^s. \end{aligned} \quad (36)$$

Because $\theta \in (1, 2)$ and $s \in (\theta, 6)$, we see that

$$\Phi_+(tv) < 0 = \Phi_+(\mathbf{0}) \quad (37)$$

for $t > 0$ small enough. So $\mathbf{0}$ is not a minimizer of Φ_+ .

By Lemma 8 and Lemma 9, we know that Φ_+ is bounded from below and satisfies the (PS) condition. It is well known that there exists a minimizer u_+ of Φ_+ (see e.g., [25], Corollary 2.5). By the claim above, we see that $u_+ \neq \mathbf{0}$ and it is a critical point of Φ_+ . As mentioned at the beginning of this section, $u_+ \geq 0$ and it is a nontrivial solution of (1).

In a similar manner, by considering $\Phi_- : H^1 \rightarrow \mathbb{R}$,

$$\begin{aligned} \Phi_-(u) &= \frac{1}{2} \int (|\nabla u|^2 + V(x)u^2) + \frac{1}{4} \int K(x)\phi_u u^2 \\ &\quad - \int G(x, u^-) + \frac{1}{s} \int h(x)(u^-)^s, \end{aligned} \quad (38)$$

we can obtain another nontrivial solution u_- , which is non-positive on $\mathbb{R}^3$. This completes the proof of Theorem 2. $\square$

4. Proofs of Theorem 3

In this section, we prove Theorem 3. Note that, in our Theorem 3, since $s \in (1, +\infty)$ is allowed to be supercritical, the usual space $H^1(\mathbb{R}^3)$ cannot be used as our framework for the study of problem (1). For this reason, motivated by [18, 19], we introduce a new space as our working space. Let $\mathcal{D}^{1,2}$ be the completion of $C_0^\infty(\mathbb{R}^3)$ under the norm

$$\|u\|_{\mathcal{D}}^2 = \int |\nabla u|^2. \quad (39)$$

For a nonnegative measurable function $l(x)$ and $1 < q < +\infty$, we define the weighted Lebesgue space

$$L_l^q(\mathbb{R}^3) := \left\{ u \text{ is measurable} : \int l(x)|u|^q < \infty \right\}, \quad (40)$$

and it is associated with the seminorm

$$|u|_{q,l} = \left(\int l(x)|u|^q \right)^{1/q}. \quad (41)$$

Motivated by [18, 19], let E be the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm

$$\|u\|_E = \|u\| + |u|_{s,h}. \quad (42)$$

Then, E is a Banach space.

Lemma 11. *If (V) and (g_1) are satisfied, then we have the compact embedding $\mathcal{D}^{1,2}(\mathbb{R}^3) \hookrightarrow L_b^r(\mathbb{R}^3)$. Furthermore, we also have the compact embedding $E \hookrightarrow L_b^r(\mathbb{R}^3)$.*

Proof. By our assumption on b , using the results in [26] (see ([26], page 255)), the embedding $\mathcal{D}^{1,2} \hookrightarrow L_b^r$ is well defined and compact. The compactness of $E \hookrightarrow L_b^r$ follows from the continuity of $E \hookrightarrow \mathcal{D}^{1,2}$. $\square$

Lemma 12 ([19], Lemma 2.2). *Given $\alpha, \beta > 0$, there is $C > 0$ such that, for any $u \in E$*

$$\alpha \int (|\nabla u|^2 + V(x)u^2) + \beta \int h(x)|u|^s \leq C(\|u\|_E^2 + \|u\|_E^s), \quad (43)$$

and for $\|u\|_E \geq 1$, we have

$$\alpha \int (|\nabla u|^2 + V(x)u^2) + \beta \int h(x)|u|^s \geq C\|u\|_E^2. \quad (44)$$

Now, let us define the variational functional corresponding to problem (1). We set $\Phi : E \rightarrow \mathbb{R}$ as

$$\begin{aligned} \Phi(u) &= \frac{1}{2} \int (|\nabla u|^2 + V(x)u^2) + \frac{1}{4} \int K(x)\phi_u u^2 \\ &\quad - \int G(x, u) + \frac{1}{s} \int h(x)|u|^s. \end{aligned} \quad (45)$$

By Lemma 11, all the integrals in (45) are well defined and converge; we know that the weak solutions of problem (1) correspond to the critical points of C^1 functional $\Phi : E \rightarrow \mathbb{R}$ with derivative given by

$$\begin{aligned} \langle \Phi'(u), v \rangle &= \int (\nabla u \nabla v + V(x)uv) + \int K(x)\phi_u uv \\ &\quad - \int g(x, u)v + \int h(x)|u|^{s-2}uv. \end{aligned} \quad (46)$$

Lemma 13. *Under the assumptions of Theorem 3, the functional Φ is coercive.*

Proof. Because S is the best Sobolev constant

$$S = \inf_{u \in \mathcal{D}^{1,2} \setminus \{0\}} \frac{\|u\|_{\mathcal{D}}^2}{\|u\|_6^2}, \quad (47)$$

using (g_1) and Hölder inequality, we get

$$\int G(x, u) \leq \frac{1}{r} \int b(x)|u|^r \leq \frac{1}{r} S^{-\frac{r}{2}} |b|_{r_0} \|u\|_{\mathcal{D}}^r \leq C \|u\|_E^r. \quad (48)$$

For $\|u\|_E$ large enough, (48) together with Lemma 12 give that

$$\begin{aligned} \Phi(u) &= \frac{1}{2} \int (|\nabla u|^2 + V(x)u^2) + \frac{1}{4} \int K(x)\phi_u u^2 - \int G(x, u) + \frac{1}{s} \int h(x)|u|^s \\ &\geq C \|u\|_E^2 - C \|u\|_E^r \longrightarrow +\infty, \quad \text{as } \|u\|_E \longrightarrow +\infty, \end{aligned} \quad (49)$$

because $r < 2$. This implies that Φ is coercive on E . $\square$

In general, to prove the (PS) condition, the reflexivity of the space is needed. However, we do not know whether E is reflexive, but we can still prove the following.

Lemma 14. *Under the assumptions of Theorem 3, the functional Φ satisfies the (PS) condition.*

Proof. From Lemma 13, we can deduce that every (PS) sequence $\{u_n\}$ of Φ is bounded in E , and $\{u_n\}$ is also bounded in H^1 . Therefore, we can assume that for some $u \in E$, up to a subsequence

$$\begin{aligned} u_n &\rightharpoonup u \text{ in } H^1, \\ u_n &\longrightarrow u \text{ in } L_{\text{loc}}^t(\mathbb{R}^3), \quad t \in [2, 6), \\ u_n &\longrightarrow u \quad \text{a.e. in } \mathbb{R}^3 \end{aligned} \quad (50)$$

First, we show that $\Phi'(u) = 0$. For any $\varphi \in C_0^\infty(\mathbb{R}^3)$, since

$$\langle \Phi'(u_n), \varphi \rangle = o(1) \|\varphi\|_E, \quad (51)$$

we have

$$\begin{aligned} &\int (\nabla u_n \nabla \varphi + V(x)u_n \varphi) + \int K(x)\phi_{u_n} u_n \varphi - \int g(x, u_n) \varphi \\ &+ \int h(x)|u_n|^{s-2} u_n \varphi = o(1) \|\varphi\|_E. \end{aligned} \quad (52)$$

Now, we claim that

$$\int K(x)\phi_{u_n} u_n \varphi \longrightarrow \int K(x)\phi_u u \varphi, \quad (53)$$

$$\int h(x)|u_n|^{s-2} u_n \varphi \longrightarrow \int h(x)|u|^{s-2} u \varphi, \quad (54)$$

$$\int g(x, u_n) \varphi \longrightarrow \int g(x, u) \varphi. \quad (55)$$

Indeed, by (14), we see that ϕ_{u_n} is bounded in $\mathcal{D}^{1,2}$, hence up to a subsequence $\phi_{u_n} \rightharpoonup \phi_u$ in $\mathcal{D}^{1,2}$, we have

$$\int K(x)\phi_{u_n} u \varphi \longrightarrow \int K(x)\phi_u u \varphi, \quad \text{as } n \longrightarrow \infty. \quad (56)$$

Moreover, by the Hölder inequality, we get

$$\left| \int K(x)\phi_{u_n} (u_n - u) \varphi \right| \leq |K|_\infty \|\phi_{u_n}\|_6 \|u_n - u\|_{L^{12/5}(\Omega)} \|\varphi\|_{12/5} \longrightarrow 0, \quad (57)$$

where $\Omega = \text{supp } \varphi$. Therefore, (56) and (57) give that

$$\begin{aligned} &\int K(x)\phi_{u_n} u_n \varphi - \int K(x)\phi_u u \varphi = \int K(x)(\phi_{u_n} - \phi_u) u \varphi \\ &+ \int K(x)\phi_{u_n} (u_n - u) \varphi \longrightarrow 0. \end{aligned} \quad (58)$$

Thus, (53) holds.

Next, we verify (54) and (55). It is easy to see that the sequence $\{h^{(s-1)/s}|u_n|^{s-2}u_n\}$ is bounded in $L^{s/(s-1)}(\mathbb{R}^3)$. Since $u_n \longrightarrow u$ a.e. in $\mathbb{R}^3$; applying the Brezis-Lieb lemma, up to a subsequence, we have

$$h^{(s-1)/s}|u_n|^{s-2}u_n \rightharpoonup h^{(s-1)/s}|u|^{s-2}u \text{ in } L^{s/(s-1)}(\mathbb{R}^3). \quad (59)$$

Moreover, $h^{1/s}\varphi \in L^s(\mathbb{R}^3)$; thus, we have (54). Similarly, using (g_1) and the Lebesgue theorem, we have (55). Letting $n \longrightarrow \infty$ in (52), we have

$$\int (\nabla u \nabla \varphi + V(x)u \varphi) + \int K(x)\phi_u u \varphi - \int g(x, u) \varphi + \int h(x)|u|^{s-2} u \varphi = 0, \quad (60)$$

that is, $\Phi'(u) = 0$.

Next, we prove $u_n \longrightarrow u$ in H^1 . By Lemma 11 and the fact that $\langle \Phi'(u_n), u_n \rangle = o(1) \|u_n\|_E$ and $\langle \Phi'(u), u \rangle = 0$, we have

$$\begin{aligned} &\lim_{n \longrightarrow \infty} \left(\|u_n\|^2 + \int K(x)\phi_{u_n} u_n^2 + \int h(x)|u_n|^s \right) \\ &= \lim_{n \longrightarrow \infty} \int g(x, u_n) u_n = \int g(x, u) u \\ &= \|u\|^2 + \int K(x)\phi_u u^2 + \int h(x)|u|^s. \end{aligned} \quad (61)$$

On the other hand, by $-\Delta \phi_u = K(x)u^2$, the Hölder inequality, and Sobolev inequality, we get $\|\phi_{u_n}\|_{\mathcal{D}} \leq C|u_n|_{12/5}^2$, so we can see that ϕ_{u_n} is bounded in $\mathcal{D}^{1,2}(\mathbb{R}^3)$; hence, we can assume

$$\phi_{u_n} \rightharpoonup \phi_u \text{ in } \mathcal{D}^{1,2}(\mathbb{R}^3). \quad (62)$$

Hence, by the weak lower semicontinuity of the norm $\|\cdot\|_{\mathcal{D}}$, we get

$$\int K(x)\phi_u u^2 = \|\phi_u\|_{\mathcal{D}}^2 \leq \liminf_{n \rightarrow \infty} \|\phi_{u_n}\|_{\mathcal{D}}^2 = \liminf_{n \rightarrow \infty} \int K(x)\phi_{u_n} u_n^2. \quad (63)$$

By Fatou's lemma, we have

$$\int h(x)|u|^s \leq \liminf_{n \rightarrow \infty} \int h(x)|u_n|^s. \quad (64)$$

It follows from (63) and (64) that

$$\begin{aligned} \lim_{n \rightarrow \infty} \left(\|u_n\|^2 + \int K(x)\phi_{u_n} u_n^2 + \int h(x)|u_n|^s \right) \\ \geq \liminf_{n \rightarrow \infty} \|u_n\|^2 + \liminf_{n \rightarrow \infty} \int K(x)\phi_{u_n} u_n^2 + \liminf_{n \rightarrow \infty} \int h(x)|u_n|^s \\ \geq \liminf_{n \rightarrow \infty} \|u_n\|^2 + \int K(x)\phi_u u^2 + \int h(x)|u|^s. \end{aligned} \quad (65)$$

From (61) and (65), we see that

$$\|u\|^2 \geq \liminf_{n \rightarrow \infty} \|u_n\|^2. \quad (66)$$

From this and the weak lower semicontinuity of the norm $\|\cdot\|$, we see that up to a subsequence,

$$\|u_n\|^2 \longrightarrow \|u\|^2, \quad (67)$$

then $u_n \longrightarrow u$ in H^1 .

Finally, we show that $u_n \longrightarrow u$ in E . It suffices to show that

$$\int h(x)|u_n|^s \longrightarrow \int h(x)|u|^s. \quad (68)$$

In fact, from $u_n \longrightarrow u$ in H^1 , we have $u_n \longrightarrow u$ in $L^{12/5}(\mathbb{R}^3)$ and $\phi_{u_n} \longrightarrow \phi_u$ in $L^6(\mathbb{R}^3)$. Hence, by Hölder inequality and Sobolev inequality, we get

$$\begin{aligned} \left| \int K(x)\phi_{u_n} u_n^2 - \int K(x)\phi_u u^2 \right| &\leq \int K(x)\phi_{u_n} |u_n^2 - u^2| \\ &+ \int K(x) |\phi_{u_n} - \phi_u| u^2 \leq |K|_{\infty} \|\phi_{u_n} - \phi_u\|_6 \|u_n - u\|_{12/5} \|u_n + u\|_{12/5} \\ &+ \int K(x) |\phi_{u_n} - \phi_u| u^2 \leq C \|u_n - u\|_{12/5} \\ &+ |K|_{\infty} \int |\phi_{u_n} - \phi_u| u^2 \longrightarrow 0. \end{aligned} \quad (69)$$

Using (61), (67), and (69) we get (68). Thus,

$$\|u_n - u\|_E = \|u_n - u\| + \|u_n - u\|_{s,h} \longrightarrow 0. \quad (70)$$

The proof is completed. $\square$

Having verified the (PS) condition, we investigate the geometry of Φ . First, we note that obviously, Φ is even (by our assumption on g).

Let

$$\Omega := \{x \in \mathbb{R}^3 : b(x) = 0\}, \quad (71)$$

and we define

$$Y := \{u \in E : u(x) = 0 \text{ a.e. } x \in \Omega\}. \quad (72)$$

Then, Y is an infinitely dimensional Banach space under the norm $\|\cdot\|_E$. Therefore, from [19], Lemma 3.3, we know that the seminorm $|u|_{r,b} = (\int b(x)|u|^r)^{1/r}$ is a norm on Y .

Let Σ be the class of the closed and symmetric (with respect to the origin) subsets of $E \setminus \{0\}$. For $A \in \Sigma$, we define the genus $\gamma(A)$ by

$$\gamma(A) := \min \{m \in \mathbb{N} | \exists \varphi \in C(A, \mathbb{R}^m \setminus \{0\}), \text{ such that } \varphi(x) = -\varphi(-x)\}. \quad (73)$$

If such a minimum does not exist, we define $\gamma(A) = +\infty$. The main properties of the genus can be found in [27, 28]; we omit them here.

Lemma 15. *Given $m \in \mathbb{N}$, there is $\varepsilon = \varepsilon(m)$ such that*

$$\gamma(\{u \in E | \Phi(u) \leq -\varepsilon\}) \geq m. \quad (74)$$

Proof. Given $m \in \mathbb{N}$, let X_m be a m -dimensional subspace of E ; as in the proof of Theorem 2, we can choose $\varepsilon = \varepsilon(m) > 0$ and $\eta > 0$ such that $\Phi(u) \leq -\varepsilon$, if $\|u\| = \eta$. (for η small enough).

Denote $S := \{u \in X_m | \|u\| = \eta\}$; S is a sphere in X_m . Then, we have

$$S \subset \{u \in E | \Phi(u) \leq -\varepsilon\}, \quad (75)$$

so, by the monotonicity property of genus, we have

$$\gamma(\{u \in E | \Phi(u) \leq -\varepsilon\}) \geq \gamma(S) = m. \quad (76)$$

The proof is completed. $\square$

Let

$$\Sigma_m := \{A \in \Sigma | \gamma(A) \geq m\}. \quad (77)$$

From Lemma 15, we can define a sequence of real numbers

$$c_m := \inf_{A \in \Sigma_m} \sup_{u \in A} \Phi(u). \quad (78)$$

Then,

$$c_1 \leq c_2 \leq \dots \leq c_m \leq c_{m+1} \leq \dots. \quad (79)$$

By Lemma 13, Φ is coercive and bounded from below.

That is, $c_m > -\infty$, for $\forall m \in \mathbb{N}$. For $c \in \mathbb{R}$, denote $K_c := \{u \in E \mid \Phi(u) = c, \Phi'(u) = 0\}$. Then, a standard argument (see [29]) gives the following.

Lemma 16. *All the c_m are critical values of Φ . Furthermore, if $c = c_m = c_{m+1} = \dots = c_{m+\tau}$, then $\gamma(K_c) \geq \tau + 1$.*

Proof of Theorem 3. Because Φ is even and note that by Lemma 15, $\Phi^{-\varepsilon} \in \Sigma_m$, thus $c_m \leq -\varepsilon(m) < 0$, for $\forall m \in \mathbb{N}$. This final Lemma 16 gives the existence of infinitely many critical points of Φ . So problem (1) has infinitely many solutions. This completed the proof of Theorem 3. $\square$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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