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Research Article

Convergence Results for Total Asymptotically Nonexpansive Monotone Mappings in Modular Function Spaces

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In this article, we consider an extensive class of monotone nonexpansive mappings. We use S-iteration to approximate the fixed point for monotone total asymptotically nonexpansive mappings in the settings of modular function space.

1. Introduction

In 1965, the existence results for nonexpansive mapping were initiated by Browder [1], Kirk [2], and Göhde [3] independently. The idea about asymptotically nonexpansive mappings was introduced by Goebel and Kirk [4] in 1972. Fixed point results of nonexpansive mapping were extended for monotone case by Bachar and Khamsi [5] in 2015. Alfuraidan and Khamsi [6] extended the concept of asymptotically nonexpansive for the case of monotone in 2018. Alber et al. [7] introduced the concept of total asymptotically nonexpansive mappings that generalizes family of mapping that are the extension of asymptotically nonexpansive mappings in 2006. Example 2 of [8] and Example 3.1 of [9] show that total asymptotically nonexpansive mappings properly contain the asymptotically nonexpansive mappings.

The notation for modular function (MF) space was initiated in 1950 by Nakano [10], which was further generalized by Musielak and Orlicz [11] in 1959. In 1990, Khamsi et al. [12] were the first who initiated fixed point theory in MF space. Alfuraidan, Bachar, and Khamsi [13] in 2017 extended results of Goebel and Kirk [4] for monotone asymptotically nonexpansive mappings in MF spaces using Mann iteration process.

In this article, we extend the notion of monotone total asymptotically nonexpansive mappings in MF space and generalize the results of Alfuraidan and Khamsi presented in [6, 13]. We use S-iteration process to approximate the fixed point, which is fastly convergent than the classic Picard [14], Mann [15], and Ishikawa [16] iterative processes.

2. Preliminaries

Firstly, we have the definitions of δ-ring and σ-algebra with examples.

Definition 1. Suppose $\Sigma \neq \emptyset$, and R be a nonempty family of subsets of Σ, then R is called ring of sets if $A, B \in R$, satisfies

(i) $A \cup B \in \mathrm{R}$

(ii) $A \setminus B \in \mathrm{R}$

A ring of sets R is called δ -ring of sets if for any sequence of sets $\{A_n\} \in \mathrm{R}$ implies $\cup_{n=1}^{\infty} A_n \in \mathrm{R}$.

Example 2. Let R be the collection of all finite subsets of $\mathbb{N}$, and then R is a ring, but not δ-ring.

Definition 3. Assume that $\Sigma \neq \varnothing$, a collection $\mathscr{A}$ of subsets of Σ is called algebra of sets if $A, B \in \mathscr{A}$, satisfies

(i) $A \cup B \in \mathscr{A}$

(ii) $A' \in \mathscr{A}$, whenever A is in $\mathscr{A}$

An algebra of sets $\mathscr{A}$ is called σ -algebra of sets if for every sequence of sets $\{A_n\} \in \mathscr{A}$ implies $\cup_{n=1}^{\infty} A_n \in \mathscr{A}$.

Example 4. For any set Σ, $P(\Sigma)$ and $\{\phi, \Sigma\}$ are σ-algebras.

In the following, we list some basic concepts of the MF space presented by Kozlowski [12].

Definition 5. Suppose that Σ be a vector space,

(a) A functional $\mu : \Sigma \longrightarrow [0,\infty]$ is known as modular if for $\mathrm{u}, \tau \in \Sigma$, μ, satisfies

  (i) $\mu(\mathrm{u}) = 0$ if and only if $\mathrm{u} = 0$

  (ii) $\mu(\gamma \mathrm{u}) = \mu(\mathrm{u})$ with $|\gamma| = 1$

  (iii) $\mu(\gamma \mathrm{u} + \nu\tau) \leq \mu(\mathrm{u}) + \mu(\tau)$, if $\gamma + \nu = 1$ and $\gamma \geq 0$, $\nu \geq 0$

(b) If condition (iii) is replaced by

  (i) $\mu(\gamma \mathrm{u} + \nu\tau) \leq \gamma\mu(\mathrm{u}) + \nu\mu(\tau)$, if $\gamma + \nu = 1$ and $\gamma \geq 0$, $\nu \geq 0$, then μ is known as convex modular

(c) A modular μ defines a respective MF space, that is, the vector space Σ_μ given by

$$\Sigma_\mu = \{\mathrm{u} \in \Sigma\, ; \mu(\lambda \mathrm{u}) \longrightarrow 0 \mathrm{as} \lambda \longrightarrow 0\} \tag{1}$$

Definition 6. A subset $A \in \mathscr{A}$ is said to be μ-null if $\mu(\nu 1_A) = 0$ (the notation 1_A represents the characteristic of A), for any $\nu \in \Sigma$, and a property $\mu(w)$ holds μ-almost everywhere (μ – *a.e.*) if the set $\{w \in \Sigma\, ; \mu(w)$ does not hold$\}$ is μ-null.

A property is considered to hold almost everywhere (a.e) if there is a set of points where this property fails to hold has measure zero.

Definition 7. Let M_∞ stands for the class of all extended functions which are also measurable. A convex and even function $\mu : M_\infty \longrightarrow [0,\infty]$ is said to be regular modular if

(1) $\mu(\mathrm{u}) = 0 \Rightarrow \mathrm{u} = 0 \mu$-almost everywhere

(2) $|\mathrm{u}(t)|'|\tau(t)|$ for all $t \in \Omega \Rightarrow \mu(\mathrm{u})'\mu(\tau)$, where $\mathrm{u}, \tau \in M_\infty \mu$ is monotone

(3) $|\mathrm{u}_n(t)|\uparrow|\mathrm{u}(t)|$ for all $t \in \Omega \Rightarrow \mu(\mathrm{u}_n)\uparrow\mu(\mathrm{u})$, where $\mathrm{u}, \in M_\infty \mu$ has Fatou property

Consider

$$M = \{\mathrm{u} \in M_\infty\, ; |\mathrm{u}(t)| < \infty \mu - \text{almost everywhere}\}. \tag{2}$$

The *MF space* L_μ is defined as

$$L_\mu = \{\mathrm{u} \in M\, ; \mu(\lambda \mathrm{u}) \longrightarrow 0 \mathrm{as} \lambda \longrightarrow 0\}. \tag{3}$$

Following few useful definitions are taken from [17, 18]. From onwards, we assume μ as a convex regular modular.

Definition 8.

(i) $\{\tau_n\}$ is termed as μ-convergent to τ if

$$\lim_{n \longrightarrow \infty} \mu(\tau_n - \tau) = 0. \tag{4}$$

(ii) A sequence $\{\tau_n\}$ is termed as μ -Cauchy if

$$\lim_{m,n \longrightarrow \infty} \mu(\tau_n - \tau_m) = 0. \tag{5}$$

(iii) Let $K \subset L_\mu$ be μ-closed if for any sequence $\{\tau_n\} \in K$, μ-converge to $\tau \Rightarrow \tau \in K$

(iv) Let $K \subset L_\mu$ be μ-bounded if its μ-diameter sup $\{\mu(\tau - h)\, ; h \in K\} < \infty$

Definition 9. Suppose that Σ be a vector space, μ is said to satisfy the Δ_2-condition, if $\sup_{n \geq 1} \mu(2u_n, D_k) \longrightarrow 0$ as $k \longrightarrow \infty$ whenever $\{D_k\}$ decreases to ϕ and $\sup_{n \geq 1} \mu(u_n, D_k) \longrightarrow 0$ *as* $k \longrightarrow \infty$.

Remark 10. Consider μ-convergence implies μ-Cauchy if and only if it satisfies the Δ_2-condition.

Definition 11. Let $r > 0$ and $\varepsilon > 0$. Define

$$\delta_\mu(r, \varepsilon) = \inf \left\{ 1 - \frac{1}{r} \mu\left(\frac{\mathrm{u} + \tau}{2}\right) ; (\mathrm{u}, \tau) \in D(r, \varepsilon) \right\}, \tag{6}$$

where

$$D(r, \varepsilon) = \left\{ (\mathrm{u}, \tau)\, ; \mathrm{u}, \tau \in L_\mu, \mu(\mathrm{u}) \leq r, \mu(\tau) \leq r, \mu(\mathrm{u} - \tau) \geq r\varepsilon \right\}. \tag{7}$$

(a) μ is said to satisfy condition (UC) if whenever $R > 0$ and $\varepsilon > 0$, we have $\delta_p(R, \varepsilon) > 0$

(b) μ is considered to satisfy condition (UUC) if whenever $s > 0$ and $\varepsilon > 0$, $\eta(s, \varepsilon) > 0$ exists such that

$$\delta_{\mu}(R, \varepsilon) > \eta(s, \varepsilon) > 0, for R > s. \quad (8)$$

(c) μ is considered to satisfy condition (SC) if whenever any $\tau, h \in L_{\mu}$ with $\mu(\tau) = \mu(h)$ and

$$\mu(\gamma\tau + (1-\gamma)h) = \gamma\mu(\tau) + (1-\gamma)\mu(h), \text{for some} \gamma \in (0,1), \quad (9)$$

where $u = \tau$.

Following definition of μ-type function will be used in the main result taken from [18].

Definition 12. Let $K \subset L_{\mu}$, and a mapping $\tau : K \longrightarrow [0,\infty]$ is said to be μ-type if a sequence $\{\tau_m\} \in L_{\mu}$ exists such that

$$\tau(u) = \limsup_{n \longrightarrow \infty} \mu(\tau_m - u), \quad (10)$$

for any $u \in K$. Any sequence $\{u_n\}$ in K is said to be a minimizing sequence of τ if

$$\lim_{n \longrightarrow \infty} \tau(u_n) = \inf \{\tau(u) ; u \in K\}. \quad (11)$$

Following are the definitions of monotone and monotone asymptotically nonexpansive mapping in modular space, and useful remark about property (R), given in [13].

Definition 13. A mapping $\Gamma : K \longrightarrow K$, where K be a nonempty subset of L_{μ}, is said to be

(i) Monotone if

$$\Gamma(u)'\Gamma(\tau)\mu - \text{a.e.whenever } u'\tau\mu - \text{a.e.,for } u, \tau \in K. \quad (12)$$

(ii) Monotone asymptotically nonexpansive if Γ is monotone, and there exists $\{L_n\} \subset [1,+\infty)$ such that $\lim_{n\longrightarrow\infty} L_n = 1$, and

$$\mu(\Gamma^n \tau - \Gamma^n h) \leq L_n \mu(\tau - h), \text{for } u, \tau \in K, \quad (13)$$

such that $\tau' h\mu$-a.e. and $n \geq 1$. Also τ is said to be fixed point if $\Gamma\tau = \tau$.

Remark 14. Let $K \neq \phi$ be a μ-bounded, convex, and μ-closed subset of L_{μ} where μ is a convex regular modular. Let $\{u_n\}$ be a monotonically increasing sequence in K (due to the convexity and μ-closedness of order intervals in L_{μ}), then property (R) will imply that

$$\cap_{n\geq 1} \left\{ u \in K ; u_n' u\mu - \text{a.e} \right\} \neq \phi. \quad (14)$$

The following Lemmas taken from [19] will be used in main result.

Lemma 15. *Let $K \neq \phi$ be a μ-bounded, convex, and μ-closed subset of L_{μ} where μ is a convex regular modular satisfying condition (UUC). Then, every μ-type minimizing sequence defined on K will be μ-convergent, and the limit will not depend upon the minimizing sequence.*

Lemma 16. *Let μ be a convex regular modular satisfying condition (UUC). If there exists $R > 0$ and $\gamma \in (0, 1)$ with*

$$\limsup_{n\longrightarrow\infty} \mu(u_n) \leq R, \limsup_{n\longrightarrow\infty} \mu(\tau_n) \leq R \text{ and } \lim_{n\longrightarrow\infty} \mu(\gamma u_n + (1-\gamma)\tau_n) = R, \quad (15)$$

then we have

$$\lim_{n\longrightarrow\infty} \mu(u_n - \tau_n) = 0. \quad (16)$$

The μ-distance from $u \in L_{\mu}$ to $K \subset L_{\mu}$ is given as

$$dist_{\mu}(u, K) = \inf \{\mu(u - h): h \in K\}. \quad (17)$$

Following Lemma taken from [9] will be used in the existence result.

Lemma 17. *Suppose $\{l_n\}$, $\{m_n\}$ and $\{\delta_n\}$ be sequences of nonnegative satisfying*

$$l_{n+1} \leq (1 + \delta_n) l_n + m_n \forall n \geq 1. \quad (18)$$

If $\Sigma\delta_n < \infty$ and $\Sigma m_n < \infty$, then $\lim_{n\longrightarrow\infty} l_{n+1}$ exists.

Following is the definition of condition (I) taken from [20].

Definition 18. Let $K \neq \phi$ be a subset of L_{μ}, and a mapping $\Gamma : K \longrightarrow K$ is assumed to fulfill the condition (I) if a nondecreasing function

$$l : [0,\infty) \longrightarrow [0,\infty) \text{ with } l(0) = 0 \text{ and } l(r) > 0, \quad (19)$$

exists for all $r \in (0,\infty)$, such that

$$\mu(u - \Gamma u) \geq l(\text{dist}_{\mu}(u, F_{\mu}(\Gamma))), \quad (20)$$

for all $u \in K$.

3. Fixed Point Results for Monotone Total Asymptotically Nonexpansive Mapping

Now, we will define monotone total asymptotically nonexpansive mapping in modular space.

Definition 19. Let $K \neq \phi$ be a subset of L_μ where μ is a convex regular modular. A self map Γ of K is said to be monotone total asymptotically nonexpansive mapping if there exists nonnegative sequences $\{\zeta_n\}$ and $\{\xi_n\}$ with $\zeta_n \longrightarrow 0$, $\xi_n \longrightarrow 0$, as $n \longrightarrow \infty$, and a strictly increasing continuous function

$$\phi : [0,\infty) \longrightarrow [0,\infty) \; with \; \phi(0) = 0, \tag{21}$$

such that

$$\mu(\Gamma^n \tau - \Gamma^n h) \leq \mu(\tau - h) + \zeta_n \phi(\mu(\tau - h)) + \xi_n \text{forall} n \geq 1. \tag{22}$$

There exists a constant $M^* > 0$ such that $\phi(\lambda) \leq M^* \lambda$ for $\lambda > 0$, then

$$\mu(\Gamma^n \tau - \Gamma^n h) \leq (1 + M^* \zeta_n)\mu(\tau - h) + \xi_n, \tag{23}$$

for every $\tau, h \in K$ such that τ and h are comparable μ-a.e.

Theorem 20. *Let $K \neq \phi$ be a μ-bounded and μ-closed subset of L_μ where μ is a convex regular modular satisfying condition (UUC). Let a self map Γ of K be a μ-continuous monotone total asymptotically nonexpansive mapping. Assume that there exists $u_0 \in K$, such that $u_0{}'\Gamma(u_0)$ or $(\Gamma(u_0){}'u_0)\mu$ -a.e. Then, Γ has a fixed point u such that $u_0{}'u$ or $(u'u_0)\mu$-a.e.*

Proof. Assume that $\mathrm{u}_0{}'\Gamma(\mathrm{u}_0)\mu$-a.e. Since Γ is monotone, then we have

$$\Gamma^n \mathrm{u}_0{}'\Gamma^{n+1}\mathrm{u}_0, \tag{24}$$

for every $n \in \mathbb{N}$, and the sequence $\{\Gamma^n \mathrm{u}_0\}$ is monotone increasing. From the above Remark,

$$K_\infty = \cap_{n\geq 1}\left\{\mathrm{u} \in K\,;\, \mathrm{u}_n{}'\mathrm{u}\mu - \mathrm{a.e}\right\} \neq \phi. \tag{25}$$

Consider the μ-type function $\tau : K_\infty \longrightarrow [0,+\infty)$ define by

$$\tau(h) = \lim_{n\longrightarrow\infty} \sup \mu(\Gamma^n \mathrm{u}_0 - h), \text{for any } h \in K_\infty, \tag{26}$$

$$\tau_0 = \inf\,\{\tau(h)\,;\, h \in K_\infty\}. \tag{27}$$

Let $\{\tau_n\}$ be a minimizing sequence of τ, from the Lemma $\{\tau_n\}\mu$-converges to $\tau \in K_\infty$. We have to show that τ is the fixed point of Γ. Since $h \in K_\infty$, we have $\Gamma^m(h) \in K_\infty$, for every $m \in \mathbb{N}$, which implies

$$\begin{aligned}\tau(\Gamma^m(h)) &= \lim_{n\longrightarrow\infty} \sup\, \mu(\Gamma^n \mathrm{u}_0 - \Gamma^m h) \\ &\leq \lim_{n\longrightarrow\infty} \sup\, [\mu(\Gamma^n \mathrm{u}_0 - h) + \mu_m \phi(\mu(\Gamma^n \mathrm{u}_0 - h)) + \xi_m] \\ &= \tau(h) + \mu_m \lim_{n\longrightarrow\infty} \sup\, (\phi(\mu(\Gamma^n \mathrm{u}_0 - h))) + \xi_m.\end{aligned} \tag{28}$$

In particular, we have

$$\begin{aligned}\tau(\Gamma^m(\tau_n)) &= \lim_{n\longrightarrow\infty} \sup\, \mu(\Gamma^n x_0 - \Gamma^m \tau_n) \\ &\leq \tau(\tau_n) + \mu_m \lim_{n\longrightarrow\infty} \sup\, (\phi(\mu(\Gamma^n x_0 - h))) + \xi_m,\end{aligned} \tag{29}$$

for $n, m \in \mathbb{N}$. As Γ is total asymptotically nonexpansive, so $\mu_m \longrightarrow 0$, $\xi_m \longrightarrow 0$, when $m \longrightarrow \infty$. Hence,

$$\lim_{m\longrightarrow\infty} \tau(\Gamma^m(\tau_n)) = \tau(\tau_n). \tag{30}$$

The sequence $\{\Gamma^{n+p}(\tau_n)\}$ is a minimizing sequence in K_∞, for any $p \in \mathbb{N}$. By Lemma 15, $\{\Gamma^{n+p}(\tau_n)\}$ is μ-converge to τ, for any $p \in \mathbb{N}$. Since Γ is μ-continuous and $\{\Gamma^n(\tau_n)\}$ is μ-convergent to τ, then $\{\Gamma^{n+1}(\tau_n)\}$ is μ-convergent to $\Gamma\tau$ and τ. Since μ-limit of any μ-convergent is unique, we have $\Gamma\tau = \tau$; also, $\tau \in K_\infty$, we have $\mathrm{u}_0{}'\tau$, hence proved. □

Example 21. Let f be an extended real valued function defined on a measureable set D, such that $f(x) = c$ for all $x \in D$. The function f is measureable if the set

$$\{x \in D : f(x) > \alpha\} = \begin{cases} D & \text{if } \alpha < c \\ \phi & \text{if } \alpha \geq c \end{cases} \tag{31}$$

is measureable. And the measureability of above set follows directly from the measureability of D and ϕ. So, a constant function is a measureable function. Now, we define a set of extended real valued functions as

$$M_\infty = \left\{f : f : D \longrightarrow \bar{\mathbb{R}} \text{with} f(x) = c\right\}. \tag{32}$$

Define a function $\mu : M_\infty \longrightarrow [0,\infty)$ by $\mu(f) = f(x)$ for all $f \in M_\infty$, which clearly it is well defined.

Firstly, we need to show that μ is a convex function. For this, we show M_∞ that is a convex set. Consider

$$\begin{aligned}(\lambda f + (1-\lambda)g)(x) &= (\lambda f)(x) + ((1-\lambda)g)(x), \text{ Point wise addition} \\ &= \lambda.f(x) + (1-\lambda).g(x), \text{ Scaler multiplication} \\ &= \lambda c_1 + (1-\lambda)c_2,\ As\, f, g \in M_\infty, \lambda \in (0,1) = c_3,\end{aligned} \tag{33}$$

which implies $\lambda f + (1-\lambda)g \in M_\infty$. Hence, M_∞ is a convex set. Now, for every $f, g \in M_\infty$, it is easy to prove that

$$\mu(\lambda f + (1-\lambda)g) = \lambda.\mu(f) + (1-\lambda).\mu(g), \quad (34)$$

which further implies that μ is a convex function. Now, we check the properties of regular modular.

(1) If $\mu(f) = 0 \Rightarrow f(x) = 0 \Rightarrow c = 0$, which further implies $f = 0$

(2) If

$$\begin{aligned} f(t) \le g(t), \text{for all } t \in D \\ c_1 \le c_2, \end{aligned} \quad (35)$$

as $\mu(f) = f(t) = c_1$ and $\mu(g) = g(t) = c_2$. So, $\mu(f) \le \mu(g)$. Thus, μ is monotone.

(3) Clearly, μ is strongly convergent which implies weak convergence

Hence, μ is convex regular modular. Define

$$M = \{f \in M_\infty : |f(x)| < \infty\}, \text{and } L_\mu = \left\{ \begin{array}{c} f \in M : \mu(\lambda f)(x) = (\lambda f)(x) \\ = \lambda.f(x) = \lambda.c \longrightarrow 0, \text{as} \lambda \longrightarrow 0 \end{array} \right\}, \quad (36)$$

and a subset

$$K = \{f \in M : f(x) = c \in [0, 2] \; with \; \mu(\lambda f) \longrightarrow 0 \; as \; \lambda \longrightarrow 0\} \quad (37)$$

of L_μ. Clearly, K is μ-bounded and μ-closed. Let a mapping $\Gamma : K \longrightarrow K$ be defined by $\Gamma(f) = \alpha f$, where $\alpha \in (0, 1)$. Let $(\xi_n)_{n\in\mathbb{N}} = 1/2n$, $(\eta_n)_{n\in\mathbb{N}} = 2/3n^2$ be any positive sequences and $\xi_n, \eta_n \longrightarrow 0$ as $n \longrightarrow \infty$. Define a strictly increasing function $\varphi : [0,\infty) \longrightarrow [0,\infty)$ by $\varphi(x) = x/2$, with $\varphi(0) = 0$. Consider

$$\mu(\Gamma^n(\tau) - \Gamma^n(h)) = \mu(\alpha^n\tau - \alpha^n h) = (\alpha^n\tau - \alpha^n h)(x) = \alpha^n(c_1 - c_2),$$

$$\begin{aligned} \mu(\tau - h) + \eta_n\varphi(\mu(\tau - h)) + \xi_n &= (\tau - h)(x) + \frac{2}{3n^2}\frac{\mu(\tau - h)}{2} + \frac{1}{2n} \\ &= \left(1 + \frac{1}{3n^2}\right)(\tau(x) - h(x)) + \frac{1}{2n} \\ &= \left(1 + \frac{1}{3n^2}\right)(c_1 - c_2) + \frac{1}{2n}. \end{aligned} \quad (38)$$

Clearly,

$$\mu(\Gamma^n(\tau) - \Gamma^n(h)) \le \mu(\tau - h) + \eta_n\varphi(\mu(\tau - h)) + \xi_n. \quad (39)$$

Also, there exists a constant $M^* = 1$, $\varphi(\lambda) = \lambda/2 < 1.\lambda$, and

$$\mu(\Gamma^n(\tau) - \Gamma^n(h)) \le (1 + M^*\eta_n)\mu(\tau - h) + \xi_n. \quad (40)$$

So, Γ is monotone asymptotically nonexpansive mapping. Since all conditions of theorem are satisfied; thus, Γ has a fixed point, since $\Gamma(f) = \alpha f$ implies $(1-\alpha)f = 0$. Thus, $f = 0$. Hence, the 0 function is a fixed point of Γ.

4. Convergence Analysis

Let $K \ne \phi$ be a convex subset of L_μ where μ is a convex regular modular.We modify S-iteration in MF space is defined as

$$\begin{cases} \mathrm{u}_1 \in K \\ y_l = \nu_l\Gamma^l\mathrm{u}_l + (1-\nu_l)\mathrm{u}_l, \\ \mathrm{u}_{l+1} = \gamma_l\Gamma^l y_l + (1-\gamma_l)\Gamma^l\mathrm{u}_l, \end{cases} \quad (41)$$

for $l \in \mathbb{N}$, where $\{\gamma_l\}$ and $\{\nu_l\}$ are sequences in $(0, 1)$.

Theorem 22. *Let $K \ne \phi$ be a μ-bounded subset of L_μ where μ is a convex regular modular satisfying condition (UUC). Let a self map Γ of K be a monotone total asymptotically nonexpansive mapping with $u(\Gamma) \ne \phi$. Assume that there exists $u_0 \in K$, such that $u_0{}'\Gamma(u_0)$ or $(\Gamma(u_0)'u_0)\mu$ -a.e. If the sequence $\{u_l\}$ is defined by (41) where $0 < a'\gamma_l$, $\nu_l{}'b < 1$, then Γ has a fixed point u such that $u_0{}'u$ or $(u'u_0)\mu$ -a.e. Then, the following holds*

(a) $\lim_{l\longrightarrow\infty} \mu(u_l - u)$ exist for $u \in u(\Gamma)$.

(b) $\lim_{l\longrightarrow\infty} \mu(\Gamma^l u_l - u_l) = 0$.

Proof. Let $\mathrm{u} \in \mathrm{u}(\Gamma)$, and assume that $\mathrm{u}_0{}'\Gamma(\mathrm{u}_0)\mu$-a.e. Using (41)

$$\begin{aligned} \mu(y_l - \mathrm{u}) &= \mu\left(\left(\nu_l\Gamma^l\mathrm{u}_l + (1-\nu_l)\mathrm{u}_l\right) - \mathrm{u}\right) \\ &\le \nu_l\mu\left(\Gamma^l\mathrm{u}_l - \mathrm{u}\right) + (1-\nu_l)\mu(\mathrm{u}_l - \mathrm{u}), \end{aligned} \quad (42)$$

using (22), we have

$$\begin{aligned} \mu(y_l - \mathrm{u}) &\le \nu_l[\mu(\mathrm{u}_l - \mathrm{u}) + \zeta_l\phi(\mu(\mathrm{u}_l - \mathrm{u})) + \xi_l] + (1-\nu_l)\mu(\mathrm{u}_l - \mathrm{u}) \\ &= \nu_l\mu(\mathrm{u}_l - \mathrm{u}) + \nu_l\zeta_l\phi(\mu(\mathrm{u}_l - \mathrm{u})) + \nu_l\xi_l + \mu(\mathrm{u}_l - \mathrm{u}) - \nu_l\mu(\mathrm{u}_l - \mathrm{u}) \\ &= \nu_l\zeta_l\phi(\mu(\mathrm{u}_l - \mathrm{u})) + \nu_l\xi_l + \mu(\mathrm{u}_l - \mathrm{u}), \end{aligned} \quad (43)$$

upon using (23), and we get

$$\mu(y_l - \mathrm{u}) \le (1 + \nu_l\zeta_l M^*)\mu(\mathrm{u}_l - \mathrm{u}) + \nu_l\xi_l. \quad (44)$$

Now,

$$\begin{aligned} \mu(\mathrm{u}_{l+1} - \mathrm{u}) = \mu\left(\left(\gamma_l\Gamma^l y_l + (1-\gamma_l)\Gamma^l\mathrm{u}_l\right) - \mathrm{u}\right) &\le \gamma_l\mu\left(\Gamma^l y_l - \mathrm{u}\right) \\ &+ (1-\gamma_l)\mu\left(\Gamma^l\mathrm{u}_l - \mathrm{u}\right), \end{aligned} \quad (45)$$

using (22), and we have

$$\mu(\mathrm{u}_{l+1}-\mathrm{u}) \leq \gamma_l[\mu(y_l-\mathrm{u})+\zeta_l\phi(\mu(y_l-\mathrm{u}))+\xi_l]+(1-\gamma_l) \cdot [\mu(\mathrm{u}_l-\mathrm{u})+\zeta_l\phi(\mu(\mathrm{u}_l-\mathrm{u}))+\xi_l], \tag{46}$$

upon using (23), and we get

$$\mu(\mathrm{u}_{l+1}-\mathrm{u}) \leq (1+\delta_l)\mu(\mathrm{u}_l-\mathrm{u})+b_l\xi_l. \tag{47}$$

where

$$\begin{aligned} \delta_l &= (\gamma_l\nu_l+\gamma_l\nu_l M^*\zeta_l+1)M^*\zeta_l, \\ b_l &= (\gamma_l\nu_l+\gamma_l\nu_l M^*\zeta_l+1). \end{aligned} \tag{48}$$

Using Lemma 17, $\lim_{l\longrightarrow\infty}\mu(\mathrm{u}_l-\mathrm{u})$ exists for $\mathrm{u}\in \mathrm{u}(\Gamma)$. For part (b), we have to show that

$$\lim_{l\longrightarrow\infty}\mu\left(\Gamma^l\mathrm{u}_l-\mathrm{u}\right)=0. \tag{49}$$

Assume that

$$\lim_{l\longrightarrow\infty}\mu(\mathrm{u}_l-\mathrm{u})=c\geq 0. \tag{50}$$

Case 1. If $c=0$, then the conclusion is trivial.
Case 2. For $c>0$, we know that

$$\mu(y_l-\mathrm{u})=(1+M^*\nu_l\zeta_l)\mu(\mathrm{u}_l-\mathrm{u})+\nu_l\xi_l. \tag{51}$$

Taking lim sup on both sides of (50),

$$\limsup_{l\longrightarrow\infty}\mu(y_l-\mathrm{u})\leq c. \tag{52}$$

Also,

$$\mu\left(\Gamma^l\mathrm{u}_l-\mathrm{u}\right)=\mu\left(\Gamma^l\mathrm{u}_l-\Gamma^l\mathrm{u}\right)\leq\mu(\mathrm{u}_l-\mathrm{u})+\zeta_l\phi(\mu(\mathrm{u}_l-\mathrm{u}))+\xi_l \tag{53}$$

applies $\limsup_{l\longrightarrow\infty}$ on both sides:

$$\limsup_{l\longrightarrow\infty}\mu\left(\Gamma^l\mathrm{u}_l-\mathrm{u}\right)\leq c. \tag{54}$$

Also,

$$\mu\left(\Gamma^l y_l-\mathrm{u}\right)\leq\mu(y_l-\mathrm{u})+\zeta_l\phi(\mu(y_l-\mathrm{u}))+\xi_l. \tag{55}$$

Taking lim sup on both sides,

$$\limsup_{l\longrightarrow\infty}\mu\left(\Gamma^l y_l-\mathrm{u}\right)\leq c. \tag{56}$$

Now,

$$\lim_{l\longrightarrow\infty}\mu(x_{l+1}-\mathrm{u})=\lim_{l\longrightarrow\infty}\mu\left(W\left(\Gamma^l\mathrm{u}_l,\Gamma^l y_l,\gamma_l\right)-\mathrm{u}\right)=c. \tag{57}$$

By using Lemma 16 and from (54) and (56), we have

$$\lim_{l\longrightarrow\infty}\mu\left(\Gamma^l\mathrm{u}_l-\Gamma^l y_l\right)=0. \tag{58}$$

From (41) and (58),

$$\begin{aligned} \mu\left(x_{l+1},\Gamma^l\mathrm{u}_l\right) &= \mu\left(W\left(\Gamma^l\mathrm{u}_l,\Gamma^l y_l,\gamma_l\right)-\Gamma^l\mathrm{u}_l\right) \\ &\leq (1-\gamma_l)\mu\left(\Gamma^l\mathrm{u}_l-\Gamma^l\mathrm{u}_l\right)+\gamma_l\mu\left(\Gamma^l\mathrm{u}_l-\Gamma^l y_l\right), \end{aligned} \tag{59}$$

taking $\lim_{l\longrightarrow\infty}$, and we have

$$\lim_{l\longrightarrow\infty}\mu\left(x_{l+1}-\Gamma^l\mathrm{u}_l\right)=0. \tag{60}$$

Similarly,

$$\lim_{l\longrightarrow\infty}\mu\left(x_{l+1}-\Gamma^l y_l\right)=0. \tag{61}$$

Next,

$$\begin{aligned} \mu(x_{l+1}-\mathrm{u}) &\leq \mu\left(x_{l+1}-\Gamma^l y_l\right)+\mu\left(\Gamma^l y_l-\mathrm{u}\right) \\ &\leq \mu\left(x_{l+1}-\Gamma^l y_l\right)+\mu(y_l-\mathrm{u})+\zeta_l\phi(\mu(y_l-\mathrm{u}))+\xi_l, \end{aligned} \tag{62}$$

taking $\liminf_{l\longrightarrow\infty}$, and we get

$$c\leq\liminf_{l\longrightarrow\infty}\mu(y_l-\mathrm{u}). \tag{63}$$

From (52) and (63), we get

$$c=\lim_{l\longrightarrow\infty}\mu(y_l-\mathrm{u})=\lim_{l\longrightarrow\infty}\mu\left(W\left(x_l,\Gamma^l x_l,\nu_l\right)-\mathrm{u}\right). \tag{64}$$

By using Lemma 16, we have

$$\lim_{l\longrightarrow\infty}\mu\left(\Gamma^l\mathrm{u}_l-\mathrm{u}_l\right)=0, \tag{65}$$

hence proved. □

Data Availability

There is no any data available.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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