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Research Article

Fixed Points and Continuity for a Pair of Contractive Maps with Application to Nonlinear Volterra Integral Equations

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In this paper, we have established and proved fixed point theorems for the Boyd-Wong-type contraction in metric spaces. In particular, we have generalized the existing results for a pair of mappings that possess a fixed point but not continuous at the fixed point. We can apply this result for both continuous and discontinuous mappings. We have concluded our results by providing an illustrative example for each case and an application to the existence and uniqueness of a solution of nonlinear Volterra integral equations.

1. Introduction and Preliminaries

Continuity is an ideal property which is sometimes difficult to be fulfilled especially in some daily life applications. For instance, most neural network systems like bar code scanning, speech recognition, and handwritten digit recognition lack the continuity property. These neural network systems are some excellent prototypes for learning discontinuity phenomena. Here, we transform different kinds of day to day real-world phenomena into threshold functions which satisfies our desirable continuity of the weaker form and a new type of contraction to provide a solution to some daily life applications. Therefore, it is desirable to relax continuity assumptions because, in some applications, the function may not be continuous. One can see more literature on the topic [1–5].

In 1969, Kannan [6] proved the following fixed point theorem for discontinuous mapping:

Theorem 1 [6]. *If a self mapping T of a complete metric space (X, d) satisfies the condition*

$$d(Tx, Ty) \leq a[d(x, Tx) + d(y, Ty)], \quad 0 \leq a < \frac{1}{2}, \tag{1}$$

for each $x, y \in X$, then, T has a unique fixed point.

This theorem gave rise to the question of continuity of contractive mappings at their fixed points. In the Kannan contractive condition, continuity of mapping T was not required for the existence of a fixed point.

In 1971, Ciric [7] (see also [8]) introduced the notion of orbital continuity, which is as follows:

Definition 2 (see [7]). Let (X, d) be a metric space and $T : X \longrightarrow X$ be a self mapping. Then, the set $O(x, T) = \{T^n x : n = 0, 1, 2, 3, \cdots,\}$ is called the orbit of T at x and T is called orbitally continuous if for any sequence $\{x_n \subset O(x, T)\}$, $x_n \longrightarrow z$ implies that $Tx_n \longrightarrow Tz$ as $n \longrightarrow \infty$.

In 2017, Pant and Pant [9] introduced the notion of k-continuity which is as follows:

Definition 3 [9]. A mapping $T : X \longrightarrow X$ is called k-continuous for $k = 1, 2, 3, \cdots$, if $T^k x_n \longrightarrow Tt$ whenever a sequence $\{x_n\}$ is in X such that $T^{k-1} x_n \longrightarrow t$.

Continuity of T implies orbital continuity, but the converse is not true (see [7]).

The following are the examples of k-continuity:

Example 1. Let $X = [0, 3]$ be equipped with the usual metric and $T : X \longrightarrow X$ be defined by

$$Tx = \begin{cases} 1, & \text{if } 0 \le x \le 1, \\ 0, & \text{if } x \in (1,3]. \end{cases} \tag{2}$$

Then, $Tx_n \longrightarrow t \Rightarrow T^2 x_n \longrightarrow t$, since $Tx_n \longrightarrow t$ implies that $t = 0$ or $t = 1$ and $T^2 x_n = 1$ for all n, that is, $T^2 x_n \longrightarrow 1 = Tt$. Hence, T is 2-continuous. However, T is discontinuous at $x = 1$.

Example 2. Let $X = [0, 5]$ be equipped with the usual metric and $T : X \longrightarrow X$ be defined by

$$Tx = \begin{cases} 1, & \text{if } 0 \le x \le 1, \\ 0, & \text{if } 1 < x \le 4, \\ \frac{x}{4}, & \text{if } 4 < x \le 5. \end{cases} \tag{3}$$

Then, $T^2 x_n \longrightarrow t \Rightarrow T^3 x_n \longrightarrow Tt$, since $T^2 x_n \longrightarrow t$ implies $t = 0$ or $t = 1$ and $T^3 x_n = 1 = Tt$ for each n. Hence, T is 3-continuous. However, $Tx_n \longrightarrow t$ does not imply that $T^2 x_n \longrightarrow Tt$, that is, T is not 2-continuous.

Example 3 [9]. Let $X = [0, 2]$ be equipped with the usual metric and $T : X \longrightarrow X$ be defined by

$$Tx = \begin{cases} \frac{(1+x)}{2}, & \text{if } 0 \le x \le 1, \\ 0, & \text{if } x > 1. \end{cases} \tag{4}$$

Then, it can be verified that T is 2-continuous but not continuous. Moreover, T^k is discontinuous for each positive integer k. Thus 2-continuity of T does not imply continuity of T^2. In general, k-continuity of T does not imply continuity of T^n.

Example 4 [9]. Let $X = [0, 3] \cup (4, 5)$ be equipped with the usual metric and $T : X \longrightarrow X$ be defined by

$$Tx = \begin{cases} 1, & \text{if } 0 \le x \le 1, \\ 0, & \text{if } 1 < x \le 3, \\ \frac{x}{4}, & \text{if } 4 < x < 5. \end{cases} \tag{5}$$

Then, T^2 is continuous but T is not 2-continuous. If we consider the sequence $\{x_n\}$ given by $x_n = 4 + 1/n$, then, $Tx_n \longrightarrow 1$ but $T^2 x_n \longrightarrow 0 \ne T1$. Hence, T is not 2-continuous.

From the above examples, one can see that continuity of T^k and k-continuity of T are independent conditions when $k > 1$. It is easy to see that 1-continuity is equivalent to continuity and

$$\text{Continuity} \Rightarrow 2 - \text{continuity} \Rightarrow 3 - \text{continuity} \Rightarrow \cdots. \tag{6}$$

Definition 4 [10]. Let $\{x_n\}$ be a sequence in a metric space (X, d). Then,

(i) A sequence $\{x_n\}$ in X converges to a point $x \in X$ if and only if $\lim_{n \longrightarrow \infty} d(x, x_n) = \lim_{n \longrightarrow \infty} d(x_n, x_n)$

(ii) A sequence $\{x_n\}$ is called a Cauchy sequence if there exists $\varepsilon > 0$ such that for all $n, m > N$, we have $d(x_n, x_m) < \varepsilon$ for some integers $N \ge 0$, that is $\lim_{n,m \longrightarrow +\infty} d(x_n, x_m)$ exists and it is finite

(iii) A metric space (X, d) is complete if every Cauchy sequence $\{x_n\}$ converges to a point $x \in X$ such that $d(x, x) = \lim_{n,m \longrightarrow +\infty} d(x_n, x_m)$

Pant and Pant [9] proved the following theorem by employing a new type of $(\varepsilon - \delta)$ condition.

Theorem 5 [9]. *Let f be a self mapping of a complete metric space (X, d) such that*

(i) $d(fx, fy) < \max \{d(x, fx), d(y, fy)\}$, $\max \{d(x, fx), d(y, fy)\} > 0$

(ii) Given that $\varepsilon > 0$, there exists a $\delta > 0$ such that $\varepsilon < \max \{d(x, fx), d(y, fy)\} \le \varepsilon + \delta \Rightarrow d(fx, fy) \le \varepsilon$

If f is k-continuous or f^k is continuous for some $k \ge 1$ or f is orbitally continuous, then, f possesses a unique fixed point.

2. Main Results

Pant and Pant [9] used $\varepsilon - \delta$ and k-continuity property to prove the above fixed point theorem for one self map. In this section, we are extending Theorem 5 for a pair of self maps using $\varepsilon - \delta$ conditions as follows:

Theorem 6. *Let X be a nonempty set and let d to be a metric on X. Let T and S be self mappings of a complete metric space (X, d) satisfying*

(i) $d(Tx, Sy) < M(x, y)$, where

$$M(x, y) = \max \left\{ d(x, y), d(x, Tx), d(y, Sy), \frac{d(x, Sy) + d(y, Tx)}{2} \right\}. \tag{7}$$

(ii) Given that $\varepsilon > 0$, there exists a $\delta > 0$ such that $\varepsilon < M(x, y) \le \varepsilon + \delta \Rightarrow d(Tx, Sy) \le \varepsilon$

If T and S are k-continuous or T^k and S^k are continuous for some $k \ge 1$ or T and S are orbitally continuous, then, T and S have a unique common fixed point.

Proof. Let x_0 be any point in X. Define a sequence $\{x_n\}$ in X for $n = 0, 1, 2, \cdots$, as $x_{2n+1} = Tx_{2n}$ and $x_{2n+2} = Sx_{2n+1}$, for all integers $n \ge 0$.

If $x_{2n}=x_{2n+1}$ for some n, then, $x_{2n}=Tx_{2n}$, that is x_{2n} is a fixed point of T. Similarly, if there exists an integer $N\geq 0$ such that $x_{2N+1}=x_{2N+2}$, then, x_{2n+1} is a fixed point of S. This concludes the proof. □

Otherwise, we suppose that $x_{2n}\neq x_{2n+1}$, for all integers $n\geq 0$. Let $d_{2n}=d(x_{2n},x_{2n+1})$; obviously, $d_{2n+1}=d(x_{2n+1},x_{2n+2})$.

Then, by using equation (7) with $x=x_{2n}$ and $y=x_{2n+1}$, we have

$$d(x_{2n+1},x_{2n+2})<M(x_{2n},x_{2n+1}), \tag{8}$$

where

$$\begin{aligned} M(x_{2n},x_{2n+1}) &= \max\left\{d(x_{2n},x_{2n+1}), d(x_{2n},Tx_{2n}), d(x_{2n+1},Sx_{2n+1}), \frac{d(x_{2n},Sx_{2n+1})+d(x_{2n+1},Tx_{2n})}{2}\right\},\\ &= \max\left\{d(x_{2n},x_{2n+1}), d(x_{2n},x_{2n+1}), d(x_{2n+1},x_{2n+2}), \frac{d(x_{2n},x_{2n+2})+d(x_{2n+1},x_{2n+1})}{2}\right\}. \end{aligned} \tag{9}$$

Since,

$$\begin{aligned} &\frac{d(x_{2n},x_{2n+2})+d(x_{2n+1},x_{2n+1})}{2}\\ &\quad\leq \frac{d(x_{2n},x_{2n+1})+d(x_{2n+1},x_{2n+2})+d(x_{2n+1},x_{2n+1})}{2}\\ &\quad= \frac{d(x_{2n},x_{2n+1})+d(x_{2n+1},x_{2n+2})}{2}, \end{aligned} \tag{10}$$

then,

$$M(x_{2n},x_{2n+1})=\max\{d(x_{2n},x_{2n+1}), d(x_{2n+1},x_{2n+2})\}. \tag{11}$$

Thus,

$$\begin{aligned} d(x_{2n+1},x_{2n+2}) &= d(Tx_{2n},Sx_{2n+1})\\ &\leq \max\{d(x_{2n},x_{2n+1}), d(x_{2n+1},x_{2n+2})\}. \end{aligned} \tag{12}$$

Obviously, if $\max\{d(x_{2n},x_{2n+1}), d(x_{2n+1},x_{2n+2})\}=d(x_{2n+1},x_{2n+2})$, we have a contradiction and so $\max\{d(x_{2n},x_{2n+1}), d(x_{2n+1},x_{2n+2})\}=d(x_{2n},x_{2n+1})$. Therefore,

$$d(x_{2n+1},x_{2n+2})<d(x_{2n},x_{2n+1}), \tag{13}$$

which implies that the sequence $\{d_{2n}\}$ is decreasing to a non-negative real number, say ε, for all integers $n\geq 0$. We claim that $\varepsilon=0$. In contrary, suppose that $\varepsilon>0$. Taking the limit as $n\longrightarrow\infty$ in (13), we obtain

$$\varepsilon<d_{2n}\leq\varepsilon+\delta\Longrightarrow d_{2n+1}\leq\varepsilon, \tag{14}$$

which is a contradiction; hence, we conclude that $\varepsilon=0$ and

$$\lim_{n\longrightarrow\infty}(d_{2n})=\lim_{n\longrightarrow\infty}d(x_{2n},x_{2n+1})=0. \tag{15}$$

Now, we need to show that a sequence $\{x_{2n}\}$ in X is a Cauchy sequence. In equation (15), it is sufficient to show that a subsequence $\{x_{2n(r)}\}$ is a Cauchy sequence. On the contrary, we claim that $\{x_{2n(r)}\}$ is not a Cauchy sequence. Therefore, there exists $\varepsilon>0$ and a sequence of integers $m(r), n(r)$ such that

$$d\left(x_{2n(r)},x_{2m(r)}\right)\geq\varepsilon, \tag{16}$$

for all $n(r)>m(r)\geq r$ for some $r\geq 0$.

Furthermore, suppose that $m(r)$ is the smallest integer which is chosen in such away that (16) holds so that we have

$$dx_{2n(r)}=d\left(x_{2n(r)},x_{2m(r)-1}\right)<\varepsilon. \tag{17}$$

Now, for all $n(r)>m(r)$, we have

$$d\left(x_{2n(r)},x_{2m(r)}\right)\leq d\left(x_{2n(r)},x_{2m(r)-1}\right)+d\left(x_{2m(r)-1},x_{2m(r)}\right). \tag{18}$$

As $r\longrightarrow\infty$ in (18) and considering (15) and (17), we see that

$$d\left(x_{2n(r)},x_{2m(r)}\right)\longrightarrow\varepsilon. \tag{19}$$

By similar computations, we see that,

$$dx_{2n(r)-1}=d\left(x_{2n(r)-1},x_{2m(r)-1}\right)\longrightarrow\varepsilon. \tag{20}$$

To show it, we shall prove that $M(x_{2n(r)-1},x_{2m(r)-1})\leq\varepsilon+\delta$. Then, by using equation (7) with $x=x_{2n(r)-1}$ and $y=x_{2m(r)-1}$, we have

$$\begin{aligned} d\left(x_{2n(r)},x_{2m(r)}\right) &= d\left(Tx_{2n(r)-1},Sx_{2m(r)-1}\right)\\ &\leq M\left(x_{2n(r)-1},x_{2m(r)-1}\right), \end{aligned} \tag{21}$$

where

$$\begin{aligned} M\left(x_{2n(r)-1}, x_{2m(r)-1}\right) &= \max\left\{ d\left(x_{2n(r)-1}, x_{2m(r)-1}\right),\right. \\ & d\left(x_{2n(r)-1}, Tx_{2n(r)-1}\right), d\left(x_{2m(r)-1}, Sx_{2m(r)-1}\right), \\ & \left.\frac{d\left(x_{2n(r)-1}, Sx_{2m(r)-1}\right) + d\left(x_{2m(r)-1}, Tx_{2n(r)-1}\right)}{2}\right\} \\ &= \max\left\{ d\left(x_{2n(r)-1}, x_{2m(r)-1}\right),\right. \\ & d\left(x_{2n(r)-1}, x_{2n(r)}\right), d\left(x_{2m(r)-1}, x_{2m(r)}\right), \\ & \left.\frac{d\left(x_{2n(r)-1}, x_{2m(r)}\right) + d\left(x_{2m(r)-1}, x_{2n(r)}\right)}{2}\right\} \\ &= d\left(x_{2n(r)-1}, x_{2m(r)-1}\right). \end{aligned} \tag{22}$$

As $r \longrightarrow \infty$ in (22) and considering (19) and (20), then, (21) becomes

$$\varepsilon < d_{2n(r)-1} \leq \varepsilon + \delta \Longrightarrow d_{2nr} \leq \varepsilon, \tag{23}$$

which is a contradiction. Hence, $\{x_{2n}\}$ in X is a Cauchy sequence and

$$\lim_{n,m \longrightarrow \infty} d(x_{2n}, x_{2m}) = 0. \tag{24}$$

Since X is complete, there exists a point $t \in X$ such that $x_{2n} \longrightarrow t$. Furtheremore, for each $k \geq 1$, we have $T^k x_{2n} \longrightarrow Tt$. Thus, $t = Tt$ as $T^k x_{2n} \longrightarrow t$. Hence, t is a fixed point of T.

Again, for $x_{2n+1} \longrightarrow t$ and for each $k \geq 1$, we have $S^k x_{2n+1} \longrightarrow St$. Hence, $t = St$ as $S^k x_{2n+1} \longrightarrow t$. Therefore, t is a fixed point of S.

In addition, assume that T^k and S^k are k-continuous for some positive integer k. Then, we have $\lim_{n \longrightarrow \infty} d(x_{2n}, t) = 0$. Therefore,

$$\begin{aligned} \lim_{n \longrightarrow \infty} T^k x_{2n} &= \lim_{n \longrightarrow \infty} Tx_{2n} = \lim_{n \longrightarrow \infty} x_{2n+1} = t, \\ \lim_{n \longrightarrow \infty} S^k x_{2n+1} &= \lim_{n \longrightarrow \infty} Sx_{2n+1} = \lim_{n \longrightarrow \infty} x_{2n+2} = t. \end{aligned} \tag{25}$$

Here, we will prove that t is a fixed point of S. Contrarily, suppose that $t \neq St$.

Now,

$$d(x_{2n+1}, St) = d(Tx_{2n}, St) \leq M(x_{2n}, t), \tag{26}$$

where

$$\begin{aligned} M(x_{2n}, t) &= \max\left\{ d(x_{2n}, t), d(x_{2n}, Tx_{2n}), d(t, St), \frac{d(x_{2n}, St) + d(t, Tx_{2n})}{2}\right\} \\ &= \max\left\{ d(x_{2n}, t), d(x_{2n}, x_{2n+1}), d(t, St), \frac{d(x_{2n}, St) + d(t, x_{2n+1})}{2}\right\}. \end{aligned} \tag{27}$$

As $n \longrightarrow \infty$ in (27), we see that

$$M(x_{2n}, t) \longrightarrow d(t, St). \tag{28}$$

Applying the limit as $n \longrightarrow \infty$ in (26), we have

$$d(t, St) \leq d(t, St) < d(t, St), \tag{29}$$

which is a contradiction. Hence, $St = t$.

Now, suppose that T is orbitally continuous. Since $x_{2n} \longrightarrow t$, orbital continuity implies that $Tx_{2n} \longrightarrow Tt$ or $Sx_{2n+1} \longrightarrow St$. This gives $t = Tt$ as $Tx_{2n} \longrightarrow t$ or $t = St$ as $Sx_{2n+1} \longrightarrow t$. Thus, t is a fixed point of T and S.

Next, we will show that a point t is a unique common fixed of T and S. In contrary, suppose that $t \in X$ and $y^* \in X$ are two different common fixed points of T and S, respectively. Thus, $d(t, y^*) > 0$.

Now,

$$d = d \leq M, \tag{30}$$

where

$$\begin{aligned} M(t, y^*) &= \max\left\{ d(t, y^*), d(t, Tt), d(y^*, Sy^*), \frac{d(t, Sy^*) + d(y^*, Tt)}{2}\right\} \\ &= \max\left\{ d(t, y^*), d(t, t), d(y^*, y^*), \frac{d(t, y^*) + d(y^*, t)}{2}\right\} \\ &= d(x^*, y^*). \end{aligned} \tag{31}$$

Hence,

$$d(t, y^*) = d(Tt, Sy^*) < Md(t, y^*) < d(t, y^*), \tag{32}$$

which is a contradiction. Therefore, T and S have a unique common fixed point, that is $t = y^*$.

To prove that any fixed point of T is also a fixed point of S , conversely, we suppose to the contrary that $t = Tt$ and $t \neq St$. Now,

$$d(Tt, St) = d(t, St) \leq M(t, St), \tag{33}$$

where

$$\begin{aligned} M(t, St) &= \max\left\{d(t, St), d(t, Tt), d(St, S^2t), \frac{d(t, S^2t) + d(St, Tt)}{2}\right\} \\ &= \max\left\{d(t, St), d(t, t), d(St, S^2t), \frac{d(t, S^2t) + d(St, t)}{2}\right\} \\ &= d(t, St). \end{aligned} \quad (34)$$

Thus,

$$d(Tt, St) = d(t, St) \leq M(t, St) < d(t, St), \quad (35)$$

which is a contradiction. Therefore, $t = Tt = St$. In a similar way, it is easy to show that any fixed point of S is also a fixed point of T.

Remark 7. If we set $S = T$, we get an improved version of the Bisht and Pant [11] theorem as a corollary for the case $a = 1$ as follows:

Corollary 8. *Let X be a nonempty set and let d to be a metric on X. Let T be self mapping of a complete metric space (X, d) satisfying*

(i) $d(Tx, Ty) < M(x, y)$, where

$$M(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\right\}. \quad (36)$$

(ii) Given that $\varepsilon > 0$, there exists a $\delta > 0$ such that $\varepsilon < M(x, y) \leq \varepsilon + \delta \Rightarrow d(Tx, Ty) \leq \varepsilon$

If T is k-continuous or T^k is continuous for some $k \geq 1$ or T is orbitally continuous, then, T has a fixed point.

Corollary 9. *The conclusions of Theorem 6 remain true, if we replace $M(x, y)$ in the contractive equation (7) by any one of the following:*

(i) $M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Sy)\}$.

(ii) $M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Sy), \frac{d(x, Sy) + d(y, Tx)}{2}\}$.

(iii) $M(x, y) = \max\{d(x, y), d(x, Sx), d(y, Ty), \frac{d(x, Tx) + d(y, Sy)}{2}\}$.

The following example shows the generality of Theorem 6 over Theorem 5.

Example 5. Let $X = [0, 3]$ be equipped with the usual metric. Let S and T be self mappings on X, i.e., $T, S : X \longrightarrow X$ defined by

$$Tx = \begin{cases} \frac{1 + x}{2}, & \text{if } 0 \leq x \leq 1, \\ 0, & \text{if } x \in (1,3]. \end{cases}$$
$$Sx = \begin{cases} x^2, & \text{if } 0 \leq x \leq 1, \\ 0, & \text{if } 1 < x \leq 2, \\ \frac{x}{2}, & \text{if } 2 < x \leq 3. \end{cases} \quad (37)$$

Hence, T and S satisfy all the conditions of the above theorem and have a unique fixed point $x = 1$; and S and T are discontinuous at $x = 1$. The mapping T is 2-continuous and S is 3-continuous at $x = 1$. S and T are orbitally continuous. Tt can be easily verified using the following cases:

Case 1. Now, we have

$$d(Tx, Sy) < M(x, y) \quad (38)$$

where

$$M(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Sy), \frac{d(x, Sy) + d(y, Tx)}{2}\right\}. \quad (39)$$

For $x, y \leq 1$ and $0 \leq x \leq 1$, we get

$$\begin{aligned} &d(Tx, Sy) = d\left(\frac{1 + x}{2}, y^2\right) = \left|\frac{1 + x}{2} - y^2\right| = \left|\frac{x - 2y^2 + 1}{2}\right|, \\ &d(x, y) = d(x, y) = |x - y|, \\ &d(x, Tx) = d\left(x, \frac{1 + x}{2}\right) = \left|x - \frac{1 + x}{2}\right| = \left|\frac{x - 1}{2}\right|, \\ &d(y, Sy) = d(y, y^2) = |y - y^2|, \\ &d(x, Sy) = d(x, y^2) = |x - y^2|, \\ &d(y, Tx) = d\left(y, \frac{1 + x}{2}\right) = \left|y - \frac{1 + x}{2}\right| = \left|\frac{2y - x - 1}{2}\right|, \\ &M(x, y) = \max\left\{|x - y|, \left|\frac{x - 1}{2}\right|, |y - y^2|, \frac{2|x - y^2| + |2y - x - 1|}{4}\right\} \\ &= |x - y|. \end{aligned} \quad (40)$$

By using (i) of Theorem 6, we have

$$d(Tx, Sy) < M(x, y) \Longrightarrow \left|\frac{x - 2y^2 + 1}{2}\right| < |x - y|, \quad (41)$$

which is a contradiction. Hence, T and S are discontinuous at $x = 1$.

Remark 10. In case of $x = 1$ and $y = 1$, both functions are discontinuous at this point but have the property of k-continuity. For more details, see Example 1.

Case 2. Next we have,

$$d(Tx, Sy) < M(x, y) \tag{42}$$

where

$$M(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Sy), \frac{d(x,Sy)+d(y,Tx)}{2}\right\}. \tag{43}$$

For $x, y > 2 \Longrightarrow 2 < x \le 3$, we get

$$\begin{aligned} d(Tx,Sy) &= d\left(0, \frac{y}{2}\right) = \left|0 - \frac{y}{2}\right| = \frac{y}{2}, \\ d(x,y) &= d(x,y) = |x-y|, \\ d(x,Tx) &= d(x,0) = |x-0| = |x|, \\ d(y,Sy) &= d\left(y, \frac{y}{2}\right) = \left|y - \frac{y}{2}\right| = \frac{y}{2}, \\ d(x,Sy) &= d\left(x, \frac{y}{2}\right) = \left|x - \frac{y}{2}\right| = \frac{2x-y}{2}, \\ d(y,Tx) &= d(y,0) = |y-0| = y, \end{aligned}$$

$$\begin{aligned} M(x,y) &= \max\left\{d(x,y), d(x,0), d\left(y,\frac{y}{2}\right), \frac{d(x,(y/2)) + d(y,0)}{2}\right\}, \\ &= \max\left\{|x-y|, x, \frac{y}{2}, \frac{((2x-y)/2)+y}{2}\right\} \\ &= \max\left\{|x-y|, x, \frac{y}{2}, \frac{2x+y}{4}\right\} \\ &= x. \end{aligned} \tag{44}$$

By using (i) of Theorem 6, we have

$$d(Tx,Sy) < M(x,y) \Longrightarrow \frac{y}{2} < x. \tag{45}$$

Thus, conditions (i) of Theorem 6 satisfy for all $x, y > 2$. This shows that T and S are continuous.

Case 3. Next, we have

$$d(Tx, Sy) < M(x, y) \tag{46}$$

where

$$M(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Sy), \frac{d(x,Sy)+d(y,Tx)}{2}\right\}. \tag{47}$$

For $x \le 1 \Longrightarrow y \le 2$, we have

$$\begin{aligned} d(Tx,Sy) &= d\left(\frac{1+x}{2}, 0\right) = \left|\frac{1+x}{2} - 0\right| = \frac{x+1}{2}, \\ d(x,y) &= d(x,y) = |x-y|, \\ d(x,Tx) &= d\left(x, \frac{1+x}{2}\right) = \left|x - \frac{1+x}{2}\right| = \frac{x-1}{2}, \\ d(y,Sy) &= d(y,0) = |y-0| = y, \\ d(x,Sy) &= d(x,0) = |x-0| = x, \\ d(y,Tx) &= d\left(y, \frac{1+x}{2}\right) = \left|y - \frac{1+x}{2}\right| = \frac{2y-x-1}{2}, \\ M(x,y) &= \max\left\{d(x,y), d\left(x, \frac{1+x}{2}\right), d(y,0), \frac{d(x,0)+d(y,((1+x)/2))}{2}\right\}, \\ &= \max\left\{|x-y|, \frac{x-1}{2}, y, \frac{x+((2y-x-1)/2)}{2}\right\}, \\ &= \max\left\{|x-y|, \frac{x-1}{2}, y, \frac{x+2y-1}{4}\right\} \\ &= y. \end{aligned} \tag{48}$$

By using (i) of Theorem 6, we have

$$d(Tx,Sy) < M(x,y) \Longrightarrow \frac{x+1}{2} < y. \tag{49}$$

Thus, conditions (i) of Theorem 6 satisfy for all $x \le 1, y \le 2$. This shows that T and S are continuous. Also, $T1 = 1$ and $S2 = 0$. Then, $TSx_{2n} \longrightarrow t \Longrightarrow T^2Sx_{2n} \longrightarrow 1 = TSt$ for each n. Hence, TS is 3-continuous.

Therefore, T and S satisfy condition (ii) of Theorem 6 with $\delta = 1 - \varepsilon$ if $\varepsilon < 1$ and $\delta = 1$ for $\varepsilon \ge 1$. To see this, consider Case 1, Case 2, and Case 3 as follows:

By Case 1, using condition (ii) of Theorem 6, $\varepsilon < 1$ and $\delta = 1 - \varepsilon$, we get

$$\begin{aligned} &\varepsilon < M(x,y) < \varepsilon + \delta \Rightarrow d(Tx,Sy) < \varepsilon, \\ &\Rightarrow \varepsilon < \frac{x-1}{2} < \varepsilon + \delta \Rightarrow \left|\frac{x-2y^2+1}{2}\right| < \varepsilon, \\ &\Rightarrow \varepsilon < \frac{x-1}{2} < \varepsilon + 1 - \varepsilon \Rightarrow \left|\frac{x-2y^2+1}{2}\right| < \varepsilon, \\ &\Rightarrow \varepsilon < 0 < 1 \Rightarrow 0 < \varepsilon. \end{aligned} \tag{50}$$

which is a contradiction.

By Case 2, using (ii) of Theorem 6, $\delta = 1$ for $\varepsilon \ge 1$, we get

$$\begin{aligned} &\varepsilon < M(x,y) < \varepsilon + \delta \Rightarrow d(Tx,Sy) < \varepsilon, \\ &\Rightarrow \varepsilon < x < \varepsilon + \delta \Rightarrow \left|\frac{y}{2}\right| < \varepsilon, \\ &\Rightarrow \varepsilon < x < \varepsilon + 1 \Rightarrow \left|\frac{y}{2}\right| < \varepsilon, \end{aligned} \tag{51}$$

satisfying for all $x, y > 2$.

However, this example is not applicable to the conditions imposed in Theorem 5.

Remark 11. It can be seen from the above example that T and S are threshold operation that models firing of a neuron, a function of two diodes, and also a low-pass filter that allows low voltages to pass but not higher voltages (e.g., noise in music systems).

One of the fundamental tools for nonlinear analysis is the Banach fixed point theorem [12]. As a result of its usefulness and applications, this theorem has been massively investigated and generalized by different researchers. One of the important generalization of the Banach fixed point theorem is the Boyd and Wong [13] fixed point theorem. A mapping T satisfying

$$d(Tx, Ty) \le \phi(d(x, y)), \quad \forall x, y \in M, \tag{52}$$

whereby (M, d) is a complete metric space and a mapping $\phi : [0,\infty) \longrightarrow [0,\infty)$ is upper semicontinuous from the right on $[0, \infty)$ such that $\phi(t) < t$, $\forall t > 0$. Consequently, T has a unique fixed point $z \in M$ and $d(T^n x, z) \longrightarrow 0$ as $n \longrightarrow \infty$, $\forall x \in M$. Pant and Pant [9] proved the following theorem for the Boyd and Wong type fixed point theorem in complete metric spaces:

Theorem 12 [9]. *Let T be a mapping of a complete metric space (X, d) into itself satisfying*

$$d(Tx, Ty) \le \phi(\max \{d(x, Tx), d(y, Ty)\}), \tag{53}$$

for all $x, y \in X$, where the function $\phi : [0,\infty) \longrightarrow [0,\infty)$ is such that $\phi(t) < t$ for each $t > 0$. If ϕ is upper semicontinuous in the open interval $(0, d(T^k(X)))$, then, T has a unique fixed point.

Now, we will demonstrate an example to explain the above theorem:

Example 6. Let $X = [0, 3]$ with usual metric $d(x, y) = |x - y|$ for all $x, y \in X$. Let a mapping $T : X \longrightarrow X$ be defined by

$$Tx = \begin{cases} 0, & x \in [0, 1) \\ 1, & x \in [1, 3]. \end{cases} \tag{54}$$

Also, define $\phi : [0,\infty) \longrightarrow [0,\infty)$ as

$$\phi(t) = \begin{cases} \dfrac{1 + t}{2}, & t > 1, \\ \dfrac{t}{2}, & t \le 1. \end{cases} \tag{55}$$

It is clear that the mapping T satisfies the criteria of Theorem 12 with a unique fixed point $T = 1$ but it is discontinuous at this fixed point. Also, we observe that $d(T(X)) = 1$ and ϕ is continuous on $(0, 1)$.

Here, we present an extension of Theorem 12 for a pair of maps to obtain a unique common fixed point.

Theorem 13. *Let X be a nonempty set and let d be a metric on X. Let T and S be self mappings of a complete metric space (X, d) satisfying*

$$d(Tx, Sy) \le \phi\{M(x, y)\}, \tag{56}$$

for all $x, y \in X$, where the mapping $\phi : [0,\infty) \longrightarrow [0,\infty)$ is such that $\phi(t) < t$ for all $t > 0$ and

$$M(x, y) = \max \left\{ d(x, y), d(x, Tx), d(y, Sy), \frac{d(x, Sy) + d(y, Tx)}{2} \right\}. \tag{57}$$

If ϕ is upper semicontinuous on $(0, d(T^k(X)))$ and $(0, d(S^k(X)))$ for $k = 0, 1, 2, \cdots$, then, T and S have a unique common fixed point.

Proof. Let $x_0 \in X$. Define a sequence $\{x_n\}$ in X as $x_{2n+1} = Tx_{2n}$ and $x_{2n+2} = Sx_{2n+1}$, for all integers $n \ge 0$. If we assume that there exists a nonnegative integer n_0 such that, $x_{2n_0} = x_{2n_0+1}$, then, $x_{2n} = x_{2n+1} = Tx_{2n}$; this implies that x_{2n} is a fixed point of T. Similarly, if there exists an integer $N \ge 0$ such that $x_{2N+1} = x_{2N+2}$, then, x_{2n+1} is a fixed point of S. This concludes the proof. ☐

Otherwise, we suppose that $x_{2n} \ne x_{2n+1}$, for all integers $n \ge 0$. Let $\mu_{2n} = d(x_{2n}, x_{2n+1})$, obviously, $\mu_{2n+1} = d(x_{2n+1}, x_{2n+2})$. From (84), we have

$$d(x_{2n+1}, x_{2n+2}) = d(Tx_{2n}, Sx_{2n+1}) \le \phi(M(x_{2n}, x_{2n+1})), \tag{58}$$

where

$$M(x_{2n}, x_{2n+1}) = \max \left\{ d(x_{2n}, x_{2n+1}), d(x_{2n}, Tx_{2n}), d(x_{2n+1}, Sx_{2n+1}), \frac{d(x_{2n}, Sx_{2n+1}) + d(x_{2n+1}, Tx_{2n})}{2} \right\}, \tag{59}$$

Using equations (8) and (12), we have the following:

$$M(x_{2n}, x_{2n+1}) = \max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\}. \quad (60)$$

Thus,

$$d(x_{2n+1}, x_{2n+2}) = d(Tx_{2n}, Sx_{2n+1}) \leq \phi(\max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\}). \quad (61)$$

If we take $\max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\} = d(x_{2n+1}, x_{2n+2})$, then,

$$\begin{aligned} d(x_{2n+1}, x_{2n+2}) &= d(Tx_{2n}, Sx_{2n+1}) \\ &\leq \phi\{d(x_{2n+1}, x_{2n+2})\} \\ &< d(x_{2n+1}, x_{2n+2}), \end{aligned} \quad (62)$$

which is a contradiction. Hence, $\max = \{d(x_{2n}, x_{2n+1})\}, d(x_{2n+1}, x_{2n+2}) = d(x_{2n}, x_{2n+1})$.Therefore,

$$d(x_{2n+1}, x_{2n+2}) = d(Tx_{2n}, Sx_{2n+1}) \leq \phi\{d(x_{2n}, x_{2n+1})\} < d(x_{2n}, x_{2n+1}), \quad (63)$$

which implies that the sequence $\{\mu_{2n}\}$ is decreasing to a nonnegative real number say δ, for all integers $n \geq 0$. We claim that $\delta = 0$. In contrary, suppose that $\delta > 0$. Taking the limit as $n \longrightarrow \infty$ in (63), we obtain

$$0 < \delta \leq \phi(\delta) < \delta, \quad (64)$$

which is a contradiction; hence, we conclude that $\delta = 0$ and

$$\lim_{n\longrightarrow\infty} (\mu_{2n}) = \lim_{n\longrightarrow\infty} d(x_{2n}, x_{2n+1}) = 0. \quad (65)$$

Now, we need to show that a sequence $\{x_n\}$ in X is a Cauchy sequence. We claim otherwise. Therefore, there exists $\varepsilon > 0$ and a sequence of integers $m(r), n(r)$ such that

$$d\left(x_{n(r)}, x_{m(r)}\right) \geq \varepsilon, \quad (66)$$

for all $n(r) > m(r) \geq r$ for some $r \geq 0$.

Furthermore, suppose that $m(r)$ is the smallest integer which is chosen in such a way that (66) holds so that we have

$$d\left(x_{(r)}, x_{m(r)-1}\right) < \varepsilon. \quad (67)$$

Now, for all $n(r) > m(r)$, we have

$$\begin{aligned} d\left(x_{n(r)}, x_{m(r)}\right) &\leq d\left(x_{n(r)}, x_{m(r)-1}\right) + d\left(x_{m(r)-1}, x_{m(r)}\right) \\ &\leq d\left(x_{n(r)}, x_{m(r)-1}\right) + d\left(x_{m(r)-1}, x_{m(r)}\right). \end{aligned} \quad (68)$$

As $r \longrightarrow \infty$ in (68) and considering (65) and (67), we see that

$$d\left(x_{n(r)}, x_{m(r)}\right) \longrightarrow \varepsilon. \quad (69)$$

By similar computations, we see that

$$d\left(x_{n(r)-1}, x_{m(r)-1}\right) \longrightarrow \varepsilon. \quad (70)$$

Thus,

$$d\left(x_{n(r)}, x_{m(r)}\right) = d\left(Tx_{n(r)-1}, Sx_{m(r)-1}\right) \leq \phi\left(M\left(x_{n(r)-1}, x_{m(r)-1}\right)\right), \quad (71)$$

where

$$\begin{aligned} M\left(x_{n(r)-1}, x_{m(r)-1}\right) &= \max\left\{d\left(x_{n(r)-1}, x_{m(r)-1}\right), d\left(x_{n(r)-1}, Tx_{n(r)-1}\right), d\left(x_{m(r)-1}, Sx_{m(r)-1}\right), \frac{d\left(x_{n(r)-1}, Sx_{m(r)-1}\right) + d\left(x_{m(r)-1}, Tx_{n(r)-1}\right)}{2}\right\} \\ &= \max\left\{d\left(x_{n(r)-1}, x_{m(r)-1}\right), d\left(x_{n(r)-1}, x_{n(r)-1}\right) d\left(x_{m(r)-1}, x_{m(r)-1}\right), \frac{d\left(x_{n(r)-1}, x_{m(r)-1}\right) + d\left(x_{m(r)-1}, x_{n(r)-1}\right)}{2}\right\}, \end{aligned} \quad (72)$$

As $r \longrightarrow \infty$ in (72) and considering (69) and (70), then, (71) becomes

$$0 < \varepsilon \le \phi(\varepsilon) < \varepsilon, \tag{73}$$

which is a contradiction. Hence, $\{x_n\}$ in X is a Cauchy sequence and $\lim_{n,m \longrightarrow \infty} d(x_n, x_m) = 0$.

Because X is complete, we can pick a point $z \in X$ such that $\lim_{n \longrightarrow \infty} d(x_{2n}, z) = 0$. Here, we will prove that z is a fixed point of S. In contrary, suppose that $z \ne Sz$.

Now,

$$d(x_{2n+1}, Sz) = \phi(d(Tx_{2n}, Sz)) \le \phi(M(x_{2n}, z)). \tag{74}$$

where

$$\begin{aligned} M(x_{2n}, z) &= \max\left\{d(x_{2n}, z), d(x_{2n}, Tx_{2n}), d(z, Sz), \frac{d(x_{2n}, Sz) + d(z, Tx_{2n})}{2}\right\} \\ &= \max\left\{d(x_{2n}, z), d(x_{2n}, Tx_{2n}), d(z, Sz), \frac{d(x_{2n}, Sz) + d(z, x_{2n+1})}{2}\right\}. \end{aligned} \tag{75}$$

As $n \longrightarrow \infty$ in (75), we see that

$$M(x_{2n}, z) \longrightarrow d(z, Sz). \tag{76}$$

Applying the limit as $n \longrightarrow \infty$ in (74), we have

$$d(z, Sz) \le \phi(d(z, Sz)) < d(z, Sz), \tag{77}$$

which is a contradiction. Hence, $Sz = z$.

Now, we will show that a point z is a unique common fix of T and S. In contrary, suppose that $z \in X$ and $w \in X$ are two different common fixed points of T and S, respectively. Thus, $d(z, w) > 0$.

Now,

$$d(z, z) = d(Tz, Sw) \le \phi(M(z, w)) \tag{78}$$

where

$$\begin{aligned} M(z, w) &= \max\left\{d(z, w), d(z, Tz), d(w, Sw), \frac{d(z, Sw) + d(w, Tz)}{2}\right\} \\ &= \max\left\{d(z, w), d(z, z), d(w, w), \frac{d(z, w) + d(w, z)}{2}\right\} \\ &= d(z, w). \end{aligned} \tag{79}$$

Hence,

$$d(z, w) = d(Tz, Sw) \le \phi(d(z, w)) < d(z, w), \tag{80}$$

which is a contradiction. Therefore, T and S have a unique common fixed point, that is $z = w$.

To prove that any fixed point of T is also a fixed point of S, conversely, we suppose to the contrary that $z = Tz$ and $z \ne Sz$.

Now,

$$d(Tz, Sz) = d(z, Sz) \le \phi(M(z, Sw)) \tag{81}$$

where

$$\begin{aligned} M(z, Sw) &= \max\left\{d(z, Sz), d(z, Tz), d(Sz, S^2z), \frac{d(z, S^2z) + d(Sz, Tz)}{2}\right\} \\ &= \max\left\{d(z, Sz), d(z, z), d(Sz, S^2z), \frac{d(z, S^2z) + d(Sz, z)}{2}\right\} \\ &= d(z, Sz). \end{aligned} \tag{82}$$

Thus,

$$d(Tz, Sz) = d(z, Sz) \le \phi(d(z, Sz)) < d(z, Sz), \tag{83}$$

which is a contradiction. Therefore, $z = Tz = Sz$. In a similar way, it is easy to show that any fixed point of S is also a fixed point of T.

On setting $S = T$, we get the following corollary:

Corollary 14. *Let X be a nonempty set and let d be a metric on X. Let T be a self mapping of a complete metric space (X, d) satisfying*

$$d(Tx, Ty) \le \phi\{M(x, y)\}, \tag{84}$$

for all $x, y \in X$, where the mapping $\phi : [0,\infty) \longrightarrow [0,\infty)$ is such that $\phi(t) < t$ for all $t > 0$ and

$$M(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\right\}. \tag{85}$$

If ϕ is upper semicontinuous on $(0, d(T^k(X)))$ for $k = 0, 1, 2, \cdots$, then, T has a fixed point.

Example 7. Let $X = [0, 3]$ be equipped with the usual metric. Let S and T be self mappings on X, i.e., $T, S : X \longrightarrow X$ defined by

$$\begin{aligned} Tx &= \begin{cases} \dfrac{1 + x}{2}, & \text{if } 0 \le x \le 1, \\ 0, & \text{if } x \in (1,3], \end{cases} \\ Sx &= \begin{cases} x^2, & \text{if } 0 \le x \le 1, \\ 0, & \text{if } 1 < x \le 2, \\ \dfrac{x}{2}, & \text{if } 2 < x \le 3. \end{cases} \end{aligned} \tag{86}$$

Define $\phi = (1 + t)/2$.

Now, we have

$$d(Tx, Sy) < \phi\{M(x, y)\}, \tag{87}$$

where

$$M(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Sy), \frac{d(x,Sy)+d(y,Tx)}{2}\right\}, \tag{88}$$

For $x, y \le 1$, using $\phi = (1+t)/2$, we have

$$\begin{aligned} d(Tx,Sy) &= \left|\frac{x-2y^2+1}{2}\right|, \\ M(x,y) &= \left|\frac{x-1}{2}\right|, \\ d(Tx,Sy) &\le \phi(M(x,y)), \\ \left|\frac{x-2y^2+1}{2}\right| &\le \phi\left(\left|\frac{x-1}{2}\right|\right), \\ 0 &\le \frac{1+x}{2}, \end{aligned} \tag{89}$$

which is true.

For $x, y > 2$, $2 \le x \le 3$ and using $\phi = (1+t)/2$, we have

$$\begin{aligned} d(Tx,Sy) &= \frac{y}{2}, \\ M(x,y) &= x, \\ d(Tx,Sy) &\le \phi(M(x,y)), \\ \frac{y}{2} \le \phi(x), &\quad \frac{y}{2} \le \frac{1+x}{2}. \end{aligned} \tag{90}$$

For $x \le 1$, $y \le 2$ and using $\phi = (1+t)/2$, we have

$$\begin{aligned} d(Tx,Sy) &= \frac{x+1}{2}, \\ M(x,y) &= y, \\ d(Tx,Sy) &\le \phi(M(x,y)), \\ \frac{x+1}{2} &\le \phi(y), \\ \frac{x+1}{2} &\le \frac{1+y}{2}, \end{aligned} \tag{91}$$

Hence, all conditions imposed in Theorem 13 are satisfied. Thus, T and S satisfy all the conditions of the above theorem and has a unique fixed point $x = 1$; S and T are discontinuous at $x = 1$. The mapping T is 2-continuous and S is 3-continuous at $x = 1$.

3. Fixed Points of Nonexpansive Mappings

Bisht and Pant [11] proved a fixed point theorem (see Theorem 15 [11]) for nonexpansive mapping. In this section, we are extending the result due to Bisht and Pant [11] for a pair of self mappings.

In what follows, we shall denote

$$P((x,y)) = \max\left\{d(x,y), d(x,Tx), d(y,Sy), a\left[\frac{d(x,Tx)+d(y,Sy)}{2}\right], b\left[\frac{d(x,Sy)+d(y,Tx)}{2}\right]\right\}, \quad 0 \le a, b < 1. \tag{92}$$

We will use this expression in the following theorem.

Theorem 15. *Let X be a nonempty set and let d be a metric on X. Let T and S be self mappings of a complete metric space (X, d) such that for any $x, y \in X$*

(i) For any given $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that $\varepsilon < P(x,y) < \varepsilon + \delta$ implies $d(Tx,Sy) \le \varepsilon$

(ii) $d(Tx,Sy) \le P(x,y)$

Then, T and S have a unique common fixed point, say z and $T^n x \longrightarrow z$ as well as $S^n x \longrightarrow z$ for each $x \in X$.

Proof. Let x_0 be any point in X. Define a sequence $\{x_n\}$ in X as $x_{2n+1} = T^n x_0 = Tx_{2n}$ and $x_{2n+2} = Sx_{2n+1}$ for all integers $n \ge 0$. Then, on following the proof of Theorem 6, we can easily prove that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists a point $z \in X$ such that $x_n \longrightarrow z$ as $n \longrightarrow \infty$. Also, $Tx_n \longrightarrow z$ and $Sx_n \longrightarrow z$ as $n \longrightarrow \infty$. We claim that $Tz = z$. For if $Tz \ne z$, we get

$$\begin{aligned} d(Tz,Sx_n) \le \max\Big\{&d(z,x_n), d(z,Tx_n), d(x_n,Sx_n), \\ &a\left[\frac{d(z,Tz)+d(x_n,Sx_n)}{2}\right], b\left[\frac{d(z,Sx_n)+d(x_n,Tz)}{2}\right]\Big\}. \end{aligned} \tag{93}$$

On letting $n \longrightarrow \infty$, this yields, $d(Tz,Sz) \le \max\{a[d(Tz,z)+d(z,Sz)]/2, b[d(Sz,z)+d(z,Tz)]/2\} < \max\{d(z,Tz), d(z,Sz)\}$, which is a contradiction since $0 \le a, b < 1$. Thus, z is a common fixed point of T and S. ☐

Remark 16. By setting $S = T$, one can get Theorem 15 of Bist and Pant [11].

Corollary 17. *Let X be a nonempty set and let d be a metric on X. Let T be a self mapping of a complete metric space (X, d) such that for any $x, y \in X$*

(i) *For any given $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that $\varepsilon < P(x, y) < \varepsilon + \delta$ implies $d(Tx, Ty) \leq \varepsilon$*

(ii) $d(Tx, Ty) \leq P(x, y)$

Then, T has a unique fixed point, say z and $T^n x \longrightarrow z$ for each $x \in X$.

Example 8. Let $X = [-1, 1]$ be equipped with usual metric and $T, S : X \longrightarrow X$ be defined by

$$Tx = \begin{cases} 0, & \text{if } x \in [-1, 0], \\ \dfrac{-1}{2}, & \text{if } x \in (0,1], \end{cases}$$
$$Sx = \begin{cases} \dfrac{x}{2}, & \text{if } x \in [-1, 0], \\ \dfrac{-1}{4}, & \text{if } x \in (0,1]. \end{cases} \quad (94)$$

To verify our contraction condition, let us consider the following two cases:

Case 1. For $x, y \in [-1, 0]$, we have

$$\begin{aligned} d(x, y) &= |x - y|, \\ d(x, Tx) &= d(x, 0) = |x - 0| = x, \\ d(y, Sy) &= \left|y, \frac{y}{2}\right| = \left|y - \frac{y}{2}\right| = \frac{y}{2}, \\ d(x, Sy) &= d\left(x, \frac{y}{2}\right) = \left|x - \frac{y}{2}\right| = \frac{2x - y}{2}, \\ d(y, Tx) &= d(y, 0) = |y - 0| = y, \\ d(Tx, Sy) &= d\left(0, \frac{y}{2}\right) = \frac{y}{2}. \end{aligned} \quad (95)$$

Thus, we have,

$$\begin{aligned} P(x, y) &= \max\left\{d(x, y), d(x, Tx), d(y, Sy), a\left[\frac{d(x, Tx) + d(y, Sy)}{2}\right], b\left[\frac{d(x, Sy) + d(y, Tx)}{2}\right]\right\}, \\ P(x, y) &= \max\left\{|x - y|, x, \frac{y}{2}, a\left[\frac{x + (y/2)}{2}\right], b\left[\frac{(2x - y/2) + y}{2}\right]\right\}, \\ P(x, y) &= \max\left\{|x - y|, x, \frac{y}{2}, a\left[\frac{2x + y}{4}\right], b\left[\frac{2x + y}{4}\right]\right\}, = |x - y|, \quad \text{for } 0 \leq a, b \leq 1. \end{aligned} \quad (96)$$

Using (i) and (ii) of Theorem 15, we have

$$\begin{aligned} \varepsilon < P(x, y) < \varepsilon + \delta &\Rightarrow d(Tx, Sy) \leq \varepsilon, \\ \varepsilon < |x - y| < \varepsilon + \delta &\Rightarrow \frac{y}{2} \leq \varepsilon, \\ d(Tx, Sy) \leq P(x, y) &\Rightarrow \frac{y}{2} \leq |x - y|. \end{aligned} \quad (97)$$

Case 2. For $x, y \in [0, 1]$, we have

$$\begin{aligned} d(x, y) &= |x - y| = |1 - 0| = 1, \\ d(x, Tx) &= d(1, T(0)) = \left|1 - \left(\frac{-1}{2}\right)\right| = \frac{3}{2}, \\ d(y, Sy) &= |0 - S0| = \left|0 - \left(-\frac{1}{4}\right)\right| = \frac{1}{4}, \\ d(x, Sy) &= d(1, S0) = \left|1 - \left(-\frac{1}{4}\right)\right| = \frac{5}{4}, \\ d(y, Tx) &= d(0, T(1)) = \left|0 - \left(-\frac{1}{2}\right)\right| = \frac{1}{2}, \\ d(Tx, Sy) &= d(T(1), S0) = d\left(\frac{-1}{2}, \left(\frac{-1}{4}\right)\right) = \frac{1}{4}. \end{aligned} \quad (98)$$

Thus, we get

$$\begin{aligned} P(x, y) &= \left\{d(x, y), d(x, Tx), d(y, Sy),\right. \\ &\quad \left. a\left[\frac{d(x, Tx) + d(y, Sy)}{2}\right], b\left[\frac{d(x, Sy) + d(y, Tx)}{2}\right]\right\}, \\ P(1, 0) &= \max\left\{1, \frac{3}{2}, \frac{1}{4}, a\left[\frac{(3/2) + (1/4)}{2}\right], b\left[\frac{(5/4) + (1/2)}{2}\right]\right\}, \\ P(x, y) &= P(1, 0), = \max\left\{1, \frac{3}{2}, \frac{1}{4}, \frac{7}{8a}, \frac{7}{8b}\right\}, = \frac{3}{2}, 0 \leq a, b \leq 1. \end{aligned} \quad (99)$$

Finally, by (i) and (ii) of Theorem 15, we get

$$\begin{aligned} \varepsilon < P(x, y) < \varepsilon + \delta &\Rightarrow d(Tx, Sy) \leq \varepsilon, \\ \varepsilon < \frac{3}{2} < \varepsilon + \delta &\Rightarrow \frac{1}{4} \leq \varepsilon, \\ d(Tx, Sy) \leq P(x, y) &\Rightarrow \frac{1}{4} \leq \frac{3}{2}. \end{aligned} \quad (100)$$

Thus, we have $d(Tx, Sy) \leq P(x, y)$. Hence, the contraction condition of Theorem 15 is satisfied, and 0 is the common fixed point of T and S. Also, T is not continuous but is 2-continuous. Similarly, S is not continuous at 0 but is 2-continuous.

4. The Existence Solution of Nonlinear Volterra Integral Equation

The integral equation method is very useful for solving many problems in several applied fields like mathematical economics and optimal control theory because problems in these areas are often reduced to integral equations.

Integral equations appear in several forms. However, in this section, we are interested in the integral equation, namely, the Volterra integral differential equation which is of the form

$$u^n(t, x) = f(t, x) + \int_a^x K(x, t, u(t))dt, \tag{101}$$

where $u^n = d^n u/dx^n$.

Now, we present the application of Theorem 6 to study the existence and uniqueness of the solution to nonlinear Volterra integral equations.

The following integral equation is inspired by [14–17].

$$\begin{aligned} u(x, y) = f(x, y) &+ \int_0^x g(x, y, \varepsilon, u(\varepsilon, y))d\varepsilon \\ &+ \int_0^x \int_0^y h(x, y, \sigma, \tau, u(\sigma, \tau))d\varepsilon d\sigma, \end{aligned} \tag{102}$$

where f, g, h are given functions and u is the unknown function to be found.

Let $C(T, S)$ be the class of continuous functions from set T to set S. We denote $E = \mathbb{R}^+ \times \mathbb{R}^+, E_1 = \{f(x, y, s): 0 \leq s \leq x < \infty, y \in \mathbb{R}^+\}$ and $E_2 = \{f(x, y, s, t): 0 \leq s \leq x < \infty, 0 \leq t \leq y < \infty\}$. We denote that $f \in C(E, \mathbb{R}), g \in C(E_1 \times \mathbb{R}, \mathbb{R})$, and $h \in C(E_2 \times \mathbb{R}, \mathbb{R})$.

Let X be the space of functions $z \in C(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R})$ satisfying

$$|z(x, t)| = O\left(e^{\lambda(x+y)}\right), \tag{103}$$

where λ is a positive constant, that is,

$$|z(x, y)| \leq M_0\left(e^{\lambda(x+y)}\right), \tag{104}$$

for constant $M_0 > 0$. Let $(X, \|.\|)$ be a Banach space. Define a norm in the space X by

$$|z|_X = \sup_{(x,y)\in X}\left[|z(x, y)|e^{(-\lambda(x+y))}\right]. \tag{105}$$

Define the mapping $T, S : X \times X \longrightarrow [0,\infty)$ by

$$\begin{aligned} T^k u(x, y) = f(x, y) &+ \int_0^x g(x, y, \varepsilon, u(\varepsilon, y))d\varepsilon \\ &+ \int_0^x \int_0^y h(x, y, \sigma, \tau, u(\sigma, \tau))d\varepsilon d\sigma, \end{aligned} \tag{106}$$

$$\begin{aligned} S^k u(x, y) = f(x, y) &+ \int_0^x g(x, y, \varepsilon, v(\varepsilon, y))d\varepsilon \\ &+ \int_0^x \int_0^y h(x, y, \sigma, \tau, v(\sigma, \tau))d\varepsilon d\sigma, \end{aligned} \tag{107}$$

for $u, v \in X$. We assume that T^k and S^k are k-continuous for some positive integer k. For sufficiently large values of k, the mappings T^k and S^k are contraction, where T and S are noncontraction if $(x - a) > 1$, for $x > a$.

Now, we prove our results by establishing the existence of a common fixed point for a pair of self mappings:

Theorem 18. *Suppose that equation (102) satisfies the following conditions:*

(i) For the continuous functions $f, g \in X$, we have

$$\begin{aligned} |g(x, y, \varepsilon, u(\varepsilon, y)) - g(x, y, \varepsilon, v(\varepsilon, y))| &\leq L_1(x, y, \varepsilon)|u - v|, \\ |h(x, y, \sigma, \tau, u(\sigma, \tau)) - h(x, y, \sigma, \tau, v(\sigma, \tau))| &\leq L_2(x, y, \sigma, \tau)|u - v|, \end{aligned} \tag{108}$$

where $L_1 \in C(E_1, [0,\infty))$ and $L_2 \in C(E_2, [0,\infty))$

(ii) There exist a nonnegative constant γ such that $\gamma < 1$ and

$$\int_0^x L_1(x, y, \varepsilon)e^{\lambda(x+y)} + \int_0^x \int_0^y L_2(x, y, \sigma, \tau)e^{\lambda(\sigma+\tau)} \leq \delta, \tag{109}$$

for all $x, y, \varepsilon, \sigma, \tau \in E_1 \cup E_2$, and

$$\gamma = \frac{[\lambda L_1 + L_2]e^{\lambda(x+y)} - L_1\lambda e^{\lambda y} - 2L_2 e^{\lambda x} + L_2}{\lambda^2}. \tag{110}$$

Then, the nonlinear Volterra integral equation (102) has a unique common solution in $E_1 \cup E_2$.

Proof. Let $T^k, S^k : X \longrightarrow X$ be two operators such that $T^k \in X$ and $S^k \in X$. Now, we verify that T^k and S^k are contractive maps in X. Let $u, v \in X$. On the contrary, we claim that neither T^k nor S^k are contractive maps in X. From (106) and

(107), using condition (i) and (ii) of Theorem 18, we have

$$\begin{aligned}\left\|T^k u - S^k v\right\| &= f(x,y) + \int_0^x g(x,y,\varepsilon,u(\varepsilon,y))d\varepsilon \\ &\quad + \int_0^x\int_0^y h(x,y,\sigma,\tau,u(\sigma,\tau))d\varepsilon d\sigma \\ &\quad - f(x,y) - \int_0^x g(x,y,\varepsilon,v(\varepsilon,y))d\varepsilon \\ &\quad - \int_0^x\int_0^y h(x,y,\sigma,\tau,v(\sigma,\tau))d\varepsilon d\sigma, \\ &\le \int_0^t |g(x,y,\varepsilon,u(\varepsilon,y)) - g(x,y,\varepsilon,v(\varepsilon,y))|d\varepsilon \\ &\quad + \int_0^x\int_0^y |h(x,y,\sigma,\tau,u(\sigma,\tau)) - h(x,y,\sigma,\tau,v(\sigma,\tau))|, \\ &\le \left[\int_0^x L_1(x,y,\varepsilon)e^{\lambda(x+y)} + \int_0^x\int_0^y L_2(x,y,\sigma,\tau)e^{\lambda(\sigma+\tau)}\right]\|u-v\|_X, \\ &\le \left[\frac{1}{\lambda}L_1\left[e^{\lambda(x+y)} - e^{\lambda y}\right] + \frac{1}{\lambda^2}L_2\left[e^{\lambda(x+y)} - 2e^{\lambda x} + 1\right]\right]\|u-v\|_X, \\ &\le \left[\frac{[\lambda L_1 + L_2]e^{\lambda(x+y)} - L_1\lambda e^{\lambda y} - 2L_2 e^{\lambda x} + L_2}{\lambda^2}\right]\|u-v\|_X, \\ &\Rightarrow \left\|T^k u - S^k v\right\| \le \gamma\|u-v\|_X \\ &\Rightarrow d(Tu,Sv) \le \gamma M(u,v),\end{aligned} \tag{111}$$

which is a contradiction. Hence, u is a common fix of T and S and also a solution to integral equation (102).

From (111), let $\gamma = 1$ and using (i) of Theorem 6, where

$$M(u,v) = \max\left\{d(u,v), d(u,Tv), d(v,Sv), \frac{d(u,Sv) + d(v,Tu)}{2}\right\}, \tag{112}$$

we have

$$d(Tu,Sv) < M(u,v). \tag{113}$$

Thus, Theorem 6 is satisfied. □

Data Availability

No data are required for this research article.

Additional Points

Code Availability. There is no coding used for this research article.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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References

[1] L. J. Cromme and I. Diener, "Fixed point theorems for discontinuous mappings," *Mathematical Programming*, vol. 51, pp. 257–267, 1991.

[2] L. J. Cromme, "Fixed point theorems for discontinuous functions and applications," *Nonlinear Analysis*, vol. 30, no. 3, pp. 1527–1534, 1997.

[3] R. Kannan, "Some results on fixed points," *Bulletin of the Calcutta Mathematical Society*, vol. 60, pp. 71–76, 1968.

[4] R. P. Pant, "Discontinuity and fixed points," *Journal of Mathematical Analysis and Applications*, vol. 240, pp. 284–289, 1999.

[5] B. E. Rhoades, "Contractive definitions and continuity," *Contemporary Mathematics*, vol. 72, pp. 233–245, 1988.

[6] R. Kannan, "Some results on fixed points-II," *The American Mathematical Monthly*, vol. 76, pp. 405–408, 1969.

[7] L. B. Ciric, "On contraction type mappings," *Math Balkanica*, vol. 1, pp. 52–57, 1971.

[8] L. B. Ciric, "A generalization of Banachs contraction principle," *Proceedings of the American Mathematical society*, vol. 45, no. 2, pp. 267–274, 1974.

[9] A. Pant and R. P. Pant, "Fixed points and continuity of contractive maps," *Univerzitet u Nišu*, vol. 31, no. 11, pp. 3501–3506, 2017.

[10] E. Kreyszig, *Introductory Functional Analysis with Applications*, John Wiley & Sons, USA, 1978.

[11] R. K. Bisht and R. P. Pant, "A remark on discontinuity at fixed points," *Journal of Mathematical Analysis and Applications*, vol. 445, pp. 1239–1242, 2017.

[12] S. Banach, "Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales," *Fundamenta Mathematicae*, vol. 3, pp. 133–181, 1922.

[13] D. W. Boyd and J. S. Wong, "On nonlinear contractions," *Proceedings of the American Mathematical Society*, vol. 20, no. 2, pp. 458–464, 1969.

[14] M. Abbas, A. Latif, and Y. I. Suleiman, "Fixed points for cyclic R-contractions and solution of nonlinear Volterra integro-differential equations," *Fixed Point Theory and Applications*, vol. 2016, 2016.

[15] C. Corduneanu, *Integral Equations and Applications*, vol. 148, Cambridge University Press, 1991.

[16] H. K. Nashine, R. P. Pathak, P. S. Somvanshi, S. Pantelic, and P. Kumam, "Solutions for a class of nonlinear Volterra integral and integro-differential equation using cyclic (ψ,ϕ, θ)-contraction," *Advances in Difference Equations*, vol. 2013, 2013.

[17] B. G. Pachpatte, "On a nonlinear Volterra integral equation in two variables," *Sarajevo Journal of Mathematics*, vol. 6, no. 18, pp. 59–73, 2010.