

Research Article

Asymptotic Behavior of Solutions to Free Boundary Problem with Tresca Boundary Conditions

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In this paper, we study the asymptotic behavior of an incompressible Herschel-Bulkley fluid in a thin domain with Tresca boundary conditions. We study the limit when the ε tends to zero, we prove the convergence of the unknowns which are the velocity and the pressure of the fluid, and we obtain the limit problem and the specific Reynolds equation.

1. Introduction

In 1926, the model of Herschel-Bulkley fluid introduced is called a non-Newtonian fluid, whose flow properties differ in any way from those of any Newtonian fluids. There are many phenomena in nature and industry exhibiting the behavior of the Herschel-Bulkley fluid medium and has been used in various publications to describe the flow of metals, plastic solids, and some polymers. The literature concerning this topic is extensive (see e.g., [1–14]). Further, let us mention the works which is realized by many authors in this area, for example, (see [2, 4, 9, 10, 13–21]).

This paper is to discuss the asymptotic behavior of steady flow of Herschel-Bulkley fluid in a three-dimensional thin layer with Tresca boundary conditions.

The paper is organized as follows. In Section 2, we introduce some notations, preliminaries, and the mechanical problem of the steady flow of Herschel-Bulkley fluid in a three-dimensional thin layer. In Section 3, we investigate some estimates and convergence theorem. To this aim, we use the change of variable x_3/ε , to transform the initial prob-

lem posed in the domain Ω^ε into a new problem posed on a fixed domain Ω independent of the parameter ε . Finally, the a priori estimate allows us to pass to the limit when ε tends to zero, and we prove the convergence results and limit problem with a specific weak form of the Reynolds equation and two-dimensional constitutive equation of the model flow.

2. Problem Statement and Variational Formulation

Let ω be fixed region in plan $x = (x_1, x_2) \in \mathbb{R}^2$. We assume that ω has a Lipschitz boundary and is the bottom of the fluid domain. The upper surface Γ_1^ε is defined by $x_3 = \varepsilon h(x)$ where ($0 < \varepsilon < 1$) is a small parameter that will tend to zero and h a smooth bounded function such that $0 < h_* < h(x) < h^*$ for all $(x, 0) \in \omega$ and Γ_L^ε the lateral surface. We denote by Ω^ε the domain of the flow:

$$\Omega^\varepsilon = \{(x, x_3) \in \mathbb{R}^3 : (x, 0) \in \omega, 0 < x_3 < \varepsilon h(x)\}. \quad (1)$$

The boundary of Ω^ε is Γ^ε . We have $\Gamma^\varepsilon = \overline{\Gamma}_1^\varepsilon \cup \overline{\Gamma}_L^\varepsilon \cup \overline{\omega}$ where $\overline{\Gamma}_L^\varepsilon$ is the lateral boundary.

(i) The law of conservation of momentum is defined by

$$u^\varepsilon \nabla u^\varepsilon = \operatorname{div}(\sigma^\varepsilon) + f^\varepsilon \text{ in } \Omega^\varepsilon, \quad (2)$$

where $\operatorname{div}(\sigma^\varepsilon) = (\sigma_{ij}^\varepsilon)$ and $f^\varepsilon = (f_i^\varepsilon)_{1 \leq i \leq 3}$ denote the body forces.

(ii) The stress tensor σ^ε is decomposed as follows

$$\begin{cases} \sigma_{ij}^\varepsilon = \tilde{\sigma}_{ij}^\varepsilon - p^\varepsilon \delta_{ij}, \\ \tilde{\sigma}^\varepsilon = \alpha^\varepsilon \frac{D(u^\varepsilon)}{|D(u^\varepsilon)|} + \mu |D(u^\varepsilon)|^{r-2} D(u^\varepsilon) \text{ if } D(u^\varepsilon) \neq 0, \\ |\tilde{\sigma}^\varepsilon| \leq \alpha^\varepsilon \text{ if } D(u^\varepsilon) = 0. \end{cases} \quad (3)$$

where $\alpha^\varepsilon \geq 0$ is the yield stress, $\mu > 0$ is the constant viscosity, u^ε is the velocity field, p^ε is the pressure, δ_{ij} is the Kronecker symbol, $1 < r \leq 2$ and $D(u^\varepsilon) = 1/2(\nabla u^\varepsilon + (\nabla u^\varepsilon)^T)$. For any tensor $D = (d_{ij})$, the notation $|D|$ represents the matrix norm: $|D| =$

$$\left(\sum_{i,j} d_{ij} d_{ij} \right)^{1/2}.$$

(iii) The incompressibility equation

$$\operatorname{div}(u^\varepsilon) = 0 \text{ in } \Omega^\varepsilon. \quad (4)$$

Our boundary conditions is described as

(iv) At the surface $\Gamma_1^\varepsilon \cup \Gamma_L^\varepsilon$, we assume that

$$u^\varepsilon = 0. \quad (5)$$

(v) On ω , there is a no-flux condition across ω so that

$$u^\varepsilon \times n = 0. \quad (6)$$

(vi) The tangential velocity on ω is unknown and satisfies Tresca boundary conditions:

$$\begin{cases} |\sigma_\tau^\varepsilon| < k^\varepsilon u_\tau^\varepsilon = 0 \\ |\sigma_\tau^\varepsilon| = k^\varepsilon \exists \lambda \geq 0, u_\tau^\varepsilon = -\lambda \sigma_\tau^\varepsilon \end{cases} \text{ in } \omega. \quad (7)$$

Here, k^ε is the friction yield coefficient and $|\cdot|$ is the Euclidean norm in \mathbb{R}^2 ; $n = (n_1, n_2, n_3)$ is the unit outward normal to Γ_1^ε , and

$$\begin{aligned} u_n^\varepsilon &= u^\varepsilon \cdot n = u_i^\varepsilon n_i, \\ u_{\tau_i}^\varepsilon &= u_i^\varepsilon - u_n^\varepsilon n_i, \\ \sigma_n^\varepsilon &= (\sigma \cdot n) n = \sigma_{ij}^\varepsilon n_i n_j, \\ \sigma_{\tau_i}^\varepsilon &= \sigma_{ij}^\varepsilon n_j - \sigma_n^\varepsilon n_i. \end{aligned} \quad (8)$$

In order to, we observe that

$$\begin{aligned} K^\varepsilon &= \left\{ \varphi \in W^{1,r}(\Omega^\varepsilon)^3 : \varphi = 0 \text{ on } \Gamma_1^\varepsilon \cup \Gamma_L^\varepsilon, \varphi \cdot n = 0 \text{ on } \omega \right\}, \\ K_{\operatorname{div}}^\varepsilon &= \left\{ \varphi \in K^\varepsilon : \operatorname{div}(\varphi) = 0 \right\}, \\ L_0^{r'}(\Omega^\varepsilon) &= \left\{ q \in L^{r'}(\Omega^\varepsilon) : \int_{\Omega^\varepsilon} q dx dx_3 = 0 \right\}. \end{aligned} \quad (9)$$

A formal application of Green's formula, using (1)–(6), leads to the following weak formulation:

Find a velocity field $u^\varepsilon \in K_{\operatorname{div}}^\varepsilon$ and $p^\varepsilon \in L_0^{r'}(\Omega^\varepsilon)$, ($1/r + 1/r' = 1$) such that:

$$\begin{aligned} a(u^\varepsilon, \varphi - u^\varepsilon) + B(u^\varepsilon, u^\varepsilon, \varphi - u^\varepsilon) - (p^\varepsilon, \operatorname{div} \varphi) + j(\varphi) \\ - j(u^\varepsilon) \geq (f^\varepsilon, \varphi - u^\varepsilon), \end{aligned} \quad (10)$$

for all $\varphi \in K^\varepsilon$, where

$$\begin{aligned} a(u^\varepsilon, \varphi - u^\varepsilon) &= \mu \int_{\Omega^\varepsilon} |D(u^\varepsilon)|^{r-2} D(u^\varepsilon) D(\varphi) dx dx_3, \\ B(u^\varepsilon, u^\varepsilon, \varphi) &= \sum_{i=1}^3 \int_{\Omega^\varepsilon} u_i^\varepsilon \frac{\partial u^\varepsilon}{\partial x_i} \varphi dx dx_3, \\ (p^\varepsilon, \operatorname{div} \varphi) &= \int_{\Omega^\varepsilon} p^\varepsilon \operatorname{div} \varphi dx dx_3, \\ j(\varphi) &= \int_{\omega} k^\varepsilon |\varphi| dx + \alpha^\varepsilon \int_{\Omega^\varepsilon} |D(\varphi)| dx dx_3, \\ (f^\varepsilon, \varphi) &= \int_{\Omega^\varepsilon} f^\varepsilon \varphi dx dx_3 = \sum_{i=1}^3 \int_{\Omega^\varepsilon} f_i^\varepsilon \varphi_i dx dx_3. \end{aligned} \quad (11)$$

As in [6, 8], we can show that this variational problem has a unique solution.

Now, we state some the following results (see, [15]).

$$\|\nabla u^\varepsilon\|_{L^r(\Omega^\varepsilon)} \leq C \|D(u^\varepsilon)\|_{L^r(\Omega^\varepsilon)} \quad (\text{Korn inequality}), \quad (12)$$

$$\|u^\varepsilon\|_{L^r(\Omega^\varepsilon)} \leq \varepsilon h^* \left\| \frac{\partial u^\varepsilon}{\partial z} \right\|_{L^r(\Omega^\varepsilon)} \quad \text{for } i = 1, 2 \quad (\text{Poincaré' inequality}), \quad (13)$$

$$ab \leq \frac{a^r}{r} + \frac{b^{r'}}{r'}, \forall (a, b) \in \mathbb{R}^2 \quad (\text{Young inequality}). \quad (14)$$

3. Change of the Domain and Study of Convergence

Here, we apply the technique of scaling in Ω^ε on the coordinate x_3 . With the variables $z = x_3/\varepsilon$, we get

$$\Omega = \{(x, z) \in \mathbb{R}^3 : (x, 0) \in \omega, 0 < z < h(x)\}. \quad (15)$$

Next, we denote by $\Gamma = \bar{\Gamma}_1 \cup \bar{\Gamma}_L \cup \bar{\omega}$ its boundary, then, we define the following functions in Ω :

$$\begin{aligned} \hat{u}_i^\varepsilon(x, z) &= u_i^\varepsilon(x, x_3), \quad i = 1, 2, \\ \hat{u}_3^\varepsilon(x, z) &= \varepsilon^{-1} u_3^\varepsilon(x, x_3), \\ \hat{p}^\varepsilon(x, z) &= \varepsilon^r p^\varepsilon(x, x_3). \end{aligned} \quad (16)$$

Now, we assume that

$$\hat{f}(x, z) = \varepsilon^r f^\varepsilon(x, x_3), \quad \hat{\alpha} = \varepsilon^{r-1} \alpha^\varepsilon, \quad \hat{k} = \varepsilon^{r-1} k^\varepsilon, \quad (17)$$

and we consider the sets

$$\begin{aligned} K(\Omega) &= \left\{ \hat{\varphi} \in (W^{1,r}(\Omega))^3 : \hat{\varphi} = 0 \text{ on } \Gamma_1 \cup \Gamma_L; \hat{\varphi} \cdot n = 0 \text{ on } \omega \right\}, \\ K_{\text{div}}(\Omega) &= \{ \hat{\varphi} \in K(\Omega) : \text{div } \hat{\varphi} = 0 \}, \\ V_z &= \left\{ \hat{\varphi} \in (L^r(\Omega))^2; \frac{\partial \hat{\varphi}_i}{\partial z} \in L^r(\Omega) : \hat{\varphi} = 0 \text{ on } \Gamma_1 \cup \Gamma_L \right\}, \\ \tilde{V}_z &= \left\{ \hat{\varphi} \in V_z : \hat{\varphi} \text{ satisfy } (D') \right\}, \end{aligned} \quad (18)$$

where the condition (D') is given by

$$\begin{aligned} (D') \int_{\omega} \left(\hat{\varphi}_1 \frac{\partial \theta}{\partial x_1} + \hat{\varphi}_2 \frac{\partial \theta}{\partial x_2} \right) dx dz &= 0, \\ \text{for all } \hat{\varphi} \in (L^r(\Omega))^2 \text{ and } \theta \in C_0^\infty(\Omega). \end{aligned} \quad (19)$$

By injecting the new data, unknown factors in (10) and after multiplication by ε^{r-1} , we deduce that

$$\begin{aligned} a_0(\hat{u}^\varepsilon, \hat{\varphi} - \hat{u}^\varepsilon) + B_0(\hat{u}^\varepsilon, \hat{u}^\varepsilon, \hat{\varphi}) - (\hat{p}^\varepsilon, \text{div } (\hat{\varphi} - \hat{u}^\varepsilon)) \\ + j_0(\hat{\varphi}) - j_0(\hat{u}^\varepsilon) \geq (\hat{f}, \hat{\varphi} - \hat{u}^\varepsilon), \forall \hat{\varphi} \in K(\Omega), \end{aligned} \quad (20)$$

where

$$\begin{aligned} a_0(\hat{u}^\varepsilon, \hat{\varphi} - \hat{u}^\varepsilon) &= \sum_{i,j=1}^2 \int_{\Omega} \left[\varepsilon^2 \mu |\tilde{D}(\hat{u}^\varepsilon)|^{r-2} \left(\frac{1}{2} \left(\frac{\partial \hat{u}_i^\varepsilon}{\partial x_j} + \frac{\partial \hat{u}_j^\varepsilon}{\partial x_i} \right) \right) \right] \\ &\quad \cdot \frac{\partial(\hat{\varphi}_i - \hat{u}_i^\varepsilon)}{\partial x_j} dx dz + \sum_{i=1}^2 \int_{\Omega} \mu |\tilde{D}(\hat{u}^\varepsilon)|^{r-2} \end{aligned}$$

$$\begin{aligned} &\left(\frac{1}{2} \left(\frac{\partial \hat{u}_i^\varepsilon}{\partial z} + \varepsilon^2 \frac{\partial \hat{u}_3^\varepsilon}{\partial x_i} \right) \right) \frac{\partial(\hat{\varphi}_i - \hat{u}_i^\varepsilon)}{\partial z} dx dz \\ &+ \int_{\Omega} \left(\mu |\tilde{D}(\hat{u}^\varepsilon)|^{r-2} \varepsilon^2 \frac{\partial \hat{u}_3^\varepsilon}{\partial z} \right) \frac{\partial(\hat{\varphi}_3 - \hat{u}_3^\varepsilon)}{\partial z} dx dz \\ &+ \sum_{j=1}^2 \int_{\Omega} \varepsilon^2 \mu |\tilde{D}(\hat{u}^\varepsilon)|^{r-2} \left(\frac{1}{2} \left(\varepsilon^2 \frac{\partial \hat{u}_3^\varepsilon}{\partial x_j} + \frac{\partial \hat{u}_j^\varepsilon}{\partial z} \right) \right) \\ &\quad \cdot \frac{\partial(\hat{\varphi}_3 - \hat{u}_3^\varepsilon)}{\partial x_j} dx dz, \end{aligned}$$

$$(\hat{p}^\varepsilon, \text{div } (\hat{\varphi} - \hat{u}^\varepsilon)) = \int_{\Omega^\varepsilon} \hat{p}^\varepsilon \text{div } (\hat{\varphi} - \hat{u}^\varepsilon) dx dz,$$

$$\begin{aligned} B_0(\hat{u}^\varepsilon, \hat{u}^\varepsilon, \hat{\varphi}) &= \sum_{i,j=1}^2 \int_{\Omega} \varepsilon^2 \hat{u}_i^\varepsilon \frac{\partial \hat{u}_j^\varepsilon}{\partial x_i} \hat{\varphi} dx dz + \sum_{i=1}^2 \int_{\Omega} \varepsilon^4 \hat{u}_i^\varepsilon \frac{\partial \hat{u}_3^\varepsilon}{\partial x_i} \hat{\varphi} dx dz \\ &+ \sum_{j=1}^2 \int_{\Omega} \varepsilon^2 \hat{u}_3^\varepsilon \frac{\partial \hat{u}_j^\varepsilon}{\partial z} \hat{\varphi}_j dx dz + \int_{\Omega} \varepsilon^4 \hat{u}_3^\varepsilon \frac{\partial \hat{u}_3^\varepsilon}{\partial z} \hat{\varphi}_3 dx dz, \end{aligned}$$

$$j_0(\hat{\varphi}) = \hat{\alpha} \int_{\Omega} |\tilde{D}(\hat{\varphi})| dx dz + \int_{\omega} \hat{k} |\hat{\varphi}| dx,$$

$$(\hat{f}, \hat{\varphi} - \hat{u}^\varepsilon) = \sum_{j=1}^2 \int_{\Omega} \hat{f}_j (\hat{\varphi}_j - \hat{u}_j^\varepsilon) dx dz + \int_{\Omega} \varepsilon \hat{f}_3 (\hat{\varphi}_3 - \hat{u}_3^\varepsilon) dx dz,$$

$$\begin{aligned} |\tilde{D}(\hat{u}^\varepsilon)| &= \left(\frac{1}{4} \sum_{i,j=1}^2 \varepsilon^2 \left(\frac{\partial \hat{u}_i^\varepsilon}{\partial x_j} + \frac{\partial \hat{u}_j^\varepsilon}{\partial x_i} \right)^2 \right. \\ &\quad \left. + \frac{1}{2} \sum_{i=1}^2 \left(\frac{\partial \hat{u}_i^\varepsilon}{\partial z} + \varepsilon^2 \frac{\partial \hat{u}_3^\varepsilon}{\partial x_i} \right)^2 + \varepsilon^2 \left(\frac{\partial \hat{u}_3^\varepsilon}{\partial z} \right)^2 \right)^{1/2}. \end{aligned} \quad (21)$$

We now establish the estimates for the velocity field \hat{u}^ε and the pressure \hat{p}^ε in Ω .

Theorem 1. *Let $(\hat{u}^\varepsilon, \hat{p}^\varepsilon) \in K_{\text{div}}(\Omega) \times L_0^r(\Omega)$ be the solution of variational problem (20), then there exists a constant $C > 0$ independent of ε such that:*

$$\begin{aligned} \sum_{i,j=1}^2 \left\| \varepsilon \frac{\partial \hat{u}_i^\varepsilon}{\partial x_j} \right\|_{L^r(\Omega)}^r + \left\| \varepsilon \frac{\partial \hat{u}_3^\varepsilon}{\partial z} \right\|_{L^r(\Omega)}^r \\ + \sum_{i=1}^2 \left(\left\| \frac{\partial \hat{u}_i^\varepsilon}{\partial z} \right\|_{L^r(\Omega)}^r + \left\| \varepsilon^2 \frac{\partial \hat{u}_3^\varepsilon}{\partial x_i} \right\|_{L^r(\Omega)}^r \right) \leq C. \end{aligned} \quad (22)$$

Proof. Choosing $\varphi = 0$ as test function in inequality (10), we get

$$\begin{aligned} a(u^\varepsilon, u^\varepsilon) + B(u^\varepsilon, u^\varepsilon, u^\varepsilon) + \alpha^\varepsilon \int_{\Omega^\varepsilon} |D(u^\varepsilon)| dx dx_3 \\ + \int_{\omega} k^\varepsilon |u^\varepsilon| dx \leq (f^\varepsilon, u^\varepsilon), \end{aligned} \quad (23)$$

and because $B(u^\varepsilon, u^\varepsilon, u^\varepsilon) = 0$, we obtain

$$a(u^\varepsilon, u^\varepsilon) + \alpha^\varepsilon \int_{\Omega^\varepsilon} |D(u^\varepsilon)| dx dx_3 + \int_{\omega} k^\varepsilon |u^\varepsilon| dx \leq (f^\varepsilon, u^\varepsilon). \quad (24)$$

Using now (13) and (14) will yield after some algebra

$$(f^\varepsilon, u^\varepsilon) \leq \varepsilon h^* \|\nabla u^\varepsilon\|_{L^r(\Omega^\varepsilon)} \|f^\varepsilon\|_{L^{r'}(\Omega^\varepsilon)} \leq \frac{1}{2} \mu C_k \|\nabla u^\varepsilon\|_{L^r(\Omega^\varepsilon)}^r + \frac{(\varepsilon h^*)^{r'}}{r'(1/2\mu r C_k)^{r'/r}} \|f^\varepsilon\|_{L^{r'}(\Omega^\varepsilon)}^{r'}. \quad (25)$$

From (24) and (25), we deduce that

$$a(u^\varepsilon, u^\varepsilon) + \alpha^\varepsilon \int_{\Omega^\varepsilon} |D(u^\varepsilon)| dx dx_3 + \int_{\omega} k^\varepsilon |u^\varepsilon - s| dx \leq \frac{1}{2} \mu C_k \|\nabla u^\varepsilon\|_{L^r(\Omega^\varepsilon)}^r + \frac{(\varepsilon h^*)^{r'}}{r'(1/2\mu r C_k)^{r'/r}} \|f^\varepsilon\|_{L^{r'}(\Omega^\varepsilon)}^{r'}. \quad (26)$$

We multiply (26) by ε^{r-1} , we get

$$\varepsilon^{r-1} a(u^\varepsilon, u^\varepsilon) + \widehat{\alpha} \int_{\Omega^\varepsilon} |\tilde{D}(\widehat{u}^\varepsilon)| dx dz + \int_{\omega} \widehat{k} |\widehat{u}^\varepsilon| dx \leq \frac{1}{2} \mu C_k \varepsilon^{r-1} \|\nabla u^\varepsilon\|_{L^r(\Omega^\varepsilon)}^r + \varepsilon^{r-1} \frac{(\varepsilon h^*)^{r'}}{r'(1/2\mu r C_k)^{r'/r}} \|f^\varepsilon\|_{L^{r'}(\Omega^\varepsilon)}^{r'}. \quad (27)$$

Now, since $\varepsilon^{r'} \|f^\varepsilon\|_{L^{r'}(\Omega^\varepsilon)}^{r'} = \varepsilon^{1-r'} \|\widehat{f}\|_{L^{r'}(\Omega)}^{r'}$, it follows that

$$\varepsilon^{r-1} a(u^\varepsilon, u^\varepsilon) + \widehat{\alpha} \int_{\Omega} |\tilde{D}(\widehat{u}^\varepsilon)| dx dz + \int_{\omega} \widehat{k} |\widehat{u}^\varepsilon| dx \leq \frac{1}{2} \mu C_k \varepsilon^{r-1} \|\nabla u^\varepsilon\|_{L^r(\Omega^\varepsilon)}^r + \frac{(h^*)^{r'}}{r'(1/2\mu r C_k)^{r'/r}} \|\widehat{f}\|_{L^{r'}(\Omega)}^{r'}. \quad (28)$$

According to Korn's inequality and (28), such that C_K independent of ε , we have

$$\frac{1}{2} \mu C_K \varepsilon^{r-1} \|\nabla u^\varepsilon\|_{L^r(\Omega^\varepsilon)}^r + \widehat{\alpha} \int_{\Omega} |\tilde{D}(\widehat{u}^\varepsilon)| dx dz + \int_{\omega} \widehat{k} |\widehat{u}^\varepsilon| dx \leq \frac{(h^*)^{r'}}{r'(1/2\mu r C_k)^{r'/r}} \|\widehat{f}\|_{L^{r'}(\Omega)}^{r'}. \quad (29)$$

Using (29), we deduce (22), with $C = (1/2\mu C_K)^{-1} (h^*)^{r'}/r'(1/2\mu r C_k)^{r'/r} \|\widehat{f}\|_{L^{r'}(\Omega)}^{r'}$, and

$$\begin{aligned} \varepsilon^{r-1} \|\nabla u^\varepsilon\|_{L^r(\Omega^\varepsilon)}^r &= \|\nabla \widehat{u}^\varepsilon\|_{L^r(\Omega)}^r = \sum_{i,j=1}^2 \left\| \varepsilon \frac{\partial \widehat{u}_i^\varepsilon}{\partial x_j} \right\|_{L^r(\Omega)}^r + \left\| \varepsilon \frac{\partial \widehat{u}_3^\varepsilon}{\partial z} \right\|_{L^r(\Omega)}^r \\ &+ \sum_{i=1}^2 \left(\left\| \frac{\partial \widehat{u}_i^\varepsilon}{\partial z} \right\|_{L^r(\Omega)}^r + \left\| \varepsilon^2 \frac{\partial \widehat{u}_3^\varepsilon}{\partial x_i} \right\|_{L^r(\Omega)}^r \right). \end{aligned} \quad (30)$$

Theorem 2. Under the conditions in Theorem (1), there exists a constant $C' > 0$ independent of ε such that

$$\left\| \frac{\partial \widehat{p}^\varepsilon}{\partial x_i} \right\|_{W^{-1,r'}(\Omega)} \leq C' \quad \text{for } i = 1, 2, \quad (31)$$

$$\left\| \frac{\partial \widehat{p}^\varepsilon}{\partial z} \right\|_{W^{-1,r'}(\Omega)} \leq \varepsilon C'. \quad (32)$$

Proof. To get the first estimate on the pressure in (31)–(32), we choose in (20), $\widehat{\varphi} = \widehat{u}^\varepsilon + \psi$, $\psi \in W_0^{1,r}(\Omega)^3$, to obtain

$$\begin{aligned} a_0(\widehat{u}^\varepsilon, \psi) + B_0(\widehat{u}^\varepsilon, \widehat{u}^\varepsilon, \psi) - (\widehat{p}^\varepsilon, \operatorname{div} \psi) \\ + \widehat{\alpha} \int_{\Omega} |\tilde{D}(\widehat{u}^\varepsilon + \psi)| dx dz - \widehat{\alpha} \int_{\Omega} |\tilde{D}(\widehat{u}^\varepsilon)| dx dz \\ \geq (\widehat{f}^\varepsilon, \psi), \\ (\widehat{p}^\varepsilon, \operatorname{div} \psi) \leq a_0(\widehat{u}^\varepsilon, \psi) + B_0(\widehat{u}^\varepsilon, \widehat{u}^\varepsilon, \psi) \\ + \widehat{\alpha} \int_{\Omega} |\tilde{D}(\widehat{u}^\varepsilon + \psi)| dx dz - \widehat{\alpha} \int_{\Omega} |\tilde{D}(\widehat{u}^\varepsilon)| dx dz \\ - (\widehat{f}^\varepsilon, \psi). \end{aligned} \quad (33)$$

Keeping in mind that $|\tilde{D}(\widehat{u}^\varepsilon + \psi)| \leq \sqrt{2} (|\tilde{D}(\widehat{u}^\varepsilon)| + |\tilde{D}(\psi)|)$, it follows that

$$\begin{aligned} (\widehat{p}^\varepsilon, \operatorname{div} \psi) \leq a_0(\widehat{u}^\varepsilon, \psi) + B_0(\widehat{u}^\varepsilon, \widehat{u}^\varepsilon, \psi) \\ + \widehat{\alpha} \sqrt{2} \int_{\Omega} |\tilde{D}(\psi)| dx dz + (\sqrt{2} - 1) \widehat{\alpha} \int_{\Omega} \\ \cdot |\tilde{D}(\widehat{u}^\varepsilon)| dx dz - \int_{\Omega} \widehat{f} \psi dx dz. \end{aligned} \quad (34)$$

Using Hölder formula, we get

$$\begin{aligned} (\widehat{p}^\varepsilon, \operatorname{div} \psi) \leq \mu \|D(\widehat{u}^\varepsilon)\|_{L^r(\Omega)}^{r/r'} \|\psi\|_{W^{1,r}(\Omega)^3} + \|\widehat{u}^\varepsilon\|_{W^{1,r}(\Omega)}^2 \\ \cdot \|\psi\|_{W^{1,r}(\Omega)^3} + \widehat{\alpha} \sqrt{2} |\Omega|^{1/r'} \|\psi\|_{W^{1,r}(\Omega)^3} \\ + (\sqrt{2} - 1) \widehat{\alpha} \|\tilde{D}(\widehat{u}^\varepsilon)\|_{L^r(\Omega)^3} \\ + \|\widehat{f}\|_{L^{r'}(\Omega)^3} \|\psi\|_{W^{1,r}(\Omega)^3}. \end{aligned} \quad (35)$$

By similar arguments, we choose in (20) $\widehat{\varphi} = \widehat{u}^\varepsilon - \psi$ and $\psi \in W_0^{1,r}(\Omega)^3$ to obtain

$$\begin{aligned}
-(\widehat{p}^\varepsilon, \operatorname{div} \psi) &\leq \mu \|D(\widehat{u}^\varepsilon)\|_{L^r(\Omega)}^{r/r'} \|\psi\|_{W^{1,r}(\Omega)^3} \\
&\quad + C \|\widehat{u}^\varepsilon\|_{W^{1,r}(\Omega)^3}^2 \|\psi\|_{W^{1,r}(\Omega)^3} + \widehat{\alpha} \sqrt{2} |\Omega|^{1/r'} \\
&\quad \cdot \|\psi\|_{W^{1,r}(\Omega)^3} + (\sqrt{2} - 1) \widehat{\alpha} \|\widehat{D}(\widehat{u}^\varepsilon)\|_{L^r(\Omega)^3} \\
&\quad + \|\widehat{f}\|_{L^r(\Omega)^3} \|\psi\|_{W^{1,r}(\Omega)^3}.
\end{aligned} \tag{36}$$

We combine now (35) and (36) to see that

$$\begin{aligned}
|(\widehat{p}^\varepsilon, \operatorname{div} \psi)| &\leq \mu \|D(\widehat{u}^\varepsilon)\|_{L^r(\Omega)}^{r/r'} \|\psi\|_{W^{1,r}(\Omega)^3} + \|\widehat{u}^\varepsilon\|_{W^{1,r}(\Omega)^3}^2 \\
&\quad \cdot \|\psi\|_{W^{1,r}(\Omega)^3} + \widehat{\alpha} \sqrt{2} |\Omega|^{1/r'} \|\psi\|_{W^{1,r}(\Omega)^3} \\
&\quad + (\sqrt{2} - 1) \widehat{\alpha} \|\widehat{D}(\widehat{u}^\varepsilon)\|_{L^r(\Omega)^3} \\
&\quad + \|\widehat{f}\|_{L^r(\Omega)^3} \|\psi\|_{W^{1,r}(\Omega)^3}.
\end{aligned} \tag{37}$$

Next, for $i = 1, 2$, we choose $\psi = (\psi_1, 0, 0)$ then $\psi = (0, \psi_2, 0)$ in the inequality (37) and using (22), we find

$$\begin{aligned}
\left| \int_{\Omega} \frac{\partial \widehat{p}^\varepsilon}{\partial x_i} \psi dx dz \right| &\leq \left(C_1 + \widehat{\alpha} \sqrt{2} |\Omega|^{1/r'} + \|\widehat{f}_i\|_{L^r(\Omega)^3} \right) \\
&\quad \cdot \|\psi\|_{W^{1,r}(\Omega)^3} + (\sqrt{2} - 1) \widehat{\alpha} C,
\end{aligned} \tag{38}$$

where $|\Omega| = \operatorname{mes}(\Omega)$. Then, (31) holds for $i = 1, 2$.

To get (32), we take $\psi = (0, 0, \psi_3)$ in the inequality (37) to see that

$$\begin{aligned}
\frac{1}{\varepsilon} \left| \int_{\Omega} \frac{\partial \widehat{p}^\varepsilon}{\partial z} \psi dx dz \right| &\leq \left(C_1 + \widehat{\alpha} \sqrt{2} |\Omega|^{1/r'} + \|\widehat{f}_3\|_{L^r(\Omega)^3} \right) \\
&\quad \cdot \|\psi\|_{W^{1,r}(\Omega)^3} + (\sqrt{2} - 1) \widehat{\alpha} C.
\end{aligned} \tag{39}$$

The question which naturally arises is to know what will be the asymptotic behavior of the fluid when the thickness of the thin film is very small. Mathematically, it is about knowing: do the speed field and the pressure admit a limit when ε tends towards zero and what is the limit problem who should check this limit?

The answer to the first question is given in Theorem (3). However, the answer to the second question will be dealt with in Theorems (4), (7), and (8).

Theorem 3. *Under the same assumptions as in Theorem (1) and Theorem (2), there exist $u^* = (u_1^*, u_2^*) \in \widetilde{V}_z$ and $p^* \in L_0^r(\Omega)$ such that:*

$$\widehat{u}_i^\varepsilon \rightharpoonup u_i^*, \quad i = 1, 2 \quad \text{weakly in } \widetilde{V}_z, \tag{40}$$

$$\varepsilon \frac{\partial \widehat{u}_i^\varepsilon}{\partial x_j} \rightharpoonup 0, \quad i, j = 1, 2 \quad \text{weakly in } L^r(\Omega), \tag{41}$$

$$\varepsilon \frac{\partial \widehat{u}_3^\varepsilon}{\partial z} \rightharpoonup 0, \quad \text{weakly in } L^r(\Omega), \tag{42}$$

$$\varepsilon^2 \frac{\partial \widehat{u}_3^\varepsilon}{\partial x_i} \rightharpoonup 0, \quad i = 1, 2 \quad \text{weakly in } L^r(\Omega), \tag{43}$$

$$\widehat{u}_3^\varepsilon \rightharpoonup 0, \quad \text{weakly in } L^r(\Omega), \tag{44}$$

$$\widehat{p}^\varepsilon \rightharpoonup p^*, \quad \text{weakly in } L^r(\Omega), \quad p^* \text{ depend only of } x. \tag{45}$$

Proof. By Theorem (1), there exists a constant C independent of ε such that

$$\left\| \frac{\partial \widehat{u}_i^\varepsilon}{\partial z} \right\|_{L^r(\Omega)} \leq C, \quad i = 1, 2, \tag{46}$$

and using Poincaré's inequality, we deduce that

$$\|\widehat{u}_i^\varepsilon\|_{L^r(\Omega)} \leq h^* \left\| \frac{\partial \widehat{u}_i^\varepsilon}{\partial z} \right\|_{L^r(\Omega)}, \quad i = 1, 2. \tag{47}$$

that is to say, $\widehat{u}_i^\varepsilon$ is bounded in V_z , $i = 1, 2$, this implies the existence of \widehat{u}_i^* in V_z such that $\widehat{u}_i^\varepsilon$ converges to \widehat{u}_i^* in $L^r(\Omega)$. The same, the inequality (22), we give

$$\varepsilon \left\| \frac{\partial \widehat{u}_i^\varepsilon}{\partial x_j} \right\|_{L^r(\Omega)} \leq C, \tag{48}$$

so $\varepsilon \partial \widehat{u}_i^\varepsilon / \partial x_j$ converges to $\partial \widehat{u}_i^* / \partial x_j$ and as $\|\widehat{u}_i^\varepsilon\|_{L^r(\Omega)} \leq C$, then $\partial \widehat{u}_i^\varepsilon / \partial x_j$ converges weakly to $\partial \widehat{u}_i^* / \partial x_j$; which gives the converges weakly of $\partial \widehat{u}_i^\varepsilon / \partial x_j$ to 0 in $L^r(\Omega)$.

Well thanks to the inequality: $\varepsilon^2 \|\partial \widehat{u}_3^\varepsilon / \partial x_j\|_{L^r(\Omega)} \leq C$, we have the convergence $\varepsilon^2 \partial \widehat{u}_3^\varepsilon / \partial x_j \rightarrow \partial \widehat{u}_3^* / \partial x_j$ and $\varepsilon \partial \widehat{u}_3^\varepsilon / \partial x_j \rightarrow \partial \widehat{u}_3^* / \partial x_j$. This shows that $\partial \widehat{u}_3^\varepsilon / \partial x_j$ converges weakly to 0 in $L^r(\Omega)$. Finally, using (31) and (32), we get (45).

4. Study of the Limit Problem

In this section, we give both the equations satisfied by p^* and u^* in Ω and the inequalities for the trace of the velocity $u^*(x, 0)$ and the stress $\partial u^* / \partial z(x, 0)$ on ω .

Theorem 4. *With the same assumptions of Theorem (3) the solution (u^*, p^*) satisfying the following relations*

$$\begin{aligned} & \mu \sum_{i=1}^2 \int_{\Omega} \frac{1}{2} \left(\frac{1}{2} \sum_{i=1}^2 \left(\frac{\partial u_i^*}{\partial z} \right)^2 \right)^{r-2/2} \frac{\partial(u_i^*)}{\partial z} \frac{\partial(\widehat{\varphi}_i - u_i^*)}{\partial z} dx dz \\ & - \int_{\Omega} p^*(x) \left(\frac{\partial \widehat{\varphi}_1}{\partial x_1} + \frac{\partial \widehat{\varphi}_2}{\partial x_2} \right) dx dz \\ & + \widehat{\alpha} \frac{\sqrt{2}}{2} \int_{\Omega} \left(\left| \frac{\partial \widehat{\varphi}}{\partial z} \right| - \left| \frac{\partial u^*}{\partial z} \right| \right) dx dz + \int_{\omega} \widehat{k} (|\widehat{\varphi}| - |u^*|) dx \\ & \geq \sum_{i=1}^2 \int_{\Omega} \widehat{f}_i (\widehat{\varphi}_i - u_i^*) dx dz, \forall \widehat{\varphi} \in W_{\Gamma_1 \cup \Gamma_L}, \end{aligned} \tag{49}$$

where

$$W_{\Gamma_1 \cup \Gamma_L} = \{ \widehat{\varphi} = (\widehat{\varphi}_1, \widehat{\varphi}_2) \in W^{1,r}(\Omega)^2, \widehat{\varphi} = 0 \text{ on } \Gamma_1 \cup \Gamma_L \}. \tag{50}$$

The proof of this theorem is based on the following lemma.

Lemma 5 (Minty). *Let E be a Banach spaces, $T : E \rightarrow E'$ a monotone and hemicontinuous operator, and $J : E \rightarrow]-\infty, +\infty]$ a proper and convex functional. Let $u \in E$ and $f \in E'$. Then, the followings assertions are equivalent:*

$$\begin{aligned} \langle Tu ; v - u \rangle_{E' \times E} + J(v) - J(u) & \geq \langle f ; v - u \rangle_{E' \times E'}, \forall v \in E, \\ \langle Tv ; v - u \rangle_{E' \times E} + J(v) - J(u) & \geq \langle f ; v - u \rangle_{E' \times E'}, \forall v \in E. \end{aligned} \tag{51}$$

Proof. By using Minty's Lemma (5) and the fact that $\text{div}(\widehat{u}^\varepsilon) = 0$ in Ω , then (20) is equivalent to

$$\begin{aligned} & a_0(\widehat{\varphi}, \widehat{\varphi} - \widehat{u}^\varepsilon) + B_0(\widehat{\varphi}, \widehat{\varphi}, \widehat{\varphi} - \widehat{u}^\varepsilon) \\ & - \sum_{i=1}^2 \left(\widehat{p}^\varepsilon, \frac{\partial \widehat{\varphi}_i}{\partial x_i} \right) - \left(\widehat{p}^\varepsilon, \frac{\partial \widehat{\varphi}_3}{\partial z} \right) + j_0(\widehat{\varphi}) - j_0(\widehat{u}^\varepsilon) \\ & \geq \sum_{i=1}^2 \int_{\Omega} \widehat{f}_i (\widehat{\varphi}_i - \widehat{u}_i^\varepsilon) dx dz + \int_{\Omega} \varepsilon \widehat{f}_3 (\widehat{\varphi}_3 - \widehat{u}_3^\varepsilon) dx dz. \end{aligned} \tag{52}$$

Using Theorem (3) and the fact that j_0 is convex and lower semicontinuous, ($\liminf j_0(\widehat{u}^\varepsilon) \geq j_0(u^*)$), we obtain

$$\begin{aligned} & \mu \sum_{i=1}^2 \int_{\Omega} \frac{1}{2} \left(\frac{1}{2} \sum_{i=1}^2 \left(\frac{\partial \widehat{\varphi}_i}{\partial z} \right)^2 \right)^{r-2/2} \frac{\partial(\widehat{\varphi}_i)}{\partial z} \frac{\partial(\widehat{\varphi}_i - u_i^*)}{\partial z} dx dz \\ & - \int_{\Omega} p^* \left(\frac{\partial \widehat{\varphi}_1}{\partial x_1} + \frac{\partial \widehat{\varphi}_2}{\partial x_2} \right) dx dz - \int_{\Omega} p^* \frac{\partial \widehat{\varphi}_3}{\partial z} dx dz \\ & + j_0(\widehat{\varphi}) - j_0(u^*) \geq \sum_{j=1}^2 \int_{\Omega} \widehat{f}_j (\widehat{\varphi}_j - u_j^*) dx dz, \end{aligned} \tag{53}$$

and as $\int_{\Omega} p^* \partial \widehat{\varphi}_3 / \partial z dx dz = 0$, because p^* independent of z , we deduce that

$$\begin{aligned} & \mu \sum_{i=1}^2 \int_{\Omega} \frac{1}{2} \left(\frac{1}{2} \sum_{i=1}^2 \left(\frac{\partial \widehat{\varphi}_i}{\partial z} \right)^2 \right)^{r-2/2} \frac{\partial(\widehat{\varphi}_i)}{\partial z} \frac{\partial(\widehat{\varphi}_i - u_i^*)}{\partial z} dx dz \\ & - \int_{\Omega} p^* \left(\frac{\partial \widehat{\varphi}_1}{\partial x_1} + \frac{\partial \widehat{\varphi}_2}{\partial x_2} \right) dx dz + j_0(\widehat{\varphi}) - j_0(u^*) \\ & \geq \sum_{j=1}^2 \int_{\Omega} \widehat{f}_j (\widehat{\varphi}_j - u_j^*) dx dz. \end{aligned} \tag{54}$$

Using again Minty's Lemma for the second time, thus, (54) is equivalent to (49).

Theorem 6. *The variational inequality (49) is equivalent the following system*

$$\begin{aligned} & \mu \int_{\Omega} \left(\frac{1}{2} \right)^{r/2} \left| \frac{\partial u^*}{\partial z} \right|^r dx dz + \widehat{\alpha} \int_{\Omega} \left| \frac{\partial u^*}{\partial z} \right| dx dz + \int_{\omega} \widehat{k} |u^*| dx \\ & = \int_{\Omega} \widehat{f} u^* dx dz, \\ & \mu \int_{\Omega} \left(\frac{1}{2} \right)^{r/2} \left| \frac{\partial u^*}{\partial z} \right|^{r-2} \frac{\partial u^*}{\partial z} \frac{\partial \widehat{\psi}}{\partial z} dx dz + \widehat{\alpha} \int_{\Omega} \left| \frac{\partial \widehat{\psi}}{\partial z} \right| dx dz \\ & + \int_{\omega} \widehat{k} |\widehat{\psi}| dx \geq \int_{\Omega} \widehat{f} \widehat{\psi} dx dz, \forall \widehat{\psi} \in \Sigma(K), \end{aligned} \tag{55}$$

where

$$\Sigma(K) = \left\{ \widehat{\psi} = (\widehat{\psi}_1, \widehat{\psi}_2) \in H^1(\Omega)^2 : \widehat{\psi} \text{ satisfy } (D') \right\}. \tag{56}$$

Theorem 7. *Let us set*

$$\sigma^* = \tilde{\sigma}^* - \nabla p^* \text{ and } \tilde{\sigma}^* = \left(\frac{1}{2} \right)^{r/2} \mu \left| \frac{\partial u^*}{\partial z} \right|^{r-2} \frac{\partial u^*}{\partial z} + \widehat{\alpha} \pi, \tag{57}$$

then

$$-\frac{\partial}{\partial z} \left[\frac{1}{2} \mu \left(\frac{1}{2} \sum_{i=1}^2 \left(\frac{\partial u_i^*}{\partial z} \right)^2 \right)^{r-2/2} \frac{\partial u^*}{\partial z} + \frac{\sqrt{2}}{2} \widehat{\alpha} \frac{\partial u^* / \partial z}{|\partial u^* / \partial z|} \right] = \widehat{f} - \nabla p^*, \tag{58}$$

in $W^{-1,r'}(\Omega)^2$, where $\pi \in L^\infty(\Omega)^2$ and $\|\pi\|_{\infty, \Omega} \leq 1$.

Proof. For the proof of this theorem, we follow the same steps as in [13] (Theorem (9)).

Theorem 8. Under the assumptions of preceding theorems, u^* and p^* satisfy the following inequality

$$\int_{\omega} \left[\frac{h^3}{12} \nabla p^* + \tilde{F} + \mu \int_0^h \int_0^y A^*(x, \zeta) \frac{\partial u^*(x, \xi)}{\partial \xi} d\xi dy \right. \\ \left. + \hat{\alpha} \frac{\sqrt{2}}{2} \int_0^h \int_0^y \left| \frac{\partial u^*}{\partial z} \right| (x, \xi) d\xi dy \right. \\ \left. - \frac{h\mu}{2} \int_0^h A^*(x, \zeta) \frac{\partial u^*(x, \xi)}{\partial \xi} d\xi \right. \\ \left. - \frac{\hat{\alpha} \sqrt{2} h}{4} \int_0^h \left| \frac{\partial u^*}{\partial z} \right| (x, \xi) d\xi \right] \cdot \nabla \varphi(x) dx = 0, \tag{59}$$

for all $\varphi \in W^{1,r}(\omega)$, where

$$\tilde{F}(x) = \int_0^h F(x, y) dy - \frac{h}{2} F(x, h), \quad F(x, y) = \int_0^y \int_0^{\xi} \tilde{f}(x, t) dt d\xi, \\ A^*(x, \xi) = \frac{1}{2} \left(\frac{1}{2} \sum_{i=1}^2 \left(\frac{\partial u^*}{\partial z} (x, \xi) \right)^2 \right)^{r-2/2}. \tag{60}$$

Proof. The proof can be found in [13].

The uniqueness of the limit velocity and pressure are given by the following theorem.

Theorem 9. The solution (u^*, p^*) in $V_z \times L_0^r(\omega)$ of inequality (49) is unique.

Proof. Let $(u^{*,1}, p^{*,1})$ and $(u^{*,2}, p^{*,2})$ be two solutions of (49); taking $\varphi = u^{*,2}$ and $\varphi = u^{*,1}$, respectively, as test function in (59), we get

$$\mu \sum_{i=1}^2 \int_{\Omega} \left(\frac{1}{2} \right)^{r/2} \left(\sum_{i=1}^2 \left(\frac{\partial u_i^{*,1}}{\partial z} \right)^2 \right)^{r-2/2} \frac{\partial u_i^{*,1}}{\partial z} \frac{\partial}{\partial z} (u_i^{*,1} - u_i^{*,2}) dx dz \\ \leq \mu \sum_{i=1}^2 \int_{\Omega} \left(\frac{1}{2} \right)^{r/2} \left(\sum_{i=1}^2 \left(\frac{\partial u_i^{*,2}}{\partial z} \right)^2 \right)^{r-2/2} \cdot \frac{\partial u_i^{*,2}}{\partial z} \frac{\partial}{\partial z} (u_i^{*,1} - u_i^{*,2}) dx dz. \tag{61}$$

Keeping in mind that for every $x, y \in \mathbb{R}^n$

$$\left(|x|^{r-2} x - |y|^{r-2} y, x - y \right) \geq (r-1) (|x| + |y|)^{r-2} |x - y|^2, \forall 1 < r \leq 2, \tag{62}$$

we obtain

$$\int_{\Omega} \left[\left| \frac{\partial u^{*,1}}{\partial z} \right| + \left| \frac{\partial u^{*,2}}{\partial z} \right| \right]^{r-2} \left| \frac{\partial u^{*,1}}{\partial z} - \frac{\partial u^{*,2}}{\partial z} \right|^2 dx dz = 0, \tag{63}$$

where $|\partial u^{*,j}/\partial z| = \left(\sum_{i=1}^2 (\partial u_i^{*,j}/\partial z)^2 \right)^{1/2}$, $j = 1, 2$.

Using Hölder's inequality, we deduce

$$\int_{\Omega} \left[\frac{\partial}{\partial z} (u^{*,1} - u^{*,2}) \right]^r dx dz \\ \leq C \left(\int_{\Omega} \left[\left| \frac{\partial u^{*,1}}{\partial z} \right| + \left| \frac{\partial u^{*,2}}{\partial z} \right| \right]^{r-2} \left| \frac{\partial u^{*,1}}{\partial z} - \frac{\partial u^{*,2}}{\partial z} \right|^2 dx dz \right)^{r/2} \\ \times \left(\int_{\Omega} \left[\left| \frac{\partial u^{*,1}}{\partial z} \right| + \left| \frac{\partial u^{*,2}}{\partial z} \right| \right]^r dx dz \right)^{2-r/2}, \tag{64}$$

from (63) and (64), we deduce that $\|u^{*,1} - u^{*,2}\|_{V_z} = 0$.

Finally, to prove the uniqueness of the pressure, we use equation (59) with the two pressures $p^{*,1}$ and $p^{*,2}$, we find

$$\int_{\omega} \frac{h^3}{12} \nabla (p^{*,1} - p^{*,2}) \nabla \varphi dx = 0. \tag{65}$$

Taking $\varphi = p^{*,1} - p^{*,2}$ and using Poincaré inequality, we obtain $\|p^{*,1} - p^{*,2}\|_{L^r(\omega)} = 0$. Then, $p^{*,1} = p^{*,2}$.

5. Conclusion

In this work, the asymptotic behavior of an incompressible Herschel-Bulkley fluid in a thin domain with Tresca boundary conditions is considered, where we prove the convergence of the unknowns which are the velocity and the pressure of the fluid when the ε tends to zero. In addition, the limit problem and the specific Reynolds equation are studied. The aim of our next study is to complement and improve our current results, which is to weaken the hypotheses of fixed point theory by using the following concepts: weak contractual applications, applications that verify some characteristics, normal global operating system, and closed graph applications. We will state and give conclusions on the fixed point theory using the concepts mentioned in recent references. On the other hand, we will study the uniform convergent behavior of a series of designations, of Banach space towards itself, with fixed points, or nearly fixed points in order to show some results in the fixed point theory applied on the problem studied in this paper. With the help of these results, we will introduce some applications and provide some examples and some notes regarding weak contraction mappings. In addition, we will mention and give some results of the fixed point theory of weak contraction mappings by using the studied algorithm in ([15, 22–35]).

Data Availability

No data were used to support the study.

Conflicts of Interest

The authors declare that they have no competing interests.

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