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*Research Article*

# The Existence of Nontrivial Solutions to a Class of Quasilinear Equations

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In this paper, we study the following quasilinear equation: $-\text{div}\,(\phi(|\nabla u|)\nabla u) + \phi(|u|)u = f(u)$ in $\mathbb{R}^N$, where $\phi \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ and $\Phi(t) = \int_0^t s\phi(|s|)ds$. In the Orlicz-Sobolev space, by variational methods and a minimax theorem, we prove the equation has a nontrivial solution.

## 1. Introduction

In this paper, we consider the following quasilinear elliptic equation:

$$-\Delta_\Phi u + \phi(|u|)u = f(u) \text{ in } \mathbb{R}^N, \tag{1}$$

where $N \geq 3$ is the dimension of space and $\phi \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ satisfies the following conditions:

($\phi_1$) $\lim_{t \longrightarrow 0} \phi(t)t = 0$, $\lim_{t \longrightarrow +\infty} \phi(t)t = +\infty$, $(\phi(t)t)' > 0$, for $t > 0$

($\phi_2$) There exist $p, q \in (1, N), q < p < q^* \equiv Nq/(N - q)$, such that $q - 2 \leq (\phi'(t)t)/(\phi(t)) \leq p - 2$, for $t > 0$

$\Phi$ is a continuous function on $\mathbb{R}$ defined by

$$\Phi(t) = \int_0^t \phi(s)sds, \quad \text{for } t > 0, \tag{2}$$

and $\Phi$-Laplace is defined by

$$\Delta_\Phi u = \text{div}\,(\phi(|\nabla u|)\nabla u). \tag{3}$$

Let

$$\Phi_*^{-1}(t) = \int_0^t \frac{\Phi^{-1}(s)}{s^{(N+1)/N}} ds. \tag{4}$$

The quasilinear elliptic equation is an important partial differential equation; in recent years, many researchers have studied the following equation:

$$-\Delta_p u = f(x, u) \text{ in } \Omega, \tag{5}$$

where $\Omega \subset \mathbb{R}^N$ is an open set, $1 < p < N$. Under some growth conditions, many people proved the existence and multiplicity of solutions to (5) and several mathematicians also obtained the bifurcation and asymptotic properties. By variational method and maximal principle, Guo [1] and Guo-Webb [2] obtained the existence and uniqueness of the solution to (5) and they also considered the partial symmetric properties of the solutions. Guedda-Veron [3] used topology and spectrum analysis to study the bifurcation and multiplicity of the solutions. We point out that by constructing the pseudogradient vectors of $p$-Laplace operator, Zhang-Li [4] firstly obtained the sign-changing solutions to (5) and see also Zhang-Chen-Li [5], for more results of $p$-Laplace equations, one can see [6–8] and the references therein. By variational method and minimax theorem, Li-Guo [9] and

Li-Liang [10] studied the $p-q$-Laplace equation and they obtained the existence and multiple solutions.

In [11], Franchi-Lanconelli-Serrin studied the quasilinear equation (1), and in weighted Sobolev space, they considered the existence and uniqueness of the solution to (1). In fact, the function $\phi$ is a general elliptic equation. For example, if $\phi(u)\equiv 1$, then (1) is the Laplace equation; if $\phi(u)=u^{p-2}$, then (1) is the $p$-Laplace equation; $\phi(u)=1/\sqrt{1+p^2}$, then (1) is the curvature equation. The Orlicz-Sobolev space is a kind of general norm space and one can study the quasilinear equations in this space. Carvalho-Silva-Goulart [12] and Carvalho-Silva-Goncalves-Goulart [13] considered (1) in the Orlicz-Sobolev space, by variational method and concentration-compactness theorem. They obtained the existence of (1), and they also studied the famous problem, i.e., concave and convex terms.

In this paper, let $f(t)$ satisfy the following conditions:

$(f_1)$ $f\in C^1(\mathbb{R},\mathbb{R})$. For $t\geq 0, f(t)\geq 0$, and for $t<0, f(t)=0$

$(f_2)$ $\lim_{t\longrightarrow 0^+}(f(t)/t\phi(t))=0$, $\lim_{t\longrightarrow +\infty}(f(t)/t\phi(t))=+\infty$

$(f_3)$ There exists a constant $\lambda\in(0,1)$, such that $\lim_{t\longrightarrow +\infty}(f(t)/g(t))=0$, with $g(t)\equiv(t\phi(t))^{\lambda}(\Phi_*{}'(t))^{1-\lambda}$ for $t>0$

$(f_4)$ $(p-1)f(t)<f'(t)t$ for all $t>0$

$(f_5)$ $f(t)$ satisfies the Ambrosetti-Rabinowitz condition: there exists a $\theta>0$ such that

$$0\leq F(t)\equiv\int_0^t f(s)ds\leq\frac{1}{p+\theta}f(t)t,\quad \text{for } t>0 \tag{6}$$

Under the preceding conditions, by a variational method, we obtain the existence of a nontrivial solution to (1). We follow the idea in [10] to obtain the existence of nontrivial solution. Our conditions $(f_1)$-$(f_5)$ are slightly different from what is in [10]. Conditions $(f_1)$ and $(f_2)$ are standard. Our condition $(f_3)$ is the improvement of $(f_3)$ in [10], which is used to obtain the superlinear growth of $f$ at $t\longrightarrow+\infty$. The condition $(f_4)$ is to guarantee that $A$ is a Finsler manifold which is used to obtain a special minimizing sequence (see Lemma 14–Lemma 16). Condition $(f_5)$ is the compactness condition.

*Remark 1.* The condition $(\phi_2)$ implies that for $t>0$,

$$q\leq\frac{\phi(t)t^2}{\Phi(t)}\leq p. \tag{7}$$

*Remark 2.* By $(f_2)$ and (7), we have that

$$\lim_{t\longrightarrow 0^+}\frac{F(t)}{\Phi(t)}=\lim_{t\longrightarrow 0^+}\frac{f(t)}{t\phi(t)}=0\,; \tag{8}$$

$$\lim_{t\longrightarrow 0^+}\frac{tf(t)}{\Phi(t)}=\lim_{t\longrightarrow 0^+}\frac{tf(t)}{t^2\phi(t)}\frac{t^2\phi(t)}{\Phi(t)}=0. \tag{9}$$

This paper is organized as follows. In Section 2, we recall some results of the Orlicz-Sobolev space; in Section 3, we list and prove some preliminary results; and in Section 4, we prove our main theorem.

## 2. Orlicz-Sobolev Spaces

In this section, we recall some useful knowledge for the Orlicz space and Orlicz-Sobolev space and give some inequalities on $\Phi$. The reader can refer to [14, 15] for more details.

By condition $(\phi_1)$ and the definition of $\Phi$, the function $\Phi$ is a $N$-function (see [14] for the definition of $N$-function). The complementary of $\Phi$ is defined by

$$\tilde{\Phi}(s)=\max_{t\geq 0}(st-\Phi(t)), \tag{10}$$

for $s\geq 0$. It is easy to see that $\Phi$ and $\tilde{\Phi}$ are complementary $N$-functions. By (7) (Chapter 8 [14]), the function $\Phi$ and $\tilde{\Phi}$ satisfies the following inequality:

$$\tilde{\Phi}\left(\frac{\Phi(t)}{t}\right)<\Phi(t),\quad \text{for } t>0. \tag{11}$$

By Lemma 4, Lemma 5, and Lemma 8 below, one can see that $\Phi$, $\tilde{\Phi}$, and $\Phi_*$ satisfy $\Delta_2$-condition (see [14] for the definition of $\Delta_2$-condition).

For any $\Omega\subset\mathbb{R}^N$, under the assumptions $(\phi_1)$ and $(\phi_2)$, the Orlicz space $L_\Phi(\Omega)$ contains all measurable functions $u:\Omega\longrightarrow\mathbb{R}$ which satisfy

$$\int_\Omega\Phi(|u(x)|)dx<\infty, \tag{12}$$

and the Luxemburg norm on $L_\Phi(\Omega)$ is defined by

$$\|u\|_{\Phi,\Omega}=\inf\left\{\lambda>0\mid\int_\Omega\Phi\left(\frac{|u(x)|}{\lambda}\right)dx\leq 1\right\}. \tag{13}$$

For any integer $m\geq 1$, the corresponding Orlicz-Sobolev space $W^mL_\Phi(\Omega)$ is defined by

$$W^mL_\Phi(\Omega)=\{u\in L_\Phi(\Omega)\mid D^\alpha u\in L_\Phi(\Omega),|\alpha|\leq m\}, \tag{14}$$

and the norm on $W^mL_\Phi(\Omega)$ is defined by

$$\|u\|_{m,\Phi,\Omega}=\|u\|_{\Phi,\Omega}+\sum_{|\alpha|\leq m}\|D^\alpha u\|_{\Phi,\Omega}. \tag{15}$$

Let $E_\Phi(\Omega)$ denote the closure in $L_\Phi(\Omega)$ of function u which are bounded on $\Omega$ and have bounded support in $\bar{\Omega}$. The space $W^mE_\Phi(\Omega)(m\in\mathbb{N},m\geq 1)$ is defined by

$$W^mE_\Phi(\Omega)=\{u\in E_\Phi(\Omega)\mid D^\alpha u\in E_\Phi(\Omega),|\alpha|\leq m\}. \tag{16}$$

In the following, for simplicity, we denote $\|u\|_{\Phi,\mathbb{R}^N}$ and $\|u\|_{m,\Phi,\mathbb{R}^N}$ by $\|u\|_\Phi$ and $\|u\|_{m,\Phi}$. Then, the Orlicz-Sobolev space has the following properties.

**Theorem 3** (Theorem 8.20 and Theorem 8.31 [14]).

(a) *The spaces $L_\Phi(\Omega)$ is reflexive if and only if both $(\Phi, \Omega)$ and $(\tilde{\Phi}, \Omega)$ are $\Delta$-regular*

(b) *If $(\Phi, \Omega)$ and $(\tilde{\Phi}, \Omega)$ are $\Delta$-regular, then $W^m E_\Phi(\Omega) = W^m L_\Phi(\Omega)$ is reflexive*

(c) *$C_0^\infty(\mathbb{R}^N)$ is dense in $W^m E_\Phi(\mathbb{R}^N)$. Thus, $W^m E_\Phi(\mathbb{R}^N) = W_0^m L_\Phi(\mathbb{R}^N)$*

*Since $\Phi$ and $\tilde{\Phi}$ are complementary N-functions, the following generalize Hölder inequality (see [14]) holds:*

$$\left|\int_\Omega u(x)v(x)dx\right| \le 2\|u\|_{\Phi,\Omega}\|v\|_{\tilde{\Phi},\Omega}, \quad \textit{for any } u \in L_\Phi(\Omega), v \in L_{\tilde{\Phi}}(\Omega). \tag{17}$$

*Next, we recall some inequalities on $\Phi$.*

**Lemma 4** (Lemma 2.1 [15]). *Let $\zeta_0(t) = \min\{t^q, t^p\}$, $\zeta_1(t) = \max\{t^q, t^p\}$, $t \ge 0$. Then,*

$$\begin{aligned} &\zeta_0(t)\Phi(\rho) \le \Phi(\rho t) \le \zeta_1(t)\Phi(\rho), \quad \textit{for } \rho, t \ge 0, \\ &\zeta_0(\|u\|_\Phi) \le \int_{\mathbb{R}^N} \Phi(|u|)dx \le \zeta_1(\|u\|_\Phi), \quad \textit{for } u \in L_\Phi(\mathbb{R}^N). \end{aligned} \tag{18}$$

**Lemma 5** (Lemma 2.2 [15]). *Let $\zeta_2(t) = \min\{t^{q^*}, t^{p^*}\}$, $\zeta_3(t) = \max\{t^{q^*}, t^{p^*}\}$, $t \ge 0$. Then,*

$$\zeta_2(t)\Phi_*(\rho) \le \Phi_*(\rho t) \le \zeta_3(t)\Phi_*(\rho), \quad \textit{for } \rho, t \ge 0, \tag{19}$$

$$\zeta_2\left(\|u\|_{\Phi_*}\right) \le \int_{\mathbb{R}^N} \Phi_*(|u|)dx \le \zeta_3\left(|u\|_{\Phi_*}\right), \quad \textit{for } u \in L_{\Phi_*}(\mathbb{R}^N). \tag{20}$$

**Lemma 6** (Lemma 2.3 [15]). *$\Phi(t)$ increases essentially more slowly than $\Phi_*(t)$ near infinity, i.e.,*

$$\lim_{t \longrightarrow +\infty} \frac{\Phi(kt)}{\Phi_*(t)} = 0, \tag{21}$$

*for every constant $k > 0$.*

**Lemma 7** (Lemma 2.4 [15]). *(19) is equivalent to*

$$q^* \le \frac{\Phi_*{}'(t)t}{\Phi_*(t)} \le p^*. \tag{22}$$

**Lemma 8** (Lemma 2.5 [15]). *Let $\zeta_4(t) = \min\{t^{q/(q-1)}, t^{p/(p-1)}\}$, $\zeta_5(t) = \max\{t^{q/(q-1)}, t^{p/(p-1)}\}$, $t \ge 0$. Then,*

$$\begin{aligned} &\zeta_4(t)\tilde{\Phi}(\rho) \le \tilde{\Phi}(\rho t) \le \zeta_5(t)\tilde{\Phi}(\rho), \quad \textit{for } \rho, t \ge 0, \\ &\zeta_4(\|u\|_{\tilde{\Phi}}) \le \int_{\mathbb{R}^N} \tilde{\Phi}(|u|)dx \le \zeta_5(\|u\|_{\tilde{\Phi}}), \quad \textit{for } u \in L_{\tilde{\Phi}}(\mathbb{R}^N). \end{aligned} \tag{23}$$

*We recall some imbedding results in the Orlicz-Sobolev space.*

**Theorem 9** (Theorem 8.12 [14]). *The imbedding $L_B(\mathbb{R}^N) \subset L_A(\mathbb{R}^N)$ holds if and only if $B$ dominates $A$ globally, i.e., there exists a constant $k > 0$ such that $A(t) \le B(kt)$ holds for all $t \ge 0$.*

**Theorem 10** (Theorem 8.36 [14]). *Let $\Omega$ be an arbitrary domain in $\mathbb{R}^N$. If the N-function $\Phi(t)$ satisfies*

$$\begin{aligned} &\int_0^1 \frac{\Phi^{-1}(t)}{t^{(N+1)/N}} dt < \infty, \\ &\int_1^\infty \frac{\Phi^{-1}(t)}{t^{(N+1)/N}} dt = \infty, \end{aligned} \tag{24}$$

*then $W_0^m L_\Phi(\Omega) \subset L_{\Phi_*}(\Omega)$, for any integer $m \ge 1$. Moreover, if $\Omega_0$ is a bounded subdomain of $\Omega$, then the imbedding $W_0^m L_\Phi(\Omega) \subset L_B(\Omega_0)$ exists and is compact for any N-function $B$ increasing essentially more slowly than $\Phi_*$ near infinity.*

Since $\Phi$ and $\tilde{\Phi}$ satisfy $\Delta_2$-condition, so by Theorem 3, $W^m L_\Phi(\mathbb{R}^N) = W_0^m L_\Phi(\mathbb{R}^N)$ is reflexive. Using Lemma 4 and Theorem 10, one can see that

$$W^m L_\Phi(\mathbb{R}^N) = W_0^m L_\Phi(\mathbb{R}^N) \subset L_{\Phi_*}(\mathbb{R}^N). \tag{25}$$

By (7), (11), Lemma 4, and Lemma 8, we can see that for $u \in W^1 L_\Phi(\mathbb{R}^N)$ and $\Omega \subset \mathbb{R}^N$, there exists some $C > 0$ such that

$$\begin{aligned} \|\phi(|u|)u\|_{\tilde{\Phi},\Omega} &\le \zeta_4^{-1}\left(\int_\Omega \tilde{\Phi}(\phi(|u|)|u|)dx\right) \\ &\le \zeta_4^{-1}\left(\int_\Omega \tilde{\Phi}\left(p\frac{\Phi(|u|)}{|u|}\right)dx\right) \\ &\le C\zeta_4^{-1}\left(\int_\Omega \Phi(|u|)dx\right) \\ &\le C\zeta_4^{-1}\left(\zeta_1\left(\|u\|_{\Phi,\Omega}\right)\right). \end{aligned} \tag{26}$$

## 3. Preliminary Results

In this section, we prove some preliminary results for future use.

To study the existence of solution to (1), we first study its energy functional. It is clear that the functional defining on $W^1L_\Phi(\mathbb{R}^N)$ is given by

$$I(u)=\int_{\mathbb{R}^N}\Phi(|\nabla u|)+\Phi(|u|)dx-\int_{\mathbb{R}^N}F(u)dx. \quad (27)$$

It is easy to see that under the assumptions $(f_1)$-$(f_5)$ and $(\phi_1)$-$(\phi_3)$, the functionals $I$ is of $C^1$. For any $u,v\in W^1L_\Phi(\mathbb{R}^N)$, we have

$$\left\langle I'(u),v\right\rangle=\int_{\mathbb{R}^N}\phi(|\nabla u|)\nabla u\cdot\nabla v+\phi(|u|)uvdx-\int_{\mathbb{R}^N}f(u)vdx. \quad (28)$$

The $\langle\cdot,\cdot\rangle$ is the dual pairing between $W^1L_\Phi(\mathbb{R}^N)$ and its dual space $(W^1L_\Phi(\mathbb{R}^N))^*$.

By condition $(f_2)$ and $(f_3)$, we have that for any $\varepsilon>0$, there exists a constant $C_\varepsilon>0$ such that

$$f(t)\le\varepsilon t\phi(t)+C_\varepsilon g(t),\quad \text{for } t>0. \quad (29)$$

Hence, it follows that for any $\varepsilon>0$, there exists a constant $C_\varepsilon>0$ such that

$$F(t)\le\varepsilon\Phi(t)+C_\varepsilon tg(t),\quad \text{for } t>0. \quad (30)$$

**Lemma 11.** *Suppose $(f_5)$ holds. Then, there exists a positive constant $C>0$ such that for any $u\in\mathbb{R}^+$, $t\in\mathbb{R}^+$,*

$$F(tu)\begin{cases}\le F(u)t^{p+\theta}, & \text{for } 0<t<1,\\ \ge F(u)t^{p+\theta}, & \text{for } t\ge 1.\end{cases} \quad (31)$$

*Proof.* By $(f_5)$, one has that, for $s\in\mathbb{R}^+$,

$$\frac{p+\theta}{s}\le\frac{F'(s)}{F(s)}. \quad (32)$$

Integrating the last inequality from $u$ to $tu$, we get the result in the lemma. This ends the proof.

Set

$$A=\left\{u\in W^1L_\Phi(\mathbb{R}^N)\mid\left\langle I'(u),u\right\rangle=0,u\not\equiv 0\right\}. \quad (33)$$

Then, we show that $A$ is not an empty set.

**Lemma 12.** *Suppose that $(\phi_1)$-$(\phi_3)$ and $(f_1)$-$(f_5)$ hold. Then, $A\neq\varnothing$.*

*Proof.* By $(f_1)$ and $(f_2)$, $f\not\equiv 0$. Hence, there exists a constant $a_0>0$ such that $f(a_0)>0$. Notice that $C_0^\infty(\mathbb{R}^N)\subset W^1L_\Phi(\mathbb{R}^N)$. We choose $u_0\in C_0^\infty(\mathbb{R}^N)$ such that $u_0\ge 0$, $u_0\not\equiv 0$, and $\sup\{u_0(x)\mid x\in\mathbb{R}^N\}>a_0$. Let

$$h(t)=\left\langle I'(tu_0),tu_0\right\rangle. \quad (34)$$

It is easy to show that $h(t)$ is continuous. For $t>1$, by (7), $(f_5)$, and Lemma 11, we have

$$\begin{aligned}h(t)&=\int_{\mathbb{R}^N}\phi(t|\nabla u_0|)t^2|\nabla u_0|^2+\phi(tu_0)t^2u_0^2dx-\int_{\mathbb{R}^N}f(tu_0)tu_0dx\\&\le p\int_{\mathbb{R}^N}\Phi(t|\nabla u_0|)+\Phi(tu_0)dx-(p+\theta)\int_{\mathbb{R}^N}F(tu_0)dx\\&\le pt^p\int_{\mathbb{R}^N}\Phi(|\nabla u_0|)+\Phi(u_0)dx-(p+\theta)t^{p+\theta}\int_{\mathbb{R}^N}F(u_0)dx.\end{aligned} \quad (35)$$

Using the definition of $u_0$, one has that $\int_{\mathbb{R}^N}F(u_0)dx>0$. It follows that

$$\lim_{t\longrightarrow+\infty}h(t)=-\infty. \quad (36)$$

For $0<t<1$, by (7), we have

$$h(t)\ge q\int_{\mathbb{R}^N}\Phi(t|\nabla u_0|)+\Phi(tu_0)dx-\int_{\mathbb{R}^N}\frac{f(tu_0)tu_0}{\Phi(tu_0)}\Phi(tu_0)dx. \quad (37)$$

By (9), we can see

$$\lim_{t\longrightarrow 0^+}\frac{f(tu_0)tu_0}{\Phi(tu_0)}=0. \quad (38)$$

It follows that, for any $0<\varepsilon<1$, there exists some $\tilde{t}>0$ such that, for any $t\in(0,\tilde{t})$,

$$\frac{f(tu_0)tu_0}{\Phi(tu_0)}<\varepsilon. \quad (39)$$

Hence,

$$h(t)\ge(q-\epsilon)\int_{\mathbb{R}^N}\Phi(t|\nabla u_0|)+\Phi(tu_0)dx>0. \quad (40)$$

Therefore, there exists some $t_0>0$ such that $h(t_0)=0$. This ends the proof.

**Lemma 13.** *For any $\epsilon>0$, there exists some $C_\epsilon>0$ such that*

$$\int_{\mathbb{R}^N}|u|g(|u|)dx\le\epsilon\int_{\mathbb{R}^N}\Phi(|u|)dx+C_\epsilon\int_{\mathbb{R}^N}\Phi_*(|u|)dx, \quad (41)$$

*for any $u\in W^1L_\Phi(\mathbb{R}^N)$.*

*Proof.* By Lemma 7 and (7), for $t>0$

$$\begin{aligned}tg(t)&=\left(t^2\phi(t)\right)^\lambda\left(t\Phi_*'(t)\right)^{1-\lambda}\\&\le p^\lambda p^{*(1-\lambda)}(\Phi(t))^\lambda(\Phi_*(t))^{1-\lambda}.\end{aligned} \quad (42)$$

It follows from Young's inequality that, for any $\epsilon > 0$, there exists $C_\epsilon > 0$ such that

$$tg(t) \le \epsilon \Phi(t) + C_\epsilon \Phi_*(t). \tag{43}$$

Hence, for $u \in W^1 L_\Phi(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} |u| g(|u|)\,dx \le \epsilon \int_{\mathbb{R}^N} \Phi(|u|)\,dx + C_\epsilon \int_{\mathbb{R}^N} \Phi_*(|u|)\,dx. \tag{44}$$

This ends the proof.
Set

$$I_0 = \inf\{I(u) \mid u \in A\}. \tag{45}$$

**Lemma 14.** *Suppose* $(\phi 1)$*-*$(\phi 3)$ *and* $(f_1)$*-*$(f_5)$ *hold. Then,* $I_0 > 0$*.*

*Proof.* For any $u \in A$, let $T(u) = \int_{\mathbb{R}^N} \Phi(|\nabla u|) + \Phi(|u|)\,dx$. Using (7) and $(f_5)$, we obtain that

$$\begin{aligned} I(u) &= T(u) - \int_{\mathbb{R}^N} F(u)\,dx \ge T(u) - \frac{1}{p+\theta} \int_{\mathbb{R}^N} f(u) u\,dx \\ &= T(u) - \frac{1}{p+\theta} \int_{\mathbb{R}^N} \phi(|\nabla u|)|\nabla u|^2 + \phi(|u|) u^2\,dx \\ &\ge \left(1 - \frac{p}{p+\theta}\right) T(u) \ge 0. \end{aligned} \tag{46}$$

Hence,

$$I_0 = \inf\{I(u) \mid u \in A\} \ge 0. \tag{47}$$

By (7), (29), (30), and Lemma 13, for any $0 < \epsilon_1 < \epsilon_2$, one has that there exist two positive constants $C_{\epsilon_1}$ and $C_{\epsilon_2}$ such that

$$\begin{aligned} I(u) &\le \frac{1}{q} \int_{\mathbb{R}^N} \phi(|\nabla u|)|\nabla u|^2 + \phi(|u|) u^2\,dx - \int_{\mathbb{R}^N} F(u)\,dx \\ &= \frac{1}{q} \int_{\mathbb{R}^N} f(u) u\,dx - \int_{\mathbb{R}^N} F(u)\,dx \\ &\le \epsilon_1 \int_{\mathbb{R}^N} \Phi(|u|)\,dx + C_{\epsilon_1} \int_{\mathbb{R}^N} |u| g(|u|)\,dx \\ &\le \epsilon_2 \int_{\mathbb{R}^N} \Phi(|u|)\,dx + C_{\epsilon_2} \int_{\mathbb{R}^N} \Phi_*(|u|)\,dx. \end{aligned} \tag{48}$$

Choose $\epsilon_2$ small enough such that $1 - p/(p+\theta) - \epsilon_2 > 0$. Then, we obtain that

$$\left(1 - \frac{p}{p+\theta} - \epsilon_2\right) T(u) \le C_{\epsilon_2} \int_{\mathbb{R}^N} \Phi_*(|u|)\,dx. \tag{49}$$

By (25), Lemma 4, and Lemma 5, we have that

$$\begin{aligned} &\int_{\mathbb{R}^N} \Phi_*(|u|)\,dx \\ &\quad \le \zeta_3\left(\|u\|_{\Phi_*}\right) \le C\zeta_3\left(\|u\|_{1,\Phi}\right) \\ &\quad \le C\zeta_3\left(\zeta_0^{-1}\left(\int_{\mathbb{R}^N} \Phi(|u|)\,dx\right) + \zeta_0^{-1}\left(\int_{\mathbb{R}^N} \Phi(|\nabla u|)\,dx\right)\right). \end{aligned} \tag{50}$$

By the definition of $\zeta_0$ and $\zeta_3$, we have

$$\begin{aligned} &\zeta_3\left(\zeta_0^{-1}\left(\int_{\mathbb{R}^N} \Phi(|u|)\,dx\right) + \zeta_0^{-1}\left(\int_{\mathbb{R}^N} \Phi(|\nabla u|)\,dx\right)\right) \\ &\quad \le \zeta_3\Bigg(\left(\int_{\mathbb{R}^N} \Phi(|u|)\,dx\right)^{1/p} + \left(\int_{\mathbb{R}^N} \Phi(|u|)\,dx\right)^{1/q} \\ &\qquad + \left(\int_{\mathbb{R}^N} \Phi(|\nabla u|)\,dx\right)^{1/p} + \left(\int_{\mathbb{R}^N} \Phi(|\nabla u|)\,dx\right)^{1/q}\Bigg) \\ &\quad \le C\zeta_3\left((T(u))^{1/p} + (T(u))^{1/q}\right) \\ &\quad \le C \max\left\{(T(u))^{p*/p} + (T(u))^{p*/q}, (T(u))^{q*/p} + (T(u))^{q*/q}\right\}. \end{aligned} \tag{51}$$

Combining (49)–(51), there exists a positive constant $C_{\epsilon_2}' > 0$, such that

$$\begin{aligned} &\max\left\{(T(u))^{p*/p-1} + (T(u))^{p*/q-1}, (T(u))^{q*/p-1} + (T(u))^{q*/q-1}\right\} \\ &\quad \ge C_{\epsilon_2}'\left(1 - \frac{p}{p+\theta} - \epsilon_2\right), \end{aligned} \tag{52}$$

which means $T(u) > C$ for some $C > 0$. It follows that $I_0 > (1 - p/(p+\theta))C$. This completes the proof.

For $u \in W^1 L_\Phi(\mathbb{R}^N)$, set

$$\begin{aligned} H(u) &= \left\langle I'(u), u \right\rangle \\ &= \int_{\mathbb{R}^N} \phi(|\nabla u|)|\nabla u|^2 + \phi(|u|) u^2\,dx - \int_{\mathbb{R}^N} f(u) u\,dx. \end{aligned} \tag{53}$$

It is easy to see that $H$ is a $C^1$-functional. For any $u \in A$, one has that $u^+ \equiv 0$. If $u^+ \equiv 0$, then $f(u) \equiv 0$ by $(f_1)$. Noting that $u \equiv 0$, one has that $H(u) > 0$, which contradicts with $u \in A$. By $(\phi_2)$ and $(f_4)$

$$\begin{aligned} &\left\langle H'(u), u \right\rangle \\ &\quad = \int_{\mathbb{R}^N} \phi'(|\nabla u|)|\nabla u|^3 + 2\phi(|\nabla u|)|\nabla u|^2 \\ &\qquad + \phi'(|u|)|u|^3 + 2\phi(|u|) u^2\,dx - \int_{\mathbb{R}^N} f(u) u + f'(u) u^2\,dx \\ &\quad \le p \int_{\mathbb{R}^N} \phi(|\nabla u|)|\nabla u|^2 + \phi(|u|) u^2\,dx - \int_{\mathbb{R}^N} f(u) u + f'(u) u^2\,dx \\ &\quad = \int_{\mathbb{R}^N} \left((p-1) f(u^+) - f'(u^+) u^+\right) u^+\,dx < 0. \end{aligned} \tag{54}$$

Hence, $A$ is a closed and a complete submanifold of $W^1 L_\Phi(\mathbb{R}^N)$ with the natural Finsler structure (see [16]). Using Lemma 2.14 [10] with $n=1, e_1 = u/\langle G'(u), u\rangle$, one has the following.

**Lemma 15.** *For any $u \in A$, $v \in W^1 L_\Phi(\mathbb{R}^N)$, we have*

$$\langle dI|_A(u), \pi v\rangle = \langle I(u), v\rangle, \tag{55}$$

*with $\pi$ is the projection from $W^1 L_\Phi(\mathbb{R}^N)$ to $T_u A$.*

By Lemma 12, $A$ is not empty, and by Lemma 14, $I$ is bounded from below. Hence, by Lemma 2.15 of [10] and Lemma 15, we can get the following result.

**Lemma 16.** *There exists a sequence $\{u_n\} \subset A$, such that,*

$$\lim_{n\longrightarrow\infty} I(u_n) = I_0 \text{ and } \lim_{n\longrightarrow\infty} I'(u_n) = 0 \text{ in } \left(W^1 L_\Phi(\mathbb{R}^N)\right)^*. \tag{56}$$

**Lemma 17.** *Let $\{\rho_n\} \subset L^1(\mathbb{R}^N)$ be a bounded sequence and $\rho_n \geq 0$, then there exists a subsequence, still denoted by $\{\rho_n\}$, such that one of the following two possibilities occurs:*

*(1) (Vanishing): $\lim_{n\longrightarrow\infty} \sup_{y\in\mathbb{R}^N} \int_{y+B_R} \rho_n(x)dx = 0$ for all $0 < R < +\infty$*

*(2) (Nonvanishing): there exists $\alpha > 0$, $0 < R < +\infty$ and $\{y_n\} \subset \mathbb{R}^N$, such that $\lim_{n\longrightarrow\infty} \int_{y_n+B_R} \rho_n(x)dx \geq \alpha > 0$*

**Lemma 18.** *Suppose $\{u_n\}$ is a bounded sequence in $W^1 L_\Phi(\mathbb{R}^N)$, and for some $r > 0$,*

$$\lim_{n\longrightarrow+\infty} \sup_{y\in\mathbb{R}^N} \int_{y+B_r} \Phi(|u_n|)dx = 0. \tag{57}$$

*Then, $\lim_{n\longrightarrow\infty} \int_{\mathbb{R}^N} g(|u_n|)|u_n|dx = 0$.*

*Proof.* By Lemma 7, (7), and Hölder inequality, for any $\tau \in (0,1)$,

$$\begin{aligned}
&\int_{B(y,r)} \left(\phi(|u_n|)u_n^2\right)^\tau \left(\Phi_*'(|u_n|)|u_n|\right)^{1-\tau} dx \\
&\leq \left(\int_{B(y,r)} \phi(|u_n|)u_n^2 dx\right)^\tau \left(\int_{B(y,r)} \Phi_*'(|u_n|)|u_n| dx\right)^{1-\tau} \\
&\leq p^\tau p^{*(1-\tau)} \left(\int_{B(y,r)} \Phi(|u_n|)dx\right)^\tau \left(\int_{B(y,r)} \Phi_*(|u_n|)dx\right)^{1-\tau}.
\end{aligned} \tag{58}$$

Noting that $\{u_n\}$ is bounded in $W^1 L_\Phi(\mathbb{R}^N)$, there exists a constant $M > 0$ such that

$$\begin{aligned}
&\int_{\mathbb{R}^N} \Phi\left(\frac{|u_n|}{M}\right) dx \leq 1, \\
&\int_{\mathbb{R}^N} \Phi\left(\frac{|\nabla u_n|}{M}\right) dx \leq 1.
\end{aligned} \tag{59}$$

Then, it follows from imbedding theorem (25) and Lemma 4 that

$$\begin{aligned}
&\|u\|_{\Phi^*,B(y,r)} \\
&\leq C_1\left(\|u\|_{\Phi,B(y,r)} + \|\nabla u\|_{\Phi,B(y,r)}\right) \\
&\leq C_2\left(\left\|\frac{u}{M}\right\|_{\Phi,B(y,r)} + \left\|\frac{|\nabla u|}{M}\right\|_{\Phi,B(y,r)}\right) \\
&\leq C_2\left[\left(\int_{B(y,r)} \Phi\left(\frac{|u|}{M}\right)dx\right)^{1/p} + \left(\int_{B(y,r)} \Phi\left(\frac{|\nabla u|}{M}\right)dx\right)^{1/p}\right],
\end{aligned} \tag{60}$$

for some positive constants $C_1$ and $C_2$. Thus, by Lemma 5,

$$\begin{aligned}
&\left(\int_{B(y,r)} \Phi_*(|u_n|)dx\right)^{1-\tau} \\
&\leq \zeta_3^{1-\tau}\left(\|u\|_{\Phi^*,B(y,r)}\right) \\
&\leq C_3\left[\zeta_3^{1-\tau}\left(\left(\int_{B(y,r)} \Phi\left(\frac{|u|}{M}\right)dx\right)^{1/p}\right) + \zeta_3^{1-\tau}\left(\left(\int_{B(y,r)} \Phi\left(\frac{|\nabla u|}{M}\right)dx\right)^{1/p}\right)\right] \\
&\leq C_3\left[\left(\int_{B(y,r)} \Phi\left(\frac{|u|}{M}\right)dx\right)^{q^*(1-\tau)/p} + \left(\int_{B(y,r)} \Phi\left(\frac{|\nabla u|}{M}\right)dx\right)^{q^*(1-\tau)/p}\right] \\
&\leq C_4\left[\left(\int_{B(y,r)} \Phi(|u|)dx\right)^{q^*(1-\tau)/p} + \left(\int_{B(y,r)} \Phi(|\nabla u|)dx\right)^{q^*(1-\tau)/p}\right],
\end{aligned} \tag{61}$$

for some positive constants $C_3$ and $C_4$. Letting $1-\tau = p/q^*$, it follows from (58) and (61) that

$$\begin{aligned}
&\int_{B(y,r)} \left(\phi(|u_n|)u_n^2\right)^\tau \left(\Phi_*'(|u_n|)|u_n|\right)^{1-\tau} dx \\
&\leq C\left(\int_{B(y,r)} \Phi(|u_n|)dx\right)^\tau \left[\int_{B(y,r)} \Phi(|u|)dx + \int_{B(y,r)} \Phi(|\nabla u|)dx\right],
\end{aligned} \tag{62}$$

for some $C>0$. Then, coving $\mathbb{R}^N$ by balls of radius $r$, in such a way that each point of $\mathbb{R}^N$ is contained in at most $N+1$ balls, we have

$$\begin{aligned}\int_{\mathbb{R}^N}&\left(\phi(|u_n|)u_n^2\right)^{\tau}\left(\Phi_*{}'(|u_n|)\,|\,u_n\,|\right)^{1-\tau}dx\\&\le (N+1)C\sup_{y\in\mathbb{R}^N}\left[\left(\int_{B(y,r)}\Phi(|u_n|)dx\right)^{\tau}\right.\\&\quad\left.\cdot\left(\int_{\mathbb{R}^N}\Phi(|u|)dx+\int_{\mathbb{R}^N}\Phi(|\nabla u|)dx\right)\right]\longrightarrow 0,\end{aligned}\tag{63}$$

as $n\longrightarrow\infty$. Noting that $\{u_n\}$ is bounded in $W^1L_\Phi(\mathbb{R}^N)$, by (2.3), we get that $\int_{\mathbb{R}^N}\phi(|u_n|)u_n^2dx$ is a bounded sequence. If $\lambda>\tau$, letting $\mu=(\lambda-\tau)/(1-\tau)$, then

$$\begin{aligned}\int_{\mathbb{R}^N}&g(|u_n|)|u_n|dx\\&\le\left(\int_{\mathbb{R}^N}\phi(|u_n|)u_n^2dx\right)^{\mu}\\&\quad\cdot\left(\int_{\mathbb{R}^N}\left(\phi(|u_n|)u_n^2\right)^{\tau}\left(\Phi_*'(|u_n|)\,|\,u_n\,|\right)^{1-\tau}dx\right)^{1-\mu}\longrightarrow 0,\end{aligned}\tag{64}$$

as $n\longrightarrow\infty$. By imbedding theorem (25) and Lemma 7, one sees that $\int_{\mathbb{R}^N}\Phi_*'(|u_n|)\,|\,u_n\,|\,dx$ is a bounded sequence. If $\lambda<\tau$, setting $\mu=(\tau-\lambda)/\tau$, then

$$\begin{aligned}\int_{\mathbb{R}^N}&g(|u_n|)|u_n|dx\\&\le\left(\int_{\mathbb{R}^N}\Phi_*'(|u_n|)\,|\,u_n\,|\,dx\right)^{\mu}\\&\quad\cdot\left(\int_{\mathbb{R}^N}\left(\phi(|u_n|)u_n^2\right)^{\tau}\left(\Phi_*'(|u_n|)\,|\,u_n\,|\right)^{1-\tau}dx\right)^{1-\mu}\longrightarrow 0,\end{aligned}\tag{65}$$

as $n\longrightarrow\infty$. This ends the proof.

**Theorem 19** (Vitali's convergence theorem [17]). *Let $(X,\mathscr{A},\mu)$ be a measure space and let $1\le p<\infty$. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence in $L^p(X,\mathscr{A},\mu)$ and let $f$ be an $\mathscr{A}$ -measurable function such that $f$ is finite $\mu$-a.e. and $f_n\longrightarrow f\,\mu$-a.e. Then, $f\in L^p(X,\mathscr{A},\mu)$ and $\|f-f_n\|_p\longrightarrow 0$ if and only if*

*(i) For each $\varepsilon>0$, there exists a set $A_\varepsilon\in\mathscr{A}$ such that $\mu(A_\varepsilon)<\infty$ and $\int_{A_\varepsilon^c}|f_n|^pd\mu<\varepsilon$ for all $n\in\mathbb{R}^N$*

*(ii) $\lim_{\mu(E)\longrightarrow 0}\int_E|f_n|^pd\mu=0$*

*uniformly in n, i.e., for each $\varepsilon>0$, there is a $\delta>0$ such that $E\in\mathscr{A}$ and $\mu(E)<\delta$ imply $\int_E|f_n|^pd\mu<\varepsilon$ for all $n\in\mathbb{N}$.*

## 4. Main Result

**Theorem 20.** *Suppose conditions $(\phi_1)$-$(\phi_3)$ and $(f_1)$-$(f_5)$ hold. Then, (1) has a nonnegative weak solution.*

*Proof.* By Lemma 16, there exists a sequence $\{u_n\}\subset A$, such that

$$I(u_n)\longrightarrow I_0, I'(u_n)\longrightarrow 0\text{in}\left(W^1L_\Phi(\mathbb{R}^N)\right)^*.\tag{66}$$

*Step 1.* There exists some constant $C>0$ such that $\|u_n\|_{1,\Phi}\le C$. By (66) and (7), one has

$$\begin{aligned}I_0+o_n(1)&=\int_{\mathbb{R}^N}\Phi(|u_n|)+\Phi(|\nabla u_n|)dx\\&\quad-\frac{1}{p+\theta}\int_{\mathbb{R}^N}\phi(|u_n|)u_n^2+\phi(|\nabla u_n|)|\nabla u_n|^2dx\\&\quad+\int_{\mathbb{R}^N}\frac{1}{p+\theta}f(u_n)u_n-F(u_n)dx\\&\ge\left(1-\frac{p}{p+\theta}\right)\int_{\mathbb{R}^N}\Phi(|u_n|)+\Phi(|\nabla u_n|)dx\\&\quad+\int_{\mathbb{R}^N}\frac{1}{p+\theta}f(|u_n|)|u_n|-F(|u_n|)dx\\&\ge\left(1-\frac{p}{p+\theta}\right)\int_{\mathbb{R}^N}\Phi(|u_n|)+\Phi(|\nabla u_n|)dx.\end{aligned}\tag{67}$$

In the last step, we use condition $(f_5)$. Hence, by Lemma 4, there exists some constant $C>0$ such that $\|u_n\|_{1,\Phi}\le C$.

*Step 2.* There exist $R>0$, $\alpha>0$, and $\{y_n\}\subset\mathbb{R}^N$, such that

$$\int_{y_n+B_R}\Phi(|u_n|)dx\ge\alpha>0.\tag{68}$$

By Step 1, $\Phi(|u_n|)$ is bounded in $L^1(\mathbb{R}^N)$. Lemma 17 shows that one of two situations (vanishing and nonvanishing) occur. We need only to show that the vanishing case is impossible. If vanishing occurs, by Lemma 18, $\lim_{n\longrightarrow\infty}\int_{\mathbb{R}^N}g(|u_n|)|u_n|dx=0$. It follows from (29) and (30) that

$$\lim_{n\longrightarrow\infty}\int_{\mathbb{R}^N}f(|u_n|)|u_n|dx=\lim_{n\longrightarrow\infty}\int_{\mathbb{R}^N}F(|u_n|)dx=0.\tag{69}$$

Notice $u_n\in A$, we obtain

$$\begin{aligned}0&<I_0\\&\le\limsup_{n\longrightarrow\infty}\left[\frac{1}{q}\int_{\mathbb{R}^N}\phi(|u_n|)u_n^2+\phi(|\nabla u_n|)|\nabla u_n|^2dx-\int_{\mathbb{R}^N}F(|u_n|)dx\right]\\&=\limsup_{n\longrightarrow\infty}\int_{\mathbb{R}^N}\frac{1}{q}f(|u_n|)|u_n|-F(|u_n|)dx=0.\end{aligned}\tag{70}$$

This is a contradiction. Hence, only nonvanishing is possible, i.e., there exist $R>0$, $\alpha>0$, and $\{y_n\}\subset\mathbb{R}^N$, such that

$$\int_{y_n+B_R}\Phi(|u_n|)dx\ge\alpha>0. \tag{71}$$

*Step 3.* There exists $\bar{u}\in W^1L_\Phi(\mathbb{R}^N)$, such that $\nabla\bar{u}_n\longrightarrow\nabla\bar{u}$ a.e. in $\mathbb{R}^N$, where $\bar{u}_n(x)=u_n(x+y_n)$.

For any $\psi\in W^1L_\Phi(\mathbb{R}^N)$, set $\psi_n(x)=\psi(x-y_n)$. Then, $\|\psi\|_{1,\Phi}=\|\psi_n\|_{1,\Phi}$, and

$$\begin{aligned}\left|\left\langle I'(\bar{u}_n),\psi\right\rangle\right|&=\left|\left\langle I'(u_n),\psi_n\right\rangle\right|\\&\le\left\|I'(u_n)\right\|_{(W^1L_\Phi(\mathbb{R}^N))^*}\|\psi_n\|_{1,\Phi}\\&=\left\|I'(u_n)\right\|_{(W^1L_\Phi(\mathbb{R}^N))^*}\|\psi\|_{1,\Phi}\longrightarrow0,\end{aligned} \tag{72}$$

as $n\longrightarrow\infty$. Since $\|\bar{u}_n\|_{1,\Phi}=\|\bar{u}_n\|_{1,\Phi}$, so $\{\bar{u}_n\}$ is a bounded sequence in $W^1L_\Phi(\mathbb{R}^N)$. Hence, there exists $\bar{u}\in W^1L_\Phi(\mathbb{R}^N)$ and $\bar{u}\equiv0$ such that some sequence of $\{\bar{u}_n\}$, still denoted by $\{\bar{u}_n\}$,

$$\begin{cases}\bar{u}_n\rightharpoonup\bar{u}\text{ in }W^1L_\Phi(\mathbb{R}^N),\\ \bar{u}_n\longrightarrow\bar{u}\text{ a.e.in }\mathbb{R}^N,\\ \int_\Omega g(|\bar{u}_n-\bar{u}|)|\bar{u}_n-\bar{u}|dx\longrightarrow0\text{ for any bounded subset }\Omega\subset\mathbb{R}^N.\end{cases} \tag{73}$$

Let $\eta_r\in C_0^\infty(\mathbb{R}^N)$ such that $0\le\eta\le1$, with $\eta_r=1$ in $B_r=\{x\in\mathbb{R}^N\mid|x|\le r\}$, $\eta_r=0$ in $B_{2r}^c$, and $|\nabla\eta_r|\le C/r$ for some $C>0$. Then, it is easy to see that

$$\left\langle I'(\bar{u}_n)-I'(\bar{u}),(\bar{u}_n-\bar{u})\eta_r\right\rangle=o_n(1). \tag{74}$$

On the other hand,

$$\begin{aligned}&\left\langle I'(\bar{u}_n)-I'(\bar{u}),(\bar{u}_n-\bar{u})\eta_r\right\rangle\\&=\int_{B_{2r}}\eta_r(\phi(|\bar{u}_n|)\bar{u}_n-\phi(|\bar{u}|)\bar{u})(\bar{u}_n-\bar{u})dx\\&\quad+\int_{B_{2r}}\eta_r(\phi(|\nabla\bar{u}_n|)\nabla\bar{u}_n-\phi(|\nabla\bar{u}|)\nabla\bar{u})\cdot\nabla(\bar{u}_n-\bar{u})dx\\&\quad+\int_{B_{2r}\setminus B_r}(\phi(|\nabla\bar{u}_n|)\nabla\bar{u}_n-\phi(|\nabla\bar{u}|)\nabla\bar{u})\cdot\nabla\eta_r(\bar{u}_n-\bar{u})dx\\&\quad-\int_{B_{2r}}\eta_r(f(\bar{u}_n)-f(\bar{u}))(\bar{u}_n-\bar{u})dx.\end{aligned} \tag{75}$$

By (17), (26), and Lemma 4,

$$\begin{aligned}&\left|\int_{B_{2r}\setminus B_r}(\phi(\nabla\bar{u}_n)\nabla\bar{u}_n-\phi(\nabla\bar{u})\nabla\bar{u})\cdot\nabla\eta_r(\bar{u}_n-\bar{u})dx\right|\\&\le\frac{C}{r}\||\phi(\nabla\bar{u}_n)\nabla\bar{u}_n-\phi(\nabla\bar{u})\nabla\bar{u}|\|_{\tilde{\Phi},B_{2r}\setminus B_r}\|\bar{u}_n-\bar{u}\|_{\Phi,B_{2r}\setminus B_r}\\&\le\frac{C}{r}\left[\zeta_4^{-1}\left(\int_{B_{2r}\setminus B_r}\Phi(\nabla\bar{u}_n)dx\right)+\zeta_4^{-1}\left(\int_{B_{2r}\setminus B_r}\Phi(\nabla\bar{u})dx\right)\right]\\&\quad\cdot\left[\zeta_0^{-1}\left(\int_{B_{2r}\setminus B_r}\Phi(\bar{u}_n)dx\right)+\zeta_0^{-1}\left(\int_{B_{2r}\setminus B_r}\Phi(\bar{u})dx\right)\right].\end{aligned} \tag{76}$$

Noting that $\bar{u}_n$ is bounded in $W^1L_\Phi(\mathbb{R}^N)$, for any $\varepsilon>0$, there exists $r_0>0$ such that for $r>r_0$,

$$\left|\int_{B_{2r}\setminus B_r}(\phi(\nabla\bar{u}_n)\nabla\bar{u}_n-\phi(\nabla\bar{u})\nabla\bar{u})\cdot\nabla\eta_r(\bar{u}_n-\bar{u})dx\right|\le\varepsilon. \tag{77}$$

For fix $r$ and $\delta>0$, by the Egoroff theorem, there exists $E_\delta\subset B_{2r}$ such that $m(E_\delta)<\delta$, and $(f(\bar{u}_n)-f(\bar{u}))(\bar{u}_n-\bar{u})\longrightarrow0$ uniformly on $B_{2r}\setminus E_\delta$ as $n\longrightarrow+\infty$. Thus, for any $\varepsilon>0$, $\delta>0$, there exists $n_0>0$ such that, for $n\ge n_0$,

$$\left|\int_{B_{2r}\setminus E_\delta}\eta_r(f(\bar{u}_n)-f(\bar{u}))(\bar{u}_n-\bar{u})dx\right|\le\varepsilon. \tag{78}$$

By $(f_2)$ and $(f_3)$, there exists some $C>0$ such that

$$f(t)\le\begin{cases}Ct\phi(t)\text{ for }0<t<1,\\ Cg(t)\text{ for }t\ge1.\end{cases} \tag{79}$$

Let $1<\mu<q^*/(p\lambda+(1-\lambda)q^*)$ be a fixed constant. Then, by Lemma 4, Lemma 5, and Lemma 7, for $t\ge1$,

$$\begin{aligned}(tf(t))^\mu&\le C\left(t^2\phi(t)\right)^{\lambda\mu}\left(t\Phi_*'(t)\right)^{(1-\lambda)\mu}\\&\le C(\Phi(t))^{\lambda\mu}(\Phi_*(t))^{(1-\lambda)\mu}\le Ct^{\lambda\mu p}(\Phi_*(t))^{(1-\lambda)\mu}\\&\le C(\Phi_*(t))^{\lambda\mu p/q^*+(1-\lambda)\mu}\le C\Phi_*(t),\end{aligned} \tag{80}$$

for some $C>0$. Set $E_{\delta,1}=\{x\in E_\delta\mid|\bar{u}(x)|<1\}$, and $E_{\delta,2}=E_\delta\setminus E_{\delta,1}$. Using Hölder inequality, we obtain

$$\begin{aligned}&\left|\int_{E_\delta}\eta_r(f(\bar{u}_n)-f(\bar{u}))(\bar{u}_n-\bar{u})dx\right|\\&\le C\left(\int_{E_\delta}|(f(\bar{u}_n)-f(\bar{u}))(\bar{u}_n-\bar{u})|^\mu dx\right)^{1/\mu}(m(E_\delta))^{1-1/\mu}\\&\le C\left(\int_{E_\delta}|f(\bar{u}_n)\bar{u}_n|^\mu+|f(\bar{u})\bar{u}|^\mu dx\right)^{1/\mu}\delta^{1-1/\mu}.\end{aligned} \tag{81}$$

Since $u_n$ and $u$ is bounded in $W^1L_\Phi(\mathbb{R}^N)$, so there exists a constant $C>0$ such that

$$\int_{\mathbb{R}^N}\Phi_*(|u_n|)dx\le C,\int_{\mathbb{R}^N}\Phi_*(|u|)dx\le C. \quad (82)$$

Therefore, for some $C>0$,

$$\begin{aligned}\int_{E_\delta}|f(\bar u)\bar u|^\mu dx\le C\int_{E_{\delta,1}}\left(\bar u^2\phi(|\bar u|)\right)^\mu dx+C\int_{E_{\delta,2}}\Phi_*(|\bar u|)dx\le C,\\ \int_{E_\delta}|f(\bar u_n)\bar u_n|^\mu dx\le C\int_{E_{\delta,1}}\left(\bar u_n^2\phi(|\bar u_n|)\right)^\mu dx+C\int_{E_{\delta,2}}\Phi_*(|\bar u_n|)dx\le C.\end{aligned} \quad (83)$$

Choosing $\delta>0$ small enough, one has

$$\left|\int_{E_\delta}\eta_r(f(\bar u_n)-f(\bar u))(\bar u_n-\bar u)dx\right|\le\varepsilon. \quad (84)$$

Hence, such that for $n>n_0$,

$$\left|\int_{B_{2r}}\eta_r(f(|\bar u_n|)-f(|\bar u|))(\bar u_n-\bar u)dx\right|\le 2\epsilon. \quad (85)$$

Since $(\phi(|\bar u_n|)\bar u_n-\phi(|\bar u|)\bar u)(\bar u_n-\bar u)\ge 0$ and $(\phi(|\nabla\bar u_n|)\nabla\bar u_n-\phi(|\nabla\bar u|)\nabla\bar u)\cdot\nabla(\bar u_n-\bar u)\ge 0$, it follows from (74)–(85) that for any $\epsilon>0$, there exist $r_0$ and $n_1\ge n_0$ such that for $n>n_1$ and $r\ge r_0$,

$$\int_{B_r}(\phi(|\nabla\bar u_n|)\nabla\bar u_n-\phi(|\nabla\bar u|)\nabla\bar u)\cdot\nabla(\bar u_n-\bar u)dx\le C\epsilon. \quad (86)$$

It follows that, passing to a subsequence if necessary, $(\phi(|\nabla\bar u_n|)\nabla\bar u_n-\phi(|\nabla\bar u|)\nabla\bar u)\cdot\nabla(\bar u_n-\bar u)\longrightarrow 0$ a.e. in $B_r$. Using Lemma 6 [18], we get that

$$\nabla\bar u_n\longrightarrow\nabla\bar u\ \text{a.e.in}\ B_r. \quad (87)$$

Since $r$ is arbitrary, one can see that $\nabla\bar u_n\longrightarrow\nabla\bar u$ a.e. in $\mathbb{R}^N$.

*Step 4.* $I'(\bar u)=0$, $\bar u\in A$.
We first show that $\phi(|\nabla\bar u_n|)\nabla\bar u_n\rightharpoonup\phi(|\nabla\bar u|)\nabla\bar u$ in $L_{\tilde\Phi}(\mathbb{R}^N)$, i.e., for any $\psi\in L_\Phi(\mathbb{R}^N)$,

$$\lim_{n\longrightarrow\infty}\int_{\mathbb{R}^N}\phi(|\nabla\bar u_n|)|\nabla\bar u_n|\psi dx=\int_{\mathbb{R}^N}\phi(|\nabla\bar u|)|\nabla\bar u|\psi dx. \quad (88)$$

Since $\nabla\bar u_n$ is bounded in $W^1L_\Phi(\mathbb{R}^N)$, by (26), there exists some $M>0$ such that

$$\|\phi(|\nabla\bar u_n|)|\nabla\bar u_n|\|_{\tilde\Phi}\le M. \quad (89)$$

Then, for any $\epsilon>0$, there exist $r>0$ and $\delta>0$ such that for any $m(E)\le\delta$,

$$\begin{aligned}\int_{B_r^c}\phi(|\nabla\bar u_n|)|\nabla\bar u_n|\psi dx\le 2\|\phi(|\nabla\bar u_n|)|\nabla\bar u_n|\|_{\tilde\Phi}\|\psi\|_{\Phi,B_r^c}\le\epsilon,\\ \int_{E}\phi(|\nabla\bar u_n|)|\nabla\bar u_n|\psi dx\le 2\|\phi(|\nabla\bar u_n|)|\nabla\bar u_n|\|_{\tilde\Phi}\|\psi\|_{\Phi,E}\le\epsilon.\end{aligned} \quad (90)$$

Then, by Theorem 19,

$$\phi(\nabla\bar u_n)\nabla\bar u_n\rightharpoonup\phi(|\nabla\bar u|)|\nabla\bar u|\ \text{in}\ \left(W^1L_\Phi(\mathbb{R}^N)\right)^*. \quad (91)$$

Similarly, we can have

$$\phi(|\bar u_n|)\bar u_n\rightharpoonup\phi(|\bar u|)\bar u, f(\bar u_n)\rightharpoonup f(\bar u),\ \text{in}\ \left(W^1L_\Phi(\mathbb{R}^N)\right)^*. \quad (92)$$

It follows from (91) and (92) that for any $\psi\in W^1L_\Phi(\mathbb{R}^N)$,

$$0=\lim_{n\longrightarrow\infty}\left\langle I'(\bar u_n),\psi\right\rangle=\lim_{n\longrightarrow\infty}\left\langle I'(\bar u),\psi\right\rangle. \quad (93)$$

In particular, taking $\psi=\bar u$, we get that $\langle I'(\bar u),\bar u\rangle=0$. Since $\bar u\equiv 0$, so $\bar u\in A$.

*Step 5.* $I(\bar u)=I_0$.
By the definition of $\bar u_k$, we obtain that

$$I(\bar u_n)=I(u_n),\left\langle I'(\bar u_n),\bar u_n\right\rangle=\left\langle I'(u_n),u_n\right\rangle=o_n(1). \quad (94)$$

Hence, by (7), condition $(f_5)$, and Fatou's lemma,

$$\begin{aligned}I(\bar u)\ge I_0&=\lim_{n\longrightarrow\infty}I(\bar u_n)\\ &=\lim_{n\longrightarrow\infty}\left[\int_{\mathbb{R}^N}\Phi(|\bar u_n|)+\Phi(|\nabla\bar u_n|)dx-\int_{\mathbb{R}^N}F(\bar u_n)dx\right]\\ &=\lim_{n\longrightarrow\infty}\left[\int_{\mathbb{R}^N}\Phi(|\bar u_n|)+\Phi(|\nabla\bar u_n|)-\frac{1}{p}\phi(|\bar u_n|)\bar u_n^2\right.\\ &\quad\left.-\frac{1}{p}\phi(|\nabla\bar u_n|)|\nabla\bar u_n|^2dx+\int_{\mathbb{R}^N}\frac{1}{p}f(\bar u_n)-F(\bar u_n)dx\right]\\ &\ge\left[\int_{\mathbb{R}^N}\Phi(|\bar u|)+\Phi(|\nabla\bar u|)-\frac{1}{p}\phi(|\bar u|)\bar u^2\right.\\ &\quad\left.-\frac{1}{p}\phi(|\nabla\bar u|)|\nabla\bar u|^2dx+\int_{\mathbb{R}^N}\frac{1}{p}f(\bar u)\bar u-F(\bar u)dx\right]\\ &=\left[\int_{\mathbb{R}^N}\Phi(|\bar u|)+\Phi(|\nabla\bar u|)dx-\int_{\mathbb{R}^N}\frac{1}{p}F(\bar u)dx\right]=I(\bar u).\end{aligned} \quad (95)$$

It follows that $I(\bar u)=I_0$. By $(f_1)$, we get that $\bar u^-=\max\{-\bar u,0\}=0$. Hence, $\bar u\ge 0$. This ends the proof.

## Data Availability

The data used to support the findings of this study are included within the article.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

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