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Research Article

A Study of Fourth-Order Hankel Determinants for Starlike Functions Connected with the Sine Function

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In this paper, upper bounds for the fourth-order Hankel determinant $H_4(1)$ for the function class $\mathcal{S}_s^*$ associated with the sine function are given.

1. Introduction

Let $\mathcal{A}$ denote the class of functions f which are analytic in the open unit disk $\mathbb{D} = \{z : |z|<1\}$ of the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots (z \in \mathbb{D}), \tag{1}$$

and let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of univalent functions.

Suppose that $\mathcal{P}$ is the class of analytic functions p normalized by

$$p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots \tag{2}$$

and satisfying the condition

$$\mathfrak{R}(p(z)) > 0 (z \in \mathbb{D}). \tag{3}$$

Assume that f and g are two analytic functions in $\mathbb{D}$. Then, we say that the function g is subordinate to the function f, and we write

$$g(z) \prec f(z) (z \in \mathbb{D}), \tag{4}$$

if there exists a Schwarz function $\omega(z)$ with $\omega(0) = 0$ and $|\omega(z)| <1$, such that (see [1])

$$g(z) = f(\omega(z)) (z \in \mathbb{D}). \tag{5}$$

In 2018, Cho et al. [2] introduced the following function class S_s^*:

$$S_s^* := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec (1 + \sin z) (z \in \mathbb{D}) \right\}, \tag{6}$$

which implies that the quantity $(zf'(z))/(f(z))$ lies in an eight-shaped region in the right-half plane.

In 1976, Noonan and Thomas [3] stated the q^{th} Hankel determinant for $q \geq 1$ and $n \geq 1$ of functions f as follows:

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_n & \cdots & a_{n+q-2} \\ \vdots & \vdots & & \vdots \\ a_{n+q-1} & a_{n+q-2} & \cdots & a_n \end{vmatrix} (a_1 = 1). \tag{7}$$

In particular, we have

$$H_2(1)=\begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix}=a_3-a_2^2\,(a_1=1,n=1,q=2),$$
$$H_2(2)=\begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix}=a_2a_4-a_3^2\,(n=2,q=2),$$
$$H_3(1)=\begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}(n=1,q=3), \tag{8}$$
$$H_4(1)=\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ a_2 & a_3 & a_4 & a_4 \\ a_3 & a_4 & a_5 & a_6 \\ a_4 & a_5 & a_6 & a_7 \end{vmatrix}\;(n=1,q=4).$$

Since $f\in\mathcal{S}, a_1=1$, thus

$$\begin{aligned} H_4(1)=&\,a_7\left\{a_3\left(a_2a_4-a_3^2\right)-a_4(a_4-a_2a_3)+a_5\left(a_3-a_2^2\right)\right\}\\ &-a_6\left\{a_3(a_2a_5-a_3a_4)-a_4(a_5-a_2a_4)+a_6\left(a_3-a_2^2\right)\right\}\\ &+a_5\left\{a_3\left(a_3a_5-a_4^2\right)-a_5(a_5-a_2a_4)+a_6(a_4-a_2a_3)\right\}\\ &+a_5\left\{a_3\left(a_3a_5-a_4^2\right)-a_5(a_5-a_2a_4)+a_6(a_4-a_2a_3)\right\}\\ &-a_4\left\{a_4\left(a_3a_5-a_4^2\right)-a_5(a_2a_5-a_3a_4)+a_6(a_4-a_2a_3)\right\}. \end{aligned} \tag{9}$$

We note that $H_2(1)$ is the well-known Fekete-Szegö functional (see [4–6]).

In recent years, many papers have been devoted to finding upper bounds for the second-order Hankel determinant $H_2(2)$ and the third-order Hankel determinant $H_3(1)$, whose elements are various classes of analytic functions; it is worth mentioning that [7–20]. For instance, Murugusundaramoorthy and Bulboacă [21] defined a new subclass of analytic functions $M\mathfrak{L}_c^a(\lambda,\phi)$ and got upper bounds for the Fekete-Szegö functional and the Hankel determinant of order two for $f\in M\mathfrak{L}_c^a(\lambda,\phi)$. Islam et al. [22] examined the q-analog of starlike functions connected with a trigonometric sine function and discussed some interesting geometric properties, such as the well-known problems of Fekete-Szegö, the necessary and sufficient condition, the growth and distortion bound, closure theorem, and convolution results with partial sums for this class. Zaprawa et al. [23] obtained the bound of the third Hankel determinant for the univalent starlike functions. Very recently, Arif et al. [24] studied the problem of fourth Hankel determinant $H_4(1)$ for the first time for the class of bounded turning functions and successfully obtained the bound of $H_4(1)$. Recently, Khan et al. [25] discussed some classes of functions with bounded turning which are connected to the sine functions and obtained upper bounds for the third- and fourth-order Hankel determinants related to such classes. Inspired by the aforementioned works, in this paper, we mainly investigate upper bounds for the fourth-order Hankel determinant $H_4(1)$ for the function class $\mathcal{S}_s^*$ associated with the sine function.

2. Main Results

By proving our desired results, we need the following lemmas.

Lemma 1 (see [26]). *If $p(z)\in\mathcal{P}$, then exists some x,z with $|x|\le 1, |z|\le 1$, such that*

$$\begin{aligned} &2c_2=c_1^2+x\left(4-c_1^2\right),\\ &4c_3=c_1^3+2c_1x\left(4-c_1^2\right)-\left(4-c_1^2\right)c_1x^2+2\left(4-c_1^2\right)\left(1-|x|^2\right)z. \end{aligned} \tag{10}$$

Lemma 2 (see [27]). *Let $p(z)\in\mathcal{P}$, then*

$$\begin{aligned} &\left|c_1^4+c_2^2+2c_1c_3-3c_1^2c_2-c_4\right|\le 2,\\ &\left|c_1^5+3c_1c_2^2+3c_1^2c_3-4c_1^3c_2-2c_1c_4-2c_2c_3+c_5\right|\le 2,\\ &|c_1^6+6c_1^2c_2^2+4c_1^3c_3+2c_1c_5+2c_2c_4+c_3^2-c_2^3\\ &\quad-5c_1^4c_2-3c_1^2c_4-6c_1c_2c_3-c_6|\le 2,\\ &|c_n|\le 2, n=1,2,\cdots. \end{aligned} \tag{11}$$

Lemma 3 (see [28]). *Let $p(z)\in\mathcal{P}$, then we have*

$$\begin{aligned} &\left|c_2-\frac{c_1^2}{2}\right|\le 2-\frac{|c_1|^2}{2},\\ &|c_{n+k}-\mu c_nc_k|<2,\quad 0\le\mu\le 1,\\ &|c_{n+2k}-\mu c_nc_k^2|\le 2(1+2\mu). \end{aligned} \tag{12}$$

We now state and prove the main results of our present investigation.

Theorem 4. *If the function $f(z)\in\mathcal{S}_s^*$ and of the form ((1)), then*

$$\begin{aligned} &|a_2|\le 1,\\ &|a_3|\le\frac{1}{2},\\ &|a_4|\le 0.344,\\ &|a_5|\le\frac{3}{8},\\ &|a_6|\le\frac{67}{120},\\ &|a_7|\le\frac{5587}{10800}. \end{aligned} \tag{13}$$

Proof. Since $f(z)\in\mathcal{S}_s^*$, according to subordination relationship, thus there exists a Schwarz function $\omega(z)$ with $\omega(0)=0$ and $|\omega(z)|<1$, satisfying

$$\frac{zf'(z)}{f(z)} = 1 + \sin(\omega(z)). \tag{14}$$

Here,

$$\begin{aligned}\frac{zf'(z)}{f(z)} &= \frac{z+\sum_{n=2}^{\infty} na_n z^n}{z+\sum_{n=2}^{\infty} a_n z^n} = \left(1+\sum_{n=2}^{\infty} na_n z^{n-1}\right)\left[1 - a_2 z + \left(a_2^2 - a_3\right)z^2 \right.\\ &\quad \left. - \left(a_2^3 - 2a_2a_3 + a_4\right)z^3 + \left(a_2^4 - 3a_2^2a_3 + 2a_2a_4 - a_5\right)z^4 + \cdots\right]\\ &= 1 + a_2 z + \left(2a_3 - a_2^2\right)z^2 + \left(a_2^3 - 3a_2a_3 + 3a_4\right)z^3\\ &\quad + \left(4a_5 - a_2^4 + 4a_2^2a_3 - 4a_2a_4 - 2a_3^2\right)z^4\\ &\quad + \left(5a_6 - 5a_2a_5 + a_2^5 - 5a_3a_4 - 5a_2^3a_3 + 5a_2^2a_4 + 5a_2a_3^2\right)z^5\\ &\quad + \left(6a_7 - 6a_2a_6 + 6a_2^2a_5 - 6a_3a_5 + 12a_2a_3a_4 - a_2^6\right.\\ &\quad \left. - 6a_2^3a_4 - 3a_4^2 + 2a_3^3 - 9a_2^2a_3^2 + 6a_2^4a_3\right)z^6 + .\end{aligned} \tag{15}$$

Now, we define a function

$$p(z) = \frac{1+\omega(z)}{1-\omega(z)} = 1 + c_1 z + c_2 z^2 + \cdots. \tag{16}$$

It is easy to see that $p(z) \in \mathscr{P}$ and

$$\omega(z) = \frac{p(z)-1}{1+p(z)} = \frac{c_1 z + c_2 z^2 + c_3 z^3 + \cdots}{2 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots}. \tag{17}$$

On the other hand,

$$\begin{aligned}1 + \sin(\omega(z)) &= 1 + \frac{1}{2}c_1 z + \left(\frac{c_2}{2} - \frac{c_1^2}{4}\right)z^2 + \left(\frac{5c_1^3}{48} + \frac{c_3 - c_1c_2}{2}\right)z^3\\ &\quad + \left(\frac{c_4 - c_1c_3}{2} + \frac{5c_1^2c_2}{16} - \frac{c_2^2}{4} - \frac{c_1^4}{32}\right)z^4\\ &\quad + \left(\frac{c_5 - c_1c_4 - c_2c_3}{2} + \frac{5c_1^2c_3 + c_1c_2^2}{16} - \frac{c_1^3c_2}{8} + \frac{c_1^5}{3840}\right)z^5\\ &\quad + \left(\frac{c_6 - c_1c_5 - c_2c_4}{2} + \frac{5c_1c_2c_3}{8} + \frac{5c_2^3}{48} - \frac{c_3^2}{4} + \frac{5c_1^6}{512}\right.\\ &\quad \left. + \frac{c_1^4c_2}{768} - \frac{3c_1^2c_2^2}{16} + \frac{5c_1^2c_4}{16} - \frac{c_1^3c_3}{8}\right)z^6 + \cdots.\end{aligned} \tag{18}$$

Comparing the coefficients of $z, z^2, z^3, z^4, z^5, z^6$ between equations (15) and (18), we obtain

$$\begin{aligned}a_2 &= \frac{c_1}{2},\\ a_3 &= \frac{c_2}{4},\\ a_4 &= \frac{c_3}{6} - \frac{c_1c_2}{24} - \frac{c_1^3}{144},\\ a_5 &= \frac{c_4}{8} - \frac{c_1c_3}{24} + \frac{5c_1^4}{1152} - \frac{c_1^2c_2}{192} - \frac{c_2^2}{32},\end{aligned} \tag{19}$$

$$a_6 = \frac{-3c_1c_4}{80} - \frac{7c_2c_3}{120} - \frac{11c_1^5}{4800} - \frac{43c_1c_2^2}{960} + \frac{71c_1^3c_2}{5760} + \frac{c_5}{10}, \tag{20}$$

$$\begin{aligned}a_7 &= \frac{c_1^2c_4}{480} + \frac{c_1c_2c_3}{480} + \frac{833c_1^6}{691200} - \frac{41c_1^2c_2^2}{3840} - \frac{109c_1^4c_2}{11520} - \frac{c_1c_5}{30}\\ &\quad - \frac{5c_2c_4}{96} + \frac{5c_2^3}{1152} + \frac{c_6}{12} + \frac{c_1^3c_3}{144}.\end{aligned} \tag{21}$$

Applying Lemma 2, we easily get

$$\begin{aligned}|a_2| &\le 1,\\ |a_3| &\le \frac{1}{2},\\ |a_4| &= \left|\frac{c_3}{6} - \frac{c_1c_2}{24} - \frac{c_1^3}{144}\right| = \left|\frac{1}{6}\left[c_3 - \frac{c_1c_2}{3}\right] + \frac{c_1}{72}\left[c_2 - \frac{c_1^2}{2}\right]\right|.\end{aligned} \tag{22}$$

Let $c_1 = c$, $c \in [0,2]$; by using Lemma 3, we show

$$|a_4| = \left|\frac{1}{6}\left[c_3 - \frac{c_1c_2}{3}\right] + \frac{c_1}{72}\left[c_2 - \frac{c_1^2}{2}\right]\right| \le \frac{1}{3} + \frac{c\left(2 - c^2/2\right)}{72}; \tag{23}$$

also, let

$$F(c) = \frac{1}{3} + \frac{c\left(2 - c^2/2\right)}{72}; \tag{24}$$

obviously, we find

$$F'(c) = \frac{1}{36} - \frac{c^2}{48}. \tag{25}$$

Setting $F'(c) = 0$, we have $c = 2\sqrt{3}/3$, and so, $F(c)$ has a maximum value attained at $c = 2\sqrt{3}/3$, also which is

$$\begin{aligned}|a_4| &\le F\left(\frac{2\sqrt{3}}{3}\right) = \frac{1}{3} + \frac{\sqrt{3}}{162} \approx 0.344,\\ |a_5| &= \left|\frac{c_4}{8} - \frac{c_1c_3}{24} + \frac{5c_1^4}{1152} - \frac{c_1^2c_2}{192} - \frac{c_2^2}{32}\right|\\ &= \left|\frac{1}{8}\left[c_4 - \frac{c_1c_3}{3}\right] - \frac{c_1^2}{576}\left[c_2 - \frac{c_1^2}{2}\right] - \frac{c_2}{32}\left(c_2 - \frac{c_1^2}{2}\right) - \frac{7c_1^2c_2}{576}\right|.\end{aligned} \tag{26}$$

Let $c_1 = c$, $c \in [0,2]$, according to Lemma 3, we obtain

$$|a_5| \le \frac{1}{4} + \frac{5c^2\left(2 - c^2/2\right)}{576} + \frac{1}{16}\left(2 - \frac{c^2}{2}\right) + \frac{7c^2}{288}. \tag{27}$$

Putting

$$F(c) = \frac{1}{4} + \frac{5c^2\left(2 - c^2/2\right)}{576} + \frac{1}{16}\left(2 - \frac{c^2}{2}\right) + \frac{7c^2}{288}, \tag{28}$$

we get

$$F'(c) = -\frac{7c}{144} - \frac{5c^3}{288} \le 0. \quad (29)$$

Therefore, the function $F(c)$ has a maximum value attained at $c = 0$, also which is

$$|a_5| \le F(0) = \frac{3}{8},$$

$$\begin{aligned} |a_6| &= \left| \frac{-3c_1c_4}{80} - \frac{7c_2c_3}{120} - \frac{11c_1^5}{4800} - \frac{43c_1c_2^2}{960} + \frac{71c_1^3c_2}{5760} + \frac{c_5}{10} \right| \\ &= \left| \frac{1}{24}\left[c_5 - \frac{9c_1c_4}{10}\right] + \frac{7}{120}[c_5 - c_2c_3] + \frac{11c_1^3}{2400}\left[c_2 - \frac{c_1^2}{2}\right] \right. \\ &\quad \left. - \frac{43c_1c_2}{960}\left(c_2 - \frac{c_1^2}{2}\right) - \frac{211c_1^3c_2}{14400} \right|. \end{aligned} \quad (30)$$

Let $c_1 = c, c \in [0, 2]$, in view of Lemma 3, we have that

$$|a_6| \le \frac{7}{60} + \frac{1}{12} + \frac{11c^3(2 - c^2/2)}{2400} + \frac{43}{240}\left(2 - \frac{c^2}{2}\right) + \frac{211c^3}{7200}. \quad (31)$$

Taking

$$F(c) = \frac{7}{60} + \frac{1}{12} + \frac{11c^3(2 - c^2/2)}{2400} + \frac{43}{240}\left(2 - \frac{c^2}{2}\right) + \frac{211c^3}{7200}, \quad (32)$$

we obtain

$$F'(c) = \frac{277c^2}{2400} - \frac{55c^4}{4800} - \frac{c}{240}. \quad (33)$$

Thus, $c = 0$ is the root of the function $F'(c) = 0$ and $F''(0) < 0$; we are easy to see that the function $F(c)$ has a maximum value attained at $c = 0$, also which is

$$|a_6| \le F(0) = \frac{67}{120},$$

$$\begin{aligned} |a_7| &= \left| \frac{c_1^2c_4}{480} + \frac{c_1c_2c_3}{480} + \frac{833c_1^6}{691200} - \frac{41c_1^2c_2^2}{3840} - \frac{109c_1^4c_2}{11520} \right. \\ &\quad \left. - \frac{c_1c_5}{30} - \frac{5c_2c_4}{96} + \frac{5c_2^3}{1152} + \frac{c_6}{12} + \frac{c_1^3c_3}{144} \right| \\ &= \left| \frac{-37c_1^6}{691200} - \frac{25c_1^2c_2^2}{5760} - \frac{c_1c_5}{30} + \frac{c_1^2[c_4 - c_2^2]}{480} + \frac{c_1c_2[c_3 - c_1c_2]}{480} \right. \\ &\quad + \frac{c_1^3[c_3 - c_1c_2]}{144} - \frac{29c_1^4[c_2 - c_1^2/2]}{11520} + \frac{5c_2^2[c_2 - c_1^2/2]}{1152} \\ &\quad \left. + \frac{[c_6 - 5/8c_2c_4]}{12} \right|. \end{aligned} \quad (34)$$

Let $c_1 = c, c \in [0, 2]$, by virtue of Lemma 3, we have that

$$\begin{aligned} |a_7| \le \frac{1}{6} + \frac{c^2}{240} + \frac{9c}{120} + \frac{29c^4(2 - c^2/2)}{11520} + \frac{37c^6}{691200} + \frac{c^3}{72} \\ + \frac{25c^2}{1440} + \frac{5(2 - c^2/2)}{288}. \end{aligned} \quad (35)$$

Letting

$$\begin{aligned} F(c) = \frac{1}{6} + \frac{c^2}{240} + \frac{9c}{120} + \frac{29c^4(2 - c^2/2)}{11520} + \frac{37c^6}{691200} + \frac{c^3}{72} \\ + \frac{25c^2}{1440} + \frac{5(2 - c^2/2)}{288}, \end{aligned} \quad (36)$$

so we get

$$F'(c) \ge 0. \quad (37)$$

Thus, the function $F(c)$ has a maximum value attained at $c = 2$, also which is

$$|a_7| \le F(2) = \frac{5587}{10800}. \quad (38)$$

Hence, the proof is complete.

Theorem 5. *If the function $f(z) \in \mathcal{S}_s^*$ and of the form ((1)), then we have*

$$|a_3 - a_2^2| \le \frac{1}{2}. \quad (39)$$

Proof. Applying equation (21), we have

$$|a_3 - a_2^2| = \left| \frac{c_2}{4} - \frac{c_1^2}{4} \right|. \quad (40)$$

Then, by applying Lemma 1, we get

$$|a_3 - a_2^2| = \left| \frac{x(4 - c_1^2)}{8} - \frac{c_1^2}{8} \right|. \quad (41)$$

Suppose that $|x| = t, t \in [0, 1], c_1 = c, c \in [0, 2]$. Then, using the triangle inequality, we obtain

$$|a_3 - a_2^2| \le \frac{t(4 - c^2)}{8} + \frac{c^2}{8}. \quad (42)$$

Suppose

$$F(c, t) = \frac{t(4 - c^2)}{8} + \frac{c^2}{8}, \quad (43)$$

then for any $t \in (0, 1)$ and $c \in (0, 2)$, we get

$$\frac{\partial F}{\partial t} = \frac{4-c^2}{8} > 0, \tag{44}$$

which means that $F(c,t)$ is an increasing function on the closed interval [0,1] about t. Therefore, the function $F(c,t)$ can get the maximum value at $t=1$, that is,

$$\max F(c,t) = F(c,1) = \frac{\left(4-c^2\right)}{8} + \frac{c^2}{8} = \frac{1}{2}. \tag{45}$$

So, obviously,

$$\left|a_3 - a_2^2\right| \le \frac{1}{2}. \tag{46}$$

Hence, the proof is complete.

Theorem 6. *If the function $f(z) \in \mathcal{S}_s^*$ and of the form ((1)), then we have*

$$\left|a_2a_3 - a_4\right| \le \frac{1}{3}. \tag{47}$$

Proof. From (21), we have

$$\left|a_2a_3 - a_4\right| = \left|\frac{c_1c_2}{8} + \frac{c_1^3}{144} - \frac{c_3}{6} + \frac{c_1c_2}{24}\right| = \left|\frac{c_1c_2}{6} - \frac{c_3}{6} + \frac{c_1^3}{144}\right|. \tag{48}$$

Now, in view of Lemma 1, we get

$$\left|a_2a_3 - a_4\right| = \left|\frac{7c_1^3}{144} + \frac{\left(4-c_1^2\right)c_1x^2}{24} - \frac{\left(4-c_1^2\right)\left(1-|x|^2\right)z}{12}\right|. \tag{49}$$

Let $|x| = t$, $t \in [0,1]$, $c_1 = c$, $c \in [0,2]$. Then, using the triangle inequality, we deduce that

$$\left|a_2a_3 - a_4\right| \le \frac{7c^3}{144} + \frac{\left(4-c^2\right)ct^2}{24} + \frac{\left(4-c^2\right)\left(1-t^2\right)}{12}. \tag{50}$$

Assume that

$$F(c,t) = \frac{7c^3}{144} + \frac{\left(4-c^2\right)ct^2}{24} + \frac{\left(4-c^2\right)\left(1-t^2\right)}{12}. \tag{51}$$

Therefore, for any $t \in (0,1)$ and $c \in (0,2)$, we have

$$\frac{\partial F}{\partial t} = \frac{\left(4-c^2\right)t(c-2)}{12} < 0, \tag{52}$$

that is, $F(c,t)$ is an decreasing function on the closed interval [0,1] about t. This implies that the maximum value of $F(c,t)$ occurs at $t=0$, which is

$$\max F(c,t) = F(c,0) = \frac{7c^3}{144} + \frac{\left(4-c^2\right)}{12}. \tag{53}$$

Define

$$G(c) = \frac{\left(4-c^2\right)}{12} + \frac{7c^3}{144}; \tag{54}$$

we clearly see that the function $G(c)$ has a maximum value attained at $c=0$, also which is

$$\left|a_2a_3 - a_4\right| \le G(0) = \frac{1}{3}. \tag{55}$$

Hence, the proof is complete.

Theorem 7. *If the function $f(z) \in \mathcal{S}_s^*$ and of the form ((1)), then we have*

$$\left|a_2a_4 - a_3^2\right| \le \frac{1}{4}. \tag{56}$$

Proof. Let $f(z) \in \mathcal{S}_s^*$, then by equation (21), we get

$$\left|a_2a_4 - a_3^2\right| = \left|\frac{c_1c_3}{12} - \frac{c_1^2c_2}{48} - \frac{c_1^4}{288} - \frac{c_2^2}{16}\right|. \tag{57}$$

Now, in terms of Lemma 1, we obtain

$$\begin{aligned}\left|a_2a_4 - a_3^2\right| &= \left|\frac{c_1c_3}{12} - \frac{c_1^2c_2}{48} - \frac{c_1^4}{288} - \frac{c_2^2}{16}\right| \\ &= \left|-\frac{5c_1^4}{576} - \frac{x^2c_1^2\left(4-c_1^2\right)}{48} - \frac{x^2\left(4-c_1^2\right)^2}{64}\right. \\ &\quad \left.+ \frac{c_1\left(4-c_1^2\right)\left(1-|x|^2\right)z}{24}\right|.\end{aligned} \tag{58}$$

Let $|x| = t$, $t \in [0,1]$, $c_1 = c$, $c \in [0,2]$. Then, using the triangle inequality, we get

$$\left|a_2a_4 - a_3^2\right| \le \frac{t^2c^2\left(4-c^2\right)}{48} + \frac{\left(1-t^2\right)c\left(4-c^2\right)}{24} + \frac{t^2\left(4-c^2\right)^2}{64} + \frac{5c^4}{576}. \tag{59}$$

Setting

$$F(c,t) = \frac{t^2c^2\left(4-c^2\right)}{48} + \frac{\left(1-t^2\right)c\left(4-c^2\right)}{24} + \frac{t^2\left(4-c^2\right)^2}{64} + \frac{5c^4}{576}, \tag{60}$$

then, for any $t \in (0,1)$ and $c \in (0,2)$, we have

$$\frac{\partial F}{\partial t} = \frac{t\left(c^2 - 8c + 12\right)\left(4-c^2\right)}{96} > 0, \tag{61}$$

which implies that $F(c,t)$ increases on the closed interval [0,1] about t. That is, that $F(c,t)$ has a maximum value at $t=1$, which is

$$\max F(c,t)=F(c,1)=\frac{c^2\left(4-c^2\right)}{48}+\frac{\left(4-c^2\right)^2}{64}+\frac{5c^4}{576}. \quad (62)$$

Putting

$$G(c)=\frac{c^2\left(4-c^2\right)}{48}+\frac{\left(4-c^2\right)^2}{64}+\frac{5c^4}{576}, \quad (63)$$

then we have

$$G'(c)=\frac{c\left(4-c^2\right)}{24}-\frac{c^3}{24}-\frac{c\left(4-c^2\right)}{16}+\frac{5c^3}{144}. \quad (64)$$

If $G'(c)=0$, then the root is $c=0$. Also, since $G''(0)=-1/12<0$, so the function $G(c)$ can take the maximum value at $c=0$, which is

$$|a_2a_4-a_3^2|\le G(0)=\frac{1}{4}. \quad (65)$$

Hence, the proof is complete.

Theorem 8. *If the function $f(z)\in\mathcal{S}_s^*$ and of the form ((1)), then we have*

$$|a_2a_5-a_3a_4|\le\frac{11}{36}. \quad (66)$$

Proof. Let $f(z)\in\mathcal{S}_s^*$, then by using (21), we have

$$\begin{aligned}|a_2a_5-a_3a_4|&=\left|\frac{5c_1^5}{2304}+\frac{c_1c_4}{16}-\frac{c_1c_2^2}{192}-\frac{c_1^2c_3}{48}-\frac{c_1^3c_2}{1152}-\frac{c_2c_3}{24}\right|\\&=\left|-\frac{c_1^3\left[c_2-c_1^2/2\right]}{1152}-\frac{c_3\left[c_2-c_1^2/2\right]}{24}+\frac{c_1\left[c_4-c_1c_3\right]}{24}\right.\\&\quad\left.+\frac{c_1^5}{576}+\frac{c_1\left[c_4-1/4c_2^2\right]}{48}\right|.\end{aligned} \quad (67)$$

Let $c_1=c, c\in[0,2]$, according to Lemma 3, we obtain

$$|a_5|\le\frac{1}{4}+\frac{5c^2\left(2-c^2/2\right)}{576}+\frac{1}{16}\left(2-\frac{c^2}{2}\right)+\frac{7c^2}{288}. \quad (68)$$

Taking

$$F(c)=\frac{c^3\left[2-c^2/2\right]}{1152}+\frac{\left[2-c^2/2\right]}{12}+\frac{c}{8}+\frac{c^5}{576}. \quad (69)$$

Then, $\forall c\in(0,2)$, we have

$$F'(c)=\frac{c^2}{192}+\frac{c^4}{128}-\frac{c}{12}+\frac{1}{8}>0, \quad (70)$$

which implies that $F(c)$ increases on the closed interval [0,2] about c. Namely, the maximum value of $F(c)$ attains at $c=2$, also which is

$$|a_2a_5-a_3a_4|\le F(2)=\frac{11}{36}. \quad (71)$$

The proof of Theorem 8 is completed.

Theorem 9. *If the function $f(z)\in\mathcal{S}_s^*$ and of the form ((1)), then we have*

$$|a_5-a_2a_4|\le\frac{13}{32}. \quad (72)$$

Proof. Assume that $f(z)\in\mathcal{S}_s^*$, then from (21), we obtain

$$\begin{aligned}|a_5-a_2a_4|&=\left|\frac{c_1^4}{128}-\frac{c_1c_3}{8}+\frac{c_1^2c_2}{64}-\frac{c_2^2}{32}+\frac{c_4}{8}\right|\\&=\left|\frac{\left[c_1^4+c_2^2+2c_1c_3-3c_1^2c_2-c_4\right]}{32}-\frac{5c_1^2\left[c_2-c_1^2/2\right]}{64}\right.\\&\quad\left.-\frac{3\left[c_4-2/3c_1c_3\right]}{32}\right|.\end{aligned} \quad (73)$$

Next, by virtue of Lemma 3, we obtain

$$|a_5-a_2a_4|\le\frac{1}{4}+\frac{5c^2\left[2-c^2/2\right]}{64}. \quad (74)$$

Setting

$$F(c)=\frac{1}{4}+\frac{5c^2\left[2-c^2/2\right]}{64}. \quad (75)$$

Then, we have

$$F'(c)=\frac{5c}{16}-\frac{5c^3}{32}. \quad (76)$$

Let $F'(c)=0$, we get $c=0$ or $c=\sqrt{2}$ and $F'(\sqrt{2})<0$, which implies that the maximum value of $F(c)$ attains at $c=\sqrt{2}$, also which is

$$|a_5-a_2a_4|\le F\left(\sqrt{2}\right)=\frac{13}{32}. \quad (77)$$

Hence, the proof is complete.

Theorem 10. *If the function $f(z)\in\mathcal{S}_s^*$ and of the form ((1)), then we have*

$$|a_5a_3-a_4^2|\le\frac{97}{324}. \quad (78)$$

Proof. Assume that $f(z) \in \mathcal{S}_s^*$, then from (21), we obtain

$$\begin{aligned} |a_5a_3 - a_4^2| &= \left| \frac{7c_1^4c_2}{13824} + \frac{c_2c_4}{32} + \frac{c_1c_2c_3}{288} - \frac{c_2^3}{128} + \frac{c_1^3c_3}{432} - \frac{7c_1^2c_2^2}{2304} - \frac{c_3^2}{36} - \frac{c_1^6}{20736} \right| \\ &= \left| \frac{c_2[c_4 - c_1c_3/9]}{32} - \frac{c_3[c_3 - c_1c_2/4]}{36} - \frac{c_2^2[c_2 - c_1^2/2]}{128} - \frac{c_1^2c_2[c_2 - c_1^2/2]}{144} + \frac{c_1^3[c_3 - 31/32c_1c_2]}{432} - \frac{5c_1^4c_2}{6912} - \frac{c_1^6}{20736} \right|. \end{aligned} \tag{79}$$

Next, in terms of Lemma 3, we obtain

$$|a_5a_3 - a_4^2| \le \frac{1}{8} + \frac{1}{9} + \frac{[2 - c^2/2]}{32} + \frac{c^2[2 - c^2/2]}{72} + \frac{c^3}{216} + \frac{5c^4}{3456} + \frac{c^6}{20736}. \tag{80}$$

Putting

$$F(c) = \frac{1}{8} + \frac{1}{9} + \frac{[2 - c^2/2]}{32} + \frac{c^2[2 - c^2/2]}{72} + \frac{c^3}{216} + \frac{5c^4}{3456} + \frac{c^6}{20736}. \tag{81}$$

Then, for any $c \in (0, 2)$, we have $F'(c) > 0$, which means that the maximum value of $F(c)$ arrives at $t = 2$, also which is

$$|a_5a_3 - a_4^2| \le F(2) = \frac{97}{324}. \tag{82}$$

Hence, the proof is complete.

Theorem 11. *If the function $f(z) \in \mathcal{S}_s^*$ and of the form ((1)), then we have*

$$|H_4(1)| \le 0.81945. \tag{83}$$

Proof. Because of

$$\begin{aligned} H_4(1) = a_7&\{a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2)\} \\ &- a_6\{a_3(a_2a_5 - a_3a_4) - a_4(a_5 - a_2a_4) + a_6(a_3 - a_2^2)\} \\ &- a_6\{a_3(a_2a_5 - a_3a_4) - a_4(a_5 - a_2a_4) + a_6(a_3 - a_2^2)\} \\ &+ a_5\{a_3(a_3a_5 - a_4^2) - a_5(a_5 - a_2a_4) + a_6(a_4 - a_2a_3)\} \\ &- a_4\{a_4(a_3a_5 - a_4^2) - a_5(a_2a_5 - a_3a_4) + a_6(a_4 - a_2a_3)\}, \end{aligned} \tag{84}$$

then, by applying the triangle inequality, we get

$$\begin{aligned} |H_4(1)| = |a_7||a_3||&|a_2a_4 - a_3^2| + |a_7||a_4||a_4 - a_2a_3| \\ &+ |a_7||a_5||a_3 - a_2^2| + |a_6||a_3||a_2a_5 - a_3a_4| \\ &+ |a_6||a_4||a_5 - a_2a_4| + |a_6|^2|a_3 - a_2^2| \\ &+ |a_5||a_3||a_3a_5 - a_4^2| + |a_5|^2|a_5 - a_2a_4| \\ &+ |a_5||a_6||a_4 - a_2a_3| + |a_4|^2|a_3a_5 - a_4^2| \\ &+ |a_4||a_5||a_2a_5 - a_3a_4| + |a_4||a_6||a_4 - a_2a_3|. \end{aligned} \tag{85}$$

Next, substituting (13) and (39)–(78) into (85), we easily obtain the desired assertion (83).

3. Conclusion

In the present paper, we mainly get upper bounds of the fourth-order Hankel determinant $H_4(1)$ of starlike functions connected with the sine function. However, the results obtained in this paper are not sharp. In the future, we will consider the sharpness of the results. Also, we can discuss the related research of the fifth-order Hankel determinant and fifth-order Toeplitz determinant for this function class.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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