Research Article

Products of Composition and Differentiation between the Fractional Cauchy Spaces and the Bloch-Type Spaces

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The operators $DC_\alpha$ and $C_\beta D$ are defined by $DC_\alpha(f) = (f \circ \Phi)'$ and $C_\beta D(f) = f' \circ \Phi$ where $\Phi$ is an analytic self-map of the unit disc and $f$ is analytic in the disc. A characterization is provided for boundedness and compactness of the products of composition and differentiation from the spaces of fractional Cauchy transforms $F_\alpha$ to the Bloch-type spaces $B_\beta$, where $\alpha > 0$ and $\beta > 0$. In the case $\beta < 2$, the operator $DC_\alpha : F_\alpha \rightarrow B_\beta$ is compact $\Leftrightarrow$ $DC_\alpha : F_\alpha \rightarrow B_\beta$ is bounded $\Leftrightarrow$ $\Phi \in B_\beta$, $\Phi \Phi' \in B_\beta$ and $\| \Phi \|_\infty < 1$. For $\beta < 1$, $C_\beta D : F_\alpha \rightarrow B_\beta$ is compact $\Leftrightarrow$ $C_\beta D : F_\alpha \rightarrow B_\beta$ is bounded $\Leftrightarrow$ $\Phi \in B_\beta$ and $\| \Phi \|_\infty < 1$.

1. Introduction

Let $U = \{ z \in C : |z| < 1 \}$ and let $H(U)$ denote the family of functions analytic on $U$. Let $M$ denote the Banach space of complex Borel measures on $T = \{ x \in C : |x| = 1 \}$, endowed with the total variation norm. For $\alpha > 0$, the space $F_\alpha$ of fractional Cauchy transforms is the family of functions of the form

$$f(z) = \int_T \frac{1}{(1-xz)^\alpha} \, d\mu(x)(|z| < 1), \quad (1)$$

where $\mu \in M$. The principal branch of the logarithm is used here. The space $F_\alpha$ is a Banach space, with norm

$$\| f \|_{F_\alpha} = \inf \| \mu \|, \quad (2)$$

where $\mu$ varies over all measures in $M$ for which (1) holds. The families $F_\alpha$ have been studied extensively [1, 2]. Interest in these spaces was first established in connection with the classical family $S$ of normalized univalent functions. It is known that $S \subseteq F_\alpha$ for any $\alpha > 2$ [2]. The reference [2] also includes MacGregor’s construction of a function $f \in S$ with $f \notin F_2$.

Let $\beta > 0$. The Bloch-type space $B_\beta$ is the Banach space of functions analytic in $U$ such that $\sup_{z \in U} (1 - |z|^2)^\beta |f'(z)| < \infty$, with norm

$$\| f \|_{B_\beta} = |f(0)| + \sup_{z \in U} (1 - |z|^2)^\beta |f'(z)|. \quad (3)$$

The relation (1) implies that $F_\alpha \subseteq B_{\alpha+1}$, and there is a constant $C$ depending only on $\alpha$ such that $\| f \|_{B_{\alpha+1}} \leq C \| f \|_{F_\alpha}$ for all $f \in F_\alpha$.

Let $\Phi$ be an analytic self-map of $U$. The composition operator $C_\alpha\Phi$ is defined by $C_\alpha\Phi(f) = f \circ \Phi$ for $f \in H(U)$. The differentiation operator $D$ is defined by $D(f) = f'$. In this paper, the products $C_\beta D(f) = f' \circ \Phi$ and $DC_\alpha(f) = \Phi' \circ f' \circ \Phi$ are studied. Conditions on $\Phi$ are given, necessary and sufficient to imply boundedness or compactness of $C_\beta D : F_\alpha \rightarrow B_\beta$ and $DC_\alpha : F_\alpha \rightarrow B_\beta$.

Products of composition and differentiation on the Bloch space were studied by Ohno in [3]. In [4], Li and Stević studied $C_\beta D$ and $DC_\alpha$ acting between the weighted Bergman spaces and the Bloch-type spaces. In [5], Hibschweiler and Portnoy studied these operators between Bergman and Hardy spaces.
2. Preliminary Results

Fix $\alpha > 0$. For fixed $z \in U$ and for $n = 0, 1, \ldots$, the relation (1) yields a constant $C$ depending only on $n$ such that $|f^{(n)}(z)| \leq C|f|_{F_n}(1 - |z|)\alpha_n^{2n}$ [2].

For each $w \in U$, $\|1/(1 - wz)\|_{F_n} = 1$ [2].

We follow the convention that $C$ denotes a positive constant, the precise value of which will differ from one appearance to the next.

Lemma 1 and Lemma 2 will be used to develop test functions for $F_a$. Proofs appear in [6].

Lemma 1. Fix $\alpha > 0$. For $w \in U$, define

$$h_w(z) = \frac{1 - |w|^2}{(1 - wz)^{\alpha_2}} (z \in U).$$

(4)

Then, $h_w \in F_a$, and there is a constant $C$ such that $\|h_w\|_{F_a} \leq C$ for all $w \in U$.

Lemma 2. Fix $\alpha > 0$. For $w \in U$, define

$$k_w(z) = \frac{(1 - |w|^2)^2}{(1 - wz)^{\alpha_2}} (z \in U).$$

(5)

Then, $k_w \in F_a$, and there is a constant $C$ such that $\|k_w\|_{F_a} \leq C$ for all $w \in U$.

3. The Operator $DC_\Phi : F_a \rightarrow B^\beta$

In [7], Shapiro proved that the condition $\|\Phi\|_{\infty} < 1$ is necessary for $C_\Phi : X \rightarrow X$ to be compact, for Banach spaces $X$ obeying regular boundary and Möbius invariance. In particular, Shapiro’s result applies to the Lipschitz spaces and, thus, to the space $B^\beta$ when $\gamma < 1$ [8].

Theorem 3. Fix $\alpha > 0$ and $0 < \beta < 2$. Let $\Phi$ be an analytic self-map of $U$.

$$DC_\Phi : F_a \rightarrow B^\beta$$

is bounded $\iff$

$$\Phi' \in B^\beta, \Phi \Phi' \in B^\beta \text{ and } \|\Phi\|_{\infty} < 1 \iff$$

(6)

$$DC_\Phi : F_a \rightarrow B^\beta$$

is compact.

Proof. First, assume that $DC_\Phi : F_a \rightarrow B^\beta$ is bounded, that is, there is a constant $C$ such that $\|DC_\Phi(f)\|_{B^\beta} \leq C\|f\|_{F_a}$ for all $f \in F_a$. It is clear that $\Phi' = DC_\Phi(z) \in B^\beta$ and $\Phi \Phi' = DC_\Phi(z^2/2) \in B^\beta$. Thus,

$$\| (1 - |z|^2)^{\beta} |\Phi'(z)| \| \leq C,$$

(7)

$$\| (1 - |z|^2)^{\beta} |\Phi(z)\Phi'(z) + (\Phi'(z)) | \| \leq C,$$

(8)

for all $z \in U$. It follows that

$$\sup_{z \in U} (1 - |z|^2)^{\beta} |\Phi'(z)| < \infty,$$

(9)

and thus, $\Phi \in B^{\beta/2}$.

Let $w \in U$ and define

$$g_w(z) = \frac{\alpha + 1}{(1 - \Phi(w)z)^{\alpha + 1}} - \frac{\alpha(1 - |\Phi(w)|^2)}{(1 - |\Phi(w)z|^2)^{\alpha + 1}} (z \in U).$$

(10)

By Lemma 1 and the preliminary results, there is a constant $C$ independent of $w$ such that $\|g_w\|_{F_a} \leq C$, and thus,

$$\|DC_\Phi(g_w)\|_{B^\beta} = \| (g_w \circ \Phi)\Phi'\|_{B^\beta} \leq C.$$
\[
\frac{(1 - |z|^2)^\beta |f'(w)|}{(1 - |\Phi(w)|^2)^{\beta/2}} \leq C (1 - |\Phi(w)|^2)^{(\alpha - \beta + 1)/2} \rightarrow 0, \quad (18)
\]
as \(|\Phi(w)| \rightarrow 1\). Thus, \(C_{\Phi}: B^{\beta/2} \rightarrow B^{\beta/2}\) is compact \([9]\), and it follows as in \([7]\) that \(\|\Phi\|_{C_0} < 1\). It has been established that the conditions \(\Phi' \in B^\beta, \Phi'' \in B^\beta, \Phi^\prime \in B^{\beta}\), and \(\|\Phi\|_{C_0} < 1\) are necessary if \(DC_{\Phi}: F_a \rightarrow B^\beta\) is bounded.

Next, assume that \(\Phi' \in B^\beta, \Phi'' \in B^\beta, \) and \(\|\Phi\|_{C_0} < 1\). To show that \(DC_{\Phi}: F_a \rightarrow B^\beta\) is compact, let \((f_n)\) be a bounded sequence in \(F_a\) with \(f_n \rightarrow 0\) uniformly on compact subsets of \(U\) as \(n \rightarrow \infty\). It is enough to prove that \(\|DC_{\Phi}(f_n)\|_{B^\beta} \rightarrow 0\) as \(n \rightarrow \infty\). First, note that if \(|f_n'(\Phi(0))\Phi'(0)| \rightarrow 0\) as \(n \rightarrow \infty\). For \(z \in U\), \((9)\) yields

\[
(1 - |z|^2)^{\beta} |DC_{\Phi}f_n'(z)| = (1 - |z|^2)^{\beta} \left| f_n'(\Phi(z)) \Phi'(z) \right|^2 + \frac{f_n'(\Phi(z))\Phi''(z)}{(1 - |\Phi(z)|^2)^{\alpha+1}} \leq C \max_{|w| = |\Phi(z)|} |f_n'(w)| + \|\Phi''\|_{C_0} \max_{|w| = |\Phi(z)|} |f_n'(w)|.
\]

(19)

Since \(f_n' \rightarrow 0\) and \(f_n'' \rightarrow 0\) uniformly on compact subsets as \(n \rightarrow \infty\), the argument shows that \(\sup_{z \in U} (1 - |z|^2)^{\beta} |DC_{\Phi}f_n'(z)| \rightarrow 0\) as \(n \rightarrow \infty\). Thus, \(\|DC_{\Phi}(f_n)\|_{B^\beta} \rightarrow 0\) as \(n \rightarrow \infty\), and \(DC_{\Phi}: F_a \rightarrow B^\beta\) is compact, as required.

The remaining implication is clear, and the proof is complete.

**Theorem 4.** Fix \(\alpha > 0\) and \(\beta \geq 2\). Let \(\Phi\) be an analytic self-map of \(U\).

Then,

\[
DC_{\Phi}: F_a \rightarrow B^\beta \text{ is bounded } \iff \sup_{z \in U} \frac{(1 - |z|^2)^\beta |\Phi''(z)|}{(1 - |\Phi(z)|^2)^{\alpha+1}} < \infty, \quad (21)
\]

and

\[
\sup_{z \in U} \frac{(1 - |z|^2)^\beta |\Phi'(z)|^2}{(1 - |\Phi(z)|^2)^{\alpha+2}} < \infty. \quad (22)
\]

**Proof.** Fix \(\alpha, \beta\) and \(\Phi\) as described.

First, assume \((21)\) and \((22)\). Let \(f \in F_a\). By \((21)\) and the introductory remarks in Section 2,

\[
(1 - |z|^2)^\beta |f'(\Phi(z))|^2 |\Phi''(z)| \leq (1 - |z|^2)^\beta |\Phi''(z)| \frac{C\|f\|_{F_a}}{(1 - |\Phi(z)|^2)^{\alpha+1}} \leq C\|f\|_{F_a}.
\]

(23)

A similar argument using \((22)\) yields

\[
(1 - |z|^2)^\beta |f''(\Phi(z))|^2 \leq C\|f\|_{F_a}, \quad (24)
\]

for all \(z \in U\). Thus, \(\sup_{z \in U} (1 - |z|^2)^\beta |DC_{\Phi}f'(z)| \leq C\|f\|_{F_a}\).

Since \(\|DC_{\Phi}(f)(0)\| \leq C\|f\|_{F_a}\), it now follows that \(\|DC_{\Phi}(f)\|_{B^\beta} \leq C\|f\|_{F_a}\) as required.

For the converse, assume that \(\|DC_{\Phi}(f)\|_{B^\beta} \leq C\|f\|_{F_a}\) for a constant \(C\) independent of \(f \in F_a\). In particular, \(\Phi' \in B^\beta\).

The argument leading to \((16)\) remains valid for \(\beta \geq 2\). Thus, \((22)\) holds. It remains to prove \((21)\). First, note that

\[
\sup_{|\Phi(w)| \leq 1/2} \frac{(1 - |w|^2)^\beta |\Phi''(w)|}{(1 - |\Phi(w)|^2)^{\alpha+1}} \leq \frac{4}{3} \|\Phi'\|_{B^\beta} < \infty.
\]

(25)

For \(w \in U\), define

\[
H_w(z) = \frac{(\alpha + 3)(1 - |\Phi(w)|^2)}{(1 - |\Phi(w)z|^\alpha)} - \frac{(\alpha + 1)(1 - |\Phi(w)|^2)^2}{(1 - |\Phi(w)z|^\alpha^2)},
\]

(26)

for \(z \in U\). By Lemma 1 and Lemma 2, there is a constant \(C\) independent of \(w\) such that \(\|H_w\|_{F_a} \leq C\). Thus, \(\|DC_{\Phi}(H_w(z))\|_{B^\beta} \leq C\) for all \(w \in U\). It follows that

\[
\sup_{z \in U} (1 - |z|^2)^\beta \left| H_w'(\Phi(z)) \Phi''(z) + (\Phi'(z))^2 H_w''(\Phi(z)) \right| < C,
\]

(27)

for all \(w \in U\). An argument using \(H_w'(\Phi(w)) = (\alpha + 1)(\Phi(w))/(1 - |\Phi(w)|^2)^{\alpha+1}\) and \(H_w''(\Phi(w)) = 0\) yields

\[
\sup_{1/2 < |\Phi(w)|} \frac{(1 - |w|^2)^\beta |\Phi''(w)|}{(1 - |\Phi(w)|^2)^{\alpha+1}} < \infty.
\]

(28)

The relations \((25)\) and \((28)\) establish relation \((21)\), and the proof is complete.

**Theorem 5.** Fix \(\alpha > 0\) and assume \(\beta \geq 2\). Let \(\Phi\) be a self-map of \(U\) for which \(DC_{\Phi}: F_a \rightarrow B^\beta\) is bounded.

\[
DC_{\Phi}: F_a \rightarrow B^\beta \text{ is compact } \iff \lim_{|\Phi(z)| \rightarrow 0} \frac{(1 - |z|^2)^\beta |\Phi''(z)|}{(1 - |\Phi(z)|^2)^{\alpha+1}} = 0, \quad (30)
\]

and

\[
\lim_{|\Phi(z)| \rightarrow 1} \frac{(1 - |z|^2)^\beta |\Phi'(z)|^2}{(1 - |\Phi(z)|^2)^{\alpha+2}} = 0. \quad (31)
\]
Proof. First, assume that $DC_{\Phi} : F_a \rightarrow B^\beta$ is bounded and relations (30) and (31) hold. Let $(f_n)$ be a bounded sequence in $F_a$ such that $f_n \rightarrow 0$ uniformly on compact subsets of $U$. As previously noted, there is a constant $C$ depending only on $\alpha$ such that
\[
(1 - |z|^2)^\beta |f_n'(\Phi(z))||\Phi'(z)|^2 \leq C(1 - |z|^2)^\beta |\Phi'(z)|^2 \left(1 - |\Phi(z)|^2 \right)^{\alpha+1},
\tag{32}
\]
for $n = 1, 2, \ldots$ and $z \in U$. Relation (31) now implies that given $\varepsilon > 0$, there exists $r_0, 0 < r_0 < 1$, such that
\[
\sup_{|\Phi(z)| > r_0} (1 - |z|^2)^\beta |f_n''(\Phi(z))||\Phi'(z)|^2 < \varepsilon,
\tag{33}
\]
for all $n$. Since $DC_{\Phi} : F_a \rightarrow B^\beta$ is bounded, relation (9) holds, and thus
\[
(1 - |z|^2)^\beta |f_n''(\Phi(z))||\Phi'(z)|^2 < C|f_n'''(\Phi(z))|,
\tag{34}
\]
for all $z \in U$. Since $f_n''' \rightarrow 0$ uniformly on $\{w : |w| \leq r_0\}$, there exists $N > 0$ such that
\[
\sup_{|\Phi(z)| \leq r_0} (1 - |z|^2)^\beta |f_n'''(\Phi(z))||\Phi'(z)|^2 < \varepsilon,
\tag{35}
\]
for all $n > N$. The relations (33) and (35) yield
\[
\sup_{z \in U} (1 - |z|^2)^\beta |f_n''(\Phi(z))||\Phi'(z)|^2 < \varepsilon,
\tag{36}
\]
for $n > N$. A similar argument using $\Phi' \in B^\beta$ and (30) yields $N_1 > 0$ such that
\[
\sup_{z \in U} (1 - |z|^2)^\beta |f_n''(\Phi(z))||\Phi'(z)|^2 < \varepsilon,
\tag{37}
\]
for $n > N_1$. The relations (36) and (37) yield
\[
\sup_{z \in U} (1 - |z|^2)^\beta |DC_{\Phi}f_n''(z)| \rightarrow 0,
\tag{38}
\]
as $n \rightarrow \infty$.
Since $||(DC_{\Phi}f_n)(0)|| \rightarrow 0$ as $n \rightarrow \infty$, the argument yields $||DC_{\Phi}(f_n)||_{B^\beta} \rightarrow 0$ as $n \rightarrow \infty$ for any sequence $(f_n)$ as described, and therefore, $DC_{\Phi} : F_a \rightarrow B^\beta$ is compact.

For the converse, assume that $DC_{\Phi} : F_a \rightarrow B^\beta$ is compact. We may assume that $||\Phi||_{\infty} = 1$. Let $(z_n)$ be any sequence in $U$ with $|\Phi(z_n)| \rightarrow 1$ as $n \rightarrow \infty$. For $z \in U$, define
\[
f_n(z) = \frac{(\alpha + 3)(1 - |\Phi(z_n)|^2)}{(1 - |\Phi(z_n)|^2)^{\alpha+1}} - \frac{(\alpha + 1)(1 - |\Phi(z_n)|^2)^2}{(1 - |\Phi(z_n)|^2)^{\alpha+2}}.
\tag{39}
\]

By the lemmas above, $||f_n||_{F_a} \leq C$. Also, $f_n \rightarrow 0$ uniformly on compact subsets. Therefore, $||DC_{\Phi}(f_n)||_{B^\beta} \rightarrow 0$ and
\[
\sup_{z \in U} (1 - |z|^2)^\beta |f_n''(\Phi(z))|\Phi''(z) + f_n'''(\Phi(z))|\Phi'(z)|^2 \rightarrow 0,
\tag{40}
\]
as $n \rightarrow \infty$. Calculations yield $f_n'''(\Phi(z_n)) = 0$ and
\[
f_n''(\Phi(z_n)) = \frac{(\alpha + 1)|\Phi(z_n)|}{(1 - |\Phi(z_n)|^2)^{\alpha+1}}.
\tag{41}
\]
Substitution into (40) yields
\[
\frac{(1 - |z|^2)^\beta |\Phi(z_n)||\Phi''(z_n)|}{(1 - |\Phi(z_n)|^2)^{\alpha+1}} \rightarrow 0,
\tag{42}
\]
as $n \rightarrow \infty$. Since $(z_n)$ is a generic sequence with $|\Phi(z_n)| \rightarrow 1$ as $n \rightarrow \infty$, this yields the relation (30).

A similar argument using the functions
\[
g_n(z) = \frac{(\alpha + 2)(1 - |\Phi(z_n)|^2)}{(1 - |\Phi(z_n)|^2)^{\alpha+1}} - \frac{(\alpha + 1)(1 - |\Phi(z_n)|^2)^2}{(1 - |\Phi(z_n)|^2)^{\alpha+2}}
\tag{43}
\]
yields the relation (31). The details are omitted.

Theorem 3 implies that if $DC_{\Phi} : F_a \rightarrow B^\beta$ is bounded for fixed $\alpha, \beta$ with $\beta < 2$, then $DC_{\Phi} : F_\gamma \rightarrow B^\beta$ is compact for all $\gamma > 0$. The next corollary gives a related result when $\beta \geq 2$.

**Corollary 6.** Fix $\alpha > 0$ and $\beta \geq 2$. Let $\Phi$ be a self-map of $U$ and assume that $DC_{\Phi} : F_a \rightarrow B^\beta$ is bounded. Then, $DC_{\Phi} : F_\gamma \rightarrow B^\beta$ is compact for any $\gamma, 0 < \gamma < \alpha$.

**Proof.** By assumption, there is a constant $C$ such that $||DC_{\Phi}(f)||_{B^\beta} \leq C ||f||_{F_a}$ for all $f \in F_a$. Fix $\gamma$ with $0 < \gamma < \alpha$ and let $f \in F_\gamma$. Then, $f \in F_a$ and $||f||_{F_a} \leq ||f||_{F_\gamma} [2]$. Therefore, $DC_{\Phi} : F_\gamma \rightarrow B^\beta$ is compact and Theorem 5 applies.

Since $DC_{\Phi} : F_a \rightarrow B^\beta$ is bounded, (21) yields
\[
(1 - |z|^2)^\beta |\Phi''(z)| \leq C(1 - |\Phi(z)|^2)^{\alpha-\gamma},
\tag{44}
\]
and therefore,

$$\lim_{|\Phi(z)| \to 1} \frac{(1 - |z|^2)^\beta |\Phi'(z)|}{(1 - |\Phi(z)|^2)^{\alpha + 2}} = 0. \quad (45)$$

A similar argument using (22) yields

$$\lim_{|\Phi(z)| \to 1} \frac{(1 - |z|^2)^\beta |\Phi'(z)|^2}{(1 - |\Phi(z)|^2)^{\alpha + 2}} = 0. \quad (46)$$

Theorem 5 now yields $DC_\Phi : F_\gamma \to B^\beta$ is compact.

4. The Operator $C_\alpha D$

In this section, characterizations are given for self-maps $\Phi$ for which $C_\alpha D : F_\alpha \to B^\beta$ is bounded or compact. The proofs are similar to those in Section 3, so details are kept to a minimum.

**Theorem 7.** Fix $\alpha > 0$ and $0 < \beta < 1$.

$$C_\alpha D : F_\alpha \to B^\beta$$

is bounded $\iff$

$$\Phi \in B^\beta \text{ and } \|\Phi\|_{\infty} < 1 \iff$$

$$C_\alpha D : F_\alpha \to B^\beta \text{ is compact.} \quad (47)$$

**Proof.** First, assume that there is a constant $C$ independent of $f \in F_\alpha$ such that $\|C_\alpha D(f)\|_{B^\beta} \leq C\|f\|_{F_\alpha}$. In particular, $\Phi \in B^\beta$. For $w \in U$, define

$$g_w(z) = \frac{1}{(1 - \Phi(z)z)\alpha} (z \in U). \quad (48)$$

There is a constant $C$ independent of $w \in U$ such that $\|g_w\|_{F_\alpha} \leq C$, and it follows that

$$\sup_{z \in U} (1 - |z|^2)^\beta |g_w'(\Phi(z))\Phi'(z)| < C, \quad (49)$$

for all $w \in U$. The substitution $z = w$ yields

$$\sup_{z \in U} (1 - |w|^2)^\beta \frac{\alpha(\alpha + 1)|\Phi(w)|^2 |\Phi'(w)|}{(1 - |\Phi(w)|^2)^{\alpha + 2}} < C, \quad (50)$$

for all $w \in U$. Therefore,

$$\sup_{1/2 < |\Phi(w)|} \frac{(1 - |w|^2)^\beta |\Phi'(w)|}{(1 - |\Phi(w)|^2)^{\alpha + 2}} < \infty. \quad (51)$$

Since $\Phi \in B^\beta$,

$$\sup_{|\Phi(w)| = 1/2} \frac{(1 - |w|^2)^\beta |\Phi'(w)|}{(1 - |\Phi(w)|^2)^{\alpha + 2}} < \infty. \quad (52)$$

It follows that

$$\sup_{w \in U} \frac{(1 - |w|^2)^\beta |\Phi'(w)|}{(1 - |\Phi(w)|^2)^{\alpha + 2}} < \infty \quad (53)$$

and therefore

$$\sup_{w \in U} \frac{(1 - |w|^2)^\beta |\Phi'(w)|}{(1 - |\Phi(w)|^2)^\beta} < \infty. \quad (54)$$

By [9], $C_\Phi : B^\beta \to B^\beta$ is bounded. A further argument as in the proof of Theorem 3 yields that $C_\Phi : B^\beta \to B^\beta$ is compact. Since $\beta < 1$, Shapiro’s result [7] applies and yields $\|\Phi\|_{\infty} < 1$. Thus, the conditions $\Phi \in B^\beta$ and $\|\Phi\|_{\infty} < 1$ are necessary in order for $C_\alpha D : F_\alpha \to B^\beta$ to be bounded.

Next, assume $\Phi \in B^\beta$ and $\|\Phi\|_{\infty} < 1$. Let $(f_n)$ be a bounded sequence in $F_\alpha$ with $f_n \to 0$ uniformly on compact subsets of $U$. First, note that $\|f_n'(\Phi(0))\| \to 0$ as $n \to \infty$. For $z \in U$,

$$(1 - |z|^2)^\beta \left| f_n' \circ \Phi \right|' (z) \leq \|\Phi\|^\beta \max_{|w| \leq \|\Phi\|_{\infty}} |f_n''(w)|. \quad (55)$$

Since $f_n'' \to 0$ uniformly on compact subsets, the argument yields $\|C_\alpha D(f_n)\|_{B^\beta} \to 0$ and $C_\alpha D : F_\alpha \to B^\beta$ is compact.

The remaining implication is trivial, and the proof is complete.

**Theorem 8.** Fix $\alpha > 0$ and $\beta \geq 1$. Let $\Phi$ be a self-map of $U$.

$$C_\alpha D : F_\alpha \to B^\beta$$

is bounded $\iff$

$$\sup_{z \in U} \frac{(1 - |z|^2)^\beta |\Phi'(z)|}{(1 - |\Phi(z)|^2)^{\alpha + 2}} < \infty. \quad (56)$$

**Proof.** First, assume that the supremum is finite.

Let $f \in F_\alpha$. By previous remarks, $|f' \circ \Phi(0)| \leq C\|f\|_{F_\alpha}$. By an argument as in the proof of Theorem 4,

$$\left| (1 - |z|^2)^\beta \left( f' \circ \Phi \right)' (z) \right| = \left| (1 - |z|^2)^\beta f''(\Phi(z)) |\Phi'(z)| \right| \leq (1 - |z|^2)^\beta \frac{C\|f\|_{F_\alpha} |\Phi'(z)|}{(1 - |\Phi(z)|^2)^{\alpha + 2}} < \infty, \quad (57)$$

and thus, $\|C_\alpha D(f)\|_{B^\beta} \leq C\|f\|_{F_\alpha}$ as required.

To complete the proof, assume that $\|C_\alpha D(f)\|_{B^\beta} \leq C\|f\|_{F_\alpha}$ for a constant $C$ independent of $f$. The argument leading to (53) remains valid for $\beta \geq 1$. This proves the opposite implication, and the proof is complete.
Theorem 9. Fix $\alpha > 0$ and $\beta \geq 1$. Let $\Phi$ be a self-map of $U$ and assume that $C_\alpha D : F_\alpha \to B^\beta$ is bounded.

$$C_\alpha D : F_\alpha \to B^\beta \text{ is compact } \iff \lim_{|\Phi(z)| \to 1} \frac{(1 - |z|^2)^\beta |f_n''(\Phi(z))| |\Phi'(z)|}{(1 - |\Phi(z)|^2)^{\alpha + 2}} = 0.$$  \hspace{1cm} (58)

Proof. First, assume that $C_\alpha D : F_\alpha \to B^\beta$ is bounded and the limit condition holds. Let $(f_n)$ be a bounded sequence in $F_\alpha$ with $f_n \to 0$ uniformly on compact subsets as $n \to \infty$. Clearly, $|f_n''(\Phi(0))| \to 0$ as $n \to \infty$. As in previous arguments,

$$(1 - |z|^2)^\beta |f_n''(\Phi(z))| |\Phi'(z)| \leq C \frac{(1 - |z|^2)^\beta |\Phi'(z)|}{(1 - |\Phi(z)|^2)^{\alpha + 2}},$$  \hspace{1cm} (59)

for all $z \in U$. The hypothesis now implies that, given $\varepsilon > 0$, there exists $r_0, 0 < r_1 < 1$, such that

$$\sup_{|\Phi(z)| > r_0} (1 - |z|^2)^\beta |f_n''(\Phi(z))| |\Phi'(z)| < \varepsilon,$$  \hspace{1cm} (60)

for all $n$. Since $\Phi \in B^\beta$ and since $f_n'' \to 0$ uniformly on compact subsets,

$$\sup_{|\Phi(z)| \leq r_0} (1 - |z|^2)^\beta |f_n''(\Phi(z))| |\Phi'(z)| \to 0,$$  \hspace{1cm} (61)

as $n \to \infty$. By (60) and (61),

$$\sup_{z \in U} \frac{(1 - |z|^2)^\beta}{(1 - |\Phi(z)|^2)^{\alpha + 2}} |f_n\circ\Phi(z)| \to 0,$$  \hspace{1cm} (62)

as $n \to \infty$. The argument yields $||f_n \circ \Phi||_{B^\beta} \to 0$ as $n \to \infty$ for any sequence $(f_n)$ as described above. Thus, $C_\alpha D : F_\alpha \to B^\beta$ is compact.

Now, assume that $C_\alpha D : F_\alpha \to B^\beta$ is compact. We may assume that $||\Phi||_{C^1} = 1$. Let $(z_n)$ be any sequence in $U$ with $|\Phi(z_n)| \to 1$ as $n \to \infty$. For $n = 1, 2, \ldots$, define

$$f_n(z) = \frac{1 - |\Phi(z_n)|^2}{(1 - |\Phi(z_n)|^2)^{\alpha + 1}},$$  \hspace{1cm} (63)

for $z \in U$. By Lemma 1, $||f_n||_{F_\alpha} \leq C$ for all $n$. Also, $f_n \to 0$ uniformly on compact subsets. Therefore, $||C_\alpha D(f_n)||_{B^\beta} \to 0$ as $n \to \infty$. Given $\varepsilon > 0$, there exists $N > 0$ such that

$$\sup_{z \in U} (1 - |z|^2)^\beta |f_n''(\Phi(z))| |\Phi'(z)| < \varepsilon,$$  \hspace{1cm} (64)

for all $n > N$. In particular, $(1 - |z|^2)^\beta |f_n''(\Phi(z_n))| |\Phi'(z_n)| < \varepsilon$ for all $n > N$. Calculations yield

$$\frac{(1 - |z_n|^2)^\beta (\alpha + 1)(\alpha + 2)|\Phi(z_n)|^2|\Phi'(z_n)|}{(1 - |\Phi(z_n)|^2)^{\alpha + 2}} < \varepsilon,$$  \hspace{1cm} (65)

for $n > N$. Since $(z_n)$ is a generic sequence with $|\Phi(z_n)| \to 1$, it follows that

$$\lim_{|\Phi(z)| \to 1} \frac{(1 - |z|^2)^\beta |\Phi'(z)|}{(1 - |\Phi(z)|^2)^{\alpha + 2}} = 0.$$  \hspace{1cm} (66)

The proof is complete.

Assume that $C_\alpha D : F_\alpha \to B^\beta$ is bounded for fixed $\alpha > 0$ and $\beta < 1$. By Theorem 7, $\Phi \in B^\beta$ and $||\Phi||_{C^1} < 1$. It follows that $C_\alpha D : F_\gamma \to B^\beta$ is compact for any $\gamma > 0$. Corollary 10 gives a related result in the case $\beta \geq 1$.

Corollary 10. Fix $\alpha > 0$, $\beta \geq 1$ and assume that $C_\alpha D : F_\alpha \to B^\beta$ is bounded. Then, $C_\alpha D : F_\gamma \to B^\beta$ is compact for any $\gamma, 0 < \gamma < \alpha$.

Proof. Fix $0 < \gamma < \alpha$ and let $f \in F_\gamma$. Then, $f \in F_\alpha$ and $||f||_{F_\alpha} \leq ||f||_{F_\gamma}$ [2]. Therefore, $C_\alpha D : F_\gamma \to B^\beta$ is bounded and Theorem 9 applies.

By Theorem 8, there is a constant $C$ with

$$\frac{(1 - |z|^2)^\beta |\Phi'(z)|}{(1 - |\Phi(z)|^2)^{\alpha + 2}} \leq C,$$  \hspace{1cm} (67)

for all $z \in U$. Therefore,

$$\frac{(1 - |z|^2)^\beta |\Phi'(z)|}{(1 - |\Phi(z)|^2)^{\alpha + 2}} \leq C \frac{(1 - |\Phi(z)|^2)^{\alpha - \gamma}}{(1 - |\Phi(z)|^2)^{\alpha + 2}} \to 0,$$  \hspace{1cm} (68)

as $|\Phi(z)| \to 1$. By Theorem 9, $C_\alpha D : F_\gamma \to B^\beta$ is compact.

Data Availability

This manuscript does not contain any data.

Conflicts of Interest

The author declares that there is no conflict of interest regarding the publication of this paper.

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