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Research Article

Products of Composition and Differentiation between the Fractional Cauchy Spaces and the Bloch-Type Spaces

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The operators DC_Φ and $C_\Phi D$ are defined by $DC_\Phi(f) = (f \circ \Phi)'$ and $C_\Phi D(f) = f' \circ \Phi$ where Φ is an analytic self-map of the unit disc and f is analytic in the disc. A characterization is provided for boundedness and compactness of the products of composition and differentiation from the spaces of fractional Cauchy transforms F_α to the Bloch-type spaces B^β, where $\alpha > 0$ and $\beta > 0$. In the case $\beta < 2$, the operator $DC_\Phi : F_\alpha \longrightarrow B^\beta$ is compact $\Leftrightarrow DC_\Phi : F_\alpha \longrightarrow B^\beta$ is bounded $\Leftrightarrow \Phi' \in B^\beta, \Phi\Phi' \in B^\beta$ and $\|\Phi\|_\infty < 1$. For $\beta < 1$, $C_\Phi D : F_\alpha \longrightarrow B^\beta$ is compact $\Leftrightarrow C_\Phi D : F_\alpha \longrightarrow B^\beta$ is bounded $\Leftrightarrow \Phi \in B^\beta$ and $\|\Phi\|_\infty < 1$.

1. Introduction

Let $U = \{z \in C : |z| < 1\}$ and let $H(U)$ denote the family of functions analytic on U. Let M denote the Banach space of complex Borel measures on $T = \{x \in C : |x| = 1\}$, endowed with the total variation norm. For $\alpha > 0$, the space F_α of fractional Cauchy transforms is the family of functions of the form

$$f(z) = \int_T \frac{1}{(1 - \bar{x}z)^\alpha}\, d\mu(x) (|z| < 1), \tag{1}$$

where $\mu \in M$. The principal branch of the logarithm is used here. The space F_α is a Banach space, with norm

$$\|f\|_{F_\alpha} = \inf \|\mu\|, \tag{2}$$

where μ varies over all measures in M for which (1) holds. The families F_α have been studied extensively [1, 2]. Interest in these spaces was first established in connection with the classical family S of normalized univalent functions. It is known that $S \subseteq F_\alpha$ for any $\alpha > 2$ [2]. The reference [2] also includes MacGregor's construction of a function $f \in S$ with $f \notin F_2$.

Let $\beta > 0$. The Bloch-type space B^β is the Banach space of functions analytic in U such that $\sup_{z \in U}(1 - |z|^2)^\beta |f'(z)| < \infty$, with norm

$$\|f\|_{B^\beta} = |f(0)| + \sup_{z \in U}\left(1 - |z|^2\right)^\beta \left|f'(z)\right|. \tag{3}$$

The relation (1) implies that $F_\alpha \subset B^{\alpha+1}$, and there is a constant C depending only on α such that $\|f\|_{B^{\alpha+1}} \le C\|f\|_{F_\alpha}$ for all $f \in F_\alpha$.

Let Φ be an analytic self-map of U. The composition operator C_Φ is defined by $C_\Phi(f) = f \circ \Phi$ for $f \in H(U)$. The differentiation operator D is defined by $D(f) = f'$. In this paper, the products $C_\Phi D(f) = f' \circ \Phi$ and $DC_\Phi(f) = \Phi'(f' \circ \Phi)$ are studied. Conditions on Φ are given, necessary and sufficient to imply boundedness or compactness of $C_\Phi D : F_\alpha \longrightarrow B^\beta$ and $DC_\Phi : F_\alpha \longrightarrow B^\beta$.

Products of composition and differentiation on the Bloch space were studied by Ohno in [3]. In [4], Li and Stević studied $C_\Phi D$ and DC_Φ acting between the weighted Bergman spaces and the Bloch-type spaces. In [5], Hibschweiler and Portnoy studied these operators between Bergman and Hardy spaces.

2. Preliminary Results

Fix $\alpha > 0$. For fixed $z \in U$ and for $n = 0, 1, \ldots$, the relation (1) yields a constant C depending only on n such that $|f^{(n)}(z)| \leq C\|f\|_{F_\alpha}/(1-|z|^2)^{\alpha+n}$ [2].

For each $w \in U$, $\|1/(1-\bar{w}z)^\alpha\|_{F_\alpha} = 1$ [2].

We follow the convention that C denotes a positive constant, the precise value of which will differ from one appearance to the next.

Lemma 1 and Lemma 2 will be used to develop test functions for F_α. Proofs appear in [6].

Lemma 1. *Fix $\alpha > 0$. For $w \in U$, define*

$$h_w(z) = \frac{1-|w|^2}{(1-\bar{w}z)^{\alpha+1}} (z \in U). \tag{4}$$

Then, $h_w \in F_\alpha$, and there is a constant C such that $\|h_w\|_{F_\alpha} \leq C$ for all $w \in U$.

Lemma 2. *Fix $\alpha > 0$. For $w \in U$, define*

$$k_w(z) = \frac{(1-|w|^2)^2}{(1-\bar{w}z)^{\alpha+2}} (z \in U). \tag{5}$$

Then, $k_w \in F_\alpha$, and there is a constant C such that $\|k_w\|_{F_\alpha} \leq C$ for all $w \in U$.

3. The Operator $DC_\Phi : F_\alpha \longrightarrow B^\beta$

In [7], Shapiro proved that the condition $\|\Phi\|_\infty < 1$ is necessary for $C_\Phi : X \longrightarrow X$ to be compact, for Banach spaces X obeying boundary regularity and Möbius invariance. In particular, Shapiro's result applies to the Lipschitz spaces and thus, to the space B^γ when $\gamma < 1$ [8].

Theorem 3. *Fix $\alpha > 0$ and $0 < \beta < 2$. Let Φ be an analytic self-map of U.*

$$\begin{aligned} &DC_\Phi : F_\alpha \longrightarrow B^\beta \text{ is bounded} \Leftrightarrow \\ &\Phi' \in B^\beta, \Phi\Phi' \in B^\beta \text{ and } \|\Phi\|_\infty < 1 \Leftrightarrow \\ &DC_\Phi : F_\alpha \longrightarrow B^\beta \text{ is compact}. \end{aligned} \tag{6}$$

Proof. First, assume that $DC_\Phi : F_\alpha \longrightarrow B^\beta$ is bounded, that is, there is a constant C such that $\|DC_\Phi(f)\|_{B^\beta} \leq C\|f\|_{F_\alpha}$ for all $f \in F_\alpha$. It is clear that $\Phi' = DC_\Phi(z) \in B^\beta$ and $\Phi\Phi' = DC_\Phi(z^2/2) \in B^\beta$. Thus,

$$(1-|z|^2)^\beta |\Phi''(z)| \leq C, \tag{7}$$

and

$$(1-|z|^2)^\beta \left|\Phi(z)\Phi''(z) + \left(\Phi'(z)\right)^2\right| \leq C, \tag{8}$$

for all $z \in U$. It follows that

$$\sup_{z\in U} (1-|z|^2)^\beta |\Phi'(z)|^2 < \infty, \tag{9}$$

and thus, $\Phi \in B^{\beta/2}$.

Let $w \in U$ and define

$$g_w(z) = \frac{\alpha+1}{(1-\Phi(\bar{w})z)^\alpha} - \frac{\alpha(1-|\Phi(w)|^2)}{(1-\Phi(\bar{w})z)^{\alpha+1}} (z \in U). \tag{10}$$

By Lemma 1 and the preliminary results, there is a constant C independent of w such that $\|g_w\|_{F_\alpha} \leq C$, and thus, $\|DC_\Phi(g_w)\|_{B^\beta} = \|(g_w{}' \circ \Phi)\Phi'\|_{B^\beta} \leq C$. It follows that

$$\sup_{z\in U} (1-|z|^2)^\beta \left|g_w{}''(\Phi(z))\left(\Phi'(z)\right)^2 + g_w{}'(\Phi(z))\,\Phi''(z)\right| \leq C, \tag{11}$$

for all $w \in U$. Calculations yield $g_w{}'(\Phi(w)) = 0$ and

$$g_w{}''(\Phi(w)) = \frac{-\alpha(\alpha+1)\Phi(\bar{w})^2}{(1-|\Phi(w)|^2)^{\alpha+2}}. \tag{12}$$

The substitution $z = w$ in (11) now yields

$$\sup_{w\in U} (1-|w|^2)^\beta \frac{\alpha(\alpha+1)|\Phi(w)|^2|\Phi'(w)|^2}{(1-|\Phi(w)|^2)^{\alpha+2}} \leq C, \tag{13}$$

and thus,

$$\sup_{|\Phi(w)|>1/2} \frac{(1-|w|^2)^\beta |\Phi'(w)|^2}{(1-|\Phi(w)|^2)^{\alpha+2}} < \infty. \tag{14}$$

By the relation (9),

$$\sup_{|\Phi(z)|\leq 1/2} \frac{(1-|w|^2)^\beta |\Phi'(w)|^2}{(1-|\Phi(w)|^2)^{\alpha+2}} < \infty. \tag{15}$$

Thus,

$$C = \sup_{w\in U} \frac{(1-|w|^2)^\beta |\Phi'(w)|^2}{(1-|\Phi(w)|^2)^{\alpha+2}} < \infty. \tag{16}$$

It follows that

$$\sup_{w\in U} \frac{(1-|w|^2)^{\beta/2} |\Phi'(w)|}{(1-|\Phi(w)|^2)^{\beta/2}} \leq \sup_{w\in U} \frac{(1-|w|^2)^{\beta/2} |\Phi'(w)|}{(1-|\Phi(w)|^2)^{(\alpha+2)/2}} < \infty. \tag{17}$$

By Xiao's result [9], $C_\Phi : B^{\beta/2} \longrightarrow B^{\beta/2}$ is bounded. Furthermore, (16) yields

$$\frac{\left(1-|w|^2\right)^{\beta/2}\left|\Phi'(w)\right|}{\left(1-|\Phi(w)|^2\right)^{\beta/2}} \le C\left(1-|\Phi(w)|^2\right)^{(\alpha-\beta+2)/2} \longrightarrow 0, \quad (18)$$

as $|\Phi(w)| \longrightarrow 1$. Thus, $C_\Phi : B^{\beta/2} \longrightarrow B^{\beta/2}$ is compact [9], and it follows as in [7] that $\|\Phi\|_\infty < 1$. It has been established that the conditions $\Phi' \in B^\beta, \Phi\Phi' \in B^\beta$, and $\|\Phi\|_\infty < 1$ are necessary if $DC_\Phi : F_\alpha \longrightarrow B^\beta$ is bounded.

Next, assume that $\Phi' \in B^\beta$, $\Phi\Phi' \in B^\beta$, and $\|\Phi\|_\infty < 1$. To show that $DC_\Phi : F_\alpha \longrightarrow B^\beta$ is compact, let (f_n) be a bounded sequence in F_α with $f_n \longrightarrow 0$ uniformly on compact subsets of U as $n \longrightarrow \infty$. It is enough to prove that $\|DC_\Phi(f_n)\|_{B^\beta} \longrightarrow 0$ as $n \longrightarrow \infty$. First, note that $|f_n{}'(\Phi(0))\Phi'(0)| \longrightarrow 0$ as $n \longrightarrow \infty$. For $z \in U$, (9) yields

$$\begin{aligned}\left(1-|z|^2\right)^\beta\left|(DC_\Phi f_n)'(z)\right| &= \left(1-|z|^2\right)^\beta\Big|f_n{}''(\Phi(z))\left(\Phi'(z)\right)^2 \\ &\quad + f_n{}'(\Phi(z))\Phi''(z)\Big| \\ &\le C\ \max_{|w|\le\|\Phi\|_\infty}\left|f_n{}''(w)\right| \\ &\quad + \left\|\Phi'\right\|_{B^\beta}\max_{|w|\le\|\Phi\|_\infty}\left|f_n{}'(w)\right|.\end{aligned} \quad (19)$$

Since $f_n{}' \longrightarrow 0$ and $f_n{}'' \longrightarrow 0$ uniformly on compact subsets as $n \longrightarrow \infty$, the argument shows that $\sup_{z\in U}(1-|z|^2)^\beta|(DC_\Phi f_n)'(z)| \longrightarrow 0$ as $n \longrightarrow \infty$. Thus, $\|DC_\Phi(f_n)\|_{B^\beta} \longrightarrow 0$ as $n \longrightarrow \infty$, and $DC_\Phi : F_\alpha \longrightarrow B^\beta$ is compact, as required.

The remaining implication is clear, and the proof is complete.

Theorem 4. *Fix $\alpha > 0$ and $\beta \ge 2$. Let Φ be an analytic self-map of U. Then,*

$$DC_\Phi : F_\alpha \longrightarrow B^\beta \text{ is bounded} \Leftrightarrow \quad (20)$$

$$\sup_{z\in U}\frac{\left(1-|z|^2\right)^\beta\left|\Phi''(z)\right|}{\left(1-|\Phi(z)|^2\right)^{\alpha+1}} < \infty, \quad (21)$$

and

$$\sup_{z\in U}\frac{\left(1-|z|^2\right)^\beta\left|\Phi'(z)\right|^2}{\left(1-|\Phi(z)|^2\right)^{\alpha+2}} < \infty. \quad (22)$$

Proof. Fix α, β and Φ as described.

First, assume (21) and (22). Let $f \in F_\alpha$. By (21) and the introductory remarks in Section 2,

$$\begin{aligned}\left(1-|z|^2\right)^\beta\left|f'(\Phi(z))\right|\left|\Phi''(z)\right| &\le \left(1-|z|^2\right)^\beta\left|\Phi''(z)\right|\frac{C\|f\|_{F_\alpha}}{\left(1-|\Phi(z)|^2\right)^{\alpha+1}} \\ &\le C\|f\|_{F_\alpha}.\end{aligned} \quad (23)$$

A similar argument using (22) yields

$$\left(1-|z|^2\right)^\beta\left|f''(\Phi(z))\right|\left|\Phi'(z)\right|^2 \le C\|f\|_{F_\alpha}, \quad (24)$$

for all $z \in U$. Thus, $\sup_{z\in U}(1-|z|^2)^\beta|(DC_\Phi f)'(z)| \le C\|f\|_{F_\alpha}$. Since $|(DC_\Phi f)(0)| \le C\|f\|_{F_\alpha}$, it now follows that $\|DC_\Phi(f)\|_{B^\beta} \le C\|f\|_{F_\alpha}$, as required.

For the converse, assume that $\|DC_\Phi(f)\|_{B^\beta} \le C\|f\|_{F_\alpha}$ for a constant C independent of $f \in F_\alpha$. In particular, $\Phi' \in B^\beta$.

The argument leading to (16) remains valid for $\beta \ge 2$. Thus, (22) holds. It remains to prove (21). First, note that

$$\sup_{|\Phi(w)|\le 1/2}\frac{\left(1-|w|^2\right)^\beta\left|\Phi''(w)\right|}{\left(1-|\Phi(w)|^2\right)^{\alpha+1}} \le \left(\frac{4}{3}\right)^{\alpha+1}\left\|\Phi'\right\|_{B^\beta} < \infty. \quad (25)$$

For $w \in U$, define

$$H_w(z) = \frac{(\alpha+3)\left(1-|\Phi(w)|^2\right)}{\left(1-\overline{\Phi(w)}z\right)^{\alpha+1}} - \frac{(\alpha+1)\left(1-|\Phi(w)|^2\right)^2}{\left(1-\overline{\Phi(w)}z\right)^{\alpha+2}}, \quad (26)$$

for $z \in U$. By Lemma 1 and Lemma 2, there is a constant C independent of w such that $\|H_w\|_{F_\alpha} \le C$. Thus, $\|DC_\Phi(H_w)\|_{B^\beta} \le C$ for all w. It follows that

$$\sup_{z\in U}\left(1-|z|^2\right)^\beta\left|H_w{}'(\Phi(z))\Phi''(z) + \left(\Phi'(z)\right)^2 H_w{}''(\Phi(z))\right| < C, \quad (27)$$

for all $w \in U$. An argument using $H_w{}'(\Phi(w)) = (\alpha+1)\overline{\Phi(w)}/(1-|\Phi(w)|^2)^{\alpha+1}$ and $H_w{}''(\Phi(w)) = 0$ yields

$$\sup_{1/2<|\Phi(w)|}\frac{\left(1-|w|^2\right)^\beta\left|\Phi''(w)\right|}{\left(1-|\Phi(w)|^2\right)^{\alpha+1}} < \infty. \quad (28)$$

The relations (25) and (28) establish relation (21), and the proof is complete.

Theorem 5. *Fix $\alpha > 0$ and assume $\beta \ge 2$. Let Φ be a self-map of U for which $DC_\Phi : F_\alpha \longrightarrow B^\beta$ is bounded.*

$$DC_\Phi : F_\alpha \longrightarrow B^\beta \text{ is compact} \Leftrightarrow \quad (29)$$

$$\lim_{|\Phi(z)|\longrightarrow 1}\frac{\left(1-|z|^2\right)^\beta\left|\Phi''(z)\right|}{\left(1-|\Phi(z)|^2\right)^{\alpha+1}} = 0, \quad (30)$$

and

$$\lim_{|\Phi(z)|\longrightarrow 1}\frac{\left(1-|z|^2\right)^\beta\left|\Phi'(z)\right|^2}{\left(1-|\Phi(z)|^2\right)^{\alpha+2}} = 0. \quad (31)$$

Proof. First, assume that $DC_\Phi : F_\alpha \longrightarrow B^\beta$ is bounded and relations (30) and (31) hold. Let (f_n) be a bounded sequence in F_α such that $f_n \longrightarrow 0$ uniformly on compact subsets of U. As previously noted, there is a constant C depending only on α such that

$$\left(1-|z|^2\right)^\beta \left|f_n'(\Phi(z))\right|\left|\Phi'(z)\right|^2 \le C\frac{\left(1-|z|^2\right)^\beta\left|\Phi'(z)\right|^2}{\left(1-|\Phi(z)|^2\right)^{\alpha+2}}, \tag{32}$$

for $n = 1, 2, \ldots$ and $z \in U$. Relation (31) now implies that given $\varepsilon > 0$, there exists r_0, $0 < r_0 < 1$, such that

$$\sup_{|\Phi(z)|>r_0}\left(1-|z|^2\right)^\beta\left|f_n''(\Phi(z))\right|\left|\Phi'(z)\right|^2 < \varepsilon, \tag{33}$$

for all n.

Since $DC_\Phi : F_\alpha \longrightarrow B^\beta$ is bounded, relation (9) holds, and thus,

$$\left(1-|z|^2\right)^\beta\left|f_n'(\Phi(z))\right|\left|\Phi'(z)\right|^2 < C\left|f_n''(\Phi(z))\right|, \tag{34}$$

for all $z \in U$. Since $f_n'' \longrightarrow 0$ uniformly on $\{w : |w| \le r_0\}$, there exists $N > 0$ such that

$$\sup_{|\Phi(z)|\le r_0}\left(1-|z|^2\right)^\beta\left|f_n''(\Phi(z))\right|\left|\Phi'(z)\right|^2 < \varepsilon, \tag{35}$$

for all $n > N$. The relations (33) and (35) yield

$$\sup_{z\in U}\left(1-|z|^2\right)^\beta\left|f_n''(\Phi(z))\right|\left|\Phi'(z)\right|^2 < \varepsilon, \tag{36}$$

for $n > N$.

A similar argument using $\Phi' \in B^\beta$ and (30) yields $N_1 > 0$ such that

$$\sup_{z\in U}\left(1-|z|^2\right)^\beta\left|f_n'(\Phi(z))\right|\left|\Phi''(z)\right| < \varepsilon, \tag{37}$$

for $n > N_1$. The relations (36) and (37) yield

$$\sup_{z\in U}\left(1-|z|^2\right)^\beta\left|(DC_\Phi f_n)'(z)\right| \longrightarrow 0, \tag{38}$$

as $n \longrightarrow \infty$.

Since $|(DC_\Phi f_n)(0)| \longrightarrow 0$ as $n \longrightarrow \infty$, the argument yields $\|DC_\Phi(f_n)\|_{B^\beta} \longrightarrow 0$ as $n \longrightarrow \infty$ for any sequence (f_n) as described, and therefore, $DC_\Phi : F_\alpha \longrightarrow B^\beta$ is compact.

For the converse, assume that $DC_\Phi : F_\alpha \longrightarrow B^\beta$ is compact. We may assume that $\|\Phi\|_\infty = 1$. Let (z_n) be any sequence in U with $|\Phi(z_n)| \longrightarrow 1$ as $n \longrightarrow \infty$. For $z \in U$, define

$$f_n(z) = \frac{(\alpha+3)\left(1-|\Phi(z_n)|^2\right)}{\left(1-\Phi(\bar{z}_n)z\right)^{\alpha+1}} - \frac{(\alpha+1)\left(1-|\Phi(z_n)|^2\right)^2}{\left(1-\Phi(\bar{z}_n)z\right)^{\alpha+2}}. \tag{39}$$

By the lemmas above, $\|f_n\|_{F_\alpha} \le C$. Also, $f_n \longrightarrow 0$ uniformly on compact subsets. Therefore, $\|DC_\Phi(f_n)\|_{B^\beta} \longrightarrow 0$ and

$$\sup_{z\in U}\left(1-|z|^2\right)^\beta\left|f_n'(\Phi(z))\Phi''(z) + f_n''(\Phi(z))\left(\Phi'(z)\right)^2\right| \longrightarrow 0, \tag{40}$$

as $n \longrightarrow \infty$. Calculations yield $f_n''(\Phi(z_n)) = 0$ and

$$f_n'(\Phi(z_n)) = \frac{(\alpha+1)\Phi(\bar{z}_n)}{\left(1-|\Phi(z_n)|^2\right)^{\alpha+1}}. \tag{41}$$

Substitution into (40) yields

$$\frac{\left(1-|z_n|^2\right)^\beta|\Phi(z_n)|\left|\Phi''(z_n)\right|}{\left(1-|\Phi(z_n)|^2\right)^{\alpha+1}} \longrightarrow 0, \tag{42}$$

as $n \longrightarrow \infty$. Since (z_n) is a generic sequence with $|\Phi(z_n)| \longrightarrow 1$ as $n \longrightarrow \infty$, this yields the relation (30).

A similar argument using the functions

$$g_n(z) = \frac{(\alpha+2)\left(1-|\Phi(z_n)|^2\right)}{\left(1-\Phi(\bar{z}_n)z\right)^{\alpha+1}} - \frac{(\alpha+1)\left(1-|\Phi(z_n)|^2\right)^2}{\left(1-\Phi(\bar{z}_n)z\right)^{\alpha+2}} \tag{43}$$

yields the relation (31). The details are omitted.

Theorem 3 implies that if $DC_\Phi : F_\alpha \longrightarrow B^\beta$ is bounded for fixed α, β with $\beta < 2$, then $DC_\Phi : F_\gamma \longrightarrow B^\beta$ is compact for all $\gamma > 0$. The next corollary gives a related result when $\beta \ge 2$.

Corollary 6. *Fix $\alpha > 0$ and $\beta \ge 2$. Let Φ be a self-map of U and assume that $DC_\Phi : F_\alpha \longrightarrow B^\beta$ is bounded. Then, $DC_\Phi : F_\gamma \longrightarrow B^\beta$ is compact for any γ, $0 < \gamma < \alpha$.*

Proof. By assumption, there is a constant C such that $\|DC_\Phi(f)\|_{B^\beta} \le C\|f\|_{F_\alpha}$ for all $f \in F_\alpha$. Fix γ with $0 < \gamma < \alpha$ and let $f \in F_\gamma$. Then, $f \in F_\alpha$ and $\|f\|_{F_\alpha} \le \|f\|_{F_\gamma}$ [2]. Therefore, $DC_\Phi : F_\gamma \longrightarrow B^\beta$ is bounded and Theorem 5 applies.

Since $DC_\Phi : F_\alpha \longrightarrow B^\beta$ is bounded, (21) yields

$$\frac{\left(1-|z|^2\right)^\beta\left|\Phi''(z)\right|}{\left(1-|\Phi(z)|^2\right)^{\gamma+1}} \le C\left(1-|\Phi(z)|^2\right)^{\alpha-\gamma}, \tag{44}$$

and therefore,

$$\lim_{|\Phi(z)|\longrightarrow 1} \frac{\left(1-|z|^2\right)^{\beta}\left|\Phi''(z)\right|}{\left(1-|\Phi(z)|^2\right)^{\gamma+1}} = 0. \tag{45}$$

A similar argument using (22) yields

$$\lim_{|\Phi(z)|\longrightarrow 1} \frac{\left(1-|z|^2\right)^{\beta}\left|\Phi'(z)\right|^2}{\left(1-|\Phi(z)|^2\right)^{\gamma+2}} = 0. \tag{46}$$

Theorem 5 now yields $DC_\Phi : F_\gamma \longrightarrow B^\beta$ is compact.

4. The Operator $C_\Phi D$

In this section, characterizations are given for self-maps Φ for which $C_\Phi D : F_\alpha \longrightarrow B^\beta$ is bounded or compact. The proofs are similar to those in Section 3, so details are kept to a minimum.

Theorem 7. *Fix $\alpha > 0$ and $0 < \beta < 1$.*

$$\begin{gathered} C_\Phi D : F_\alpha \longrightarrow B^\beta \text{ is bounded} \Leftrightarrow \\ \Phi \in B^\beta \text{ and } \|\Phi\|_\infty < 1 \Leftrightarrow \\ C_\Phi D : F_\alpha \longrightarrow B^\beta \text{ is compact}. \end{gathered} \tag{47}$$

Proof. First, assume that there is a constant C independent of $f \in F_\alpha$ such that $\|C_\Phi D(f)\|_{B^\beta} \le C\|f\|_{F_\alpha}$. In particular, $\Phi \in B^\beta$. For $w \in U$, define

$$g_w(z) = \frac{1}{\left(1-\Phi(\bar{w})z\right)^{\alpha}} \, (z \in U). \tag{48}$$

There is a constant C independent of $w \in U$ such that $\|g_w\|_{F_\alpha} \le C$, and it follows that

$$\sup_{z\in U}\left(1-|z|^2\right)^{\beta}\left|g_w''(\Phi(z))\Phi'(z)\right| < C, \tag{49}$$

for all $w \in U$. The substitution $z = w$ yields

$$\left(1-|w|^2\right)^{\beta}\frac{\alpha(\alpha+1)|\Phi(w)|^2\left|\Phi'(w)\right|}{\left(1-|\Phi(w)|^2\right)^{\alpha+2}} < C, \tag{50}$$

for all $w \in U$. Therefore,

$$\sup_{1/2<|\Phi(w)|}\frac{\left(1-|w|^2\right)^{\beta}\left|\Phi'(w)\right|}{\left(1-|\Phi(w)|^2\right)^{\alpha+2}} < \infty. \tag{51}$$

Since $\Phi \in B^\beta$,

$$\sup_{|\Phi(w)|\le 1/2}\frac{\left(1-|w|^2\right)^{\beta}\left|\Phi'(w)\right|}{\left(1-|\Phi(w)|^2\right)^{\alpha+2}} < \infty. \tag{52}$$

It follows that

$$\sup_{w\in U}\frac{\left(1-|w|^2\right)^{\beta}\left|\Phi'(w)\right|}{\left(1-|\Phi(w)|^2\right)^{\alpha+2}} < \infty \tag{53}$$

and therefore

$$\sup_{w\in U}\frac{\left(1-|w|^2\right)^{\beta}\left|\Phi'(w)\right|}{\left(1-|\Phi(w)|^2\right)^{\beta}} < \infty. \tag{54}$$

By [9], $C_\Phi : B^\beta \longrightarrow B^\beta$ is bounded. A further argument as in the proof of Theorem 3 yields that $C_\Phi : B^\beta \longrightarrow B^\beta$ is compact. Since $\beta < 1$, Shapiro's result [7] applies and yields $\|\Phi\|_\infty < 1$. Thus, the conditions $\Phi \in B^\beta$ and $\|\Phi\|_\infty < 1$ are necessary in order for $C_\Phi D : F_\alpha \longrightarrow B^\beta$ to be bounded.

Next, assume $\Phi \in B^\beta$ and $\|\Phi\|_\infty < 1$. Let (f_n) be a bounded sequence in F_α with $f_n \longrightarrow 0$ uniformly on compact subsets of U. First, note that $|f_n'(\Phi(0))| \longrightarrow 0$ as $n \longrightarrow \infty$. For $z \in U$,

$$\left(1-|z|^2\right)^{\beta}\left|\left(f_n' \circ \Phi\right)'(z)\right| \le \|\Phi\|_{B^\beta} \max_{|w|\le\|\Phi\|_\infty}\left|f_n''(w)\right|. \tag{55}$$

Since $f_n'' \longrightarrow 0$ uniformly on compact subsets, the argument yields $\|C_\Phi D(f_n)\|_{B^\beta} \longrightarrow 0$ and $C_\Phi D : F_\alpha \longrightarrow B^\beta$ is compact.

The remaining implication is trivial, and the proof is complete.

Theorem 8. *Fix $\alpha > 0$ and $\beta \ge 1$. Let Φ be a self-map of U.*

$$\begin{gathered} C_\Phi D : F_\alpha \longrightarrow B^\beta \text{ is bounded} \Leftrightarrow \\ \sup_{z\in U}\frac{\left(1-|z|^2\right)^{\beta}\left|\Phi'(z)\right|}{\left(1-|\Phi(z)|^2\right)^{\alpha+2}} < \infty. \end{gathered} \tag{56}$$

Proof. First, assume that the supremum is finite.

Let $f \in F_\alpha$. By previous remarks, $|f'(\Phi(0))| \le C\|f\|_{F_\alpha}$. By an argument as in the proof of Theorem 4,

$$\begin{aligned} \left(1-|z|^2\right)^{\beta}\left|\left(f' \circ \Phi\right)'(z)\right| &= \left(1-|z|^2\right)^{\beta}\left|f''(\Phi(z))\right|\left|\Phi'(z)\right| \\ &\le \left(1-|z|^2\right)^{\beta}\frac{C\|f\|_{F_\alpha}\left|\Phi'(z)\right|}{\left(1-|\Phi(z)|^2\right)^{\alpha+2}} \\ &\le C\|f\|_{F_\alpha}, \end{aligned} \tag{57}$$

and thus, $\|C_\Phi D(f)\|_{B^\beta} \le C\|f\|_{F_\alpha}$ as required.

To complete the proof, assume that $\|C_\Phi D(f)\|_{B^\beta} \le C\|f\|_{F_\alpha}$ for a constant C independent of f. The argument leading to (53) remains valid for $\beta \ge 1$. This proves the opposite implication, and the proof is complete.

Theorem 9. *Fix $\alpha > 0$ and $\beta \geq 1$. Let Φ be a self-map of U and assume that $C_\Phi D : F_\alpha \longrightarrow B^\beta$ is bounded.*

$$C_\Phi D : F_\alpha \longrightarrow B^\beta \text{ is compact} \Leftrightarrow \lim_{|\Phi(z)| \longrightarrow 1} \frac{\left(1-|z|^2\right)^\beta \left|\Phi'(z)\right|}{\left(1-|\Phi(z)|^2\right)^{\alpha+2}} = 0. \tag{58}$$

Proof. First, assume that $C_\Phi D : F_\alpha \longrightarrow B^\beta$ is bounded and the limit condition holds. Let (f_n) be a bounded sequence in F_α with $f_n \longrightarrow 0$ uniformly on compact subsets as $n \longrightarrow \infty$. Clearly, $|f_n{}'(\Phi(0))| \longrightarrow 0$ as $n \longrightarrow \infty$. As in previous arguments,

$$\left(1-|z|^2\right)^\beta \left|f_n{}''(\Phi(z))\right| \left|\Phi'(z)\right| \leq C \frac{\left(1-|z|^2\right)^\beta \left|\Phi'(z)\right|}{\left(1-|\Phi(z)|^2\right)^{\alpha+2}}, \tag{59}$$

for all $z \in U$. The hypothesis now implies that, given $\varepsilon > 0$, there exists r_0, $0 < r_0 < 1$, such that

$$\sup_{|\Phi(z)|>r_0} \left(1-|z|^2\right)^\beta \left|f_n{}''(\Phi(z))\right| \left|\Phi'(z)\right| < \varepsilon, \tag{60}$$

for all n. Since $\Phi \in B^\beta$ and since $f_n{}'' \longrightarrow 0$ uniformly on compact subsets,

$$\sup_{|\Phi(z)|\leq r_0} \left(1-|z|^2\right)^\beta \left|f_n{}''(\Phi(z))\right| \left|\Phi'(z)\right| \longrightarrow 0, \tag{61}$$

as $n \longrightarrow \infty$. By (60) and (61),

$$\sup_{z\in U} \left(1-|z|^2\right)^\beta \left|\left(f_n{}' \circ \Phi\right)'(z)\right| \longrightarrow 0, \tag{62}$$

as $n \longrightarrow \infty$. The argument yields $\|f_n{}' \circ \Phi\|_{B^\beta} \longrightarrow 0$ as $n \longrightarrow \infty$ for any sequence (f_n) as described above. Thus, $C_\Phi D : F_\alpha \longrightarrow B^\beta$ is compact.

Now, assume that $C_\Phi D : F_\alpha \longrightarrow B^\beta$ is compact. We may assume that $\|\Phi\|_\infty = 1$. Let (z_n) be any sequence in U with $|\Phi(z_n)| \longrightarrow 1$ as $n \longrightarrow \infty$. For $n = 1, 2, \cdots$, define

$$f_n(z) = \frac{1-|\Phi(z_n)|^2}{\left(1-\bar{\Phi(z_n)}z\right)^{\alpha+1}}, \tag{63}$$

for $z \in U$. By Lemma 1, $\|f_n\|_{F_\alpha} \leq C$ for all n. Also, $f_n \longrightarrow 0$ uniformly on compact subsets. Therefore, $\|C_\Phi D(f_n)\|_{B^\beta} \longrightarrow 0$ as $n \longrightarrow \infty$. Given $\varepsilon > 0$, there exists $N > 0$ such that

$$\sup_{z\in U} \left(1-|z|^2\right)^\beta \left|f_n{}''(\Phi(z))\right| \left|\Phi'(z)\right| < \varepsilon, \tag{64}$$

for all $n > N$. In particular, $\left(1-|z_n|^2\right)^\beta \left|f_n{}''(\Phi(z_n))\right| \left|\Phi'(z_n)\right| < \varepsilon$ for $n > N$. Calculations yield

$$\frac{\left(1-|z_n|^2\right)^\beta (\alpha+1)(\alpha+2)|\Phi(z_n)|^2 \left|\Phi'(z_n)\right|}{\left(1-|\Phi(z_n)|^2\right)^{\alpha+2}} < \varepsilon, \tag{65}$$

for $n > N$. Since (z_n) is a generic sequence with $|\Phi(z_n)| \longrightarrow 1$, it follows that

$$\lim_{|\Phi(z)|\longrightarrow 1} \frac{\left(1-|z|^2\right)^\beta \left|\Phi'(z)\right|}{\left(1-|\Phi(z)|^2\right)^{\alpha+2}} = 0. \tag{66}$$

The proof is complete.

Assume that $C_\Phi D : F_\alpha \longrightarrow B^\beta$ is bounded for fixed $\alpha > 0$ and $\beta < 1$. By Theorem 7, $\Phi \in B^\beta$ and $\|\Phi\|_\infty < 1$. It follows that $C_\Phi D : F_\gamma \longrightarrow B^\beta$ is compact for any $\gamma > 0$. Corollary 10 gives a related result in the case $\beta \geq 1$.

Corollary 10. *Fix $\alpha > 0, \beta \geq 1$ and assume that $C_\Phi D : F_\alpha \longrightarrow B^\beta$ is bounded. Then, $C_\Phi D : F_\gamma \longrightarrow B^\beta$ is compact for any γ, $0 < \gamma < \alpha$.*

Proof. Fix $0 < \gamma < \alpha$ and let $f \in F_\gamma$. Then, $f \in F_\alpha$ and $\|f\|_{F_\alpha} \leq \|f\|_{F_\gamma}$ [2]. Therefore, $C_\Phi D : F_\gamma \longrightarrow B^\beta$ is bounded and Theorem 9 applies.

By Theorem 8, there is a constant C with

$$\frac{\left(1-|z|^2\right)^\beta \left|\Phi'(z)\right|}{\left(1-|\Phi(z)|^2\right)^{\alpha+2}} \leq C, \tag{67}$$

for all $z \in U$. Therefore,

$$\frac{\left(1-|z|^2\right)^\beta \left|\Phi'(z)\right|}{\left(1-|\Phi(z)|^2\right)^{\gamma+2}} \leq C\left(1-|\Phi(z)|^2\right)^{\alpha-\gamma} \longrightarrow 0, \tag{68}$$

as $|\Phi(z)| \longrightarrow 1$. By Theorem 9, $C_\Phi D : F_\gamma \longrightarrow B^\beta$ is compact.

Data Availability

This manuscript does not contain any data.

Conflicts of Interest

The author declares that there is no conflict of interest regarding the publication of this paper.

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