Research Article

Some Qualitative Analyses of Neutral Functional Delay Differential Equation with Generalized Caputo Operator

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In this paper, a new class of a neutral functional delay differential equation involving the generalized ψ -Caputo derivative is investigated on a partially ordered Banach space. The existence and uniqueness results to the given boundary value problem are established with the help of the Dhage's technique and Banach contraction principle. Also, we prove other existence criteria by means of the topological degree method. Finally, Ulam-Hyers type stability and its generalized version are studied. Two illustrative examples are presented to demonstrate the validity of our obtained results.

1. Introduction

Fractional calculus has demonstrated high visibility and capability in the applications of various topics linked to physics, signal processing, mechanics, electromagnetics, economics, biology, and many more [1–3]. Even recently, fractional differential equations (FDEs) have acquired particular attention because of their numerous applications in the fractional modeling [4–12]. FDEs involving hybrid nonlinearity have been gained much attention during the past few years. This class of equations arises from various mathematical and physical phenomena such as three-layer beam, electromagnetic waves, curved beam's deflection with a constant or varying cross-section, and gravity-driven [13–19].

Almeida [20] introduced a new fractional derivative, named the ψ -fractional order derivative (FOD), with respect to another function, which extended the classical fractional derivative. Therefore, the generalizations of existing results

in fractional calculus and FBVPs have been established by several mathematicians [21–25].

The qualitative analysis of FDEs such as the solution's existence and uniqueness is the most popular problems that many researchers focus on. Various fixed point theorems are considered as the most effective tools for dealing with such problems. In this work, we follow some results presented by Ragusa et al. [26, 27] concerning the qualitative properties of some suitable FDEs.

In the last decade, a new technique was developed by Dhage [28], named Dhage iteration principle, for investigating the numerical solutions' existence and approximation of integral and FDEs by constructing a sequence of successive approximations with initial lower or upper solution. Dhage [29–32] provided a generalized form of hybrid fixed point theorem in the context of a metric space having the partial order without applying any geometric condition. In Dhage's research study, with the help of the measure of

noncompactness, an algorithm for studying the solutions' existence of a certain nonlinear functional integral equation was investigated under weaker conditions. The advantages of the applied method were studied by Dhage to compare with the standard approaches that involve Banach, Schauder's, and Krasnoselskii's fixed point theorems. As a result, the iteration method due to Dhage has recently become an important tool for investigating the solution's existence and approximate results of nonlinear hybrid FDEqs that have various scientific applications such as air motion, electricity, fluid dynamics, process control with nonlinear structures, and electromagnetism. In addition, this method can be extended to other functional differential equations (FuDEqs) classes. On the other side, in recent years, the topological degree method has been considered as one of the main tools for studying the existence results to different fractional differential equations and inclusions. This method will be used in our research study to derive desired results in relation to the solutions of the proposed problem. For more details, see [33–38].

The FuDEqs' stability was first proposed by Ulam [39] and then by Hyers [40]. Later on, this type of stability and its generalization were called of the Ulam-Hyers (UH) and generalized Ulam-Hyers (GUH) type, respectively. Investigating the UH and GUH stability has been given a special attention in studying all FuDEqs kinds and FDEqs in particular [41–44].

Motivated by the novel developments in ψ -fractional calculus, the solution's existence, uniqueness, and UH stability of the proposed neutral functional differential equation (NFuDEq) is investigated in this research work. The NFuDEq is expressed as:

$$\begin{cases} {}^c D_{a^+}^{\nu;\psi} [\omega(\tau) - \mathbb{F}(\tau, \omega_\tau)] = \mathbb{H}(\tau, \omega_\tau), \tau \in J := [a, b], \\ \omega(\tau) = \phi(\tau), \quad \tau \in [a - \delta, a], \end{cases} \quad (1)$$

where the ψ -Caputo FOD, denoted by ${}^c D_{a^+}^{\nu;\psi}$, of order $\nu \in (0, 1)$, given $\mathbb{F}, \mathbb{H} : J \times \mathbb{R} \longrightarrow \mathbb{R}$ are continuous functions such that $\mathbb{F}(a, \omega_a) = 0$, and $\phi : [a - \delta, a] \longrightarrow \mathbb{R}$ is a continuous function with $\phi(a) = 0$. For any function u defined on $[a - \delta, a]$ and any $\tau \in J$, it is given by

$$u_\tau(\rho) = u(\tau + \rho), \rho \in [-\delta, 0]. \quad (2)$$

The main aim of this research work is to apply an iteration principle due to Dhage to ensure the solutions' existence along with approximation of (1) under weaker partial continuity and partial compactness type conditions.

This article is constructed as follows: some important definitions and lemmas which are needed for our results are provided in Section 2. The solutions' existence and approximation of (1) are proven in Section 3 via the Dhage iteration principle. In Section 4, a theorem, based on the coincidence degree theory for condensing maps, is established on the solutions' existence of the proposed NFuDEq (1). In Section 5, the solution's uniqueness for the NFuDEq (1) is proven by the Banach contraction principle of solutions. Moreover, we investigate the UH and GUH stability for the NFuDEq (1).

Some illustrative examples for supposed problem are provided at the end to validate our theoretical results.

2. Fundamental Preliminaries

Some important definitions, theorems, and lemmas concerning advanced fractional calculus and nonlinear analysis are stated in this section which are needed for our approach in the next parts.

Consider the space of all continuous real-valued functions $\mathcal{C} = C(J, \mathbb{R})$ endowed with the norm

$$\|\omega\|_{\mathcal{C}} = \sup_{\tau \in J} |\omega(\tau)|. \quad (3)$$

Also, $\mathcal{C}_\delta = C([-\delta, 0], \mathbb{R})$ is endowed with norm

$$\|\phi\|_{\mathcal{C}_\delta} = \sup_{\tau \in [-\delta, 0]} |\phi(\tau)|, \text{ and } \|\omega_\tau\|_{\mathcal{C}_\delta} = \sup_{\rho \in [-\delta, 0]} |\omega(\tau + \rho)|. \quad (4)$$

Consider the Banach space $\mathcal{C}_b = C([a - \delta, b], \mathbb{R})$ defined on $[a - \delta, b]$ with the norm

$$\|\omega\|_{\mathcal{C}_b} = \|\omega\|_{\mathcal{C}_\delta} + \|\omega\|_{\mathcal{C}} = \sup_{\tau \in [a - \delta, b]} |\omega(\tau)|. \quad (5)$$

The order relation $\preceq$ is defined as follows:

$$[\omega \preceq \omega \Leftrightarrow \omega(t) \leq \omega(t)] \forall t \in [a - \delta, b], \quad (6)$$

which gives a partial ordering in $\mathcal{C}_b$.

From the research study in [29], let us now state some necessary definitions and preliminary results for our research work. Assume that $\mathbb{X} = (\mathbb{X}, \preceq, \|\cdot\|)$ displays a real partial order on $\mathbb{X}$. If for ω, ω in $\mathbb{X}$, either $\omega \preceq \omega$ or $\omega \preceq \omega$, then ω and ω are termed as comparable elements, and also when all members of $\emptyset \neq \mathcal{C} \subset \mathbb{X}$ are comparable, then $\mathcal{C}$ is named either totally ordered or a chain. If there exists a nondecreasing (resp., nonincreasing) sequence $(\omega_n)_{n \in \mathbb{N}}$ and ω^* in $\mathbb{X}$ such that $\omega_n \longrightarrow \omega^*$ as $n \longrightarrow \infty$, then $\mathbb{X}$ is regular ($\omega_n \preceq \omega^*$ (resp. $\omega_n \succeq \omega^*$)) for all $n \in \mathbb{N}$. By assuming this fact that there are lower and upper bounds in $\mathbb{X}$ for every both members of $\mathbb{X}$, in that case, the partially ordered Banach space $\mathbb{X}$ is named regular and lattice.

Definition 1 (see [29]). An operator: $\mathcal{Q} : \mathbb{X} \longrightarrow \mathbb{X}$ is termed as nondecreasing or isotone if $\mathcal{Q}$ maintains the order relation $\preceq$, i.e., when $\omega \preceq \omega$, it means that $\mathcal{Q}\omega \preceq \mathcal{Q}\omega$ for all $\omega, \omega \in \mathbb{X}$.

Definition 2 (see [29]). A mapping $\mathcal{Q} : \mathbb{X} \longrightarrow \mathbb{X}$ has the compactness specification if $\mathcal{Q}(\mathbb{X})$ is a set in $\mathbb{X}$ with the relative compactness. In addition, $\mathcal{Q}$ is totally bounded if $\mathcal{Q}(S)$ has the relative compactness property in $\mathbb{X}$, where $S \subseteq \mathbb{X}$ is an arbitrary bounded set.

Every operator having the continuity and total boundedness properties will be completely continuous.

Definition 3 (see [29]). $\mathcal{Q} : \mathbb{X} \longrightarrow \mathbb{X}$ has the partial continuity property at $a \in \mathbb{X}$, if for each $\varepsilon > 0$, $\delta > 0$ exists so that

$\|Q\omega - Qa\| < \varepsilon$ whenever $\|\omega - a\| < \delta$ and ω and a are comparable. Assuming Q as an operator with the partial continuity on $\mathbb{X}$, it is well-known that Q is continuous on each chain $\mathcal{C} \subset \mathbb{X}$. Furthermore, if $Q(\mathcal{C})$ is bounded for every $\mathcal{C} \subseteq \mathbb{X}$, then Q is partially bounded. In addition, Q is uniformly partially bounded if all existing chains $Q(\mathcal{C}) \subseteq \mathbb{X}$ involve the boundedness by a bound uniquely.

Definition 4 (see [29]). $Q : \mathbb{X} \longrightarrow \mathbb{X}$ has the partial compactness if $Q(\mathcal{C}) \subset \mathbb{X}$ has the relative compactness with respect to all chains $\mathcal{C} \subseteq \mathbb{X}$. It has the partial total boundedness property if for each bounded and totally ordered set $\mathcal{C}$ contained in $\mathbb{X}$, $Q(\mathcal{C}) \subset \mathbb{X}$ possesses the relative compactness.

Every operator with the partial continuity and the partial total boundedness is named as partially completely continuous on the underlying space.

Remark 5. Assume that Q is a nondecreasing selfmap on $\mathbb{X}$ and $\mathcal{C}$ is an arbitrary chain in it. In this case, Q possesses the partial compactness or the partial boundedness specifications whenever $Q(\mathcal{C})$ is relatively compact or bounded in $\mathbb{X}$.

Definition 6 (see [28]). Regard d and $\leq$ as a metric and an order relation on $\mathbb{X}$. We say that d and $\leq$ are compatible if $\{\omega_n\}_{n \in \mathbb{N}} \subset \mathbb{X}$ is monotone, and if a subsequence $\{\omega_{n_k}\}_{k \in \mathbb{N}}$ of $\{\omega_n\}_{n \in \mathbb{N}}$ tends to ω^* , then $\{\omega_n\}_{n \in \mathbb{N}}$ tends to ω^* . Similar definition can be applied on a partially order norm space. A subset S of $\mathbb{X}$ is named Janhavi if the order relation $\leq$ and the metric d (or the norm $\|\cdot\|$) are compatible in it. Particularly, if $S = \mathbb{X}$, then we say that $\mathbb{X}$ is Janhavi metric (or Janhavi Banach space).

Definition 7 (see [29]). An operator $Q : \mathbb{X} \longrightarrow \mathbb{X}$ is $\mathfrak{D}$ -Lipschitz if there exists an upper semicontinuous nondecreasing function: $\Psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ with $\Psi(0) = 0$ such that

$$\|Q\omega - Q\omega\| \leq \Psi(\|\omega - \omega\|), \quad (7)$$

for all $\omega, \omega \in \mathbb{X}$.

Definition 8 (see [29]). The same above operator Q is termed as partially nonlinear $\mathfrak{D}$ -Lipschitz whenever a $\mathfrak{D}$ -function $\Psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ exists provide

$$\|Q\omega - Q\omega\| \leq \Psi(\|\omega - \omega\|), \quad (8)$$

$\forall \omega, \omega \in \mathbb{X}$. In addition, when Q is nonlinear $\mathfrak{D}$ -Lipschitz subject to $\Psi(\tau) < \tau$ for $\tau > 0$, in that case, Q is nonlinear $\mathfrak{D}$ -contraction.

Let us at present introduce a novel procedure, named Dhage iterative method, which is very useful for obtaining a scheme for the approximation of solutions to problems with nonlinearity.

Theorem 9 (see [29]). Let $(\mathbb{X}, \leq, \|\cdot\|)$ be a complete regular normed linear algebra via the partial order so that $\leq$ and $\|\cdot\|$

are compatible. Consider two nondecreasing operators $\mathcal{K}, \mathcal{H} : \mathbb{X} \longrightarrow \mathbb{X}$ such that

- (a) $\mathcal{K}$ is partially nonlinear $\mathfrak{D}$ -Lipschitz and partially bounded with $\mathfrak{D}$ -function $\psi_{\mathcal{K}}$
- (b) $\mathcal{H}$ has the partial continuity and the compactness
- (c) $\exists$ an element $\omega_0 \in \mathbb{X}$ such that $\omega_0 \leq \mathcal{K}\omega_0 + \mathcal{H}\omega_0$ or $\omega_0 \pm \mathcal{K}\omega_0 + \mathcal{H}\omega_0$

Then, $\mathcal{K}\omega + \mathcal{H}\omega = \omega$ possesses a solution ω^* in $\mathbb{X}$, and the sequence of the successive iterations $\{\omega_n\}_{n=0}^{\infty}$, expressed as $\omega_{n+1} = \mathcal{K}\omega_n + \mathcal{H}\omega_n$, approaches to ω^* monotonically.

Theorem 10 (see [30]). Let $\mathcal{H} : \mathbb{X} \longrightarrow \mathbb{X}$ be a nondecreasing and partially nonlinear $\mathfrak{D}$ -contraction. Assume that $\omega_0 \in \mathbb{X}$ exists with $\omega_0 \leq \mathcal{H}\omega_0$ or $\omega_0 \pm \mathcal{H}\omega_0$. If $\mathbb{X}$ is regular or $\mathcal{H}$ is continuous, then a fixed point ω^* is found, and the sequence of successive iterations $\{\mathcal{H}^n \omega_0\}$ tends to ω^* monotonically. In addition, ω^* is unique if each of both members of $\mathbb{X}$ possesses a lower and an upper bound.

Remark 11 (see [31]). Let every set contained in $\mathbb{X}$ with the partial compactness includes the compatibility specification with respect to $\leq$ and $\|\cdot\|$. Then, every compact chain of $\mathbb{X}$ is Janhavi. This implication can be simply applied to establish the existence property of solutions in our research work.

Remark 12. The regularity property of $\mathbb{X}$ in Theorem 9 can be replaced with another strong continuity condition of the operators $\mathcal{K}$ and $\mathcal{H}$ on $\mathbb{X}$ where Dhage in [28] proved this result.

Remark 13 (see [30]).

- (1) In a partially normed linear space, every compact operator has the partial compactness, and all partially compact operators has the partial total boundedness, while the converse is not valid
- (2) Each completely continuous operator has the partial complete continuity, and each partially completely continuous operator has the continuity and the partial total boundedness, while the converse is not valid

In such a situation, the hypotheses regarding to the partial continuity and the partial compactness of an operator in Theorem 9 can be replaced by the continuity and compactness of that operator.

We state here the results below given by [45–47].

Definition 14. The mapping $\kappa : \mathfrak{M}_{\mathcal{C}} \longrightarrow [0, \infty)$ is named Kuratowski measure of non-compactness (KMNC) if

$$\kappa(B) = \inf \{ \varepsilon > 0 : B \text{ can be covered by finitely many sets with } \text{DIAM}(B) \leq \varepsilon \}, \quad (9)$$

where $\mathfrak{M}_{\mathcal{C}}$ represents a class of all bounded mappings in $\mathcal{C}$.

Proposition 15. *The following are fulfilled for KMNC:*

- (1) $A \subset E \Rightarrow \kappa(A) \leq \kappa(E)$
- (2) $\kappa(A) = \kappa(\bar{A}) = \kappa(\text{conv}(A))$, where $\bar{A}$ and $\text{conv}(A)$ represent the closure and the convex hull of A , respectively
- (3) $\kappa(A + E) \leq \kappa(A) + \kappa(E)$ and $\kappa(cA) = |c| \kappa(A)$, $c \in \mathbb{R}$

Definition 16. Assume that $\mathcal{K} : A \longrightarrow \mathcal{C}$ be a continuous bounded mapping and $A \subset \mathcal{C}$. The operator $\mathcal{K}$ is said to be κ -Lipschitz if we can find a constant $\ell \geq 0$ satisfying the following condition:

$$\kappa(\mathcal{K}(B)) \leq \ell \kappa(B), \text{ for every } B \subset A. \quad (10)$$

Moreover, $\mathcal{K}$ is called strict κ -contraction subject to $\ell < 1$.

Definition 17. $\mathcal{K}$ is called κ -condensing when

$$\kappa(\mathcal{K}(B)) < \kappa(B), \quad (11)$$

for every bounded and nonprecompact subset B of A . So,

$$\kappa(\mathcal{K}(B)) \geq \kappa(B), \text{ which implies } \kappa(B) = 0. \quad (12)$$

Further, we have $\mathcal{K} : A \longrightarrow \mathcal{C}$ is Lipschitz if we can find $\ell > 0$ such that

$$\|\mathcal{K}(u) - \mathcal{K}(v)\| \leq \ell \|u - v\|, \text{ for all } u, v \in A, \quad (13)$$

if $\ell < 1$, $\mathcal{K}$ is said to be strict contraction.

The following three interesting results are based on [48]:

Proposition 18. Let $\mathcal{K}, \mathcal{H} : A \longrightarrow \mathcal{C}$ be κ -Lipschitz with constants ℓ_1 and ℓ_2 . Then, $\mathcal{K} + \mathcal{H} : A \longrightarrow \mathcal{C}$ is κ -Lipschitz with $\ell_1 + \ell_2$.

Proposition 19. Every compact mapping $\mathcal{K} : A \longrightarrow \mathcal{C}$ is κ -Lipschitz with $\ell = 0$.

Proposition 20. Every Lipschitz mapping $\mathcal{K} : A \longrightarrow \mathcal{C}$ with ℓ is κ -Lipschitz with ℓ .

Isaia [48] used the topological degree theory to introduce the following interesting results:

Theorem 21. Let $\mathcal{F} : A \longrightarrow \mathcal{C}$ be κ -condensing and

$$\Theta = \{u \in \mathcal{C} : \exists \xi \in [0, 1] \text{ s.t. } x = \xi \mathcal{F}u\}. \quad (14)$$

If $\Theta \subset \mathcal{C}$ is bounded, i.e., $r > 0$ exists subject to $\Theta \subset B_r(0)$; then, the degree

$$\deg(I - \xi \mathcal{F}, B_r(0), 0) = 1, \text{ for all } \xi \in [0, 1]. \quad (15)$$

As a result, it is found a fixed-point for $\mathcal{F}$ and all possible fixed-points of $\mathcal{F}$ are contained in $B_r(0)$.

Let $\psi \in \mathcal{C}^1 = C^1(J, \mathbb{R})$ be an increasing differentiable function such that $\psi'(\tau) \neq 0$, $\forall \tau \in J$. Now, we start by defining ψ -FODs as follows:

Definition 22 (see [2]). The ψ -Riemann-Liouville fractional integral of order $\alpha > 0$ for an integrable function $\omega : J \longrightarrow \mathbb{R}$ is given by

$$\mathbb{I}_{a^+}^{\alpha; \psi} \omega(\tau) = \frac{1}{\Gamma(\alpha)} \int_a^\tau \psi'(s) (\psi(\tau) - \psi(s))^{\alpha-1} \omega(s) ds, \quad (16)$$

where the Gamma function is denoted by Γ .

Definition 23 (see [2]). Let $n - 1 < \alpha < n$ ($n \in \mathbb{N}$), $\omega : J \longrightarrow \mathbb{R}$ be an integrable function, and $\psi \in C^n(J, \mathbb{R})$. Then, the ψ -Riemann-Liouville FOD of a function ω of order α is expressed as:

$$\mathbb{D}_{a^+}^{\alpha; \psi} \omega(\tau) = \left(\frac{D_t}{\psi'(\tau)} \right)^n \mathbb{I}_{a^+}^{n-\alpha; \psi} \omega(\tau), \quad (17)$$

where $n = [\alpha] + 1$ and $D_t = d/dt$.

Definition 24 (see [20]). For $n - 1 < \alpha < n$ ($n \in \mathbb{N}$) and $\omega, \psi \in C^n(J, \mathbb{R})$, the ψ -Caputo FOD of a function ω of order α is given by

$${}^c \mathbb{D}_{a^+}^{\alpha; \psi} \omega(\tau) = \mathbb{I}_{a^+}^{n-\alpha; \psi} \omega_{\psi}^{[n]}(\tau), \quad (18)$$

where $\omega_{\psi}^{[n]}(\tau) = (D_t/\psi'(\tau))^n \omega(\tau)$.

From the above definition, we can express ψ -Caputo FOD by the following formula:

$${}^c \mathbb{D}_{a^+}^{\alpha; \psi} \omega(\tau) = \begin{cases} \int_a^\tau \frac{\psi'(s) (\psi(\tau) - \psi(s))^{n-\alpha-1}}{\Gamma(n-\alpha)} \omega_{\psi}^{[n]}(s) ds, & \text{if } \alpha \notin \mathbb{N}, \\ \omega_{\psi}^{[n]}(\tau), & \text{if } \alpha \in \mathbb{N}. \end{cases} \quad (19)$$

Also, the ψ -Caputo FOD of order α of ω is defined as

$${}^c \mathbb{D}_{a^+}^{\alpha; \psi} \omega(\tau) = \mathbb{D}_{a^+}^{\alpha; \psi} \left[\omega(\tau) - \sum_{k=0}^{n-1} \frac{\omega_{\psi}^{[k]}(a)}{k!} (\psi(\tau) - \psi(a))^k \right]. \quad (20)$$

For more details, see ([20], Theorem 3).

Lemma 25 (see [2]). For $\alpha, \beta > 0$, and $\omega \in C(J, \mathbb{R})$, we have

$$\mathbb{I}_{a^+}^{\alpha; \psi} \mathbb{I}_{a^+}^{\beta; \psi} \omega(\tau) = \mathbb{I}_{a^+}^{\alpha+\beta; \psi} \omega(\tau), \text{ a.e. } \tau \in J. \quad (21)$$

Lemma 26 (see [22]). Assume that $\alpha > 0$. If $\omega \in C(J, \mathbb{R})$, then

$${}^c\mathbb{D}_{a^+}^{\alpha;\psi} \mathbb{I}_{a^+}^{\alpha;\psi} \omega(\tau) = \omega(\tau), \tau \in J, \quad (22)$$

and if $\omega \in C^{n-1}(J, \mathbb{R})$, then

$$\mathbb{I}_{a^+}^{\alpha;\psi} {}^c\mathbb{D}_{a^+}^{\alpha;\psi} \omega(\tau) = \omega(\tau) - \sum_{k=0}^{n-1} \frac{\bar{\omega}_{\psi}^{[k]}(a)}{k!} [\psi(\tau) - \psi(a)]^k, \tau \in J. \quad (23)$$

It is easily to deduce that

$${}^c\mathbb{D}_{a^+}^{k;\psi} \mathbb{I}_{a^+}^{\alpha;\psi} = \mathbb{I}_{a^+}^{\alpha-k;\psi}. \quad (24)$$

Lemma 27 (see [2, 20]). Let $\tau > a$, $\alpha \geq 0$, $\varsigma > 0$ and let $\chi(\tau) = \psi(\tau) - \psi(a)$. Then:

- (i) $\mathbb{I}_{a^+}^{\alpha;\psi} (\chi(\tau))^{\varsigma-1} = \Gamma(\varsigma)/\Gamma(\varsigma + \alpha) (\chi(\tau))^{\varsigma+\alpha-1}$
- (ii) ${}^c\mathbb{D}_{a^+}^{\alpha;\psi} (\chi(\tau))^{\beta-1} = \Gamma(\varsigma)/\Gamma(\varsigma - \alpha) (\chi(\tau))^{\varsigma-\alpha-1}$
- (iii) ${}^c\mathbb{D}_{a^+}^{\alpha;\psi} (\chi(\tau))^k = 0$, for all $k \in \{0, \dots, n-1\}$, $n \in \mathbb{N}$

3. Existence and Approximation Results via Dhage's Technique

The solutions' existence and approximation of problem (1) are studied in this section.

Lemma 28. Assume that $(\mathcal{E}_b, \leq, \|\cdot\|)$ is a partially ordered Banach space with the norm $\|\cdot\|$, and the order relation $\leq$ defined by (5) and (6), respectively. Then, every partially compact subset of $\mathcal{E}_b$ is Janhavi.

Proof (see [31]). Let us now discuss exactly the problem (1). $\square$

Definition 29. A function $\omega \in \mathcal{E}_b$ is a lower solution for the NFudeq (1) if:

- (1) $\omega_\tau \in \mathcal{E}_b$, $\forall \tau \in J$
- (2) the function $\tau \mapsto [\omega(\tau) - \mathbb{F}(\tau, \omega_\tau)]$ is continuously differentiable on J and settles

$$\begin{cases} {}^c\mathbb{D}_{a^+}^{\alpha;\psi} [\omega(\tau) - \mathbb{F}(\tau, \omega_\tau)] \leq \mathbb{H}(\tau, \omega_\tau), \tau \in J, \\ \omega(\tau) \leq \phi(\tau). \end{cases} \quad (25)$$

Similarly, a differentiable function $\omega \in \mathcal{E}_b$ is named an upper solution of the NFudeq (1) if the above inequality is satisfied with reverse sign.

To demonstrate the solutions' existence to (1), we state this lemma:

Lemma 30. Assume that $0 < \nu < 1$, $\phi(a) = 0$, and $g, h : J \rightarrow \mathbb{R}$ are continuous with $h(a) = 0$. The linear problem

$${}^c\mathbb{D}_{a^+}^{\nu;\psi} [\omega(\tau) - h(\tau)] = g(\tau), \quad \tau \in J; \quad \omega(\tau) = \phi(\tau), \quad \tau \in [a - \delta, a], \quad (26)$$

has a unique solution $\omega(\tau)$ defined by:

$$\begin{cases} h(\tau) + \mathbb{I}_{a^+}^{\nu;\psi} g(\tau), & \text{if } \tau \in J, \\ \phi(\tau), & \text{if } \tau \in [a - \delta, a]. \end{cases} \quad (27)$$

For the proof of Lemma 30, it is useful to refer to [2, 23, 41, 49].

With the help of the following hypothesis, we can investigate our results:

(H1) The functions $\mathbb{F}(\tau, \omega)$ and $\mathbb{H}(\tau, \omega)$ are monotone nondecreasing with respect to ω for any $\tau \in J$.

(H2) $\exists$ a $\mathfrak{D}$ -function Ψ that satisfies $\Psi(R) < R$ for $R > 0$ such that

$$0 \leq [\mathbb{F}(\tau, \omega) - \mathbb{F}(\tau, \omega)] \leq \Psi(\omega - \omega) \forall \tau \in J \text{ and } \omega, \omega \in \mathbb{R} \text{ with } \omega \geq \omega. \quad (28)$$

(H3) $\exists M > 0$ such that $|\mathbb{H}(\tau, \omega)| \leq M, \forall \tau \in J$, and $\omega \in \mathbb{R}$.

(H4) $\exists L > 0$ such that $|\mathbb{F}(\tau, \omega)| \leq L, \forall \tau \in J$, and $\omega \in \mathbb{R}$.

(H5) The FBVP (1) possesses a lower solution $x \in \mathcal{E}_b$.

(H6) $\exists$ a positive constant $L_{\mathbb{H}}$ such that

$$|\mathbb{H}(\tau, \omega) - \mathbb{H}(\tau, \omega)| \leq L_{\mathbb{H}} |\omega - \omega|, \forall \tau \in J \text{ and } \omega, \omega \in \mathbb{R} \text{ with } \omega \geq \omega. \quad (29)$$

Theorem 31. If the hypotheses (H1)-(H5) are fulfilled, then the NFudeq (1) includes a solution $\bar{\omega}^*$ formulated on $[a - \delta, b]$, and $\{\omega_n\}$ containing the successive approximations expressed as:

$$\omega_0 = x, \omega_{n+1}(\tau) := \begin{cases} \mathbb{I}_{a^+}^{\nu;\psi} \mathbb{H}(\tau, \omega_{n,\tau}) + \mathbb{F}(\tau, \omega_{n,\tau}), & \text{if } \tau \in J, \\ \phi(\tau), & \text{if } \tau \in [a - \delta, a], \end{cases} \quad (30)$$

where $\omega_{n,\tau}(\rho) = \omega_n(\tau + \rho)$, $\rho \in [-\delta, 0]$, tends to $\bar{\omega}^*$ monotonically.

Proof. Take $\mathbb{X} = \mathcal{E}_b = C([a - \delta, b], \mathbb{R})$. Then, using Lemma 28, each compact chain $\mathcal{C} \subset \mathcal{E}_b$ admits the compatibility property in $\|\cdot\|$ and $\leq$ such that $\mathcal{C}$ is Janhavi in $\mathcal{E}_b$. On the other side, $\mathcal{K}$ and $\mathcal{H}$ can be defined on $\mathcal{E}_b$ as follows:

$$\mathcal{K}(\omega)(\tau) := \begin{cases} \mathbb{F}(\tau, \omega_\tau), & \text{if } \tau \in J, \\ 0, & \text{if } \tau \in [a - \delta, a], \end{cases} \quad (31)$$

$$\mathcal{H}(\omega)(\tau) := \begin{cases} \mathbb{I}_{a^+}^{\nu;\psi} \mathbb{H}(\tau, \omega_\tau), & \text{if } \tau \in J, \\ \phi(\tau), & \text{if } \tau \in [a - \delta, a]. \end{cases} \quad (32)$$

According to the structure of integral, it is obvious that $\mathcal{K}, \mathcal{H} : \mathcal{C}_b \longrightarrow \mathcal{C}_b$ are well-defined. In addition, the studied problem (1) can be reformulated by:

$$\mathcal{K}(\omega)(\tau) + \mathcal{H}(\omega)(\tau) = \omega(\tau), \tau \in [a - \delta, b]. \quad (33)$$

To investigate the solutions' existence to this operator equation, we can sufficiently show that the operators $\mathcal{K}$ and $\mathcal{H}$ satisfy all items of Theorem 9. We follow our argument split into five steps. $\square$

Step I. $\mathcal{K}$ and $\mathcal{H}$ are nondecreasing on $\mathcal{C}_b$.

For $\omega, \omega \in \mathcal{C}_b$ with $\omega \geq \omega$, using (H1), we get

$$\begin{aligned} \mathcal{K}(\omega)(\tau) &= \begin{cases} \mathbb{F}(\tau, \omega_\tau), & \text{if } \tau \in J, \\ 0, & \text{if } \tau \in [a - \delta, a] \end{cases} \\ &\geq \begin{cases} \mathbb{F}(\tau, \omega_\tau), & \text{if } \tau \in J, \\ 0, & \text{if } \tau \in [a - \delta, a] \end{cases} = \mathcal{K}(\omega)(\tau), \end{aligned} \quad (34)$$

for all $\tau \in [a - \delta, b]$. It means that $\mathcal{K} : \mathcal{C}_b \longrightarrow \mathcal{C}_b$ is a nondecreasing operator.

Similarly, we obtain

$$\begin{aligned} \mathcal{H}(\omega)(\tau) &= \begin{cases} \mathbb{I}_{a^+}^{\nu, \psi} \mathbb{H}(\tau, \omega_\tau), & \text{if } \tau \in J, \\ \phi(\tau), & \text{if } \tau \in [a - \delta, a] \end{cases} \\ &\geq \begin{cases} \mathbb{I}_{a^+}^{\nu, \psi} \mathbb{H}(\tau, \omega_\tau), & \text{if } \tau \in J, \\ \phi(\tau), & \text{if } \tau \in [a - \delta, a] \end{cases} = \mathcal{H}(\omega)(\tau), \end{aligned} \quad (35)$$

for all $\tau \in [a - \delta, b]$. Thus, it is concluded that $\mathcal{H} : \mathcal{C}_b \longrightarrow \mathcal{C}_b$ is a nondecreasing operator.

Step II. $\mathcal{K}$ is a nonlinear $\mathfrak{D}$ -contraction on $\mathcal{C}_b$.

For $\omega, \omega \in \mathcal{C}_b$ with $\omega \geq \omega$ and by (H2), we get that

$$\begin{aligned} |\mathcal{K}\omega(\tau) - \mathcal{K}\omega(\tau)| &\leq |\mathbb{F}(\tau, \omega_\tau) - \mathbb{F}(\tau, \omega_\tau)| \\ &\leq \Psi\left(\|\omega_\tau - \omega_\tau\|_{C_\delta}\right) \\ &\leq \Psi(\|\omega - \omega\|), \end{aligned} \quad (36)$$

$\forall \tau \in [a - \delta, b]$. By taking the supremum over τ , we get

$$\|\mathcal{K}\omega - \mathcal{K}\omega\| \leq \Psi\left(\|\omega - \omega\|_{\mathcal{C}_b}\right), \quad (37)$$

$\forall \omega, \omega \in \mathcal{C}_b, \omega \geq \omega$, where $r > \Psi(r)$ for $r > 0$. Therefore, according to Definition 8, our result is derived.

Step III. $\mathcal{H}$ is partially continuous on $\mathcal{C}_b$.

Regard $\{\omega_n\}_{n \in \mathbb{N}}$ in a chain $\mathcal{C} \ni \omega_n \longrightarrow \omega$ as $n \longrightarrow \infty$. Then, $\omega_{n,s} \longrightarrow \omega_s$ for any $s \in J$ letting $n \longrightarrow \infty$. The continuity of $\mathbb{H}$ yields

$$\begin{aligned} \lim_{n \longrightarrow \infty} (\mathcal{H}\omega_n)(\tau) &= \begin{cases} \frac{1}{\Gamma(\nu)} \lim_{n \longrightarrow \infty} \int_a^\tau \psi'(s)(\psi(\tau) - \psi(s))^{\nu-1} \mathbb{H}(s, \omega_{n,s}) ds, & \text{if } \tau \in J, \\ \phi(\tau), & \text{if } \tau \in [a - \delta, a] \end{cases} \\ &= \begin{cases} \frac{1}{\Gamma(\nu)} \int_a^\tau \psi'(s)(\psi(\tau) - \psi(s))^{\nu-1} \lim_{n \longrightarrow \infty} \mathbb{H}(s, \omega_{n,s}) ds, & \text{if } \tau \in J, \\ \phi(\tau), & \text{if } \tau \in [a - \delta, a] \end{cases} \\ &= \begin{cases} \mathbb{I}_{a^+}^{\nu, \psi} \mathbb{H}(\tau, \omega_\tau), & \text{if } \tau \in J, \\ \phi(\tau), & \text{if } \tau \in [a - \delta, a] \end{cases} \\ &= \mathcal{H}\omega(\tau), \end{aligned} \quad (38)$$

$\forall \tau \in [a - \delta, b]$. Hence, $\mathcal{H}\omega_n$ converges to $\mathcal{H}\omega$ pointwise on $[a - \delta, b]$.

In the following two cases, we prove that $\{\mathcal{H}\omega_n\}_{n \in \mathbb{N}}$ is an equicontinuous sequence of functions in $\mathcal{C}_b$.

Case A. Take $\tau_1, \tau_2 \in J$, with $\tau_1 < \tau_2$. Then,

$$\begin{aligned} & |\mathcal{H}(\omega_n)(\tau_2) - \mathcal{H}(\omega_n)(\tau_1)| \\ & \leq |\mathbb{I}_{a^+}^{\nu, \psi} \mathbb{H}(\tau_2, \omega_{n, \tau_2}) - \mathbb{I}_{a^+}^{\nu, \psi} \mathbb{H}(\tau_1, \omega_{n, \tau_1})| \\ & \leq \frac{1}{\Gamma(\nu)} \int_a^{\tau_1} \psi'(s) |(\psi(\tau_1) - \psi(s))^{\nu-1} - (\psi(\tau_2) - \psi(s))^{\nu-1}| |\mathbb{H}(s, \omega_{n, s})| ds \\ & \quad + \frac{1}{\Gamma(\nu)} \int_{\tau_1}^{\tau_2} \psi'(s) (\psi(\tau_2) - \psi(s))^{\nu-1} |\mathbb{H}(s, \omega_{n, s})| ds \\ & \leq \frac{L}{\Gamma(\nu+1)} ((\psi(\tau_2) - \psi(a))^\nu - (\psi(\tau_1) - \psi(a))^\nu) \\ & \quad + 2(\psi(\tau_2) - \psi(\tau_1))^\nu, \end{aligned} \quad (39)$$

which tends to zero as $\tau_1 \rightarrow \tau_2$.

Case B. For $\tau_1, \tau_2 \in [a - \delta, a]$. Then,

$$|\mathcal{H}(\omega_n)(\tau_2) - \mathcal{H}(\omega_n)(\tau_1)| = |\phi(\tau_2) - \phi(\tau_1)| \rightarrow 0, \text{ as } \tau_1 \rightarrow \tau_2. \quad (40)$$

Clearly, if $\tau_1 \in [a - \delta, a]$ and $\tau_2 \in J$ such that $\tau_1 \rightarrow \tau_2$ has only one possibility that they are close to a at which $\mathcal{H}(\omega_n)$ is close to zero.

Thus,

$$|\mathcal{H}(\omega_n)(\tau_2) - \mathcal{H}(\omega_n)(\tau_1)| \rightarrow 0, \text{ as } \tau_1 \rightarrow \tau_2, \quad (41)$$

uniformly $\forall n \geq 1$. This proves that $\{\mathcal{H}\omega_n\}$ is equi-continuous on $[a - \delta, b]$. Thus, the pointwise convergence of $\{\mathcal{H}\omega_n\}$ on $[a - \delta, b]$ implies the uniform convergence, so $\mathcal{H}\omega_n$ converges to $\mathcal{H}\omega$ uniformly on $[a - \delta, b]$. Consequently, the selfmap $\mathcal{H}$ possesses the partial continuity on $\mathcal{C}_b$.

Step IV. $\mathcal{H}$ has the partial compactness property on $\mathcal{C}_b$.

Regard the chain $\mathcal{C}$ in $\mathcal{C}_b$ and $\omega \in \mathcal{H}(\mathcal{C})$. Then $\exists \omega \in \mathcal{C} \ni \omega = \mathcal{H}\omega$. Using hypothesis (H3), if $\tau \in [a - \delta, a]$, we have

$$|\omega(\tau)| = |(\mathcal{H}\omega)(\tau)| \leq |\phi(\tau)| \leq \|\phi\|_{\mathcal{C}_\delta} \leq \|\phi\|_{\mathcal{C}_b}. \quad (42)$$

Otherwise, if $\tau \in J$, then

$$|\mathcal{H}(\omega)(\tau)| \leq \mathbb{I}_{a^+}^{\nu, \psi} |\mathbb{H}(\tau, \omega_\tau)| \leq M \mathbb{I}_{a^+}^{\nu, \psi} (1)(\tau) \leq \frac{(\psi(\tau) - \psi(a))^\nu}{\Gamma(\nu+1)} M. \quad (43)$$

Hence,

$$\|\mathcal{H}(\omega)\|_{\mathcal{C}_b} \leq \|\phi\|_{\mathcal{C}_b} + \frac{(\psi(\tau) - \psi(a))^\nu}{\Gamma(\nu+1)} M := R, \quad (44)$$

$\forall \tau \in [a - \delta, b]$. Thus, we obtain $\|\omega\| \leq \|\mathcal{H}\omega\| \leq R$ for any $\omega \in \mathcal{H}(\mathcal{C})$. Thus, $\mathcal{H}(\mathcal{C})$ is a uniformly bounded subset of $\mathcal{C}_b$.

Let us now prove that $\mathcal{H}(\mathcal{C})$ is an equi-continuous set in $\mathcal{C}_b$. Let $\tau_1, \tau_2 \in J$, with $\tau_1 < \tau_2$. Then, according to Step III arguments, it is concluded that

$$|\omega(\tau_2) - \omega(\tau_1)| = |\mathcal{H}\omega(\tau_2) - \mathcal{H}\omega(\tau_1)| \rightarrow 0 \text{ as } \tau_1 \rightarrow \tau_2, \quad (45)$$

uniformly for any $\omega \in \mathcal{H}(\mathcal{C})$ which illustrates the equi-continuity of $\mathcal{H}(\mathcal{C})$ in $\mathcal{C}_b$. So, $\mathcal{H}(\mathcal{C})$ is compact in reference to Arzelà-Ascoli criterion. As a result, the selfmap $\mathcal{H} : \mathcal{C}_b \rightarrow \mathcal{C}_b$ admits the partial compactness property on $\mathcal{C}_b$.

Step V. ω satisfies $\omega \leq \mathcal{H}\omega + \mathcal{H}\omega$.

By (H5), W is a lower solution of the NFudeq (1) defined on $[a - \delta, b]$. Then, according to the lower solution definition, we get

$$\begin{cases} {}^c \mathbb{D}_{a^+}^{\nu, \psi} [W(\tau) - \mathbb{F}(\tau, W_\tau)] \leq \mathbb{H}(\tau, W_\tau), \tau \in J = [a, b], \\ W(\tau) \leq \phi(\tau), \tau \in [a - \delta, a]. \end{cases} \quad (46)$$

Let us integrate the above inequality from a to τ , we obtain

$$\begin{aligned} W(\tau) & \leq \begin{cases} \mathbb{I}_{a^+}^{\nu, \psi} \mathbb{H}(\tau, W_\tau) + \mathbb{F}(\tau, W_\tau), & \text{if } \tau \in J, \\ \phi(\tau), & \text{if } \tau \in [a - \delta, a]. \end{cases} \\ & = \mathcal{H}W(\tau) + \mathcal{H}W(\tau), \end{aligned} \quad (47)$$

$\forall \tau \in [a - \delta, b]$. Thus, $W \leq \mathcal{H}W + \mathcal{H}W$. Obviously, both operators $\mathcal{H}$ and $\mathcal{H}$ satisfy all of the items of Theorem 9; therefore, the operator equation $\mathcal{H}\omega + \mathcal{H}\omega = \omega$ has a solution ω^* defined on $[a - \delta, b]$. Furthermore, the sequence $\{\omega_n\}_{n=0}^\infty$ of successive approximations defined by (30) tends to ω^* monotonically. So, our proof is ended.

Remark 32. Above theorem's conclusion also remains true if the hypothesis (H5) is replaced with (H7) such that the NFudeq (1) has an upper solution: $y \in \mathcal{C}_b$.

Similarly, its proof under this replaced condition can be shown by the observation of the same arguments with some modifications.

Theorem 33. *Let (H1), (H5), and (H6) be valid. Then, the problem (1) has a unique solution $\bar{\omega}^*$ defined on $[a - \delta, b]$ provided that $\Omega(R) < R, R > 0$, where*

$$\Omega(R) = \frac{L_H(\psi(b) - \psi(a))^v R}{\Gamma(v + 1)} + \Psi(R). \quad (48)$$

Moreover, the sequence $\{\bar{\omega}_n\}$ of successive approximations defined by (30) converges monotonically to $\bar{\omega}^*$.

Proof. First, the operator: $\mathfrak{G} : \mathcal{C}_b \longrightarrow \mathcal{C}_b$ is defined by

$$\mathfrak{G}(\bar{\omega})(\tau) := \begin{cases} \mathbb{I}_{a^+}^{v,\psi} \mathbb{H}(\tau, \bar{\omega}_\tau) + \mathbb{F}(\tau, \bar{\omega}_\tau), & \text{if } \tau \in J, \\ \phi(\tau), & \text{if } \tau \in [a - \delta, a], \end{cases} \quad (49)$$

for $\tau \in [a - \delta, b]$. To prove this theorem, we establish the satisfaction of all items of Theorem 10 for $\mathfrak{G}$ in $\mathcal{C}_b$. We know that $\mathfrak{G}$ is nondecreasing and continuous.

The details are similar as in the proof of Theorem 31, so we omit them. Therefore, it is needed to be verified that $\mathfrak{G}$ is a partially $\mathfrak{D}$ -contraction on $\mathcal{C}_b$. To arrive at such an aim, by taking $\bar{\omega}, \omega \in \mathcal{C}_b$ such that $\bar{\omega} \geq \omega$, if $\tau \in [a - \delta, a]$, then it is obvious that

$$|\mathfrak{G}(\bar{\omega})(\tau) - \mathfrak{G}(\omega)(\tau)| = 0. \quad (50)$$

Otherwise, let $\tau \in J$, it follows from (H1) and (H6), that

$$\begin{aligned} & |\mathfrak{G}(\bar{\omega})(\tau) - \mathfrak{G}(\omega)(\tau)| \\ & \leq \mathbb{I}_{a^+}^{v,\psi} |\mathbb{H}(\tau, \bar{\omega}_\tau) - \mathbb{H}(\tau, \bar{\omega}_\tau)| + |\mathbb{F}(\tau, \bar{\omega}_\tau) - \mathbb{F}(\tau, \bar{\omega}_\tau)| \\ & \leq L_H \|\bar{\omega}_\tau - \bar{\omega}_\tau\|_{\mathcal{C}_\delta} \mathbb{I}_{a^+}^{v,\psi} (1)(\tau) + \Psi \left(\|\bar{\omega}_\tau - \bar{\omega}_\tau\|_{\mathcal{C}_\delta} \right) \\ & \leq \frac{L_H(\psi(b) - \psi(a))^v}{\Gamma(v + 1)} \|\bar{\omega}_\tau - \bar{\omega}_\tau\|_{\mathcal{C}_\delta} + \Psi \left(\|\bar{\omega}_\tau - \bar{\omega}_\tau\|_{\mathcal{C}_\delta} \right) \\ & \leq \Omega \left(\|\bar{\omega}_\tau - \bar{\omega}_\tau\|_{\mathcal{C}_\delta} \right), \end{aligned} \quad (51)$$

for all $\tau \in J$, where $\Omega(R) < R, R > 0$. Let us now take the supremum over τ , we get

$$\|\mathfrak{G}\bar{\omega} - \mathfrak{G}\bar{\omega}\| \leq \Omega \left(\|\bar{\omega} - \bar{\omega}\|_{\mathcal{C}_b} \right), \quad (52)$$

for all $\bar{\omega}, \bar{\omega} \in \mathcal{C}_b$, with $\bar{\omega} \geq \bar{\omega}$. As a result, $\mathfrak{G}$ is a partially nonlinear $\mathfrak{D}$ -contraction on $\mathcal{C}_b$. In addition, by using Theorem 31, it is proven that the given function x in (H5) satisfies the operator inequality $x \leq \mathfrak{G}x$ on $[a - \delta, b]$. Therefore, from Theorem 10, it is found a solution $\bar{\omega}^*$ uniquely for the NFDEq (1), and $\{\bar{\omega}_n\}$ defined by (30) tends to $\bar{\omega}^*$ monotonically. $\square$

4. Existence Result via Topological Degree Theory

The existence problem of the NFDEq (1) is investigated in this section based on the Topological Degree Theory due to Isaia [48]. Let us first introduce the following hypothesis for convenience:

(M1) The functions $\mathbb{F}$ and $\mathbb{H}$ satisfy the following growth conditions for constants $M_i, N_i > 0, i = 1, 2, p \in (0, 1)$:

$$\begin{aligned} |\mathbb{F}(t, \bar{\omega})| & \leq M_1 |\bar{\omega}|^p + N_1, \\ |\mathbb{H}(t, \bar{\omega})| & \leq M_2 |\bar{\omega}|^p + N_2, \end{aligned} \quad (53)$$

for each $t \in J$ and each $\bar{\omega} \in \mathbb{R}$.

(M2) For each $\tau \in J$, and for each, $\bar{\omega}, \omega \in \mathbb{R}$, $\exists$ constants $L_{\mathbb{F}}, L_{\mathbb{H}} > 0$, provided

$$\begin{aligned} |\mathbb{F}(\tau, \bar{\omega}) - \mathbb{F}(\tau, \bar{\omega})| & \leq L_{\mathbb{F}} |\bar{\omega} - \bar{\omega}|, \\ |\mathbb{H}(\tau, \bar{\omega}) - \mathbb{H}(\tau, \bar{\omega})| & \leq L_{\mathbb{H}} |\bar{\omega} - \bar{\omega}|. \end{aligned} \quad (54)$$

In view of Lemma 30, we consider two operators $\mathcal{K}, \mathcal{H} : \mathcal{C}_b \longrightarrow \mathcal{C}_b$ given by (31) and (32), respectively. Then, we write the integral equation (27) as an operator equation:

$$\mathcal{F}\bar{\omega}(t) = \mathcal{K}\bar{\omega}(t) + \mathcal{H}\bar{\omega}(t), t \in [a - \delta, b]. \quad (55)$$

The continuity of $\mathbb{F}$ and $\mathbb{H}$ shows that the operator $\mathcal{F}$ is well-defined, and its fixed points are the same solutions of the existing equation (27) in Lemma 30.

Lemma 34. *If (M1) and (M2) hold, then the operator $\mathcal{K}$ is Lipschitz with constant $L_{\mathbb{F}}$ and satisfies*

$$\|\mathcal{K}\bar{\omega}\| \leq (M_1 \|\bar{\omega}\|^p + N_1), \text{ for every } \bar{\omega} \in \mathcal{C}_b. \quad (56)$$

Proof. Let $\bar{\omega}, \bar{\omega} \in \mathcal{C}_b$, then we get

$$|\mathcal{K}\bar{\omega}(\tau) - \mathcal{K}\bar{\omega}(\tau)| = |\mathbb{F}(\tau, \bar{\omega}) - \mathbb{F}(\tau, \bar{\omega})| \leq L_{\mathbb{F}} \|\bar{\omega} - \bar{\omega}\|, \quad (57)$$

$\forall \tau \in J$. Let us take the supremum over τ , so we get

$$\|\mathcal{K}\bar{\omega} - \mathcal{K}\bar{\omega}\| \leq L_{\mathbb{F}} \|\bar{\omega} - \bar{\omega}\|. \quad (58)$$

Hence, $\mathcal{K} : \mathcal{C}_b \longrightarrow \mathcal{C}_b$ is a Lipschitzian on $\mathcal{C}_b$ with Lipschitz constant $L_{\mathbb{F}}$. From Proposition 20, $\mathcal{K}$ is κ -Lipschitz with constant $L_{\mathbb{F}}$. In addition, we get

$$|\mathcal{K}\bar{\omega}(t)| \leq (M_1 \|\bar{\omega}\|^p + N_1), \quad (59)$$

for every $\bar{\omega} \in \mathcal{C}_b$. This finishes the proof. $\square$

Lemma 35. *If (M1) holds, then $\mathcal{H}$ is continuous and satisfies the growth condition*

$$\|\mathcal{H}\bar{\omega}\| \leq \bar{M} \|\bar{\omega}\|^p + \bar{N}, \text{ for every } \bar{\omega} \in \mathcal{C}_b. \quad (60)$$

Proof. Choose a bounded subset $\mathbb{D}_r = \{\bar{\omega} \in \mathcal{C}_b : \|\bar{\omega}\| \leq r\} \subset \mathcal{C}_b$ and consider a sequence $\{\bar{\omega}_n\} \in \mathbb{D}_r$ via $\bar{\omega}_n \rightarrow \bar{\omega}$ by letting $n \rightarrow \infty$ in $\mathbb{D}_r$. We shall prove that $\|\mathcal{H}\bar{\omega}_n - \mathcal{H}\bar{\omega}\| \rightarrow 0$, letting $n \rightarrow \infty$. From the continuity of $\mathbb{H}$, it follows that $\mathbb{H}(s, \bar{\omega}_n) \rightarrow \mathbb{H}(s, \bar{\omega})$, as $n \rightarrow \infty$. In view of (M1), we get $|\mathcal{H}\bar{\omega}_n(\tau)| \leq \lambda(\tau)$, where

$$\lambda(\tau) = \begin{cases} |\phi(\tau)|, & \text{if } \tau \in [a - \delta, a], \\ \frac{(\psi(\tau) - \psi(a))^\nu}{\Gamma(\nu + 1)} (M_2 r^p + N_2), & \text{if } \tau \in J, \end{cases} \quad (61)$$

which is Lebesgue's integrable bounded function. The Labesgue dominated convergence theorem ensures that $\|\mathcal{H}\bar{\omega}_n - \mathcal{H}\bar{\omega}\| \rightarrow 0$, letting $n \rightarrow +\infty$, which confirms the continuity of $\mathcal{H}$.

Next, it is easy as above to deduce that

$$|\mathcal{H}\bar{\omega}(\tau)| \leq \begin{cases} \|\phi\|_{\mathcal{C}_b}, & \text{if } \tau \in [a - \delta, a], \\ \left(\frac{(\psi(b) - \psi(a))^\nu}{\Gamma(\nu + 1)} \right) (M_2 \|\bar{\omega}\|^p + N_2), & \text{if } \tau \in J. \end{cases} \quad (62)$$

Therefore,

$$\begin{aligned} \|\mathcal{H}\bar{\omega}\| &\leq \frac{(\psi(b) - \psi(a))^\nu}{\Gamma(\nu + 1)} (M_2 \|\bar{\omega}\|^p + N_2) + \|\phi\|_{\mathcal{C}_b} \\ &= \bar{M} \|\bar{\omega}\|^p + \bar{N}, \end{aligned} \quad (63)$$

where $\bar{M} = (\psi(b) - \psi(a))^\nu / \Gamma(\nu + 1) M_2$ and $\bar{N} = (\psi(b) - \psi(a))^\nu / \Gamma(\nu + 1) N_2 + \|\phi\|_{\mathcal{C}_b}$. This completes the proof. $\square$

Lemma 36. *If (M1) holds, then the operator $\mathcal{H} : \mathcal{C}_b \rightarrow \mathcal{C}_b$ is compact. As a result, $\mathcal{H}$ is κ -Lipschitz with zero constant.*

Proof. Take a bounded set $\bar{\Omega} \subset \mathbb{D}_r$. We need to establish the relative compactness of $\mathcal{H}(\bar{\Omega})$ in $\mathcal{C}_b$. For $\bar{\omega} \in \bar{\Omega}$, with the help of the estimate (63), we can obtain

$$\|\mathcal{H}\bar{\omega}\| \leq \bar{M} r^p + \bar{N}, \quad (64)$$

which shows that $\mathcal{H}(\bar{\Omega})$ is uniformly bounded.

Now, we prove the equi-continuity of $\mathcal{H}$. For $\tau \in J$, we can estimate the derivative operator using (24) as follows:

$$\begin{aligned} |(\mathcal{H}\bar{\omega})'(\tau)| &\leq \mathbb{I}_{a^+}^{\nu-1, \psi} |\mathbb{H}(\tau, \bar{\omega}(\tau))| \\ &\leq (M_2 \|\bar{\omega}\|^p + N_2) \mathbb{I}_{a^+}^{\nu-1, \psi} (1)(\tau) \\ &\leq \frac{(\psi(b) - \psi(a))^{\nu-1}}{\Gamma(\nu)} (M \|\bar{\omega}\|^p + N) := \ell. \end{aligned} \quad (65)$$

Hence, for each $\tau_1, \tau_2 \in J$ with $a < \tau_1 < \tau_2 < b$, we get

$$|(\mathcal{H}\bar{\omega})(\tau_2) - (\mathcal{H}\bar{\omega})(\tau_1)| \leq \int_{\tau_1}^{\tau_2} |(\mathcal{H}\bar{\omega})'(s)| ds \leq \ell(\tau_2 - \tau_1), \quad (66)$$

which tends to zero independently of $\bar{\omega}$ as $(\tau_2 - \tau_1) \rightarrow 0$. So, $\mathcal{H}$ is equi-continuous. The equi-continuity for the case $\tau_1, \tau_2 \in [a - \delta, a]$ is obvious. From the foregoing arguments along with Arzela-Ascoli theorem, we deduce that $\mathcal{H}$ is compact on $\mathbb{D}_r$. Thus, from Proposition 19, $\mathcal{H}$ is κ -Lipschitz with zero constant. This completes our proof. $\square$

Theorem 37. *If (M1) and (M2) hold, then the NFudeq (1) has at least one solution $\bar{\omega} \in \mathcal{C}_b$ provided that $L_F < 1$, and the set of the solutions is bounded in $\mathcal{C}_b$.*

Proof. Assume that $\mathcal{K}, \mathcal{H}, \mathcal{F}$ are the operators defined by (31), (32) and (55), respectively, which all of them are bounded and continuous, and also, by Lemma 34, $\mathcal{K}$ is κ -Lipschitz with L_F and by Lemma 36, $\mathcal{H}$ is κ -Lipschitz with constant 0. Thus, by Proposition 18, $\mathcal{F}$ is κ -Lipschitz with L_F . Hence, $\mathcal{F}$ is strict κ -contraction with $L_F > 0$. Since $L_F < 1$, $\mathcal{F}$ is κ -condensing.

Now, let us consider the following set:

$$\Theta = \{\bar{\omega} \in \mathcal{C}_b : \text{there exists } \varsigma \in [0, 1] \text{ such that } x = \varsigma \mathcal{F}\bar{\omega}\}. \quad (67)$$

We will show that the set Θ is bounded. For $\bar{\omega} \in \Theta$, we have $\bar{\omega} = \varsigma \mathcal{F}\bar{\omega} = \varsigma(\mathcal{K}(\bar{\omega}) + \mathcal{H}(\bar{\omega}))$, which implies that

$$\begin{aligned} \|\bar{\omega}\| &\leq \varsigma(\|\mathcal{K}\bar{\omega}\| + \|\mathcal{H}\bar{\omega}\|) \leq \varsigma[M_1 \|\bar{\omega}\|^p + N_1 + \bar{M} \|\bar{\omega}\|^p + \bar{N}] \\ &= \varsigma[(M_1 + \bar{M}) \|\bar{\omega}\|^p + (N_1 + \bar{N})] = \mathcal{M} \|\bar{\omega}\|^p + \mathcal{N}, \end{aligned} \quad (68)$$

where $\mathcal{M} = (M_1 + \bar{M})$ and $\mathcal{N} = (N_1 + \bar{N})$. If Θ is unbounded in $\mathcal{C}_b$, in that case, we divide the obtained inequality by $a := \|\bar{\omega}\|$ and supposing $a \rightarrow \infty$, we get

$$1 \leq \lim_{a \rightarrow \infty} \frac{\mathcal{M} a^p + \mathcal{N}}{a} = 0, \quad (69)$$

which is impossible, and Θ is bounded. Accordingly, it is found a fixed point for $\mathcal{F}$ which is interpreted as the solution of the NFudeq (1). This finishes the proof. $\square$

Remark 38. If (M1) is represented for $p = 1$, then Theorem 37 is true so that $\mathcal{M} < 1$.

5. Uniqueness Result and UH Stability

The uniqueness of the solution for the NFudeq (1) will be investigated below by using the standard Banach fixed point theorem. Moreover, The UH stability of the NFudeq (1) will be also checked.

Theorem 39. Suppose that assumption (M2) holds. Assume that

$$\Delta := \left(\frac{(\psi(b) - \psi(a))^v}{\Gamma(v+1)} L_H + L_F \right) < 1. \quad (70)$$

Then, $\exists$ a unique solution for (1) on the interval $[a - \delta, b]$.

Proof. Define the set

$$U := \left\{ \omega \in \mathcal{C}_b : \omega|_{[a-\delta, a]} \in \mathcal{C}_\delta, \omega|_J \in \mathcal{C}; {}^c\mathbb{D}_{a^+}^v \omega \in \mathcal{C} \right\}, \quad (71)$$

and the operator $\mathcal{G} : U \longrightarrow U$:

$$\mathcal{G}(\omega)(\tau) := \begin{cases} \mathbb{I}_{a^+}^{v,\psi} \mathbb{H}(\tau, \omega_\tau) + \mathbb{F}(\tau, \omega_\tau), & \text{if } \tau \in J, \\ \phi(\tau), & \text{if } \tau \in [a - \delta, a]. \end{cases} \quad (72)$$

Notice that $\mathcal{G}$ is well defined. Indeed, for $\omega \in U$, $\tau \mapsto \mathcal{G}(\omega)(\tau)$ is continuous, for any $\tau \in a - \delta, b]$. In addition, $\forall \tau \in J$, ${}^c\mathbb{D}_{a^+}^{a,\psi} [\mathcal{G}(\omega)(\tau) - \mathbb{F}(\tau, \omega_\tau)] = \mathbb{H}(\tau, \omega_\tau)$ exists, and it is continuous too due to the continuity of $\mathbb{H}$ and Lemma 26.

Now, we need to show that $\mathcal{G}$ is a contraction. If $\omega, \bar{\omega} \in U$ and $\tau \in a - \delta, a]$, then, $|\mathcal{G}(\omega)(\tau) - \mathcal{G}(\bar{\omega})(\tau)|$ equals to zero. On the contrary, for $\tau \in J$, by (M2), it is derived that

$$\begin{aligned} & |\mathcal{G}(\omega)(\tau) - \mathcal{G}(\bar{\omega})(\tau)| \\ & \leq \mathbb{I}_{a^+}^{v,\psi} |\mathbb{H}(\tau, \omega_\tau) - \mathbb{H}(\tau, \bar{\omega}_\tau)| + |\mathbb{F}(\tau, \omega_\tau) - \mathbb{F}(\tau, \bar{\omega}_\tau)| \\ & \leq L_H \|\omega_\tau - \bar{\omega}_\tau\|_{\mathcal{C}_\delta} \mathbb{I}_{a^+}^{v,\psi} (1)(\tau) + L_F \|\omega_\tau - \bar{\omega}_\tau\|_{\mathcal{C}_\delta} \\ & \leq \left(\frac{(\psi(b) - \psi(a))^v}{\Gamma(v+1)} L_H + L_F \right) \|\omega_\tau - \bar{\omega}_\tau\|_{\mathcal{C}_\delta} \\ & \leq \left(\frac{(\psi(b) - \psi(a))^v}{\Gamma(v+1)} L_H + L_F \right) \|\omega - \bar{\omega}\|_{\mathcal{C}_b}, \end{aligned} \quad (73)$$

which implies

$$\|\mathcal{G}(\omega) - \mathcal{G}(\bar{\omega})\|_{\mathcal{C}_b} \leq \Delta \|\omega - \bar{\omega}\|_{\mathcal{C}_b}. \quad (74)$$

Since $\Delta < 1$, the operator $\mathcal{G}$ is a contraction. Hence, Banach fixed point theorem shows that $\mathcal{G}$ admits a unique fixed point. This finishes the proof. $\square$

Here, we discuss the UH and GUH stability types of (1).

Definition 40. The NFudeq (1) is UH stable when $\exists c \in \mathbb{R}^+$ so that $\forall \varepsilon \in \mathbb{R}^+$ and $\forall \bar{\omega} \in \mathcal{C}_b$ satisfying

$$\begin{cases} |{}^c\mathbb{D}_{a^+}^{v,\psi} [\bar{\omega}(\tau) - \mathbb{F}(\tau, \bar{\omega}_\tau)] - \mathbb{H}(\tau, \bar{\omega}_\tau)| \leq \varepsilon, \tau \in J, \\ |\bar{\omega}(\tau) - \phi(\tau)| \leq \varepsilon, \tau \in [a - \delta, a], \end{cases} \quad (75)$$

exactly one solution $\omega \in \mathcal{C}_b$ of (1) exists with

$$\|\bar{\omega} - \omega\| \leq c\varepsilon. \quad (76)$$

Definition 41. The NFudeq (1) is GUH stable if $\exists \sigma \in C(\mathbb{R}^+, \mathbb{R}^+)$, $\sigma(0) = 0$ so that $\forall \varepsilon \in \mathbb{R}^+$ and $\forall \bar{\omega} \in \mathcal{C}_b$ satisfying (75), one and exactly one solution $\omega \in \mathcal{C}_b$ of (1) exists with

$$\|\bar{\omega} - \omega\| \leq \sigma(\varepsilon). \quad (77)$$

Remark 42. A function $\bar{\omega} \in \mathcal{C}_b$ is a solution of the inequality (75) iff $\exists$ a function $\eta \in \mathcal{C}$ such that

$$\begin{aligned} (1) \quad & |\eta(\tau)| \leq \varepsilon, \tau \in J \\ (2) \quad & {}^c\mathbb{D}_{a^+}^{v,\psi} [\bar{\omega}(\tau) - \mathbb{F}(\tau, \bar{\omega}_\tau)] = \mathbb{H}(\tau, \bar{\omega}_\tau) + \eta(\tau), \tau \in J \end{aligned}$$

Theorem 43. Suppose that (M2) and (70) hold. In this case, the solution of (1) is UH and GUH stable.

Proof. Assume that each of these two members $\varepsilon \in \mathbb{R}^+$ and $\bar{\omega} \in \mathcal{C}_b$ satisfy (75). Then, $\exists \eta \in \mathcal{C}$ such that $|\eta(\tau)| \leq \varepsilon$, $\tau \in J$, and

$$\begin{cases} {}^c\mathbb{D}_{a^+}^{v,\psi} [\bar{\omega}(\tau) - \mathbb{F}(\tau, \bar{\omega}_\tau)] = \mathbb{H}(\tau, \bar{\omega}_\tau) + \eta(\tau), \tau \in J, \\ \bar{\omega}(\tau) = \phi(\tau), \tau \in [a - \delta, a]. \end{cases} \quad (78)$$

Using Lemma 30, the NFudeq (78) has a solution given as

$$\bar{\omega}(t) := \begin{cases} \mathbb{I}_{a^+}^{v,\psi} [\mathbb{H}(\tau, \bar{\omega}_\tau) + \eta(t)] + \mathbb{F}(\tau, \bar{\omega}_\tau), & \text{if } \tau \in J, \\ \phi(\tau), & \text{if } \tau \in [a - \delta, a]. \end{cases} \quad (79)$$

$\square$

Theorem 44. Ensures the existence of a unique solution $\omega \in \mathcal{C}_b$ of the NFudeq (1) which satisfies

$$\omega(t) := \begin{cases} \mathbb{I}_{a^+}^{v,\psi} \mathbb{H}(\tau, \omega_\tau) + \mathbb{F}(\tau, \omega_\tau), & \text{if } \tau \in J, \\ \phi(\tau), & \text{if } \tau \in [a - \delta, a]. \end{cases} \quad (80)$$

Therefore, for any $\tau \in J$, we get:

$$\begin{aligned} & |\bar{\omega}(\tau) - \omega(\tau)| \\ & \leq \mathbb{I}_{a^+}^{v,\psi} |\mathbb{H}(\tau, \bar{\omega}_\tau) - \mathbb{H}(\tau, \omega_\tau)| + \mathbb{I}_{a^+}^{v,\psi} |\eta(\tau)| \\ & \quad + |\mathbb{F}(\tau, \bar{\omega}_\tau) - \mathbb{F}(\tau, \omega_\tau)| \\ & \leq \frac{(\psi(\tau) - \psi(a))^v}{\Gamma(v+1)} L_H \|\bar{\omega}_\tau - \omega_\tau\|_{\mathcal{C}_\delta} + \frac{(\psi(\tau) - \psi(a))^v}{\Gamma(v+1)} \varepsilon \\ & \quad + L_F \|\bar{\omega}_\tau - \omega_\tau\|_{\mathcal{C}_\delta} \\ & \leq \left(\frac{(\psi(b) - \psi(a))^v}{\Gamma(v+1)} L_H + L_F \right) \|\bar{\omega} - \omega\|_{\mathcal{C}_b} + \kappa \varepsilon, \end{aligned} \quad (81)$$

where $\kappa = (\psi(b) - \psi(a))^\nu / \Gamma(\nu + 1)$. Therefore, we have proved that

$$\|\bar{\omega} - \omega\|_{\mathcal{E}_b} \leq \Delta \|\bar{\omega} - \omega\|_{\mathcal{E}_b} + \kappa \varepsilon. \quad (82)$$

By the condition in Theorem 39, one can deduce that

$$\|\bar{\omega} - \omega\|_{\mathcal{E}_b} \leq \frac{\kappa}{1 - \Delta} \varepsilon. \quad (83)$$

For $c = \kappa/1 - \Delta > 0$, we infer that the solution of (1) is UH stable. In a similar manner, it is shown the existence of $\sigma \in \mathcal{C}(\mathbb{R}^{>0}, \mathbb{R}^{>0})$ by $\sigma(\varepsilon) = \kappa/1 - \Delta \varepsilon$ with $\sigma(0) = 0$. Hence, the solution of (1) is GUH stable.

6. Examples

Two illustrative examples are provided in this section to apply and validate our obtained results.

Example 45. Let us consider the NFuDq according to (1) such that

$$\begin{cases} {}^c \mathbb{D}_{a^+}^{3/4; \ln}(\tau) [\bar{\omega}(\tau) - \mathbb{F}(\tau, \bar{\omega}_\tau)] = \mathbb{H}(\tau, \bar{\omega}_\tau), \tau \in J := [1, e], \\ \bar{\omega}(\tau) = 0, \quad \tau \in [1 - \delta, 1], \end{cases} \quad (84)$$

where

$\psi(\tau) = \ln(\tau)$, $a = 1$, $b = e$, $\nu = 3/4$,
and $\mathbb{F}(\tau, u_\tau)$, $\mathbb{H}(\tau, u_\tau)$ are given as

$$\mathbb{F}(\tau, u) = \frac{\ln(\tau) \cos u}{\sqrt{100 + \ln(\tau)}}, \quad \mathbb{H}(\tau, u) = \frac{1}{(t+1)^2} \left(u + \sqrt{1 + u^2} \right). \quad (85)$$

To explain Theorem 39, let us take $\mathbb{H}(\tau, u)$ and $\mathbb{F}(\tau, u)$ given by (85) and $\mathbb{F}(1, u_1) = 0$. Clearly, the condition (M2) holds with $L_{\mathbb{F}} = 1/10$ and $L_{\mathbb{H}} = 1/4$. In addition, $\Delta \approx 0.3720 < 1$. Hence, all hypotheses of Theorem 39 are satisfied. Therefore, it is found exactly one solution for the NFuDq (84) on $[1, e]$.

Example 46. Consider the NFuDq as follows:

$$\begin{cases} {}^c \mathbb{D}_{a^+}^{1/2; \tau} [z(\tau) - \mathbb{F}(\tau, z_\tau)] = \mathbb{H}(\tau, z_\tau), \tau \in J := [0, 1], \\ z(0) = 0. \end{cases} \quad (86)$$

Notice that

$$\psi(\tau) = \tau, \phi(\tau) = 0, a = 0, b = 1, \nu = \frac{1}{2}. \quad (87)$$

To illustrate Theorem 37, let us take

$$\begin{aligned} \mathbb{H}(\tau, u_\tau) &= \frac{1}{e^{(-\pi\tau)} + 9} \left(\frac{|u(\tau)|}{1 + |u(\tau)|} \right) + \tau, \\ \mathbb{F}(\tau, v_\tau) &= \frac{\sqrt{3 + \tau^2} |v|}{10} + (1 + \tau^2). \end{aligned} \quad (88)$$

It is obvious that

$$\begin{aligned} |\mathbb{H}(\tau, u) - \mathbb{H}(\tau, v)| &\leq \frac{1}{10} \|u - v\|_{\mathcal{E}_b}, \\ |\mathbb{F}(\tau, u) - \mathbb{F}(\tau, v)| &\leq \frac{1}{5} \|u - v\|_{\mathcal{E}_b}. \end{aligned} \quad (89)$$

Hence, (M2) also holds with $L_{\mathbb{H}} = 1/10$, $L_{\mathbb{F}} = 1/5$. Further, from the above-given data, we can easily calculate

$$\left[\frac{(\psi(b) - \psi(a))^\nu}{\Gamma(\nu + 1)} \right] L_{\mathbb{H}} = 0.1128. \quad (90)$$

On the contrary, $\forall \tau \in J$, $u \in \mathbb{R}$, we get

$$|\mathbb{H}(\tau, u)| \leq 1 + \frac{1}{10} |u|, \quad |\mathbb{F}(\tau, u)| \leq 2 + \frac{1}{5} |u|. \quad (91)$$

Hence, (M1) holds with $M_1 = 1/10$, $M_2 = 1/5$, $p = N_1 = 1$, and $N_2 = 2$. In view of Theorem 37,

$$\Theta = \{u \in \mathcal{E}_b : \text{there exists } \varsigma \in [0, 1] \text{ such that } u = \varsigma \mathcal{K}u\} \quad (92)$$

is the solution set, then

$$\|u\| \leq \xi \left(\|\mathcal{K}u\|_{\mathcal{E}_b} + \|\mathcal{H}u\|_{\mathcal{E}_b} \right) \leq \mathcal{M} \|u\|_{\mathcal{E}_b} + \mathcal{N}. \quad (93)$$

Using the Matlab program, we have

$$\|u\|_{\mathcal{E}_b} \leq \frac{\mathcal{N}}{1 - \mathcal{M}} = 4.8298. \quad (94)$$

By Theorem 37, the NFuDq (86) with the data (87) and (88) has at least a solution.

7. Conclusion

In this paper, we considered and studied a fractional neutral functional delay differential equation involving a ψ -Caputo fractional derivative on a partially ordered Banach space. To do this, we proved the existence results with the help of the Dhage approximation technique, and then by topological degree method for condensing maps. We established the uniqueness result by the well-known Banach contraction principle. The different kinds of Hyers-Ulam stability were checked in the sequel. Finally, we supported the validity of our findings by providing two examples. This study can be extended to more general structures by using generalized fractional operators with singular or nonsingular kernels due to their high accuracy.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

The authors declare that the study was realized in collaboration with equal responsibility. All authors read and approved the final manuscript.

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