Research Article

Sufficiency Criteria for q -Starlike Functions Associated with Cardioid

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This article deals with the q -differential subordinations for starlike functions associated with the lemniscate of Bernoulli and cardioid domain. The primary goal of this work is to find the conditions on γ for $1 + (\gamma z \partial_q h(z))/(h^n(z)) < \sqrt{1+z}$, where $h(z)$ is analytic function and is subordinated by the function which is producing cardioid domain as its image domain while mapping the open unit disk. Along with this, certain sufficient conditions for q -starlikeness of analytic functions are determined.

1. Introduction

Consider the class A of analytic functions defined in open unit disk F with normalization condition $f(0) = 0$ and $f'(0) = 1$ which provides the Taylor series expansion of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in F. \quad (1)$$

The class S consists of functions from A which are univalent functions in F , and the class P contains the analytic functions whose codomains are bounded by the open right half plane. For more details, see [1, 2].

The concept of differential subordination plays a vital role in the study of geometric properties of analytic func-

tions. It was first introduced by Lindelof, but Littlewood [3] did the remarkable work in this field. Many researchers contributed in the study of differential subordinations. History and the development of works in the field related to differential subordination are briefly described and included in the book by Miller and Mocanu [4]. The major development in the field of differential subordination started in 1974 by Miller et al. [5].

An analytic function f is considered to be subordinated by analytic function g , denoted as $f < g$, if there exists another analytic function w with the property that $w(0) = 0$ and $|w(z)| < |z|$ such that $f(z) = g(w(z))$. Moreover, in case of univalent functions in F , we can have

$$\begin{aligned} f < g &\Leftrightarrow f(0) = g(0), \\ f(F) &\subset g(F). \end{aligned} \quad (2)$$

Recently, many mathematicians have used this concept of differential subordinations to prove many helpful results. Familiar Jack's lemma [6] has produced several advancements for the generalization of differential subordinations and found many applications in this field. The work of Ma and Minda [7] in this field is not negligible as they studied the function Φ which is analytic, and condition of normalization given for prescribed function is defined as $\Phi(0) = 1$ and $\Phi'(0) > 0$ with a positive real part. With the help of the function Φ , they introduced the following subclasses for starlike and convex functions.

$$S^*(\Phi) = \left\{ f \in A : \frac{zf'(z)}{f(z)} < \Phi(z); z \in F \right\}, \quad (3)$$

$$C(\Phi) = \left\{ f \in A : 1 + \frac{zf''(z)}{f'(z)} < \Phi(z); z \in F \right\}.$$

These subclasses helped many researchers for further studies in the field of differential subordination. Ali et al. [8] used the concept of differential subordination to prove analytic functions to be Janowski starlike. Ali et al. [9] also evaluated several differential subordinations: $1 + \gamma z(p'(z)/p^n(z))$ and found the γ for $p(z) < \sqrt{1+z}$. Raina and Sokol [10] used subordinations for coefficient estimation of starlike functions. Similar kinds of works have also been done by Sharma et al. [11] by using starlikeness for cardioid function, and Yunus et al. [12] studied for limacon.

Quantum calculus is the new branch of mathematics and is equally important for its applications both in physics and in mathematics as well. Jackson [13, 14] presented the functions of q -derivatives and q -integrals and highlighted their definitions for the first time. He also holds the credit for the systematic initiation of q -calculus. Ismail et al. [15] were the pioneers to contribute in the application of q -calculus in geometric function theory. The new form of the subclass of starlike functions $S^*(\Phi)$ with the involvement of q -derivative was introduced by Seoudy and Aouf [16]. By choosing different image domains instead of $\Phi(z)$, so many attractive subclasses of starlike functions are obtained. Mahmood et al. [17] have dealt with the class of q -starlike functions by relating them with conic domains. The most recent work related to q -starlikeness of functions is done by Srivastava et al. [18]. The contributions of Haq et al. [19] are remarkable. They proved differential subordinations with q -analogue for cardioid and limacon domain with the involvement of Janowski function and found the sufficient conditions for q -starlike functions. The q version of Jack's lemma which is the soul of our work was given by Çetinkaya and Polatoglu [20]. These recent efforts of mathematicians discussed above motivated us and provide strength to contribute in the field of differential subordinations with the involvement of its q -analogue, which is the main idea of this article. The foundation of all this work in q -analogue is the q -derivative which is defined below.

The q -derivative of a complex-valued function f , defined in the domain F , is given as follows:

$$(D_q f)(z) = \begin{cases} \frac{f(z) - f(qz)}{(1-q)z}, & z \neq 0, \\ f'(0), & z = 0, \end{cases} \quad (4)$$

where $0 < q < 1$. This implies the following:

$$\lim_{q \rightarrow 1^-} (D_q f)(z) = \lim_{q \rightarrow 1^-} \frac{f(z) - f(qz)}{(1-q)z} = f'(z), \quad (5)$$

provided the function f is differentiable in domain F . The function $D_q f$ has Maclaurin's series representation

$$(D_q f)(z) = \sum_{n=0}^{\infty} [n]_q a_n z^{n-1}, \quad (6)$$

where

$$[n]_q = \begin{cases} \frac{1-q^n}{1-q}, & n \in \mathbb{C}, \\ \sum_{k=0}^{n-1} q^k = 1 + q + q^2 + \dots + q^{n-1}, & n \in \mathbb{N}. \end{cases} \quad (7)$$

For more details about q -derivatives and recent work on it, we refer the reader to [21–25].

Definition 1. The function $f(z) \in A$ is said to be in the class $S_{q,c}^*$, if

$$\frac{z \partial_q f(z)}{f(z)} < 1 + \frac{4}{3}z + \frac{2}{3}z^2, \quad z \in F. \quad (8)$$

Lemma 2 (q -Jack's lemma, [20]). Consider an analytic function w in F with $w(0) = 0$. For a maximum value of w on the circle $|z| = 1$ at $z_0 = ae^{i\theta}$, where $\theta \in [-\pi, \pi]$, and $0 < q < 1$, then, we have

$$z_0 \partial_q w(z_0) = mw(z_0). \quad (9)$$

Here, m is real and $m \geq 1$.

By using the above lemma, we have proved our main results.

2. Main Results

Theorem 3. Assume that

$$\gamma \geq \frac{3(\sqrt{2} + 1)}{2(1-q)}, \quad (10)$$

and we define an analytic function h on F with $h(0) = 1$ which satisfies

$$1 + \gamma z \partial_q h(z) < \sqrt{1+z}. \quad (11)$$

and with this, one can have

In addition, we suppose that

$$1 + \gamma z \partial_q h(z) = \sqrt{1+w(z)}, \quad (12)$$

where w is analytic in F with $w(0) = 0$. Then,

$$h(z) < 1 + \frac{4}{3}z + \frac{2}{3}z^2. \quad (13)$$

Proof. Consider the function

$$p(z) = 1 + \gamma z \partial_q h(z), \quad (14)$$

which is analytic in F with the condition $p(0) = 1$ and the function

$$h(z) = 1 + \frac{4}{3}w(z) + \frac{2}{3}w^2(z), \quad (15)$$

where w is an analytic function in F with $w(0) = 0$. To prove the result, it would be sufficient to show that $|w(z)| \leq 1$ for

$$w(z) = p^2(z) - 1. \quad (16)$$

From (14) and (15), we deduce the following:

$$p(z) = 1 + \frac{\gamma}{3}z \partial_q w(z) \{4(1+w(z)) - 2(1-q)z \partial_q w(z)\}, \quad (17)$$

$$\begin{aligned} w(z) = p^2(z) - 1 &= \left[\frac{\gamma}{3}z \partial_q w(z) \{4(1+w(z)) - 2(1-q)z \partial_q w(z)\} \right]^2 \\ &+ 2 \left[\frac{\gamma}{3}z \partial_q w(z) \{4(1+w(z)) - 2(1-q)z \partial_q w(z)\} \right]. \end{aligned} \quad (18)$$

This implies that

$$\begin{aligned} |p^2(z) - 1| &= \left| \left[\frac{\gamma}{3}z \partial_q w(z) \{4(1+w(z)) - 2(1-q)z \partial_q w(z)\} \right]^2 \right. \\ &\quad \left. + 2 \left[\frac{\gamma}{3}z \partial_q w(z) \{4(1+w(z)) - 2(1-q)z \partial_q w(z)\} \right] \right| \\ &= \left| 2 + \frac{\gamma}{3}z \partial_q w(z) \{4(1+w(z)) - 2(1-q)z \partial_q w(z)\} \right| \\ &\quad \times \left| \frac{\gamma}{3}z \partial_q w(z) \{4(1+w(z)) - 2(1-q)z \partial_q w(z)\} \right|. \end{aligned} \quad (19)$$

Now, considering the existence of a point $z_0 \in F$ such that

$$\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1. \quad (20)$$

Now, we use the q -Jack's lemma which implies that there exist a number $m \geq 1$ such that $z_0 \partial_q w(z_0) = mw(z_0)$. This, with the consideration that $w(z_0) = e^{i\theta}$, $\theta \in [-\pi, \pi]$ for $z_0 \in F$, we have

$$\begin{aligned} |p^2(z_0) - 1| &= \left| 2 + \frac{\gamma}{3}z_0 \partial_q w(z_0) \{4(1+w(z_0)) - 2(1-q)z_0 \partial_q w(z_0)\} \right| \cdot \left| \frac{\gamma}{3}z_0 \partial_q w(z_0) \{4(1+w(z_0)) - 2(1-q)z_0 \partial_q w(z_0)\} \right| \\ &= \left| 2 + \frac{\gamma}{3}me^{i\theta} \{4(1+e^{i\theta}) - 2m(1-q)e^{i\theta}\} \right| \cdot \left| \frac{\gamma}{3}me^{i\theta} \{4(1+e^{i\theta}) - 2m(1-q)e^{i\theta}\} \right| \\ &= \sqrt{4 + \frac{4}{9}\gamma^2 m^2 (4 + (2 - m(1-q))^2) + \frac{16}{3}\gamma m \left\{1 + \frac{\gamma m}{3}(2 - m(1-q))\right\} \cos \theta + \frac{8}{3}\gamma m(2 - m(1-q)) \cos 2\theta} \\ &\quad \cdot \frac{m\gamma}{3} \sqrt{16 + (4 - 2m(1-q))^2 + 8(4 - 2m(1-q)) \cos \theta}. \end{aligned} \quad (21)$$

The function

$$\begin{aligned} G(\theta) &= \sqrt{4 + \frac{4}{9}\gamma^2 m^2 (4 + (2 - m(1-q))^2) + \frac{16}{3}\gamma m \left\{1 + \frac{\gamma m}{3}(2 - m(1-q))\right\} \cos \theta + \frac{8}{3}\gamma m(2 - m(1-q)) \cos 2\theta} \\ &\quad \cdot \frac{m\gamma}{3} \sqrt{16 + (4 - 2m(1-q))^2 + 8(4 - 2m(1-q)) \cos \theta} \end{aligned} \quad (22)$$

is clearly an even function. So, in order to find the maximum value of G , we will consider the interval $[0, \pi]$. Thus,

$$G'(\theta) = \frac{-8\gamma m \sqrt{9 + 4\gamma^2 m^2 + \gamma^2 m^2 (2 - m(1 - q))^2 + 12\gamma m \cos \theta + 4\gamma^2 m^2 (2 - m(1 - q)) \cos \theta + 6\gamma m (2 - m(1 - q)) \cos 2\theta} (4 - 2m(1 - q)) \sin \theta}{9 \sqrt{16 + (4 - 2m(1 - q))^2 + 8(4 - 2m(1 - q)) \cos \theta}} + \frac{1}{9} \frac{\gamma m \sqrt{16 + (4 - 2m(1 - q))^2 + 8(4 - 2m(1 - q)) \cos \theta} (-12\gamma m \sin \theta - 4\gamma^2 m^2 (2 - m(1 - q)) \sin \theta - 12\gamma m (2 - m(1 - q)) \sin 2\theta)}{\sqrt{9 + 4\gamma^2 m^2 + \gamma^2 m^2 (2 - m(1 - q))^2 + 12\gamma m \cos \theta + 4\gamma^2 m^2 (2 - m(1 - q)) \cos \theta + 6\gamma m (2 - m(1 - q)) \cos 2\theta}} \quad (23)$$

gives $G'(\theta) = 0$ for $\theta = 0$ and π . Also, we can see that $G'(\pi) > 0$ for $1 < m < 2.5$, which results that $G(\theta) \geq G(\pi)$. Now, consider the function

$$\begin{aligned} \Theta(m) &= \sqrt{4 + \frac{4}{9}\gamma^2 m^2 (4 + (2 - m(1 - q))^2) - \frac{16}{3}\gamma m \left\{1 + \frac{\gamma m}{3}(2 - m(1 - q))\right\} + \frac{8}{3}\gamma m (2 - m(1 - q))} \frac{m\gamma}{3} \sqrt{16 + (4 - 2m(1 - q))^2 - 8(4 - 2m(1 - q))} \\ &= \frac{4}{9}\gamma^2 m^4 (1 - q)^2 - \frac{4}{3}\gamma m^2 (1 - q). \end{aligned} \quad (24)$$

So we have

$$\Theta'(m) = \frac{16}{3}\gamma^2 m^3 (1 - q)^2 - \frac{8}{3}\gamma m (1 - q) > 0. \quad (25)$$

Thus, $\Theta(m)$ is an increasing function which gives a minimum value for $m = 1$. Then, we have

$$|p^2(z_0) - 1| \geq \frac{4}{9}\gamma^2 (1 - q)^2 - \frac{4}{3}\gamma (1 - q). \quad (26)$$

From (10), we conclude that

$$|p^2(z_0) - 1| \geq 1, \quad (27)$$

but this result contradicts (11). Hence, $|w(z)| < 1$ and this leads us to the desired result.

By taking $h(z) = z \partial_q f(z)/f(z)$, the above result reduces to the following.

Corollary 4. Let $\gamma \geq 3(\sqrt{2} + 1)/2(1 - q)$ and $f \in A$ satisfy the subordination

$$1 + \gamma z \partial_q \left(\frac{z \partial_q f(z)}{f(z)} \right) < \sqrt{1 + z}. \quad (28)$$

Then, $f(z) \in S_{q,c}^*$.

Theorem 5. Assume that

$$\gamma \geq \frac{\sqrt{2} + 1}{2(1 - q)}, \quad (29)$$

and we define an analytic function h on F with $h(0) = 1$ which satisfies

$$1 + \frac{\gamma z \partial_q h(z)}{h(z)} < \sqrt{1 + z}. \quad (30)$$

In addition, we suppose that

$$1 + \frac{\gamma z \partial_q h(z)}{h(z)} = \sqrt{1 + w(z)}, \quad (31)$$

where w is analytic in F with $w(0) = 0$. Then,

$$h(z) < 1 + \frac{4}{3}z + \frac{2}{3}z^2. \quad (32)$$

Proof. Consider the function

$$p(z) = 1 + \gamma \frac{z \partial_q h(z)}{h(z)}, \quad (33)$$

which is analytic in F with the condition $p(0) = 1$ and the function

$$h(z) = 1 + \frac{4}{3}w(z) + \frac{2}{3}w^2(z), \quad (34)$$

where w is an analytic function in F with $w(0) = 0$. Using (33) and (34), we obtain

$$p(z) = 1 + \frac{(\gamma/3)z\partial_q w(z)\{4(1+w(z)) - 2(1-q)z\partial_q w(z)\}}{1 + (4/3)w(z) + (2/3)w^2(z)}. \quad (35)$$

Proving the fact that $|w(z)| \leq 1$ will be sufficient to prove our assertion. For this, consider

$$\begin{aligned} |p^2(z) - 1| &= \left| \frac{2(3 + 4w(z) + 2w^2(z)) + \gamma z \partial_q w(z) \{4(1+w(z)) - 2(1-q)z\partial_q w(z)\}}{(3 + 4w(z) + 2w^2(z))} \right| \\ &\quad \cdot \left| \frac{\gamma z \partial_q w(z) \{4(1+w(z)) - 2(1-q)z\partial_q w(z)\}}{(3 + 4w(z) + 2w^2(z))} \right|. \end{aligned} \quad (36)$$

Considering the existence of a point $z_0 \in F$ such that

$$\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1, \quad (37)$$

we can make use of q -Jack's lemma which implies that there exists a number $m \geq 1$ such that $z_0 \partial_q w(z_0) = mw(z_0)$. Now, consider that $w(z_0) = e^{i\theta}$, $\theta \in [-\pi, \pi]$, then for $z_0 \in F$, we have

$$\begin{aligned} |p^2(z_0) - 1| &= \left| \frac{2(3 + 4e^{i\theta} + 2e^{2i\theta}) + \gamma m e^{i\theta} \{4(1 + e^{i\theta}) - 2m(1-q)e^{i\theta}\}}{(3 + 4e^{i\theta} + 2e^{2i\theta})} \right| \\ &\quad \times \left| \frac{\gamma m e^{i\theta} \{4(1 + e^{i\theta}) - 2m(1-q)e^{i\theta}\}}{(3 + 4e^{i\theta} + 2e^{2i\theta})} \right|. \end{aligned} \quad (38)$$

Now, one can easily see that the function

$$\begin{aligned} G(\theta) &= \frac{\sqrt{\Psi_1}}{\sqrt{29 + 40 \cos \theta + 29 \cos 2\theta}} \\ &\quad \cdot \frac{m\gamma \sqrt{16 + (4 - 2m(1-q))^2 + 8(4 - 2m(1-q)) \cos \theta}}{\sqrt{29 + 40 \cos \theta + 29 \cos 2\theta}} \end{aligned} \quad (39)$$

with

$$\begin{aligned} \Psi_1 &= 68 + 48\gamma m + 8\gamma m^2 + 4\gamma^2 m^4 + 32\gamma^2 m^2 - 16\gamma^2 m^3 \\ &\quad - 8\gamma m^2 q + 16\gamma^2 m^3 q - 8\gamma^2 m^4 q + 4\gamma^2 m^4 q^2 + 160 \cos \theta \\ &\quad + 48\gamma m^2 q \cos 2\theta + 32 \cos \theta \gamma m^2 q + 16 \cos \theta \gamma^2 m^3 q \\ &\quad + 96 \cos 2\theta - 48\gamma m^2 \cos 2\theta + 96\gamma m \cos 2\theta \\ &\quad + 144 \cos \theta \gamma m - 16 \cos \theta \gamma^2 m^3 + 32 \cos \theta \gamma^2 m^2 \\ &\quad - 32 \cos \theta \gamma m^2 \end{aligned} \quad (40)$$

is clearly an even function. So, in order to find the maximum value of G , we will consider the interval $[0, \pi]$. Now, we have

$$G'(\theta) = 0 \quad (41)$$

for $\theta = 0$ and π . Also, we can see that $G''(\pi) > 0$ for $m \geq 1$, thus we conclude that $G(\theta) \geq G(\pi)$. So we have the function

$$\begin{aligned} \Theta(m) &= \sqrt{4 - 8\gamma m^2 + 8\gamma m^2 q + 4\gamma^2 m^4 - 8\gamma^2 m^4 q + 4\gamma^2 m^4 q^2} \\ &\quad \cdot m\gamma(2m(1-q)) = [2\gamma m^2(1-q) - 2][2\gamma m^2(1-q)]. \end{aligned} \quad (42)$$

This gives

$$\Theta'(m) = 16\gamma^2 m^3(1-q)^2 - 8\gamma m(1-q) > 0. \quad (43)$$

Thus, $\Theta(m)$ is an increasing function which gives a minimum value for $m = 1$. Then, we have

$$|p^2(z_0) - 1| \geq 4\gamma^2(1-q)^2 - 4\gamma(1-q). \quad (44)$$

From (29), we conclude that

$$|p^2(z_0) - 1| \geq 1, \quad (45)$$

but this result contradicts (30). Hence, $|w(z)| < 1$ which provides the required result.

By taking $h(z) = z\partial_q f(z)/f(z)$, the above result reduces to the following.

Corollary 6. Let $\gamma \geq \sqrt{2} + 1/2(1-q)$ and $f \in A$ satisfy the subordination

$$1 + \gamma z \left(\frac{f(z)}{z\partial_q f(z)} \right) \partial_q \left(\frac{z\partial_q f(z)}{f(z)} \right) < \sqrt{1+z}. \quad (46)$$

Then, $f(z) \in S_{q,c}^*$.

Theorem 7. Assume that

$$\gamma \geq \frac{\sqrt{2} + 1}{2.3(1-q)}, \quad (47)$$

and we define an analytic function h on F with $h(0) = 1$ which satisfies

$$1 + \frac{\gamma z \partial_q h(z)}{h^2(z)} < \sqrt{1+z}. \quad (48)$$

In addition, we suppose that

$$1 + \frac{\gamma z \partial_q h(z)}{h^2(z)} = \sqrt{1+w(z)}, \quad (49)$$

where w is analytic in F with $w(0) = 0$. Then,

$$h(z) < 1 + \frac{4}{3}z + \frac{2}{3}z^2. \quad (50)$$

Proof. Let us define the function

$$p(z) = 1 + \gamma \frac{z \partial_q h(z)}{h^2(z)}, \quad (51)$$

which is analytic in F with the condition $p(0) = 1$ and the function

$$h(z) = 1 + \frac{4}{3}w(z) + \frac{2}{3}w^2(z), \quad (52)$$

where w is an analytic function in F with $w(0) = 0$. Using (51) and (52), we get

$$p(z) = 1 + \frac{(\gamma/3)z \partial_q w(z) \{4(1+w(z)) - 2(1-q)z \partial_q w(z)\}}{(1 + (4/3)w(z) + (2/3)w^2(z))^2}. \quad (53)$$

To prove the assertion, it would be enough to show that $|w(z)| \leq 1$. Therefore,

$$|p^2(z) - 1| = \left| 2 + \frac{(\gamma/3)z \partial_q w(z) \{4(1+w(z)) - 2(1-q)z \partial_q w(z)\}}{(1 + (4/3)w(z) + (2/3)w^2(z))^2} \right| \cdot \left| \frac{(\gamma/3)z \partial_q w(z) \{4(1+w(z)) - 2(1-q)z \partial_q w(z)\}}{(1 + (4/3)w(z) + (2/3)w^2(z))^2} \right|, \quad (54)$$

which after using (9) gives

$$|p^2(z_0) - 1| = \left| \frac{2(3 + 4w(z_0) + 2w^2(z_0))^2 + 3\gamma z_0 \partial_q w(z_0) \{4(1+w(z_0)) - 2(1-q)z_0 \partial_q w(z_0)\}}{(3 + 4w(z_0) + 2w^2(z_0))^2} \right| \times \left| \frac{3\gamma z_0 \partial_q w(z_0) \{4(1+w(z_0)) - 2(1-q)z_0 \partial_q w(z_0)\}}{(3 + 4w(z_0) + 2w^2(z_0))^2} \right|. \quad (55)$$

Now, we consider the function

$$G(\theta) = \frac{\sqrt{\Psi_2}}{(29 + 40 \cos \theta + 12 \cos 2\theta)} \cdot \frac{3\gamma m \sqrt{16 + (4 - 2m(1-q))^2 - 8(4 - 2m(1-q)) \cos \theta}}{(29 + 40 \cos \theta + 12 \cos 2\theta)}, \quad (56)$$

where

$$\begin{aligned} \Psi_2 = & 1156 + 1104\gamma m - 360\gamma m^2 - 960\gamma m^2 \cos \theta + 360\gamma m^2 q \\ & + 144\gamma^2 m^3 q - 72\gamma^2 m^4 q + 36\gamma^2 m^4 q^2 + 288\gamma^2 m^2 \\ & - 144\gamma^2 m^3 + 36\gamma^2 m^4 + 9664 \cos 2\theta + 768\gamma m \cos 3\theta \\ & + 2784\gamma m \cos 2\theta - 624\gamma m^2 \cos \theta + 3120\gamma m \cos \theta \\ & + 288\gamma^2 m^2 \cos \theta - 144\gamma^2 m^3 \cos \theta + 2304 \cos 4\theta \\ & + 7680 \cos 3\theta + 5440 \cos \theta + 624\gamma m^2 q \cos 2\theta \\ & + 960\gamma m^2 q \cos \theta + 144\gamma^2 m^3 q \cos \theta. \end{aligned} \quad (57)$$

As we see that $G(\theta)$ is an even function, so $G'(\theta) = 0$ at $\theta = 0, \pi$ and also we see that $G''(\pi) > 0$ for $m \geq 1$. Thus, we conclude that $G(\theta) \geq G(\pi)$ and we get a new function

$$\Theta(m) = \sqrt{4 - 24\gamma m^2 + 24\gamma m^2 q - 72\gamma^2 m^4 q + 36\gamma^2 m^4 q^2 + 36\gamma^2 m^4} \cdot 3\gamma m(2m(1-q)), \quad (58)$$

and we have

$$\Theta'(m) = 72\gamma^2 m^3(1-q)^2 + 12(6\gamma m^2(1-q) - 2)\gamma m(1-q) > 0. \quad (59)$$

So $\Theta(m)$ is an increasing function, and it has its minimum value at $m = 1$. Then, we have

$$|p^2(z_0) - 1| \geq (6\gamma(1-q) - 2)(6\gamma(1-q)). \quad (60)$$

Using (47), we get

$$|p^2(z_0) - 1| \geq 1, \quad (61)$$

but this result contradicts (48). Hence, $|w(z)| < 1$ which proves the required result.

By taking $h(z) = z \partial_q f(z)/f(z)$, the above result reduces to the following.

Corollary 8. Let $\gamma \geq \sqrt{2} + 1/2.3(1-q)$ and $f \in A$ satisfy the subordination

$$1 + \gamma z \left(\frac{f(z)}{z \partial_q f(z)} \right)^2 \partial_q \left(\frac{z \partial_q f(z)}{f(z)} \right) < \sqrt{1+z}. \quad (62)$$

Then, $f(z) \in S_{q,c}^*$.

Theorem 9. Assume that

$$\gamma \geq \frac{\sqrt{2}+1}{2.3^2(1-q)}, \quad (63)$$

and we define an analytic function h on F with $h(0) = 1$ which satisfies

$$1 + \frac{\gamma z \partial_q h(z)}{h^3(z)} < \sqrt{1+z}. \quad (64)$$

In addition, we suppose that

$$1 + \frac{\gamma z \partial_q h(z)}{h^3(z)} = \sqrt{1+w(z)}, \quad (65)$$

where w is analytic in F with $w(0) = 0$. Then,

$$h(z) < 1 + \frac{4}{3}z + \frac{2}{3}z^2. \quad (66)$$

Proof. Let us define the function

$$p(z) = 1 + \gamma \frac{z \partial_q h(z)}{h^3(z)}, \quad (67)$$

which is analytic in F with the condition $p(0) = 1$ and the function

$$h(z) = 1 + \frac{4}{3}w(z) + \frac{2}{3}w^2(z), \quad (68)$$

where w is an analytic function in F with $w(0) = 0$. Using (67) and (68), we obtain

$$p(z) = 1 + \frac{(\gamma/3)z \partial_q w(z) \{4(1+w(z)) - 2(1-q)z \partial_q w(z)\}}{(1 + (4/3)w(z) + (2/3)w^2(z))^3}. \quad (69)$$

To prove the result, we have to show that $|w(z)| \leq 1$. Therefore,

$$\begin{aligned} |p^2(z) - 1| &= \left| 2 + \frac{(\gamma/3)z \partial_q w(z) \{4(1+w(z)) - 2(1-q)z \partial_q w(z)\}}{(1 + (4/3)w(z) + (2/3)w^2(z))^3} \right| \\ &\quad \cdot \left| \frac{(\gamma/3)z \partial_q w(z) \{4(1+w(z)) - 2(1-q)z \partial_q w(z)\}}{(1 + (4/3)w(z) + (2/3)w^2(z))^3} \right|. \end{aligned} \quad (70)$$

Hence, by applying (9), we obtain

$$\begin{aligned} &|p^2(z_0) - 1| \\ &= \left| \frac{2(3 + 4w(z_0) + 2w^2(z_0))^3 + 9\gamma z_0 \partial_q w(z_0) \{4(1+w(z_0)) - 2(1-q)z_0 \partial_q w(z_0)\}}{(3 + 4w(z_0) + 2w^2(z_0))^3} \right| \\ &\quad \times \left| \frac{9\gamma z_0 \partial_q w(z_0) \{4(1+w(z_0)) - 2(1-q)z_0 \partial_q w(z_0)\}}{(3 + 4w(z_0) + 2w^2(z_0))^3} \right|. \end{aligned} \quad (71)$$

Now, consider the function

$$\begin{aligned} G(\theta) &= \frac{\sqrt{\Psi_3}}{(29 + 40 \cos \theta + 12 \cos 2\theta)^{3/2}} \\ &\quad \cdot \frac{9\gamma m \sqrt{16 + (4 - 2m(1-q))^2 + 8(4 - 2m(1-q)) \cos \theta}}{(29 + 40 \cos \theta + 12 \cos 2\theta)^{3/2}}, \end{aligned} \quad (72)$$

where

$$\begin{aligned} \Psi_3 &= 19652 + 18288\gamma m^2 q \cos 2\theta + 13824\gamma m^2 q \cos 3\theta \\ &\quad + 4608 \cos 4\theta \gamma m^2 q + 12384 \cos \theta \gamma m^2 q + 1296 \cos \theta \gamma^2 m^3 q \\ &\quad + 18432 \cos 5\theta \gamma m - 720\gamma m - 3384\gamma m^2 + 138720 \cos \theta \\ &\quad + 41184\gamma m \cos 2\theta + 80640\gamma m \cos 3\theta + 64512 \cos 4\theta \gamma m \\ &\quad + 3384\gamma m^2 q - 18288\gamma m^2 \cos 2\theta + 2592 \cos \theta \gamma^2 m^2 \\ &\quad + 5904 \cos \theta \gamma m - 1296 \cos \theta \gamma^2 m^3 - 12384 \cos \theta \gamma m^2 \\ &\quad - 13824\gamma m^2 \cos 3\theta - 4608 \cos 4\theta \gamma m^2 + 1296\gamma^2 m^3 q \\ &\quad - 648\gamma^2 m^4 q + 324\gamma^2 m^4 q^2 + 409632 \cos 2\theta + 647680 \cos 3\theta \\ &\quad + 578304 \cos 4\theta + 276480 \cos 5\theta + 55296 \cos 6\theta + 2592\gamma^2 m^2 \\ &\quad - 1296\gamma^2 m^3 + 324\gamma^2 m^4. \end{aligned} \quad (73)$$

The above function is clearly an even function. So in order to find its maximum value, we will consider the interval $[0, \pi]$. Now, we have $G'(\theta) = 0$ for $\theta = 0$ and π . Clearly, $G''(\pi) > 0$, and hence, we have obtained the minimum value of G at $\theta = \pi$, and thus, we conclude that $G(\theta) \geq G(\pi)$. So now consider the function

$$\Theta(m) = \sqrt{4 - 72\gamma m^2 + 72\gamma m^2 q - 648\gamma^2 m^4 q + 324\gamma^2 m^4 q^2 + 324\gamma^2 m^4} \cdot 9\gamma m \sqrt{16 + (4 - 2m(1 - q))^2 - 8(4 - 2m(1 - q))}, \quad (74)$$

which gives

$$\Theta'(m) = 648\gamma^2 m^2 (1 - q)^3 > 0. \quad (75)$$

Thus, $\Theta(m)$ is an increasing function. So for $m = 1$, it gives a minimum value. Then, we have

$$|p^2(z_0) - 1| \geq [18\gamma(1 - q) - 2][18\gamma(1 - q)]. \quad (76)$$

Using (63), we get

$$|p^2(z_0) - 1| \geq 1, \quad (77)$$

but this result contradicts (64). Hence $|w(z)| < 1$ which proves the required result.

By taking $h(z) = z\partial_q f(z)/f(z)$, the above result reduces to the following.

Corollary 10. Let $\gamma \geq \sqrt{2} + 1/2 \cdot 3^2(1 - q)$ and $f \in A$ satisfy the subordination

$$1 + \gamma z \left(\frac{f(z)}{z\partial_q f(z)} \right)^3 \partial_q \left(\frac{z\partial_q f(z)}{f(z)} \right) < \sqrt{1 + z}. \quad (78)$$

Then, $f(z) \in S_{q,c}^*$.

Theorem 11. Assume that

$$\gamma \geq \frac{\sqrt{2} + 1}{2 \cdot 3^{n-1}(1 - q)}, \quad (79)$$

and we define an analytic function h on F with $h(0) = 1$ which satisfies

$$1 + \frac{\gamma z \partial_q h(z)}{h^n(z)} < \sqrt{1 + z}. \quad (80)$$

In addition, we suppose that

$$1 + \frac{\gamma z \partial_q h(z)}{h^n(z)} = \sqrt{1 + w(z)}, \quad (81)$$

where w is analytic in F with $w(0) = 0$. Then,

$$h(z) < 1 + \frac{4}{3}z + \frac{2}{3}z^2. \quad (82)$$

We omit the proof of this result as it can be done by using a similar technique as applied in the above results.

By taking $h(z) = z\partial_q f(z)/f(z)$, the above result reduces to the following.

Corollary 12. Let $\gamma \geq \sqrt{2} + 1/2 \cdot 3^{n-1}(1 - q)$ and $f \in A$ satisfy the subordination

$$1 + \gamma z \left(\frac{f(z)}{z\partial_q f(z)} \right)^n \partial_q \left(\frac{z\partial_q f(z)}{f(z)} \right) < \sqrt{1 + z}. \quad (83)$$

Then, $f(z) \in S_{q,c}^*$.

3. Conclusion

In this article, we have worked on q -differential subordinations associated with lemniscate of Bernoulli and defined sufficient conditions for q -starlikeness related to cardioid domain. We have also determined the conditions on γ to prove the starlikeness of prescribed function such as

$$1 + \frac{\gamma z \partial_q h(z)}{(h(z))^n} < \sqrt{1 + z} \text{ for } n = 0, 1, 2, 3, \quad (84)$$

then

$$h(z) < 1 + \frac{4}{3}z + \frac{2}{3}z^2. \quad (85)$$

We can use these results to study the sufficiency criteria of other analytic functions.

Data Availability

All data generated or analyzed during this study are included within this article.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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