Hindawi
Journal of Function Spaces
Volume 2022, Article ID 1011342, 11 pages
https://doi.org/10.1155/2022/1011342

Research Article

Multiple Positive Solutions and Estimates of Extremal Values for a Nonlocal Problem with Critical Sobolev Exponent and Concave-Convex Nonlinearities

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Received 17 March 2022; Revised 3 May 2022; Accepted 21 May 2022; Published 2 June 2022

Academic Editor: Simone Secchi

We are concerned with the following nonlocal problem involving critical Sobolev exponent $\begin{cases} -(a-b\int_\Omega |\nabla u|^2 dx)\Delta u = \lambda |u|^{q-2}u + \delta |u|^2 u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$ where Ω is a smooth bounded domain in $\mathbb{R}^4$, $a, b > 0$, $1 < q < 2$, δ, and λ are positive parameters. We prove the existence of two positive solutions and obtain uniform estimates of extremal values for the problem. Moreover, the blow-up and the asymptotic behavior of these solutions are also discussed when $b \searrow 0$ and $\delta \searrow 0$. In the proofs, we apply variational methods.

1. Introduction and Main Results

In this paper, we study a new class of Kirchhoff type problem with critical exponent and concave-convex nonlinearities

$$\begin{cases} -\left(a-b\int_\Omega |\nabla u|^2 dx\right)\Delta u = \lambda |u|^{q-2}u + \delta |u|^2 u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \quad (\mathscr{P}_{b,\delta}), \tag{1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^4$ ($2^* = 4$ is the critical exponent in dimension four), $a, b > 0$, $1 < q < 2$, δ, and λ are positive parameters.

We call $(\mathscr{P}_{b,\delta})$ a Kirchhoff type problem since the presence of the term $\int_\Omega |\nabla u|^2 dx$, which means that $(\mathscr{P}_{b,\delta})$ is no longer a pointwise identity. Such nonlocal problem arises in various models concerning physical and biological systems, see, e.g., [1–3]. Among others, Kirchhoff [2] built a model defined by the equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0, \tag{2}$$

where $u = u(x,t)$ represents the lateral displacement, ρ denotes the mass density, P_0 is the initial tension, h denotes the area of the cross-section, E denotes the Young modulus of the material, and L is the length of the string. This equation is an extension of the classical D'Alembert wave equation for free vibrations of elastic strings.

Different from the traditional Kirchhoff type problem, the sign of nonlocal term included in $(\mathscr{P}_{b,\delta})$ is negative, which causes some interesting difficulties. In the past few years, much attention has been paid to the existence, multiplicity, and the behaviour of solutions for this kind of nonlocal problem but without critical growth. In particular, Yin and Liu [4] were concerned with the following problem

$$\begin{cases} -\left(a-b\int_{\Omega}|\nabla u|^2dx\right)\Delta u=|u|^{p-2}u, & x\in\Omega,\\ u=0, & x\in\partial\Omega, \end{cases} \quad (3)$$

where $1<p<2^*$ and Ω is a bounded domain in $\mathbb{R}^N$ with $N\geq 1$ and succeeded to find the problem (3) admits at least two nontrivial solutions. In [5, 6], sign-changing solutions to (3) were further obtained. When $N=3$ and the nonlinear term has an indefinite potential, Lei et al. [7] and Qian and Chao [8] established the existence of positive solution of (3) for $1<p<2$ and $3<p<6$, respectively. For the singular nonlinearity, two positive solutions to (3) with $N=3$ were proved in [9]. In our previous work [10], we obtained two positive solutions of $(\mathscr{P}_{b,\delta})$ with $\delta=0$, as well as their blow-up and asymptotic behavior when $b\searrow 0$. For more related results, we refer the interested readers to [11–15] and the references therein.

In 1994, Ambrosetti et al. [16] first studied the following critical local problem involving concave-convex nonlinearities

$$\begin{cases} -\Delta u=\lambda|u|^{q-2}u+|u|^{2^*-1}u, & x\in\Omega,\\ u=0, & x\in\partial\Omega, \end{cases} \quad (4)$$

where $1<q<2$ and $\Omega\subset\mathbb{R}^N$ is a smooth bounded domain. The authors proved that there exists $\lambda_0>0$ such that the problem (4) has two positive solutions for $\lambda\in(0,\lambda_0)$ and no positive solutions for $\lambda>\lambda_0$. Since then, many scholars have considered problems with critical exponent and concave-convex nonlinearities, see, e.g., [7, 16–21]. Also, the problem (4) of traditional Kirchhoff type is studied in [22–26] and the reference therein. An interesting question now is whether the same existence results as in [16] occur to the nonlocal problem $(\mathscr{P}_{b,\delta})$ with critical exponent. For $\lambda=0$ and $\delta=1$, Wang et al. [27] proved the existence of two positive solutions of $(\mathscr{P}_{b,\delta})$ with an additional inhomogeneous perturbation on the whole space $\mathbb{R}^4$. When $2<q<2^*$ and δ is replaced by a nonnegative function $Q(x)$, [28] showed how the shape of the graph of $Q(x)$ affects the number of positive solutions to $(\mathscr{P}_{b,\delta})$. However, there are no known existence results for $(\mathscr{P}_{b,\delta})$ provided $\lambda>0$ and $1<q<2$.

Motivated by the works described above, in the present paper, we try to prove the existence and multiplicity of positive solutions of problem $(\mathscr{P}_{b,\delta})$ when $\lambda\in(0,T^-)$ for some $T^->0$ (see Theorem 1), provide uniform estimates of extremal values λ^* for problem $(\mathscr{P}_{b,\delta})$ (see Theorem 2), and obtain the blow-up and asymptotic behavior of these positive solutions when $b\searrow 0$ and $\delta\searrow 0$ (see Theorem 3).

Denote by $H_0^1(\Omega)$ the standard Sobolev space endowed with the standard norm $\|\cdot\|$. Let $|\cdot|_p$ be the norm of the space $L^s(\Omega)$. Denote by $\longrightarrow$ ($\rightharpoonup$) the strong (weak) convergence. C and C_i denote various positive constants whose exact values are not important. Let μ_1 be the positive principal eigenvalue of the operator $-\Delta$ on Ω with corresponding positive principal eigenfunction e_1. Denote by S the best constant in the Sobolev embedding $H_0^1(\Omega)^\circ L^{2^*}(\Omega)$, namely,

$$S=\inf_{u\in H_0^1(\Omega)\setminus\{0\}}\frac{\|u\|^2}{|u|_4^2}>0. \quad (5)$$

It is well known that the weak solutions of problem ($\mathscr{P}_{b,\delta}$) correspond to the critical points of the following energy functional

$$I_{b,\delta}(u)=\frac{a}{2}\|u\|^2-\frac{b}{4}\|u\|^4-\frac{\lambda}{q}|u|_q^q-\frac{\delta}{4}|u|_4^4. \quad (6)$$

Moreover, we easily see that $I_{b,\delta}\in C^1(H_0^1(\Omega),\mathbb{R})$.
Define the manifold

$$\begin{aligned}\mathscr{M}_{b,\delta}&=\left\{u\in H_0^1(\Omega)\colon \left\langle I'_{b,\delta}(u),u\right\rangle=0\right\}\\ &=\left\{u\in H_0^1(\Omega)\colon a\|u\|^2=b\|u\|^4+\lambda|u|_q^q+\delta|u|_4^4\right\},\end{aligned} \quad (7)$$

and decompose $\mathscr{M}_{b,\delta}$ into three subsets as follows:

$$\begin{aligned}\mathscr{M}_{b,\delta}^0&=\left\{u\in\mathscr{M}_{b,\delta}: a(2-q)\|u\|^2-b(4-q)\|u\|^4-\delta(4-q)|u|_4^4=0\right\},\\ \mathscr{M}_{b,\delta}^+&=\left\{u\in\mathscr{M}_{b,\delta}: a(2-q)\|u\|^2-b(4-q)\|u\|^4-\delta(4-q)|u|_4^4>0\right\},\\ \mathscr{M}_{b,\delta}^-&=\left\{u\in\mathscr{M}_{b,\delta}: a(2-q)\|u\|^2-b(4-q)\|u\|^4-\delta(4-q)|u|_4^4<0\right\}.\end{aligned} \quad (8)$$

Set

$$\begin{aligned}T_1&=\frac{2aS^{q/2}}{(4-q)|\Omega|^{(4-q)/4}}\left(\frac{a(2-q)}{\left(b+\delta S^{-2}\right)(4-q)}\right)^{(2-q)/2},\\ T_2&=\frac{2q(aS)^{(4-q)/2}}{(2-q)(4-q)\left(bS^2+\delta\right)|\Omega|^{(4-q)/4}}\left[\frac{\delta(2-q)}{q}\right]^{q/2},\\ T^-&=\min\left\{T_1,T_2\right\}.\end{aligned} \quad (9)$$

Our main results are as follows.

Theorem 1. *Assume that $\lambda\in(0,T^-)$, then problem $(\mathscr{P}_{b,\delta})$ has at least two positive solutions $u_*\in\mathscr{M}_{b,\delta}^+$ and $U_*\in\mathscr{M}_{b,\delta}^-$ with $\|u_*\|<\|U_*\|$.*

Theorem 2. *Let*

$$\lambda^*=\sup\left\{\lambda>0:(\mathscr{P}_{b,\delta})\ \textit{has at least two positive solutions}\right\}. \quad (10)$$

Then, we have

$$0 < T^- \leq \lambda^* \leq T^+ < \infty, \tag{11}$$

where T^- is defined as above and

$$T^+ = \frac{2a\mu_1}{4-q}\left[\frac{(2-q)a\mu_1}{(4-q)\delta}\right]^{1/2} + 1. \tag{12}$$

Theorem 3. *Assume that $\{b_n\}$ and $\{\delta_n\}$ are two sequences satisfying $b_n \searrow 0$ and $\delta_n \searrow 0$ as $n \longrightarrow \infty$. Let u_n and U_n be the two positive solutions of $(\mathscr{P}_{b,\delta})$ corresponding to b_n and δ_n obtained in Theorem 1 with $u_n \in \mathscr{M}^+_{b_n,\delta_n}$ and $U_n \in \mathscr{M}^-_{b_n,\delta_n}$, then passing to a subsequence if necessary,*

(i) $\|U_n\| \longrightarrow \infty$ *as* $n \longrightarrow \infty$

(ii) $u_n \longrightarrow \bar{u}$ *in* $H_0^1(\Omega)$ *as* $n \longrightarrow \infty$, *where $\bar{u}$ is a positive ground state solution of the problem*

$$\begin{cases} -a\Delta u = \lambda |u|^{q-2}u, & x \in \Omega, (\mathscr{P}_{0,0}), \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{13}$$

Remark 4. The multiplicity result of $(\mathscr{P}_{b,\delta})$ with $\delta = 0$ has been proved by [10]. So, our result presented in Theorem 1 can be viewed as an extension of [10] considering the subcritical case where $\delta = 0$. In particular, we provide uniform estimates of extremal values λ^* for the problem, which are observed for the first time in the studies of such nonlocal problem like $(\mathscr{P}_{b,\delta})$.

Remark 5. Comparing with [16], which considered problem $(\mathscr{P}_{b,\delta})$ with $b = 0$, we in this paper investigate the nonlocal case of $b \neq 0$. Moreover, unlike [22–24, 26], where the nonlocal term is positive, here we study the case of negative sign of nonlocal term and additionally obtain a bound from above for the parameter.

The plan of this paper is as follows. In Section 2, we give some preliminaries. Section 3 is devoted to the Proof of Theorem 1. In Section 4, we prove Theorems 2 and 3. In the proof of our main results, we use variational methods, and they are inspired by [10, 16]. However, in the present paper, we encounter some new difficulties due to the critical growth and nonlocal term. Firstly, compared with [10], the calculations here are more delicate and difficult since we now face the critical problem $(\mathscr{P}_{b,\delta})$. Secondly, to provide the bound from above for λ^* of $(\mathscr{P}_{b,\delta})$ involving nonlocal term, we need to develop some techniques applied in [16] where dealt with local case. Thirdly, in order to obtain the asymptotic behavior of the solutions of $(\mathscr{P}_{b,\delta})$ as in the work of [10], we add the condition of $\delta \searrow 0$ and conduct some new analysis.

2. Preliminaries

Lemma 6. *Let $\lambda \in (0, T_1)$. Then, $\mathscr{M}^{\pm}_{b,\delta} \neq \varnothing$ and $\mathscr{M}^0_{b,\delta} = \{0\}$.*

Proof. A simple calculation shows that

$$\partial I_{b,\delta}\phi\partial t(tu) = t^{q-1}\left(at^{2-q}\|u\|^2 - bt^{4-q}\|u\|^4 - \lambda|u|_q^q - \delta t^{4-q}|u|_4^4\right). \tag{14}$$

For any $u \in H_0^1(\Omega) \setminus \{0\}$, $t > 0$, set

$$\begin{aligned} \psi(t) &= at^{2-q}\|u\|^2 - bt^{4-q}\|u\|^4 - \delta t^{4-q}|u|_4^4, t > 0, \\ \psi_1(t) &= at^{2-q}\|u\|^2 - t^{4-q}\left(b + \delta S^{-2}\right)\|u\|^4, t > 0. \end{aligned} \tag{15}$$

Since $1 < q < 2$, it is clear that $\lim_{t\longrightarrow 0^+}\psi_1(t) = 0$ and $\lim_{t\longrightarrow +\infty}\psi_1(t) = -\infty$. Moreover, $\psi_1(t)$ is concave and achieves its maximum at the point $t_{\max} = [a(2-q)\|u\|^2/(b+\delta S^{-2})(4-q)\|u\|^4]^{1/2}$ with

$$\psi_1(t_{\max}) = (24-q)(2-q4-q)^{2-q/2}\left[\frac{\left(a\|u\|^2\right)^{4-q}}{\left(\left(b+\delta S^{-2}\right)\|u\|^4\right)^{2-q}}\right]^{1/2}. \tag{16}$$

By Hölder and Sobolev inequalities, for $\lambda \in (0, T_1)$, we obtain

$$\lambda|u|_q^q \leq \lambda|\Omega|^{4-q4}S^{-q/2}\|u\|^q < \psi_1(t_{\max}) \leq \psi(t_{\max}). \tag{17}$$

From which we infer that there exist two constants $t^+ = t^+(u)$ and $t^- = t^-(u)$ satisfying $t^+ > t_{\max} > t^- > 0$ and

$$\begin{aligned} \psi(t^+) &= \lambda|u|_q^q = \psi(t^-), \\ \psi'(t^+) &< 0 < \psi'(t^-). \end{aligned} \tag{18}$$

This gives that $t^+u \in \mathscr{M}^-_{b,\delta}$ and $t^-u \in \mathscr{M}^+_{b,\delta}$.

In what follows, we prove that $\mathscr{M}^0_{b,\delta} = \{0\}$. Suppose to the contrary that there is $w \in \mathscr{M}^0_{b,\delta}$ with $w \neq 0$. By $w \in \mathscr{M}^0_{b,\delta}$, we have

$$a(2-q)\|w\|^2 = b(4-q)\|w\|^4 + \delta(4-q)|w|_4^4. \tag{19}$$

As a consequence, by Sobolev inequality,

$$\begin{aligned} a(2-q)\|w\|^2 &\leq b(4-q)\|w\|^4 + \delta(4-q)S^{-2}\|w\|^4 \\ &= \left(b+\delta S^{-2}\right)(4-q)\|w\|^4. \end{aligned} \tag{20}$$

Moreover, we can also infer from $w \in \mathscr{M}^0_{b,\delta}$ that $-2a\|w\|^2 + \lambda(4-q)|w|_q^q = 0$ and so

$$\lambda|w|_q^q = \frac{2a}{4-q}\|w\|^2. \tag{21}$$

Combining (20) and (21), for $\lambda \in (0, T_1)$, we conclude that

$$0 < \left(\frac{2}{4-q}\right)\left(\frac{2-q}{4-q}\right)^{(2-q)/2}\left[\frac{\left(a\|w\|^2\right)^{4-q}}{\left(\left(b+\delta S^{-2}\right)\|w\|^4\right)^{2-q}}\right]^{1/2} - \lambda|w|_q^q \le \left(\frac{2}{4-q}\right)\left(\frac{2-q}{4-q}\right)^{(2-q)/2} \cdot \left[\frac{\left(a\|w\|^2\right)^{4-q}}{\left(\left(a(2-q)/(4-q)\right)\|w\|^2\right)^{2-q}}\right]^{1/2} - \lambda|w|_q^q = \frac{2a}{4-q}\|w\|^2 - \lambda|w|_q^q = 0, \tag{22}$$

which is absurd. The proof of Lemma 6 is completed. □

Lemma 7. *Assume that $\lambda \in (0, T_1)$, then there is a gap structure in $\mathcal{M}_{b,\delta}$:*

$$\|u\| \le A(\lambda) < A(0) \le \|U\|, \forall u \in \mathcal{M}_{b,\delta}^{+}, U \in \mathcal{M}_{b,\delta}^{-}, \tag{23}$$

where

$$A(0) = \left(\frac{a(2-q)}{(4-q)\left(b+\delta S^{-2}\right)}\right)^{1/2}, \quad A(\lambda) = \left(\frac{\lambda(4-q)|\Omega|^{(4-q)/4}}{2aS^{q/2}}\right)^{1/(2-q)}. \tag{24}$$

Proof. In the case of $U \in \mathcal{M}_{b,\delta}^{-}$, using Sobolev inequality, we have

$$a(2-q)\|U\|^2 < b(4-q)\|U\|^4 + \delta(4-q)|U|_4^4 \le \left(b+\delta S^{-2}\right)(4-q)\|U\|^4, \tag{25}$$

which yields $\|U\| \ge A(0)$.

In the case of $u \in \mathcal{M}_{b,\delta}^{+}$, it holds

$$2a\|u\|^2 < \lambda(4-q)|u|_q^q \le \lambda(4-q)|\Omega|^{4-q/4} S^{-q/2}\|u\|^q, \tag{26}$$

which gives that $\|u\| \le A(\lambda)$. Moreover, we easily check that if $\lambda \in (0, T_1)$, then $A(\lambda) < A(0)$. □

Lemma 8. *For any $u \in \mathcal{M}_{b,\delta}^{\pm}$, there exist $\rho_u > 0$ and a differential functional $g_{\rho_u} : B_{\rho_u}(0) \longrightarrow \mathbb{R}^+$ such that*

$$g_{\rho_u}(0) = 1, g_{\rho_u}(w)(u-w) \in \mathcal{M}_{b,\delta}^{\pm}, \quad \left\langle g_{\rho_u}'(0), \phi \right\rangle = \frac{\left(2a - 4b\|u\|^2\right)\int_\Omega \nabla u \nabla \phi dx - q\lambda\int_\Omega |u|^{q-2}u\phi dx - 4\delta\int_\Omega |u|^2 u\phi dx}{a(2-q)\|u\|^2 - b(4-q)\|u\|^4 - \delta(4-q)|u|_4^4}. \tag{27}$$

Proof. Fix $u \in \mathcal{M}_{b,\delta}^{-}$ and define $F : \mathbb{R}^+ \times H \longrightarrow \mathbb{R}$ by

$$F(t, w) = at^{2-q}\|u-w\|^2 - bt^{4-q}\|u-w\|^4 - \lambda|u-w|_q^q - \delta t^{4-q}|u-w|_4^4. \tag{28}$$

Since for $u \in \mathcal{M}_{b,\delta}^{-} \subset \mathcal{M}_{b,\delta}$, one has $F(1, 0) = 0$ and

$$F_t(1, 0) = a(2-q)\|u\|^2 - b(4-q)\|u\|^4 - \delta(4-q)|u|_4^4 < 0, \tag{29}$$

then we can employ the implicit function theorem for F at the point $(1, 0)$ and derive $\bar{\rho} > 0$ and a differential functional $g = g(w) > 0$ defined for $w \in H_0^1(\Omega)$, $\|w\| < \bar{\rho}$ such that

$$g(0) = 1, g(w)(u-w) \in \mathcal{M}_{b,\delta}, \forall w \in H, \|w\| < \bar{\rho}. \tag{30}$$

In view of the continuity of g, we may choose $\rho > 0$ possibly smaller ($\rho < \bar{\rho}$) such that for any $w \in H_0^1(\Omega)$, $\|w\| < \rho$, it holds

$$g(w)(u-w) \in \mathcal{M}_{b,\delta}^{-}. \tag{31}$$

In a similar way, we can prove the case of $u \in \mathcal{M}_{b,\delta}^{+}$, and thus, Lemma 8 follows. □

Lemma 9. *If $\lambda \in (0, T_1)$, then we have*

(i) The functional $I_{b,\delta}$ is coercive and bounded from below on $\mathcal{M}_{b,\delta}$

(ii) $\inf_{\mathcal{M}_{b,\delta}^{+} \cup \mathcal{M}_{b,\delta}^{0}} I_{b,\delta} = \inf_{\mathcal{M}_{b,\delta}^{+}} I_{b,\delta} \in (-\infty, 0)$

Proof.

(i) For $u \in \mathcal{M}_{b,\delta}$, using Hölder's inequality, we obtain

$$I_{b,\delta}(u) = I_{b,\delta}(u) - \frac{1}{4}\left\langle I'_{b,\delta}(u), u\right\rangle = \frac{a}{4}\|u\|^2 - \lambda\left(\frac{1}{q}-\frac{1}{4}\right)|u|_q^q \geq \frac{a}{4}\|u\|^2 - \lambda\left(\frac{1}{q}-\frac{1}{4}\right)|\Omega|^{(4-q)/4}S^{-q/2}\|u\|^q. \quad (32)$$

This proves the conclusion (i).

(ii) For $u \in \mathcal{M}_{b,\delta}^+$, it holds

$$I_{b,\delta}(u) = I_{b,\delta}(u) - \frac{1}{q}\left\langle I'_{b,\delta}(u), u\right\rangle = a\left(\frac{1}{2}-\frac{1}{q}\right)\|u\|^2 + b\left(\frac{1}{q}-\frac{1}{4}\right)\|u\|^4 + \delta\left(\frac{1}{q}-\frac{1}{4}\right)|u|_4^4 < \frac{-a(2-q)\|u\|^2 + b(4-q)\|u\|^4 + \delta(4-q)|u|_4^4}{4q} < 0. \quad (33)$$

Combining this and Lemma 6, we have that $\inf_{\mathcal{M}_{b,\delta}^+\cup\mathcal{M}_{b,\delta}^0} I_{b,\delta} = \inf_{\mathcal{M}_{b,\delta}^+} I_{b,\delta} < 0$. Furthermore, we deduce from (i) that $\inf_{\mathcal{M}_{b,\delta}^+\cup\mathcal{M}_{b,\delta}^0} I_{b,\delta} \neq -\infty$. Thus, $\inf_{\mathcal{M}_{b,\delta}^+\cup\mathcal{M}_{b,\delta}^0} I_{b,\delta} \in (-\infty, 0)$. □

Lemma 10. *If $\lambda \in (0, T_1)$, then $\mathcal{M}_{b,\delta}^+ \cup \mathcal{M}_{b,\delta}^0$ and $\mathcal{M}_{b,\delta}^-$ are closed.*

Proof. Let $\{U_n\}$ be a sequence in $\mathcal{M}_{b,\delta}^-$ such that $U_n \longrightarrow U_0$ in $H_0^1(\Omega)$. Since $\{U_n\} \subset \mathcal{M}_{b,\delta}^- \subset \mathcal{M}_{b,\delta}$, we have

$$a\|U_0\|^2 - b\|U_0\|^4 = \lim_{n\longrightarrow\infty}\left(a\|U_n\|^2 - b\|U_n\|^4\right) = \lim_{n\longrightarrow\infty}\left(\lambda|U_n|_q^q + \delta|U_n|^4\right) = \lambda|U_0|_q^q + \delta|U_0|^4,$$

$$a(2-q)\|U_0\|^2 - b(4-q)\|U_0\|^4 - \delta(4-q)|U_0|^4 = \lim_{n\longrightarrow\infty}\left[a(2-q)\|U_n\|^2 - b(4-q)\|U_n\|^4 - \delta(4-q)|U_n|^4\right] \leq 0, \quad (34)$$

namely, $U_0 \in \mathcal{M}_{b,\delta}^- \cup \mathcal{M}_{b,\delta}^0$. For $\lambda \in (0, T_1)$, it then follows from Lemma 7 that $U_0 \notin \mathcal{M}_{b,\delta}^0$. In turn, we obtain $U_0 \in \mathcal{M}_{b,\delta}^-$, and so, $\mathcal{M}_{b,\delta}^-$ is closed for $\lambda \in (0, T_1)$. The same argument can prove that $\mathcal{M}_{b,\delta}^0 \cup \mathcal{M}_{b,\delta}^+$ is closed. This completes the proof of Lemma 10. □

3. Proof of Theorem 1

Lemma 11. *Suppose that $\lambda \in (0, T_1)$, then problem $(\mathcal{P}_{b,\delta})$ admits a positive solution u_* with $u_* \in \mathcal{M}_b^+$.*

Proof. By Lemmas 9 and 10, we can apply Ekeland variational principle to get a minimizing sequence $\{u_n\} \subset \mathcal{M}_{b,\delta}^+ \cup \mathcal{M}_{b,\delta}^0$ such that

$$\lim_{n\longrightarrow\infty} I_{b,\delta}(u_n) = \inf_{\mathcal{M}_{b,\delta}^+\cup\mathcal{M}_{b,\delta}^0} I_{b,\delta} < 0, \quad (35)$$

$$I_{b,\delta}(z) \geq I_{b,\delta}(u_n) - \frac{1}{n}\|z-u_n\|, \forall z \in \mathcal{M}_{b,\delta}^+ \cup \mathcal{M}_{b,\delta}^0. \quad (36)$$

Since $I_{b,\delta}(|u|) = I_{b,\delta}(u)$, we can assume that $u_n \geq 0$ in Ω. By Lemma 9, $\{u_n\}$ is bounded in $H_0^1(\Omega)$, and so, we may assume that

$$\begin{cases} u_n \rightharpoonup u_*, & \text{in } H_0^1(\Omega), \\ u_n \longrightarrow u_*, & \text{in } L^s(\Omega), \quad 1 \leq s < 4, \\ u_n \longrightarrow u_*, & \text{a.e.in } \Omega. \end{cases} \quad (37)$$

In the following, we prove that u_* is a positive solution to $(\mathcal{P}_{b,\delta})$. To this purpose, we divide the proof into five steps.

Step 1. $u_* \neq 0$.

If, to the contrary, we have $u_* = 0$. Since $u_n \in \mathcal{M}_{b,\delta}^+ \cup \mathcal{M}_{b,\delta}^0$, it follows that for n large,

$$a\|u_n\|^2 \geq \frac{4-q}{2-q}b\|u_n\|^4 + \frac{4-q}{2-q}\delta|u_n|_4^4, \quad (38)$$

and hence,

$$I_{b,\delta}(u_n) = \frac{1}{2}a\|u_n\|^2 - \frac{1}{4}b\|u_n\|^4 - \frac{1}{4}\delta|u_n|_4^4 + o(1) > \cdot\left(\frac{4-q}{2(2-q)} - \frac{1}{4}\right)b\|u_n\|^4 + \left(\frac{4-q}{2(2-q)} - \frac{1}{4}\right)\delta|u_n|_4^4 + o(1) > 0, \quad (39)$$

which contradicts with (35). Therefore, $u_* \neq 0$.

Step 2. There is a positive constant C_1 satisfying

$$2a\|u_n\|^2 - \lambda(4-q)|u_n|_q^q < -C_1. \quad (40)$$

To prove that, it suffices to check that

$$2a\limsup_{n\longrightarrow\infty}\|u_n\|^2 < \lambda(4-q)|u_*|_q^q. \quad (41)$$

In view of $u_n \in \mathcal{M}_{b,\delta}^+ \cup \mathcal{M}_{b,\delta}^0$, one has

$$2a\limsup_{n\longrightarrow\infty}\|u_n\|^2 \leq \lambda(4-q)|u_*|_q^q. \quad (42)$$

Assume to the contrary that

$$2a\limsup_{n\longrightarrow\infty}\|u_n\|^2 = \lambda(4-q)|u_*|_q^q. \quad (43)$$

Then, we can suppose $\|u_n\|^2 \longrightarrow A > 0$ as $n \longrightarrow \infty$, where A satisfies

$$\lambda|u_*|_q^q = \frac{2aA}{4-q}. \quad (44)$$

From this, we have that for $\lambda \in (0, T_1)$,

$$\begin{aligned} 0 \le & \left[\left(\frac{2}{4-q}\right)\left(\frac{2-q}{4-q}\right)^{(2-q)/2} \frac{a^{(4-q)/2}}{\left(b+\delta S^{-2}\right)^{(2-q)/2}} - \lambda|\Omega|^{(4-q)/4} S^{-q/2}\right] \\ & \cdot \|u_n\|^q \le \left(\frac{2}{4-q}\right)\left(\frac{2-q}{4-q}\right)^{(2-q)/2} \\ & \cdot \left[\frac{\left(a\|u_n\|^2\right)^{4-q}}{\left(\left(b+\delta S^{-2}\right)\|u_n\|^4\right)^{2-q}}\right]^{1/2} - \lambda|u_n|_q^q \le \left(\frac{2}{4-q}\right)\left(\frac{2-q}{4-q}\right)^{(2-q)/2} \\ & \cdot \left[\frac{\left(a\|u_n\|^2\right)^{4-q}}{\left(a\|u_n\|^2 - \lambda|u_n|_q^q\right)^{2-q}}\right]^{1/2} - \lambda|u_n|_q^q \longrightarrow \left(\frac{2}{4-q}\right)\left(\frac{2-q}{4-q}\right)^{(2-q)/2} \\ & \cdot \left[\frac{(aA)^{4-q}}{\left(((2-q)/(4-q))aA\right)^{2-q}}\right]^{1/2} - \frac{2aA}{4-q} = 0, \end{aligned} \quad (45)$$

which implies that $u_n \longrightarrow 0$ in $H_0^1(\Omega)$, contradicting $u_* \neq 0$. In turn, we deduce that (40) holds.

Step 3. $\|I'_{b,\delta}(u_n)\| \longrightarrow 0$ as $n \longrightarrow \infty$.

Let $0 < \rho < \rho_n \equiv \rho_{u_n}$, $g_n \equiv g_{u_n}$, where ρ_{u_n} and g_{u_n} are defined as Lemma 8 with $u = u_n$. Let $w_\rho = \rho v$ with $v = u/\|u\|$. Fix n and set $z_\rho = g_n(w_\rho)(u_n - w_\rho)$. Since $z_\rho \in \mathscr{M}_{b,\delta}^+$, it follows from (36) that

$$I_{b,\delta}(z_\rho) - I_{b,\delta}(u_n) \ge -\frac{1}{n}\|z_\rho - u_n\|. \quad (46)$$

By the definition of Fréchet derivative, we obtain

$$\left\langle I'_{b,\delta}(u_n), z_\rho - u_n\right\rangle + o\left(\|z_\rho - u_n\|\right) \ge -\frac{1}{n}\|z_\rho - u_n\|. \quad (47)$$

Then,

$$\left\langle I'_{b,\delta}(u_n), -w_\rho + \left(g_n(w_\rho) - 1\right)\left(u_n - w_\rho\right)\right\rangle \ge -\frac{1}{n}\|z_\rho - u_n\| + o\left(\|z_\rho - u_n\|\right), \quad (48)$$

and hence,

$$\begin{aligned} & -\rho\left\langle I'_{b,\delta}(u_n), v\right\rangle + \left(g_n(w_\rho) - 1\right)\left\langle I'_{b,\delta}(u_n), u_n - w_\rho\right\rangle \ge \\ & -\frac{1}{n}\|z_\rho - u_n\| + o\left(\|z_\rho - u_n\|\right), \end{aligned} \quad (49)$$

which yields that

$$\begin{aligned} \left\langle I'_{b,\delta}(u_n), v\right\rangle \le & \frac{1}{n}\frac{\|z_\rho - u_n\|}{\rho} + o\frac{\left(\|z_\rho - u_n\|\right)}{\rho} + \frac{g_n(w_\rho) - 1}{\rho} \\ & \cdot \left\langle I'_{b,\delta}(u_n), u_n - w_\rho\right\rangle. \end{aligned} \quad (50)$$

From Step 2, Lemma 8, and the boundedness of $\{u_n\}$, we also have

$$\begin{aligned} & \|z_\rho - u_n\| = \left\|\left(g_n(w_\rho) - 1\right)\left(u_n - w_\rho\right) - w_\rho\right\| \le \left|g_n(w_\rho) - 1\right| C_2 + \rho, \\ & \lim_{\rho \longrightarrow 0} \frac{\left|g_n(w_\rho) - 1\right|}{\rho} = \langle g_{n'}(0), v\rangle \le \|g'_n(0)\| \le C_3, \\ & \left\langle I'_{b,\delta}(u_n), u_n - w_\rho\right\rangle = \left\langle I'_{b,\delta}(u_n), -w_\rho\right\rangle = -\rho\left\langle I'_{b,\delta}(u_n), v\right\rangle. \end{aligned} \quad (51)$$

As a consequence, for fixed n, we can derive letting $\rho \longrightarrow 0$ in (50) that

$$\left\langle I'_{b,\delta}(u_n), v\right\rangle \le \frac{C}{n}, \quad (52)$$

which implies that $\|I'_{b,\delta}(u_n)\| \longrightarrow 0$ as $n \longrightarrow \infty$.

Step 4. $u_n \longrightarrow u_*$ in $H_0^1(\Omega)$.

Set $v_n = u_n - u_*$. If $\|v_n\| \longrightarrow 0$, we are done, thus assume $\|v_n\| \longrightarrow L > 0$. By $\langle I'_{b,\delta}(u_n), u_*\rangle = o(1)$ and (37),

$$0 = a\|u_*\|^2 - b\left(L^2 + \|u_*\|^2\right)\|u_*\|^2 - \lambda|u_*|_q^q - \delta|u_*|_4^4. \quad (53)$$

Moreover, from $I'_{b,\delta}(u_n) \longrightarrow 0$, the boundedness of $\{u_n\}$, and Brézis-Lieb lemma, we have that

$$\begin{aligned} o(1) = \left\langle I'_{b,\delta}(u_n), u_n\right\rangle = & a\left(\|v_n\|^2 + \|u_*\|^2\right) \\ & - b\left(\|v_n\|^4 + 2\|v_n\|^2\|u_*\|^2 + \|u_*\|^4\right) \\ & - \lambda|u_*|_q^q - \delta|v_n|_4^4 - \delta|u_*|_4^4 + o(1). \end{aligned} \quad (54)$$

Combining this and (53), we get

$$o(1) = a\|v_n\|^2 - b\|v_n\|^4 - b\|v_n\|^2\|u_*\|^2 - \delta|v_n|_4^4. \quad (55)$$

It then follows from Sobolev inequality that

$$a\|v_n\|^2 - b\|v_n\|^4 - b\|v_n\|^2\|u_*\|^2 = \delta|v_n|_4^4 + o(1) \le \delta S^{-2}\|v_n\|^4 + o(1). \quad (56)$$

Passing the limit as $n \longrightarrow \infty$, we obtain that

$$L^2 \ge \frac{S^2\left(a - b\|u_*\|^2\right)}{bS^2 + \delta} > 0. \quad (57)$$

By (53), (57), and Hölder inequality,

$$\begin{aligned} I_{b,\delta}(u_*) &= \frac{a}{2}\|u_*\|^2 - \frac{b}{4}\|u_*\|^4 - \frac{\lambda}{q}\int |u_*|^q dx - \frac{\delta}{4}|u_*|_4^4 \\ &= \frac{a}{4}\|u_*\|^2 + \frac{b}{4}L^2\|u_*\|^2 - \lambda\left(\frac{1}{q}-\frac{1}{4}\right)|u_*|_q^q \\ &\geq \frac{abS^2\|u_*\|^2}{4(bS^2+\delta)} + \frac{b}{4}L^2\|u_*\|^2 + \frac{a\delta}{4(bS^2+\delta)}\|u_*\|^2 \\ &\quad - \lambda\left(\frac{1}{q}-\frac{1}{4}\right)|\Omega|^{(4-q)/4}S^{-q/2}\|u_*\|^q. \end{aligned} \tag{58}$$

For $\xi := a\delta/4(bS^2+\delta)$ and $\eta := \lambda((1/q)-(1/4))|\Omega|^{4-q/4} S^{-q/2}$, define

$$f(t) = \xi t^2 - \eta t^q. \tag{59}$$

By easy calculation, we have that $f(t)$ achieves its minimum value at $t_{\min} = (q\eta/2\xi)^{1/(2-q)}$ and

$$f(t_{\min}) = -\frac{2-q}{2}\eta^{2/(2-q)}\left(\frac{q}{2\xi}\right)^{q/(2-q)}. \tag{60}$$

Therefore, we obtain

$$\begin{aligned} I_{b,\delta}(u_*) &\geq \frac{abS^2\|u_*\|^2}{4(bS^2+\delta)} + \frac{b}{4}L^2\|u_*\|^2 + f(t_{\min}) = \frac{abS^2\|u_*\|^2}{4(bS^2+\delta)} \\ &\quad + \frac{b}{4}L^2\|u_*\|^2 - \frac{2-q}{2}\left(\frac{\lambda(4-q)}{4q}|\Omega|^{(4-q)/4}S^{-q/2}\right)^{2/(2-q)} \\ &\quad \cdot \left[\frac{2q(bS^2+\delta)}{a\delta}\right]^{q/(2-q)}. \end{aligned} \tag{61}$$

Using (37), (53), and (61), we deduce that for $\lambda \in (0, T_1)$,

$$\begin{aligned} I_{b,\delta}(u_n) &= I_{b,\delta}(u_*) + \frac{a}{4}\|v_n\|^2 - \frac{b}{4}\|v_n\|^2\|u_*\|^2 + o(1) \\ &\geq \frac{abS^2\|u_*\|^2}{4(bS^2+\delta)} + \frac{a}{4}L^2 - \frac{2-q}{2} \\ &\quad \cdot \left(\frac{(4-q)}{4q}|\Omega|^{(4-q)/4}S^{-q/2}\right)^{2/(2-q)} \\ &\quad \cdot \left[\frac{2q(bS^2+\delta)}{a\delta}\right]^{q/(2-q)}\lambda^{2/(2-q)} + o(1) \geq \frac{a^2S^2}{4(bS^2+\delta)} \\ &\quad - \frac{2-q}{2}\left(\frac{(4-q)}{4q}|\Omega|^{(4-q)/4}S^{-q/2}\right)^{2/(2-q)} \\ &\quad \cdot \left[\frac{2q(bS^2+\delta)}{a\delta}\right]^{q/(2-q)}\lambda^{2/(2-q)} + o(1) > 0, \end{aligned} \tag{62}$$

which is a contradiction since $\lim_{n\longrightarrow\infty} I_{b,\delta}(u_n) < 0$. This implies that $\|v_n\| \longrightarrow L > 0$ is impossible. Hence, $\|v_n\| \longrightarrow 0$; that is, $u_n \longrightarrow u_*$ in $H_0^1(\Omega)$.

Step 5. u_* is a positive solution of problem $(\mathscr{P}_{b,\delta})$ and $u_* \in \mathscr{M}_{b,\delta}^+$.

From (35) and Steps 3 and 4, we have that, up to a subsequence, $u_n \longrightarrow u_*$ in $H_0^1(\Omega)$ with $I_{b,\delta}(u_*) < 0$ and $I'_{b,\delta}(u_*) = 0$. Namely, $u_* \geq 0$ is a weak nontrivial solution of problem $(\mathscr{P}_{b,\delta})$. Moreover, by Lemmas 6 and 10, we know $u_* \in \mathscr{M}_{b,\delta}^+$. Standard elliptic regularity argument and strong maximum principle provide that u_* is positive. Therefore, the proof of Lemma 11 is completed. □

Lemma 12. *Let $\lambda \in (0, T_1)$, then problem $(\mathscr{P}_{b,\delta})$ has a positive solution U_* with $U_* \in \mathscr{M}_b^-$.*

Proof. As in the proof of Lemma 11, we can prove that there exists a bounded and nonnegative sequence $\{U_n\} \subset \mathscr{M}_{b,\delta}^-$ with the properties

(i) $\lim_{n\longrightarrow\infty} I_{b,\delta}(U_n) = \inf_{\mathscr{M}_{b,\delta}^-} I_{b,\delta}$

(ii) $I_{b,\delta}(z) \geq I_{b,\delta}(U_n) - 1n\|z-u_n\|, \forall z \in \mathscr{M}_{b,\delta}^-$

(iii) $U_n \rightharpoonup U_*$ in $H_0^1(\Omega)$

(iv) $U_n \longrightarrow U_*$ in $L^s(\Omega), 2 \leq s < 4$

(v) $U_n \longrightarrow U_*$ a.e.in Ω

Without loss of generality, we may assume that $0 \in \Omega$. Let $\varphi(x) \in C_0^\infty(\Omega)$ be a cut-off function such that $0 \leq \varphi \leq 1$ in Ω and $\varphi(x) \equiv 1$ near zero. Set

$$v_\varepsilon(x) = \varphi(x)\frac{(8)^{1/2}\varepsilon}{\varepsilon^2 + |x|^2}. \tag{63}$$

By [29, 30], one has for $\varepsilon > 0$ small,

$$\begin{cases} \|v_\varepsilon\|^2 = S^2 + O(\varepsilon^2), \\ |v_\varepsilon|_4^4 = S^2 + O(\varepsilon^4), \\ |v_\varepsilon|_3^3 = O(\varepsilon). \end{cases} \tag{64}$$

In the first place, we prove the following upper bound for $\inf_{\mathscr{M}_{b,\delta}^-} I_{b,\delta}$,

$$\inf_{\mathscr{M}_{b,\delta}^-} I_{b,\delta} \leq \sup_{t>0} I_{b,\delta}(u_* + tv_\varepsilon) < I_{b,\delta}(u_*) + \frac{a^2S^2}{4(bS^2+\delta)}, \tag{65}$$

where u_* is the positive solution obtained in Lemma 11. Since $u_* \in \mathscr{M}_{b,\delta}^+$, it is easy to verify that $a - b\|u_*\|^2 > 0$. By the fact that $\langle {I_{b,\delta}}'(u_*), tv_\varepsilon\rangle = 0$, we also have

$$0 = t\left(a - b\|u_*\|^2\right)\int_\Omega \nabla u_* \nabla v_\varepsilon dx - t\lambda\int_\Omega u_*^{q-1} v_\varepsilon dx - t\delta\int_\Omega u_*^3 v_\varepsilon dx, \tag{66}$$

and hence,

$$\int_\Omega \nabla u_* \nabla v_\varepsilon dx = \frac{\lambda\int_\Omega u_*^{q-1} v_\varepsilon dx + \delta\int_\Omega u_*^3 v_\varepsilon dx}{a - b\|u_*\|^2} > 0. \tag{67}$$

Let $w_\varepsilon = u_* + Rv_\varepsilon$ with $R > 1$. It follows from (67) that

$$\|w_\varepsilon\|^2 = \|u_*\|^2 + 2R\int_\Omega \nabla u_* \nabla v_\varepsilon dx + R^2\|v_\varepsilon\|^2 \geq \|u_*\|^2 + R^2S^2 + O\left(\varepsilon^2\right). \tag{68}$$

Let $\psi(t)$ be given by Lemma 6. As can be seen from the proof of Lemma 6, there exist $\psi(t_\varepsilon) = \lambda|w_\varepsilon/\|w_\varepsilon\||_q^q$ and $\psi'(t_\varepsilon) < 0$, where $t_\varepsilon = t^+(w_\varepsilon/\|w_\varepsilon\|)$. From the structure of ψ and the fact of $|w_\varepsilon/\|w_\varepsilon\||_q^q > 0$, we easily see that t_ε is uniformly bounded by a suitable constant $C_1 > 0$, $\forall R \geq 1$, and $\forall \varepsilon > 0$.

Moreover, we have from (68) that there is $\varepsilon_1 > 0$ satisfying

$$\|w_\varepsilon\|^2 \geq \|u_*\|^2 + \frac{1}{2}R^2S^2, \forall \varepsilon \in (0, \varepsilon_1). \tag{69}$$

Therefore, we may find $R_1 \geq 1$ such that $\|w_\varepsilon\| > C_1$, $\forall R \geq R_1$, and $\forall \varepsilon \in (0, \varepsilon_1)$.

Define

$$\begin{aligned} E_1 &= \left\{u : u = 0 \text{ or } \|u\| < t^+\left(\frac{u}{\|u\|}\right)\right\}, \\ E_2 &= \left\{u : \|u\| > t^+\left(\frac{u}{\|u\|}\right)\right\}. \end{aligned} \tag{70}$$

Notice that $H_0^1(\Omega) - \mathscr{M}_{b,\delta}^- = E_1 \cup E_2$ and $\mathscr{M}_{b,\delta}^+ \subset E_1$. Because $u_* \in \mathscr{M}_{b,\delta}^+$ and the continuity of $t^+(u)$, we have that $u_* + tR_1v_\varepsilon$ for $t \in (0,1)$ must intersect $\mathscr{M}_{b,\delta}^-$. As a consequence,

$$\inf_{\mathscr{M}_{b,\delta}^-} I_{b,\delta} \leq \sup_{t>0} I_{b,\delta}(u_* + tv_\varepsilon). \tag{71}$$

Thus, to complete the proof of (65), it suffices to show that

$$\sup_{t>0} I_{b,\delta}(u_* + tv_\varepsilon) < I_{b,\delta}(u_*) + \frac{a^2S^2}{4\left(bS^2 + \delta\right)}. \tag{72}$$

By mean value theorem, there exists $\delta(x) \in [0,1]$ such that

$$\begin{aligned} (u_*(x) + tv_\varepsilon(x))^q - u_*^q(x) &= q(u_*(x) + \delta(x)tv_\varepsilon(x))^{q-1}tv_\varepsilon(x) \\ &\geq qtu_*^{q-1}(x)v_\varepsilon(x), \end{aligned} \tag{73}$$

for any $x \in \Omega$. Using (66), (67), and (73), we obtain

$$\begin{aligned} I_{b,\delta}(u_* + tv_\varepsilon) &= \frac{a}{2}\|u_*\|^2 + at\int_\Omega \nabla u_* \nabla v_\varepsilon dx + \frac{a}{2}t^2\|v_\varepsilon\|^2 - \frac{b}{4} \\ &\cdot \|u_*\|^4 - bt^2\left(\int_\Omega \nabla u_* \nabla v_\varepsilon dx\right)^2 - \frac{b}{4}t^4\|v_\varepsilon\|^4 \\ &- bt\|u_*\|^2\int_\Omega \nabla u_* \nabla v_\varepsilon dx - \frac{b}{2}t^2\|u_*\|^2\|v_\varepsilon\|^2 \\ &- bt^3\|v_\varepsilon\|^2\int_\Omega \nabla u_* \nabla v_\varepsilon dx - \lambda q\int_\Omega (u_* + tv_\varepsilon)^q dx \\ &- \frac{\delta}{4}\int_\Omega (u_* + tv_\varepsilon)^4 dx \leq I_{b,\delta}(u_*) + \frac{a}{2}t^2\|v_\varepsilon\|^2 \\ &- \frac{b}{4}t^4\|v_\varepsilon\|^4 - \frac{b}{2}t^2\|u_*\|^2\|v_\varepsilon\|^2 - \frac{\lambda}{q}\int_\Omega \\ &\cdot \left[(u_* + tv_\varepsilon)^q - u_*^q - qtu_*^{q-1}v_\varepsilon\right]dx - \frac{\delta}{4}\int_\Omega \\ &\cdot \left[(u_* + tv_\varepsilon)^4 - u_*^4 - 4tu_*^3v_\varepsilon\right]dx \leq I_{b,\delta}(u_*) \\ &+ \frac{a}{2}t^2\|v_\varepsilon\|^2 - \frac{b}{4}t^4\|v_\varepsilon\|^4 - \frac{b}{2}t^2\|u_*\|^2\|v_\varepsilon\|^2 \\ &- \frac{\delta}{4}\int_\Omega \left[(u_* + tv_\varepsilon)^4 - u_*^4 - 4tu_*^3v_\varepsilon\right]dx. \end{aligned} \tag{74}$$

To proceed, we set

$$J(v) = \frac{a}{2}\|v\|^2 - \frac{b}{4}\|v\|^4 - \frac{b}{2}\|u_*\|^2\|v\|^2 - \frac{\delta}{4}\int_\Omega \left[(u_* + v)^4 - u_*^4 - 4u_*^3v\right]dx. \tag{75}$$

Recall that, for $r, s \geq 1$, it holds

$$(r + s)^4 - r^4 - 4r^3s \geq s^4 + C_1rs^3, \tag{76}$$

for some $C_1 > 0$. By (73) and (76), we have that

$$\begin{aligned} J(tv_\varepsilon) &= \frac{a}{2}t^2\|v_\varepsilon\|^2 - \frac{b}{4}t^4\|v_\varepsilon\|^4 - \frac{b}{2}t^2\|u_*\|^2\|v_\varepsilon\|^2 - \frac{\delta}{4}\int_\Omega \\ &\cdot \left[(u_* + tv_\varepsilon)^4 - u_*^4 - 4tu_*^3v_\varepsilon\right]dx \leq \frac{a}{2}t^2\|v_\varepsilon\|^2 - \frac{b}{4}t^4 \\ &\cdot \|v_\varepsilon\|^4 - \frac{b}{2}t^2\|u_*\|^2\|v_\varepsilon\|^2 - \frac{\delta}{4}\int_\Omega \left[(tv_\varepsilon)^4 + C_1u_*(tv_\varepsilon)^3\right]dx \\ &= \frac{a}{2}t^2\|v_\varepsilon\|^2 - \frac{b}{2}t^2\|u_*\|^2\|v_\varepsilon\|^2 - \frac{b}{4}t^4\|v_\varepsilon\|^4 - \frac{\delta}{4}t^4|v_\varepsilon|_4^4 \\ &- \frac{\delta}{4}C_1t^3\int_\Omega u_*v_\varepsilon^3 dx, \end{aligned} \tag{77}$$

which implies that there exists a constant $t_1 > 0$ small enough such that

$$\sup_{0<t<t_1} I_{b,\delta}(u_* + tv_\varepsilon) < \frac{a^2}{4b}. \tag{78}$$

Thus, we only need to consider the case of $t \geq t_1$. By the same argument of Lemma 11 of [21], we have

$$\int_\Omega u_* v_\varepsilon^3 dx = (8)^{3/2} \varepsilon u_*(0) \int_{\mathbb{R}^4} \frac{1}{\left(1+|x|^2\right)^3} dx + o(\varepsilon). \tag{79}$$

Combining this and (64), we have for $\varepsilon > 0$ sufficiently small,

$$\begin{aligned} \sup_{t\geq t_1} J(tv_\varepsilon) &\leq \sup_{t>0}\left\{\frac{a}{2}t^2\|v_\varepsilon\|^2 - \frac{b}{2}t^2\|u_*\|^2\|v_\varepsilon\|^2 - \frac{b}{4}t^4\|v_\varepsilon\|^4\|v_\varepsilon\|^4 - \frac{\delta}{4}t^4|v_\varepsilon|_4^4\right\} \\ &- \frac{\delta}{4}C_1 t_1^3 \int_\Omega u_* v_\varepsilon^3 dx \leq \frac{\left(a\|v_\varepsilon\|^2 - b\|u_*\|^2\|v_\varepsilon\|^2\right)^2}{4\left(b\|v_\varepsilon\|^4 + |v_\varepsilon|_4^4\right)} \\ &- C_2\varepsilon + o(\varepsilon) = \frac{\left(aS^2 - bS^2\|u_*\|^2\right)^2}{4\left(bS^4 + \delta S^2\right)} + O\left(\varepsilon^2\right) - C_2\varepsilon + o(\varepsilon) \\ &< \frac{S^2\left(a - b\|u_*\|^2\right)^2}{4\left(bS^2 + \delta\right)} < \frac{a^2 S^2}{4\left(bS^2 + \delta\right)}, \end{aligned} \tag{80}$$

where $C_2 > 0$ is a positive constant independent of ε. This together with (74) implies that (65) holds.

In the second place, we claim that $U_* \neq 0$. If, to the contrary, we have $U_* \equiv 0$. Since $U_n \in \mathscr{M}_{b,\delta}^- \subset \mathscr{M}_{b,\delta}$, it follows that

$$a\|U_n\|^2 - b\|U_n\|^4 - \lambda|U_n|_q^q - \delta|U_n|_4^4 = 0, \tag{81}$$

and so, by Sobolev inequality

$$a\|U_n\|^2 = b\|U_n\|^4 + \delta|U_n|_4^4 + o(1) \leq \left(b + \delta S^{-2}\right)\|U_n\|^4. \tag{82}$$

Assume that $\|U_n\|^2 \longrightarrow \iota^2$. By $\{U_n\} \subset \mathscr{M}_{b,\delta}^-$ and Lemma 7, we obtain that $\iota^2 > 0$. Taking $n \longrightarrow \infty$ in (82), we have $\iota^2 \geq aS^2/(bS^2 + \delta)$, and thus

$$\begin{aligned} \inf_{\mathscr{M}_{b,\delta}^-} I_{b,\delta} &= \lim_{n\to\infty} I_{b,\delta}(U_n) = \lim_{n\to\infty}\left[I_{b,\delta}(U_n) - \frac{1}{4}\left\langle I_{b,\delta}{}'((U_n)), U_n\right\rangle\right] \\ &= \lim_{n\to\infty}\left[\frac{a}{4}\|U_n\|^2 - \lambda\left(\frac{1}{q} - \frac{1}{4}\right)|U_n|_q^q\right] = \frac{a}{4}\iota^2 \\ &\geq \frac{a^2S^2}{4\left(bS^2 + \delta\right)}, \end{aligned} \tag{83}$$

which is a contradiction with (65). Therefore, the claim follows. At this point, we may proceed as in the proof of Lemma 11 and conclude that U_* is a positive solution of problem $(\mathscr{P}_{b,\delta})$ with $U_* \in \mathscr{M}_{b,\delta}^-$. This completes the proof of Lemma 12. □

Proof of Theorem 1. Theorem 1 is an immediate consequence of Lemmas 7, 11, and 12. □

4. Proofs of Theorems 2 and 3

Proof of Theorem 2. By the definition of λ^* and Theorem 1, we easily see that $\lambda^* \geq T^-$. Hence, Proof of Theorem 2 is completed if we show that $\lambda^* \leq T^+$. To this goal, let us define the functions

$$\begin{aligned} h_\lambda(t) &= t^{q-1}\left(\delta t^{4-q} - a\mu_1 t^{2-q} + \lambda\right), t > 0, \\ \tilde{h}_\lambda(t) &= \delta t^{4-q} - a\mu_1 t^{2-q} + \lambda, t > 0. \end{aligned} \tag{84}$$

Obviously, we have that $\tilde{h}(t)$ is convex and attains its minimum at the point $t_{\min} = [(2-q)a\mu_1/(4-q)\delta]^{1/2}$ with

$$\tilde{h}_\lambda(t_{\min}) = -\frac{2a\mu_1}{4-q}\left[\frac{(2-q)a\mu_1}{(4-q)\delta}\right]^{1/2} + \lambda. \tag{85}$$

As a consequence, we can take

$$T^+ = \frac{2a\mu_1}{4-q}\left[\frac{(2-q)a\mu_1}{(4-q)\delta}\right]^{1/2} + 1, \tag{86}$$

such that

$$\tilde{h}_{T^+}(t) \geq \tilde{h}_{T^+}(t_{\min}) = 1 > 0, \forall t > 0. \tag{87}$$

This gives that

$$h_{T^+}(t) \geq t^{q-1}\tilde{h}_{T^+}(t) > 0, \forall t > 0, \tag{88}$$

namely,

$$T^+ t^{q-1} + \delta t^3 > a\mu_1 t, \forall t > 0. \tag{89}$$

Assume that any $\lambda > 0$ is such that $(\mathscr{P}_{b,\delta})$ admits a positive solution u. On the one hand, using (89) with $t = u$, multiplying by e_1, and integrating over Ω, we get

$$T^+ \int_\Omega u^{q-1} e_1 dx + \delta \int_\Omega u^3 e_1 dx > a\mu_1 \int_\Omega u e_1 dx. \tag{90}$$

On the other hand, multiplying $(\mathscr{P}_{b,\delta})$ by e_1 and integrating over Ω, there holds

$$\left(a - b\|u\|^2\right)\int_\Omega \nabla u \nabla e_1 dx = \lambda \int_\Omega u^{q-1} e_1 dx + \delta \int_\Omega u^3 e_1 dx > 0. \tag{91}$$

Since

$$a\mu_1 \int_\Omega u e_1 dx = a\int_\Omega \nabla u \nabla e_1 dx > \left(a - b\|u\|^2\right)\int_\Omega \nabla u \nabla e_1 dx, \tag{92}$$

we infer from (90) and (91) that $\lambda < T^+$. By the arbitrariness of λ and the definition of λ^*, we conclude that $\lambda^* \le T^+ < \infty$. Proof of Theorem 2 is thus completed. □

Proof of Theorem 3. Let $\{b_n\}$ and $\{\delta_n\}$ be two sequences satisfying $b_n \searrow 0$ and $\delta_n \searrow 0$ as $n \longrightarrow \infty$, and let u_n and U_n be the two positive solutions of $(\mathscr{P}_{b_n,\delta_n})$ obtained in Theorem 1 with $u_n \in \mathscr{M}^+_{b_n,\delta_n}$ and $U_n \in \mathscr{M}^-_{b_n,\delta_n}$.

Using Lemma 7 and $U_n \in \mathscr{M}^-_{b_n}$, we have that

$$\lim_{n\longrightarrow\infty} \|U_n\|^2 \ge \lim_{n\longrightarrow\infty} \left(\frac{a(2-q)}{(4-q)\left(b_n + \delta_n S^{-2}\right)} \right)^{1/2} = +\infty, \quad (93)$$

and thus, the conclusion (i) holds.

In what follows, we prove the conclusion (ii) of Theorem 3. Noting that

$$I_{b_n,\delta_n}(u_n) = \inf_{\mathscr{M}^+_{b_n,\delta_n} \cup \mathscr{M}^0_{b_n,\delta_n}} I_{b_n,\delta_n} < 0, \quad (94)$$

for all $n \in \mathbb{N}$, we obtain from Hölder inequality that

$$\begin{aligned} 0 &\ge I_{b_n,\delta_n}(u_n) - \frac{1}{4}\left\langle I'_{b_n,\delta_n}(u_n), u_n \right\rangle \ge \left(\frac{1}{2} - \frac{1}{4}\right)\|u_n\|^2 \\ &\quad - \lambda\left(\frac{1}{q} - \frac{1}{4}\right)|\Omega|^{4-q/4} S^{-q/2} \|u_n\|^q. \end{aligned} \quad (95)$$

As a consequence of $1 < q < 2$, we have that $\{u_n\}$ is bounded in $H^1_0(\Omega)$. Thus, there is a subsequence of $\{u_n\}$ (still denoted by $\{u_n\}$) such that $u_n \rightharpoonup \bar{u}$ in $H^1_0(\Omega)$ as $n \longrightarrow \infty$. Furthermore, for all $\phi \in H^1_0(\Omega)$, it holds

$$\begin{aligned} 0 &= \lim_{n\longrightarrow\infty} \left\langle I'_{b_n,\delta_n}(u_n), \phi \right\rangle = \lim_{n\longrightarrow\infty} \left[\left(a - b_n\|u_n\|^2\right) \int_\Omega \nabla u_n \nabla \phi dx \right. \\ &\quad \left. - \lambda \int_\Omega u_n^{q-1}\phi dx - \delta_n \int_\Omega u_n^3 \phi dx \right] = a\int_\Omega \nabla u_0 \nabla \phi dx - \lambda \int_\Omega u_0^{q-1}\phi dx, \end{aligned} \quad (96)$$

which provides that $\bar{u}$ is a nonnegative weak solution of problem $(\mathscr{P}_{0,0})$. Let $I_{0,0}(u)$ be the corresponding functional of $(\mathscr{P}_{0,0})$ defined by

$$I_{0,0}(u) = \frac{a}{2}\|u\|^2 - \frac{\lambda}{q}|u|_q^q. \quad (97)$$

Since

$$\begin{aligned} a\|u_n - \bar{u}\|^2 &= \left\langle I_{b_n,\delta_n}{}'(u_n) - I_{0,0}{}'(\bar{u}), u_n - \bar{u} \right\rangle + b_n \int_\Omega \\ &\quad \cdot |\nabla u_n|^2 dx \int_\Omega \nabla u_n \nabla(u_n - \bar{u}) dx + \lambda \int_\Omega \\ &\quad \left(u_n^{q-1} - \bar{u}^{q-1}\right)(u_n - \bar{u}) dx + \delta_n \int_\Omega u_n^3 (u_n - \bar{u}) dx \longrightarrow 0, \end{aligned} \quad (98)$$

as $n \longrightarrow \infty$, it follows that $u_n \longrightarrow \bar{u}$ in $H^1_0(\Omega)$.

Define $c_0 = \inf\{I_{0,0}(u) : u \in H^1_0(\Omega)\}$. It is easy to check that there exists $v_0 \in H^1_0(\Omega) \setminus \{0\}$ such that $c_0 = I_{0,0}(v_0)$ and $c_0 < 0$. As $I_{0,0}(u) \ge I_{b_n,\delta_n}(u)$ for any $u \in H^1_0(\Omega)$, we easily see that $\inf_{\mathscr{M}^+_{b_n,\delta_n} \cup \mathscr{M}^0_{b_n,\delta_n}} I_{b_n,\delta_n} \le c_0$. Set $c_{b_n,\delta_n} = I_{b_n,\delta_n}(u_n)$ and suppose that $\lim_{n\longrightarrow\infty} c_{b_n,\delta_n} = k$. We claim $k = c_0$. Otherwise, we have $k < c_0$, and hence, by $b_n \longrightarrow 0$, $\delta_n \longrightarrow 0$ as $n \longrightarrow \infty$ and $\{u_n\}$ is bounded in $H^1_0(\Omega)$; one has for large n,

$$\begin{aligned} c_0 &\le I_{0,0}(u_n) = I_{b_n,\delta_n}(u_n) + \frac{b_n}{4}\|u_n\|^4 + \frac{\delta_n}{4}|u_n|_4^4 = c_{b_n,\delta_n} \\ &\quad + \frac{b_n}{4}\|u_n\|^4 + \frac{\delta_n}{4}|u_n|_4^4 \le k + \frac{c_0 - k}{2} = \frac{c_0 + k}{2} < c_0, \end{aligned} \quad (99)$$

a contradiction. Thus, the claim follows. Then,

$$c_0 = \lim_{n\longrightarrow\infty} I_{b_n,\delta_n}(u_n) = \frac{a}{2}\|\bar{u}\|^2 - \frac{\lambda}{q}|\bar{u}|_q^q = I_{0,0}(\bar{u}), \quad (100)$$

which implies that $\bar{u}$ is a global minimum of $I_{0,0}$. This result, together with strong maximum principle proves that $\bar{u}$ is a positive ground state solution of $(\mathscr{P}_{0,0})$. Theorem 3 is thus proved. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Acknowledgments

This research was supported by the Natural Science Foundation of Fujian Province (Nos. 2021J011228 and 2021J01330).

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