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Research Article

Inverse Source Problem for Sobolev Equation with Fractional Laplacian

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In this paper, we are interested in the problem of determining the source function for the Sobolev equation with fractional Laplacian. This problem is ill-posed in the sense of Hadamard. In order to edit the instability instability of the solution, we applied the fractional Landweber method. In the theoretical analysis results, we show the error estimate between the exact solution and the regularized solution by using an a priori regularization parameter choice rule and an a posteriori regularization parameter choice rule. Finally, we investigate the convergence of the source function when fractional order $\beta \longrightarrow 1^+$.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N (N \geq 1)$ with sufficiently smooth boundary $\partial\Omega$. In this paper, we are interested to study the following pseudo-parabolic equation

$$\begin{cases} u_t - a\Delta u_t + (-\Delta)^\beta u = F(x, t), & \text{in } \Omega \times (0, T], \\ u|_{\partial\Omega} = 0, & \text{in } \Omega, \end{cases} \quad (1)$$

where $a > 0$ is the diffusion coefficient, F is the source function, and u desribe the distribution of the temperature at position x and time t. The paramater β is the fractional order of Laplacian operator with $\beta \geq 1$.

Pseudo-parabolic equations or called *Sobolev equation* describe describing various important physical phenomena, such as heat conduction involving two temperatures [1], homogeneous liquid permeability in fractured rock [2], unidirectional propagation of long waves in a nonlinearly dispersed medium [3], and its references.

Until now, the results on fractional pseudo-parabolic equations equation are not rich we can mention them in a few some few papers, for example, [1, 4–6]. From the fraction operator $(-\Delta)^\beta$ appearing in the main equation which is nonlocal, many scientists believe that it describes some physical phenomena more accurately than classical integrals differential equation. Properties of fractional operator $(-\Delta)^\beta$ have been described in detail in [1].

For equation (1) we usually divide it into three forms.

(i) The first type is an initial value problem, i.e., determining u when the initial value $u(x, 0) = u_0(x)$ and the source function F is known. The results in this category are vibrant and plentiful ([7, 8])

(ii) The second type is terminal value problem, i.e., recovering the function u from the terminal value data $u(x, T) = u_T(x)$ and the source function data F. To the best of our knowledge, there are limited results for the terminal value problem. We can list some recent papers, for example, [9–13]. In general, the terminal value problem is an ill-posed problem; namely, a solution does do not exist, and if a solution exists, it does not depend continuously on the data. The results of the regularized method for this form were recently investigated by [14, 15]

(iii) The last type is inverse source problem, i.e., recovering the source function F if we know the initial value data $u(x,0)=u_0(x)$ and the terminal data $u(x,T)=u_T(x)$

The main purpose of this paper is to determine the source function $F=\psi(t)f(x)$ with the split form when we know that

$$u(x,T)=g(x), \quad u(x,0)=0, x\in\Omega. \tag{2}$$

The question of determining the function f when we know ψ and g will be studied carefully in this paper. It is surprising that the problem of determining the source function for the pseudo-parabolic equation has not been investigated before. We detail the objective of the problem. In practice, the given data$(\psi_\delta, g_\delta)$is noisy by the observed data (ψ, g) by level $\delta>0$ such that

$$\|\psi_\delta-\psi\|_{L^\infty(0,T)}+\|g_\delta-g\|_{L^2(\Omega)}\le\delta. \tag{3}$$

Our main task here is to construct a regularized method which looking for the function f_δ and claims claim that

$$\lim\|f_\delta-f\|=0, \quad \text{when } \delta\longrightarrow 0^+, \tag{4}$$

in the appropriate norm. It can be claimed that our paper was one of the first works on the inverse source problem for the Sobolev equation.

In [7], Tuan-Long-Thinh used the Tikhonov regularization method to regularize regularized an inverse source problem for time fractional diffusion equation. They also introduced two methods, a priori and a posteriori parameter choice rules, to obtain the convergence estimate of the regularized methods. In [16], the authors studied the problem of finding the source distribution for the linear biparabolic equation when we have the final observation. Ma et al. [17] identified the unknown space-dependent source term in a time-fractional diffusion equation by applying the generalized and revised generalized Tikhonov regularization methods. There are many different regularized methods, and in this paper, we choose the fractional Landweber regularization method. The Landweber regularization method was first derived from [18] where the authors applied the filter regularization technique for solving a linear inverse problem. Up to now, the Landweber regularization method has been applied to solve many inverse problems, for example, [19–21] and references therein. This method is beneficial very useful for investigating for the linear ill-posed equation. Recently, Binh et al. [22] studied an inverse source problem for the Rayleigh–Stokes problem using the Tikhonov method.

For the reader's convenience, we would like to outline the main results and novelties of the paper briefly:

(i) The first goal of this paper is to provide the fractional Landweber method to solve this inverse space-dependent source problem for pseudo-parabolic equation. We give the ill-posedness of our inverse source problem and introduce the convergence rate of the fractional Landweber regularized solution. In addition, we obtain the convergence rate by using an apriori parameter choice rule and an a posteriori parameter choice rule. Looking back at the articles [19–21], we realize that the source functions in these papers do not depend on the time function. So, the computation is not complicated. Meanwhile, the source function of the current paper depends on the function ψ which makes the calculation more cumbersome. The presence of (3) makes our problem more clearly complex complex than [19–21]. One point to note is that the method in the article [23] can be applied to our model, but we approach it differently, in a different way.

(ii) The second interesting point in the paper is the investigation of the convergence of the source function when the order of derivative approaches 1. Comparing the difference between the source function of equation (1) with $\beta>1$ and the classical pseudo-parabolic equation $\beta=1$ will help us understand more information about problem (1).

The paper is organized as follows. Section 2 states some preliminary theoretical knowledge. In Section 3, we give the Fourier formula of the source function and also present the ill-posedness of our problem. The conditional stability for the source function source function is also discussed in the same section. Section 4 provides the fractional Landweber regularization method and states a convergence estimate under a priori assumption on the exact solution. The posteriori parameter choice rule is also shown in section 4. Finally, in Section 5, we prove the convergence of the source function in Hilbert scales space with the appropriate assumption of ψ and g.

2. Preliminary Results

Let us consider the operator $\mathscr{A}=-\Delta$ on $\mathbb{V}:=\mathscr{H}_0^1(\Omega)\cap H^2(\Omega)$, and assume that the operator $\mathscr{A}$ has the eigenvalues λ_j such that $0<\lambda_1\le\lambda_2\le\cdots\le\lambda_j\le\cdots$ which approach ∞ as j goes to ∞, The corresponding eigenfunctions are denoted by $e_j\in\mathbb{V}$. Now, let us define fractional powers of $\mathscr{A}$ and its domain. For all $s\ge 0$, we define by $\mathscr{A}^s$ the following operator:

$$\mathscr{A}^s v:=\sum_{j=1}^{\infty}\langle v,e_j\rangle\lambda_j^s e_j, v\in D(\mathscr{A}^s)=\left\{v\in L^2(\Omega): \sum_{j=1}^{\infty}\left|\langle v,e_j\rangle\right|^2\lambda_j^{2s}<\infty\right\}. \tag{5}$$

The domain $\mathbb{H}^s(\Omega) = D(\mathscr{A}^s)$ is the Banach space equipped with the norm

$$\|v\|_{D(\mathscr{A}^s)} := \left(\sum_{j=1}^{\infty} \left| \langle v, e_j \rangle \right|^2 \lambda_j^{2s} \right)^{1/2}, \quad v \in D(\mathscr{A}^s). \tag{6}$$

We introduce the following two lemmas, which are useful and helpful in the next proofs.

Lemma 1. *Let $\psi : [0, T] \longrightarrow \mathbb{R}$ such that $\psi_0 \le \psi(t) \le \psi_1$ where ψ_0 and ψ_1 are positive numbers. Let us assume that $\beta \ge 1$. Then, the following estimates are true:*

$$\begin{aligned} &\int_0^T \exp\left(-(T-s)\lambda_j^\beta (1 + a\lambda_j)^{-1}\right) \psi(s) ds \le \psi_1 \frac{1 + a\lambda_j}{\lambda_j^\beta}, \\ &\frac{1 + a\lambda_j}{\lambda_j^\beta} \left[1 - \exp\left(-T\lambda_1^\beta (1 + a\lambda_1)^{-1}\right)\right] \psi_0 \\ &\quad \le \int_0^T \exp\left(-(T-s)\lambda_j^\beta (1 + a\lambda_j)^{-1}\right) \psi(s) ds. \end{aligned} \tag{7}$$

Proof. Since $\psi(t) \le \psi_1$, we infer that

$$\begin{aligned} &\int_0^T \exp\left(-(T-s)\lambda_j^\beta (1 + a\lambda_j)^{-1}\right) \psi(s) \\ &\quad \cdot ds \le \psi_1 \int_0^T \exp\left(-(T-s)\lambda_j^\beta (1 + a\lambda_j)^{-1}\right) ds = \psi_1 \frac{1 + a\lambda_j}{\lambda_j^\beta}. \end{aligned} \tag{8}$$

□

Since $\psi(t) \ge \psi_0 > 0$, we infer that

$$\begin{aligned} &\int_0^T \exp\left(-(T-s)\lambda_j^\beta (1 + a\lambda_j)^{-1}\right) \psi(s) \\ &\quad \cdot ds \ge \psi_0 \int_0^T \exp\left(-(T-s)\lambda_j^\beta (1 + a\lambda_j)^{-1}\right) \\ &\quad \cdot ds = \frac{1 + a\lambda_j}{\lambda_j^\beta} \left[1 - \exp\left(-T\lambda_j^\beta (1 + a\lambda_j)^{-1}\right)\right] \psi_0 \cdot \end{aligned} \tag{9}$$

Let us consider the following function:

$$\Phi(z) = \frac{z^\beta}{1 + az}, \quad z > 0. \tag{10}$$

The derivative of it is equal to

$$\Phi'(z) = \frac{a\beta z^{\beta-1} + \beta z^\beta - z^\beta}{(1 + az)^2} > 0. \tag{11}$$

This implies that Φ is an increasing function on $(0, +\infty)$. Therefore, we get that

$$\lambda_j^\beta (1 + a\lambda_j)^{-1} \ge \lambda_1^\beta (1 + a\lambda_1)^{-1}. \tag{12}$$

It follows from (9) that

$$\begin{aligned} &\int_0^T \exp\left(-(T-s)\lambda_j^\beta (1 + a\lambda_j)^{-1}\right) \psi(s) \\ &\quad \cdot ds \ge \frac{1 + a\lambda_j}{\lambda_j^\beta} \left[1 - \exp\left(-T\lambda_1^\beta (1 + a\lambda_1)^{-1}\right)\right] \psi_0. \end{aligned} \tag{13}$$

The proof of the Lemma 1 is completed.

Lemma 2. *Let ψ_0, ψ_1 be positive constants such that $\psi_0 < \psi < \psi_1$. By choosing $\delta \in (0, \psi_1/4)$, and $\mathscr{B}(\psi_0, \psi_1) = \psi_1 + (\psi_0/4)$, we obtain*

$$4^{-1}\psi_0 \le |\psi_\delta(t)| \le \mathscr{B}(\psi_0, \psi_1). \tag{14}$$

Proof. The proof is completed in [26], page 4. □

3. Inverse Source Problem: Explicit Form and Ill-Posedness

Let us first give the explicit of Fourier form of the mild solution to problems (1) and (2). First, taking the inner product of both sides of (1) with $e_j(x)$, we find that

$$\begin{aligned} &\frac{d}{dt}\left(\int_\Omega u(x,t)e_j(x)dx\right) + a\lambda_j\left(\int_\Omega u(x,t)e_j(x)dx\right) \\ &\quad + \lambda_j^\beta\left(\int_\Omega u(x,t)e_j(x)dx\right) = \int_\Omega F(x,t)e_j(x)dx, \end{aligned} \tag{15}$$

and from the initial condition $u(x, 0) = 0$, we have that

$$\begin{aligned} \int_\Omega u(x,T)e_j(x)dx = &\int_0^T \exp\left(-(T-s)\lambda_j^\beta (1 + a\lambda_j)^{-1}\right) \\ &\cdot \left(\int_\Omega F(x,s)e_j(x)dx\right) ds, \end{aligned} \tag{16}$$

since $F(x, s) = \psi(s)f(x)$; we know that

$$\int_\Omega f(x)e_j(x)dx = \frac{\int_\Omega g(x)e_j(x)dx}{\int_0^T \exp\left(-(T-s)\lambda_j^\beta (1 + a\lambda_j)^{-1}\right)\psi(s)ds}. \tag{17}$$

Hence, the source function is defined as follows:

$$f(x) = \sum_{j=1}^{\infty} \left[\frac{\int_\Omega g(x)e_j(x)dx}{\int_0^T \exp\left(-(T-s)\lambda_j^\beta (1 + a\lambda_j)^{-1}\right)\psi(s)ds}\right] e_j(x). \tag{18}$$

Let us prove the ill-posedness of inverse source problems

(1) and (2). Logically, we will consider the source function problem as the problem of finding f satisfying (18). From now on, we only treat the source term (18).

Theorem 3. *The problem of determining f that satisfies (18) is ill-posed in the sense of Hadamard.*

Proof. We defined a linear operator $Y : L^2(\Omega) \longrightarrow L^2(\Omega)$ as follows:

$$Yf(x) = \sum_{j=1}^{\infty} \left(\int_0^T \exp\left(-(T-s)\lambda_j^\beta (1+a\lambda_j)^{-1}\right)\psi(s)ds \right) < f, e_j > e_j(x) = \int_\Omega k(x,\xi) f(\xi) d\xi. \tag{19}$$

□

Due to $k(x,\xi) = k(\xi, x)$, we know Y is a self-adjoint operator. Next, its compactness is explained as follows. Let us define the finite rank operators $Y_{\mathcal{N}}$ as follows:

$$Y_N f(x) = \sum_{j=1}^{N} \left(\int_0^T \exp\left(-(T-s)\lambda_j^\beta (1+a\lambda_j)^{-1}\right)\psi(s)ds \right) < f, \quad e_j > e_j(x). \tag{20}$$

By some simple calculations and using Lemma 1, we have

$$\|Y_{\mathcal{N}} f - Yf\|^2_{L^2(\Omega)} \le \underbrace{\psi_1^2 \left(\lambda_1^{-1} + a\right)^2}_{\mathscr{A}_1^2} \sum_{j=N+1}^{+\infty} \frac{\left| < f, e_j > \right|^2}{\lambda_j^{2\beta-2}}. \tag{21}$$

From (21), we have

$$\left\| \|Y_N f - Yf\|^2_{L^2(\Omega)} \le \frac{\mathscr{A}_1^2}{\lambda_N^{2\beta-2}} \right\| f\|^2_{L^2(\Omega)} \longrightarrow 0 \text{ in } L\left(L^2(\Omega); L^2(\Omega)\right) \text{ as } N \longrightarrow \infty. \tag{22}$$

Therefore, Y is a compact operator. The SVDs for the linear self-adjoint compact operator Y are

$$Y = \left(\int_0^T \exp\left(-(T-s)\lambda_j^\beta (1+a\lambda_j)^{-1}\right)\psi(s)ds \right), \tag{23}$$

and corresponding eigenvectors are e_j which is an orthonormal basis in $L^2(\Omega)$. Therefore, the inverse source problem we introduced above can be formulated as an operator equation $Yf(x) = g(x)$ where by $g(x)$ is the numerator in formula (18), and by Kirsch, we can conclude that it is ill-posed. The final time data $g_i = \lambda_i e_i$, by (18), the source term corresponding to g_i is

$$\begin{aligned} f_i(x) &= \sum_{j=1}^{\infty} \left(\int_0^T \exp\left(-(T-s)\lambda_j^\beta (1+a\lambda_j)^{-1}\right)\psi(s)ds \right)^{-1} \\ &\quad \cdot \left(\int_\Omega g(x) e_j(x) dx \right) e_j(x) \\ &\ge \underbrace{\left(\frac{1}{\lambda_1} + a \right)^{-1} \left[1 - \exp\left(-T\lambda_i^\beta (1+a\lambda_i)^{-1}\right) \right]^{-1} \psi_0^{-1} \lambda_i^\beta}_{\mathscr{A}_2}, \end{aligned} \tag{24}$$

whereby $\mathscr{A}_1$ is defined in formula (21). The input final data $g = 0$, by (18), the source term corresponding to g is $f = 0$. We have error in $L^2(\Omega)$ norm between g_m and g

$$\lim_{i \longrightarrow +\infty} \|g_i - g\|_{L^2(\Omega)} = \lim_{i \longrightarrow +\infty} \lambda_i^{-1} = 0. \tag{25}$$

Then, the error in L^2 norm between f_i and f is estimated as follows:

$$\|f_i - f\|_{L^2(\Omega)} \ge \frac{\lambda_i}{\mathscr{A}_2} \longrightarrow \lim_{i \longrightarrow +\infty} \|f_i - f\|_{L^2(\Omega)} \ge +\infty. \tag{26}$$

From (25) and (26), we deduce that the solution to problem (1) is unstable in $L^2(\Omega)$.

Next, we consider stability of the inverse source problem.

Theorem 4. *If $f \in \mathscr{D}(\mathscr{A}^s)$ such that*

$$\|f\|_{D(\mathscr{A}^s)} \le \mathscr{E}, s = \frac{k(\beta-1)}{2} \ge 0, \tag{27}$$

then we get

$$\|f\|_{L^2(\Omega)} \le \mathscr{C}_1^{-k/(k+2)} \mathscr{E}^{2/(k+2)} \|g\|^{k/(k+2)}_{L^2(\Omega)}, \tag{28}$$

where $\mathscr{C}_1 = (a[1 - \exp(-T\lambda_1^\beta (1+a\lambda_1))^{-1}]\psi_0)$.

Proof. From (18) and the Hölder inequality, it gives

$$\begin{aligned} \|f\|^2_{L^2(\Omega)} &= \sum_{j=1}^{\infty} \left[\frac{\int_\Omega g(x) e_j(x) dx}{\int_0^T \exp\left(-(T-s)\lambda_j^\beta (1+a\lambda_j)^{-1}\right)\psi(s)ds} \right]^2 \\ &\le \sum_{j=1}^{\infty} \frac{\left(\int_\Omega g(x) e_j(x)\right)^{2k/(k+2)}}{\left[\int_0^T \exp\left(-(T-s)\lambda_j^\beta (1+a\lambda_j)^{-1}\right)\psi(s)ds\right]^2} \\ &\quad \cdot \left(\int_\Omega g(x) e_j(x) dx \right)^{4/k+2} \\ &\le \left(\sum_{j=1}^{\infty} \frac{\left(\int_\Omega f(x) e_j(x) dx\right)^2}{\left(\int_0^T \exp\left(-(T-s)\lambda_j^\beta (1+a\lambda_j)^{-1}\right)\psi(s)ds\right)^k} \right)^{2/k+2} \\ &\quad \cdot \left(\sum_{j=1}^{\infty} \left(\int_\Omega g(x) e_j(x) dx \right)^2 \right)^{k/k+2}. \end{aligned} \tag{29}$$

Applying Lemma 1 and the priori boundary condition (27), we have

$$\sum_{j=1}^{\infty}\frac{\left(\int_{\Omega}f(x)e_j(x)dx\right)^2}{\left(\int_0^T \exp\left(-(T-s)\lambda_j^{\beta}\left(1+a\lambda_j\right)^{-1}\right)\psi(s)ds\right)^k} = \sum_{j=1}^{\infty}\left(\int_{\Omega}f(x)e_j(x)dx\right)^2\lambda_j^{k(\beta-1)} \cdot\left(a\left[1-\exp\left(-T\lambda_1^{\beta}(1+a\lambda_1)^{-1}\right)\right]\psi_0\right)^{-k} \le \mathscr{E}^2(\mathscr{C}_1)^{-k}. \quad (30)$$

Combining (29) to (30), one has

$$\|f\|_{L^2(\Omega)} \le \mathscr{C}_1^{-k/(k+2)}\mathscr{E}^{2/(k+2)}\|g\|_{L^2(\Omega)}^{k/(k+2)}, \quad (31)$$

where $\mathscr{C}_1=(a[1-\exp(-T\lambda_1^{\beta}(1+a\lambda_1)^{-1})]\psi_0)$. The proof of this theorem is completed. □

4. A Fractional Landweber Method and Convergent Rate

In this section, we apply the fractional Landweber regularization method to solve the inverse source problem (1) and give a convergence estimate. The construction of this method and its iterative implementation are clarified in [19]. We denote the fractional Landweber regularization solution with the observed data by

$$f_{c(\delta),\delta}(x)=\sum_{j=1}^{\infty}1-\left[1-b\left(\frac{\lambda_1^{-1}+a}{\lambda_j^{\beta-1}}\right)^2\right]^{c(\delta)} \cdot\frac{\left(\int_{\Omega}g_{\delta}(x)e_j(x)dx\right)e_j(x)}{\int_0^T \exp\left(-(T-s)\lambda_j^{\beta}\left(1+a\lambda_j\right)^{-1}\right)\psi_{\delta}(s)ds}, \quad (32)$$

$$f^d_{c(\delta),\delta}(x)=\sum_{j=1}^{\infty}\left[1-\left(1-b\left(\frac{\lambda_1^{-1}+a}{\lambda_j^{\beta-1}}\right)^2\right)^{c(\delta)}\right]^d \cdot\frac{\left(\int_{\Omega}g_{\delta}(x)e_j(x)dx\right)e_j(x)}{\int_0^T \exp\left(-(T-s)\lambda_j^{\beta}\left(1+a\lambda_j\right)^{-1}\right)\psi_{\delta}(s)ds}, \quad (33)$$

$$f^d_{c(\delta)}(x)=\sum_{j=1}^{\infty}\left[1-\left(1-b\left(\frac{\lambda_1^{-1}+a}{\lambda_j^{\beta-1}}\right)^2\right)^{c(\delta)}\right]^d \cdot\frac{\left(\int_{\Omega}g(x)e_j(x)dx\right)e_j(x)}{\int_0^T \exp\left(-(T-s)\lambda_j^{\beta}\left(1+a\lambda_j\right)^{-1}\right)\psi(s)ds}. \quad (34)$$

It is obvious to see that formulas (33) and (34) are more complicated. For simplicity, we put $\mathscr{C}_{\beta}(a,s,\lambda_j)=\exp(-(T-s)\lambda_j^{\beta}(1+a\lambda_j)^{-1})$. Expressions (33) and (34) become

$$f^d_{c,\delta}(x)=\sum_{j=1}^{\infty}\left[1-\left(1-b\left(\frac{\lambda_1^{-1}+a}{\lambda_j^{\beta-1}}\right)^2\right)^{c(\delta)}\right]^d \cdot\frac{\left(\int_{\Omega}g_{\delta}(x)e_j(x)dx\right)e_j(x)}{\int_0^T\mathscr{C}_{\beta}(a,s,\lambda_j)\psi_{\delta}(s)ds}, \frac{1}{2}<d<1, \quad (35)$$

$$f^d_{c}(x)=\sum_{j=1}^{\infty}\left[1-\left(1-b\left(\frac{\lambda_1^{-1}+a}{\lambda_j^{\beta-1}}\right)^2\right)^{c(\delta)}\right]^d \cdot\frac{\left(\int_{\Omega}g(x)e_j(x)dx\right)e_j(x)}{\int_0^T\mathscr{C}_{\beta}(a,s,\lambda_j)\psi(s)ds}, \frac{1}{2}<d<1, \quad (36)$$

where $d\in(1/2,1]$ is called the fractional parameter and $c(\delta)\ge 1$ is a regularization parameter and $b\in(0,(\lambda_1^{\beta-1}/\lambda_1^{-1}+a)^2)$. If $d=1$, it is the classical Landweber method. Next, we have the following lemmas:

Lemma 5. *For $0<\lambda<1,\tau>0$, $n\in\mathbb{N}$, let $r_n(\lambda):=(1-\lambda)^n$, we get*

$$r_n(\lambda)\lambda^{\tau}\le\theta_{\tau}(n+1)^{-\tau}, \quad (37)$$

where

$$\theta_{\tau}=\begin{cases}1, & 0\le\tau\le 1,\\ \tau^{\tau}, & \tau>1.\end{cases} \quad (38)$$

Proof. Please see in [19]. □

Lemma 6. *For $(1/2)<d<1, c(\delta)\ge 1$, choosing $b\in(0,(\lambda_1^{\beta-1}/\lambda_1^{-1}+a)^2)$ then $0<b(\lambda_1^{-1}+a/\lambda_1^{\beta-1})^2<1$, by denoting $z=b(\lambda_1^{-1}+a/\lambda_1^{\beta-1})^2$, we have the following estimates:*

$$(a)[1-(1-z)^c]^d\left(\frac{z}{b}\right)^{-1/2}\le b^{1/2}c^{1/2}, \qquad (b)(1-z)^c\left(\frac{z}{b}\right)^{\varsigma/2}\le\left(\frac{\varsigma}{2b}\right)^{\varsigma/2}c^{-\varsigma/2}. \quad (39)$$

Proof. The proof can be found in [19]. □

4.1. A Priori Parameter Choice Rule

Theorem 7. *Suppose that f is given by (18) such that $\|f\|_{\mathscr{D}(A^{k(\beta-1)})}\le\mathscr{E}$ for any $\mathscr{E}>0$. Let the data $(\psi,g,\psi_{\delta},g_{\delta})$ satisfy (3). If we choose $[c(\delta)]=(\mathscr{E}/\delta)^{2/k+1}$, then we obtain*

$$\|f^r_{c(\delta),\delta}-f\|_{L^2(\Omega)}\text{ is of order }\delta^{k/k+1}, \quad (40)$$

where $f^d_{c(\delta),\delta}$ is a regularized solution defined in (35).

Proof. By using the triangle inequality, we have

$$\|f^d_{c(\delta),\delta} - f\|_{L^2(\Omega)} \le \|f^d_{c(\delta),\delta} - f^d_{c(\delta)}\|_{L^2(\Omega)} + \|f^d_{c(\delta)} - f\|_{L^2(\Omega)}. \tag{41}$$

□

We receive $\|f^d_{c(\delta),\delta} - f^d_{c(\delta)}\|_{L^2(\Omega)}$ as follows:

$$f^d_{c(\delta),\delta}(x) - f^d_{c(\delta)}(x) = \sum_{j=1}^{\infty}\left[1-\left(1-b\left(\frac{\lambda_1^{-1}+a}{\lambda_j^{\beta-1}}\right)^2\right)^{c(\delta)}\right]^d \cdot \left(\frac{\int_\Omega g_\delta(x)e_j(x)dx e_j(x)}{\int_0^T \mathscr{C}_\beta(a,s,\lambda_j)\psi_\delta(s)ds} - \frac{\int_\Omega g(x)e_j(x)dx e_j(x)}{\int_0^T \mathscr{C}_\beta(a,s,\lambda_j)\psi(s)ds}\right). \tag{42}$$

From (42), we get

$$\|f^d_{c(\delta),\delta} - f^d_{c(\delta)}\|_{L^2(\Omega)} = \underbrace{\sum_{j=1}^{\infty}\left[1-\left(1-b\left(\frac{\lambda_1^{-1}+a}{\lambda_j^{\beta-1}}\right)^2\right)^{c(\delta)}\right]^d \left(\frac{\int_\Omega (g_\delta(x)-g(x))e_j(x)dx}{\int_0^T \mathscr{C}_\beta(a,s,\lambda_j)\psi_\delta(s)ds}\right)}_{\mathscr{I}_1} + \underbrace{\sum_{j=1}^{\infty}\left[1-\left(1-b\left(\frac{\lambda_1^{-1}+a}{\lambda_j^{\beta-1}}\right)^2\right)^{c(\delta)}\right]^d \left(\frac{\int_\Omega g(x)e_j(x)dx}{\int_0^T \mathscr{C}_\beta(a,s,\lambda_j)\psi_\delta(s)ds} - \frac{\int_\Omega g(x)e_j(x)dx}{\int_0^T \mathscr{C}_\beta(a,s,\lambda_j)\psi(s)ds}\right)}_{\mathscr{I}_2}. \tag{43}$$

Using (43), Lemma 6, and Lemma 1 and noting that $|(\lambda_1^{-1}+a)/\lambda_j^{\beta-1}|^{-1}$, we provide the estimation of $\mathscr{I}_1$ as follows:

$$\begin{aligned}\mathscr{I}_1 &\le \sum_{j=1}^{\infty}\left[1-\left(1-b\left(\frac{\lambda_1^{-1}+a}{\lambda_j^{\beta-1}}\right)^2\right)^{c(\delta)}\right]^d \left|\frac{\lambda_1^{-1}+a}{\lambda_j^{\beta-1}}\right|^{-1} \\ &\quad\times \left|\frac{\lambda_1^{-1}+a}{\lambda_j^{\beta-1}}\right| \left|\frac{\lambda_j^{\beta-1}}{\lambda_1^{-1}+a}\right| \frac{4}{\psi_0}\left(\frac{\int_\Omega (g_\delta(x)-g(x))e_j(x)dx}{[1-\exp(-T\lambda_1(1+a\lambda_1)^{-1})]]}\right) \\ &\le [c(\delta)]^{1/2} b^{1/2} 4\varepsilon \left[\psi_0\left(1-\exp\left(-T\lambda_1(1+a\lambda_1)^{-1}\right)\right)\right]^{-1}.\end{aligned} \tag{44}$$

Next, we have the estimation of $\mathscr{I}_2$

$$\begin{aligned}\mathscr{I}_2 &\le \sum_{j=1}^{\infty}\left|\frac{\int_0^T \mathscr{C}_\beta(a,s,\lambda_j)(\psi(s)-\psi_\delta)ds}{\int_0^T \mathscr{C}_\beta(a,s,\lambda_j)\psi_\delta(s)ds} \times \frac{\int_\Omega g(x)e_j(x)dx e_j(x)}{\int_0^T \mathscr{C}_\beta(a,s,\lambda_j)\psi(s)ds}\right| \\ &\le \frac{4\delta}{\psi_0}\|f\|_{L^2(\Omega)}.\end{aligned} \tag{45}$$

Combining (42) to (45), we derive that

$$\begin{aligned}\left\|f^d_{c(\delta),\delta} - f^d_{c(\delta)}\right\|_{L^2(\Omega)} &\le [c(\delta)]^{1/2} b^{1/2} 4\delta \\ &\quad\cdot \left[\psi_0\left(1-\exp\left(-T\lambda_1(1+a\lambda_1)^{-1}\right)\right)\right]^{-1} + \frac{4\delta}{\psi_0}\|f\|_{L^2(\Omega)}.\end{aligned} \tag{46}$$

Next, we give

$$\begin{aligned}\left\|f^d_{c(\delta)} - f\right\|^2_{L^2(\Omega)} &= \sum_{j=1}^{\infty}\left[1-\left[1-\left(1-b\left(\frac{\lambda_1^{-1}+a}{\lambda_j^{\beta-1}}\right)^2\right)^{c(\delta)}\right]^d\right]^2 \\ &\quad\cdot \left|\frac{\int_\Omega g(x)e_j(x)dx}{\int_0^T \mathscr{C}_\beta(a,s,\lambda_j)\psi(s)ds}\right|^2 \\ &\le \sum_{j=1}^{\infty}\left[1-\left[1-\left(1-b\left(\frac{\lambda_1^{-1}+a}{\lambda_j^{\beta-1}}\right)^2\right)^{c(\delta)}\right]^d\right]^2 \\ &\quad\cdot \lambda_j^{-k(\beta-1)}\|f\|^2_{\mathscr{D}(A^{k(\beta-1)})} \le \sum_{j=1}^{\infty}\left[1-b\left|\frac{\lambda_1^{-1}+a}{\lambda_j^{\beta-1}}\right|^2\right]^{2c(\delta)} \lambda_j^{-k(\beta-1)}\mathscr{E}^2.\end{aligned} \tag{47}$$

From estimate (13) and Lemma 1, we arrive at

$$\begin{aligned}&\int_0^T \exp\left(-(T-s)\lambda_j^\beta(1+a\lambda_j)^{-1}\right) \\ &\quad\cdot ds \ge \frac{1+a\lambda_j}{\lambda_j^\beta}\left[1-\exp\left(-T\lambda_1^\beta(1+a\lambda_1)^{-1}\right)\right] \\ &\ge \frac{a\lambda_j}{\lambda_j^\beta}\left[1-\exp\left(-T\lambda_1^\beta(1+a\lambda_1)^{-1}\right)\right] \ge a\lambda_j^{-(\beta-1)} \\ &\quad\cdot \left[1-\exp\left(-T\lambda_1^\beta(1+a\lambda_1)^{-1}\right)\right].\end{aligned} \tag{48}$$

Hence,

$$\begin{aligned}\lambda_j^{-(\beta-1)} &\le \frac{\int_0^T \exp\left(-(T-s)\lambda_j^\beta(1+a\lambda_j)^{-1}\right)ds}{a\left[1-\exp\left(-T\lambda_1^\beta(1+a\lambda_1)^{-1}\right)\right]} \\ &\le \frac{(\lambda_1^{-1}+a)}{\lambda_j^{\beta-1}a\left[1-\exp\left(-T\lambda_1^\beta(1+a\lambda_1)^{-1}\right)\right]}.\end{aligned} \tag{49}$$

The above estimate (49) implies that

$$\lambda_j^{-k(\beta-1)} \le \frac{(\lambda_1^{-1}+a)^k}{\lambda_j^{k(\beta-1)}a^k\left[1-\exp\left(-T\lambda_1^\beta(1+a\lambda_1)^{-1}\right)\right]^k}. \tag{50}$$

From observation above, using Lemma 6, we conclude that

$$\|f^d_{c(\delta)} - f\|^2_{L^2(\Omega)} \le \sum_{j=1}^{\infty} \left[1 - b\left|\frac{\lambda_1^{-1} + a}{\lambda_j^{\beta-1}}\right|^2\right]^{2c(\delta)} \cdot \left(\frac{(\lambda_1^{-1} + a)}{\lambda_j^{(\beta-1)}}\right)^k \frac{\mathscr{E}^2}{a^k\left[1 - \exp\left(-T\lambda_1^{\beta}(1 + a\lambda_1)^{-1}\right)\right]^k} \le \left(\frac{k}{2b}\right)^k [c(\delta)]^{-k} \frac{\mathscr{E}^2}{a^k\left[1 - \exp\left(-T\lambda_1^{\beta}(1 + a\lambda_1)^{-1}\right)\right]^k}. \tag{51}$$

From (47) and (50), we have

$$\|f^d_{c(\delta)} - f\|_{L^2(\Omega)} \le \left(\frac{k}{2b}\right)^{k/2} [c(\delta)]^{-k/2} \cdot \frac{\mathscr{E}^2}{a^{k/2}\left[1 - \exp\left(-T\lambda_1^{\beta}(1 + a\lambda_1)^{-1}\right)\right]^{k/2}}. \tag{52}$$

Combining (68) to (52), it can be seen

$$\|f^d_{c(\delta)} - f\|_{L^2(\Omega)} \le [c(\delta)]^{1/2} b^{1/2} 4\delta\left[\psi_0\left(1 - \exp\left(-T\lambda_1(1 + a\lambda_1)^{-1}\right)\right)\right]^{-1} + \frac{4\delta}{\psi_0}\|f\|_{L^2(\Omega)} + \left(\frac{k}{2b}\right)^{k/2} \cdot [c(\delta)]^{-k/2} \frac{\mathscr{E}}{a^{k/2}\left[1 - \exp\left(-T\lambda_1^{\beta}(1 + a\lambda_1)^{-1}\right)\right]^{k/2}}. \tag{53}$$

By substituting $c(\delta) = [(\mathscr{E}/\delta)^{2/k+1}]$ in the above expression, we deduce that

$$\|f^d_{c(\delta),\delta} - f\|_{L^2(\Omega)} \le \delta^{k/k+1}\mathscr{E}^{k/k+1}(\mathscr{L}_1 + \mathscr{L}_2), \tag{54}$$

where

$$\mathscr{L}_1 = b^{1/2} 4\left[\psi_0\left(1 - \exp\left(-T\lambda_1(1 + a\lambda_1)^{-1}\right)\right)\right]^{-1} + 4\delta^{k/k+1}\psi_0^{-1}\mathscr{C}_1^{-k/k+2}\|g\|^{k/k+2},$$
$$\mathscr{L}_2 = \left(\frac{k}{2b}\right)^{k/2} \frac{\mathscr{E}}{a^{k/2}\left[1 - \exp\left(-T\lambda_1^{\beta}(1 + a\lambda_1)^{-1}\right)\right]^{k/2}}. \tag{55}$$

The proof of Theorem 7 is completed.

4.2. A Posteriori Parameter Choice Rule. In order to obtain a posteriori convergence error estimate, we apply Morozov's discrepancy principle, which is introduced in [18]. Furthermore, we learn the analysis techniques from previous papers [19–21].

Let us assume that $\sigma > 1$ is a fixed constant. By a similar claim in [19–21], we provide that the general a posteriori rule in the following:

$$\|Yf^d_{c(\delta),\delta} - g_\delta\|_{L^2(\Omega)} \le \sigma\delta. \tag{56}$$

From here on, in this subsection, we need to assume further to further assume that $c(\delta)$ is a natural number that greater than 1. If $\|g_\delta\|_{L^2(\Omega)} \ge \sigma\delta$, then the equation (56) exists in a unique solution.

Lemma 8. *Set $\mathscr{R}(c(\delta)) = \|Yf^{\delta}_{c(\delta),\delta} - g_\delta\|_{L^2(\Omega)}$ where $0 < \delta < \| g_\delta\|_{L^2(\Omega)}$. Then, we declare that*

(a) $\mathscr{R}(c(\delta))$ is a continuous function

(b) $\mathscr{R}(c(\delta)) \longrightarrow 0$ as $c(\delta) \longrightarrow +\infty$

(c) $\mathscr{R}(c(\delta)) \longrightarrow \|g_\delta\|_{L^2(\Omega)}$ as $c(\delta) \longrightarrow 0$

(d) $\mathscr{R}(c(\delta))$ is a strictly increasing function for $c(\delta) \in (0 ,+\infty)$

Proof. The proof of Lemma 8 is simple and completely similar to that in [19–21]. Hence, we omit it here. □

Lemma 9. *Let us assume that (56) holds. Then, $c(\delta)$ satisfies*

$$c(\delta) \le \left(\frac{2\mathscr{K}^2_\beta(\psi_1, a, T, \lambda_1, \psi_0)}{\sigma^2 - 2}\right)^{1/k+1}\left(\frac{k+1}{2b}\right)\mathscr{E}^{2/k+1}\delta^{-2/k+1}. \tag{57}$$

Proof. From the definition of $c(\delta), d \in (1/2, 1]$, $0 < b\,|(\lambda_1^{-1} + a)/\lambda_j^{\beta-1}|^2 < 1$, $\|f\|_{\mathscr{D}(\mathscr{A}^{k(\beta-1)})} \le \mathscr{E}$, we have

$$\|Yf^d_{c(\delta),\delta} - g_\delta\|^2_{L^2(\Omega)} = \|\sum_{j=1}^{\infty}\left[1 - \left[1 - \left(1 - b\left|(\lambda_1^{-1} + a)\lambda_j^{1-\beta}\right|^2\right)^{c(\delta)-1}\right]^d\right] \cdot \left(\int_\Omega g_\delta(x)e_j(x)dx\right)\|^2_{L^2(\Omega)}. \tag{58}$$

Using the inequality $(a+b)^2 \le 2(a^2 + b^2)$, we derive that

$$\left\|Yf^d_{c(\delta)-1,\delta} - g_\delta\right\|^2_{L^2(\Omega)} \le \underbrace{2\left\|\sum_{j=1}^{\infty}\left[1 - \left[1 - \left(1 - b\left|(\lambda_1^{-1} + a)\lambda_j^{1-\beta}\right)\right|^2\right)^{c(\delta)-1}\right]^d\right]\left(\int_\Omega (g_\delta(x) - g(x))e_j(x)dx\right)\right\|^2_{L^2(\Omega)}}_{\mathscr{O}_1} + \underbrace{2\left\|\sum_{j=1}^{\infty}\left[1 - \left[1 - \left(1 - b\left|(\lambda_1^{-1} + a)\lambda_j^{1-\beta}\right|^2\right)^{c(\delta)-1}\right]^d\right]\left(\int_\Omega g(x)e_j(x)dx\right)\right\|^2_{L^2(\Omega)}}_{\mathscr{O}_2}. \tag{59}$$

(Step 1) Due to $[1 - [1 - (1 - b|(\lambda_1^{-1} + a)\lambda_j^{1-\beta}|^2)^{c(\delta)-1}]^d] \le 1$ and $\|g_\delta - g\|_{L^2(\Omega)} \le \delta$, from (59), the estimate of $\mathscr{O}_1$ is as follows:

$$\mathcal{O}_1 \le 2\delta^2. \tag{60}$$

(Step 2) $\mathcal{O}_2$ can be bounded as follows:

$$\begin{aligned}\mathcal{O}_2 &\le 2\| \sum_{j=1}^{\infty} \left(1 - b\left|\left(\lambda_1^{-1} + a\right)\lambda_j^{1-\beta}\right|^2\right)^{c(\delta)-1} \left|\int_0^T \mathcal{C}_\beta(a,s,\lambda_j)\psi(s)ds\right| \\ &\cdot \left(\int_\Omega f(x)e_j(x)dx\right)\|^2_{L^2(\Omega)} \le 2\| \sum_{j=1}^{\infty} \left(1 - b\left|\left(\lambda_1^{-1} + a\right)\lambda_j^{1-\beta}\right|^2\right)^{c(\delta)-1} \\ &\cdot \left|\int_0^T \mathcal{C}_\beta(a,s,\lambda_j)\psi(s)ds\right|^{k+1} \frac{\left(\int_\Omega f(x)e_j(x)dx\right)}{\left|\int_0^T \mathcal{C}_\beta(a,s,\lambda_j)\psi(s)ds\right|^k} \|^2_{L^2(\Omega)} \\ &\le 2\| \sum_{j=1}^{\infty} \left(1 - b\left|\left(\lambda_1^{-1} + a\right)\lambda_j^{1-\beta}\right|^2\right)^{c(\delta)-1} \left|\left(\lambda_1^{-1} + a\right)\lambda_j^{1-\beta}\right|^{k+1} \\ &\cdot \mathcal{K}_\beta(\psi_1, a, T, \lambda_1, \psi_0)\mathcal{E}\|^2_{L^2(\Omega)},\end{aligned} \tag{61}$$

where

$$\mathcal{K}_\beta(\psi_1, a, T, \lambda_1, \psi_0) = |\psi_1|^{k+1} \left|a\left[1 - \exp\left(-T\lambda_1^\beta(1 + a\lambda_1)^{-1}\right)\right]\psi_0\right|^{-k}. \tag{62}$$

Thanks for the two articles [24, 25], we get the following inequality:

$$(1-\omega)^s \omega^r \le r^r (s+1)^{-r}, \tag{63}$$

for $0 < \omega < 1, r > 0,$ and $s \in \mathbb{N}$. Combining (58) and (63), we deduce that

$$\sigma^2\delta^2 \le 2\delta^2 + 2\left(\frac{k+1}{2b}\right)^{k+1} \left(\frac{1}{c(\delta)}\right)^{k+1} \mathcal{K}^2_\beta(\psi_1, a, T, \lambda_1, \psi_0)\mathcal{E}^2. \tag{64}$$

This implies that

$$c(\delta) \le \left(\frac{2\mathcal{K}^2_\beta(\psi_1, a, T, \lambda_1, \psi_0)}{\sigma^2 - 2}\right)^{1/k+1} \left(\frac{k+1}{2b}\right) \mathcal{E}^{2/k+1}\delta^{-2/k+1}. \tag{65}$$

□

Theorem 10. *Let $f^d_{c(\delta),\delta}$ be the regularized solution which is defined in (33). Suppose that condition (3) is satisfied, and the parameter regularization is chosen by (56). Then we get the following estimate:*

$$\|f^d_{c(\delta),\delta} - f\|_{L^2(\Omega)} \text{ is of order } \delta^{k/k+1}. \tag{66}$$

Proof. By the triangle inequality, we receive

$$\|f^d_{c(\delta),\delta} - f\|_{L^2(\Omega)} \le \|f^d_{c(\delta),\delta} - f^d_{c(\delta)}\|_{L^2(\Omega)} + \|f^d_{c(\delta)} - f\|_{L^2(\Omega)}. \tag{67}$$

Firstly, we have

$$\begin{aligned}\|f^d_{c(\delta),\delta} - f^d_{c(\delta)}\|_{L^2(\Omega)} &\le [c(\delta)]^{1/2} b^{1/2} 4\delta \\ &\cdot \left[\psi_0\left(1 - \exp\left(-T\lambda_1(1 + a\lambda_1)^{-1}\right)\right)\right]^{-1} \\ &+ \frac{4\delta}{\psi_0}\|f\|_{L^2(\Omega)}.\end{aligned} \tag{68}$$

Secondly, we find that

$$\begin{aligned}\left\|f^d_{c(\delta)} - f\right\|_{L^2(\Omega)} &= \left\| \sum_{j=1}^{\infty} \left[1 - \left[1 - \left(1 - b\left|\left(\lambda_1^{-1} + a\right)\lambda_j^{1-\beta}\right|^2\right)^{c(\delta)}\right]^d\right] \right. \\ &\left. \cdot \frac{\left(\int_\Omega g(x)e_j(x)dx\right)e_j(\cdot)}{\left|\int_0^T \mathcal{C}_\beta(a,s,\lambda_j)\psi(s)ds\right|} \right\|_{L^2(\Omega)}.\end{aligned} \tag{69}$$

In view of Hölder inequality, we follow from (69) that

$$\left\|f^d_{c(\delta)} - f\right\|_{L^2(\Omega)} \le \mathcal{V}_1\mathcal{V}_2, \tag{70}$$

where

$$\begin{aligned}\mathcal{V}_1 &= \left\| \sum_{j=1}^{\infty} \left(1 - b\left|\left(\lambda_1^{-1} + a\right)\lambda_j^{1-\beta}\right|^2\right)^{\kappa(\delta)} \left(\int_\Omega f(x)e_j(x)dx\right)e_j(\cdot) \right\|^{1/k+1}, \\ \mathcal{V}_2 &= \left\| \sum_{j=1}^{\infty} \left[1 - \left[1 - \left(1 - b\left|\left(\lambda_1^{-1} + a\right)\lambda_j^{1-\beta}\right|^2\right)^{c(\delta)}\right]^d\right] \frac{\left(\int_\Omega g(x)e_j(x)dx\right)e_j(\cdot)}{\left|\int_0^T \mathcal{C}_\beta(a,s,\lambda_j)\psi(s)ds\right|} \right\|^{k/k+1}.\end{aligned} \tag{71}$$

□

To continue the proof, we divide it into two steps.

(Step 1) In the estimate of $\mathcal{V}_1$, we have

$$\begin{aligned}\mathcal{V}_1 &\le \| \sum_{j=1}^{\infty} \left(1 - b\left|\left(\lambda_1^{-1} + b\right)\lambda_j^{1-\beta}\right|^2\right)^{c(\delta)} \\ &\cdot \lambda_j^{k(\beta-1)} \left(\int_\Omega f(x)e_j(x)dx\right)e_j(\cdot)\|^{1/k+1}_{L^2(\Omega)}.\end{aligned} \tag{72}$$

(Step 2) In the estimate of $\mathcal{V}_2$, we have

$$\mathscr{V}_2 \leq \left(\left\| \sum_{j=1}^{\infty} \left[1 - \left[1 - \left(1 - b \left| \left(\lambda_1^{-1} + a\right) \lambda_j^{1-\beta} \right|^2 \right)^{c(\delta)} \right]^d \right] \cdot \frac{\left(\int_\Omega (g(x) - g_\delta(x)) e_j(x) dx\right) e_j(\cdot)}{\left| \int_0^T \mathscr{C}_\beta(a, s, \lambda_j) \psi(s) ds \right|} \right\|_{L^2(\Omega)} + \left\| \sum_{j=1}^{\infty} \left[1 - \left[1 - \left(1 - b \left| \left(\lambda_1^{-1} + a\right) \lambda_j^{1-\beta} \right|^2 \right)^{c(\delta)} \right]^d \right] \cdot \frac{\left(\int_\Omega g_\delta(x) e_j(x) dx\right) e_j(\cdot)}{\left| \int_0^T \mathscr{C}_\beta(a, s, \lambda_j) \psi(s) ds \right|} \right\|_{L^2(\Omega)} \right)^{k/k+1}. \quad (73)$$

From (73), we have

$$\begin{aligned} \mathscr{V}_2 &\leq \left(\left\| \sum_{j=1}^{\infty} \left(1 - b \left| \left(\lambda_1^{-1} + a\right) \lambda_j^{1-\beta} \right|^2 \right)^{c(\delta)} \right] \frac{\left(\int_\Omega (g(x) - g_\delta(x)) e_j(x) dx\right) e_j(\cdot)}{\left| \int_0^T \mathscr{C}_\beta(a, s, \lambda_j) \psi(s) ds \right|} \right\|_{L^2(\Omega)} \right. \\ &\quad \left. + \left\| \sum_{j=1}^{\infty} \left[1 - \left[1 - \left(1 - b \left| \left(\lambda_1^{-1} + a\right) \lambda_j^{1-\beta} \right|^2 \right)^{c(\delta)} \right]^d \right] \frac{\left(\int_\Omega g_\delta(x) e_j(x) dx\right) e_j(\cdot)}{\left| \int_0^T \mathscr{C}_\beta(a, s, \lambda_j) \psi(s) ds \right|} \right\|_{L^2(\Omega)} \right)^{k/k+1} \\ &\leq \delta^{k/k+1} (1+\sigma)^{k/k+1} \sup_{\lambda_j > 1} \left[\frac{\lambda_j^{\beta-1}}{\left| \psi_0 \left(\lambda_j^{-1} + a\right) \left[1 - \exp\left(-T\lambda_1 (1 + a\lambda_1)^{-1}\right)\right] \right|} \right]^{k/k+1}. \end{aligned} \quad (74)$$

Substituting (73) into (69), it gives

$$\left\| f_{c(\delta)}^d - f \right\|_{L^2(\Omega)} \leq \delta^{k/k+1} \mathscr{E}^{1/k+1} (1+\sigma)^{k/k+1} \left[\frac{1}{\psi_0 a \left[1 - \exp\left(-T\lambda_1 (1 + a\lambda_1)^{-1}\right)\right]} \right]^{k/k+1}. \quad (75)$$

Substituting (65) into (68) and combining estimate (75), we conclude that

$$\left\| f_{c(\delta),\delta}^d - f \right\|_{L^2(\Omega)} \leq \delta^{k/k+1} \mathscr{E}^{1/k+1} (\mathscr{R}_1 + \mathscr{R}_2), \quad (76)$$

where

$$\begin{aligned} \mathscr{R}_1 &= \frac{\left(2\mathscr{K}_\beta^2(\psi_1, a, T, \lambda_1, \psi_0)\right)^{k/2(k+1)} (k + 1/2b)^{1/2} b^{1/2} 4}{(\sigma^2 - 2)^{-1/2(k+1)} \left[\psi_0 \left(1 - \exp\left(-T\lambda_1 (1 + a\lambda_1)^{-1}\right)\right)\right]} \\ &\quad + 4\delta^{k/k+1} \psi_0^{-1} \mathscr{C}_1^{-k/k+2} \|g\|^{k/k+2}, \\ \mathscr{R}_2 &= (1+\sigma)^{k/k+1} \left[\frac{1}{\psi_0 a \left[1 - \exp\left(-T\lambda_1 (1 + a\lambda_1)^{-1}\right)\right]} \right]^{k/k+1}. \end{aligned} \quad (77)$$

Theorem 10 is proven.

5. Convergence of the Source Function when $\beta \longrightarrow 1$

In this section, we will first prove the convergence of the source function when $\beta \longrightarrow 1$.

Theorem 11. *Let the Cauchy data $g \in D(\mathscr{A}^{r+2\beta}(\Omega))$ for any $r \geq 0$. Let the function $\psi \in L^\infty(0, T)$. Then, we have the following estimate:*

$$\left\| f^{(\beta)}(x) - f^{(1)}(x) \right\|_{D(\mathscr{A}^r(\Omega))} \lesssim (\beta - 1)^{2-(\beta-1)\varepsilon} \|\psi\|_{L^\infty(0,T)} \|g\|_{D\left(\mathscr{A}^{r+2\beta}(\Omega)\right)}, \quad (78)$$

where $\varepsilon > 0$ satisfies that $2 - (\beta - 1)\varepsilon > 0$.

Proof. Using the inequality $|e^{-m} - e^{-n}| \leq C_\varepsilon |m - n|^\varepsilon$, we find that for $z > 0$

$$\begin{aligned} &\left| \exp\left(-h\lambda_j^\beta (1 + a\lambda_j)^{-1}\right) - \exp\left(-h\lambda_j (1 + a\lambda_j)^{-1}\right) \right| \\ &\leq C_\varepsilon h^\varepsilon \left| \frac{\lambda_j^\beta - \lambda_j}{1 + a\lambda_j} \right|^\varepsilon \leq \frac{C_\varepsilon}{a} h^\varepsilon \lambda_j^{-\varepsilon} \left| \lambda_j^\beta - \lambda_j \right|^\varepsilon. \end{aligned} \quad (79)$$

Let us recall Lemma 12 which is proved in [27]. □

Lemma 12. *Assume that $0 \leq a \leq b$ and $0 < z$. For any $\varepsilon > 0$, there always exists $\bar{C}_\varepsilon > 0$ such that*

(a) If $z < 1$ then

$$\left| z^a - z^b \right| \leq \bar{C}_\varepsilon (b - a)^\varepsilon z^{a-\varepsilon} \quad (80)$$

(b) If $z \geq 1$ then

$$\left| z^a - z^b \right| \leq \bar{C}_\varepsilon (b - a)^\varepsilon z^{b+\varepsilon} \quad (81)$$

Let us divide the set of natural number into two sets in the following:

$$\mathbb{N} = \mathbb{N}_1 \cup \mathbb{N}_2, \quad (82)$$

where

$$\begin{aligned} \mathbb{N}_1 &= \left\{ j \in \mathbb{R}, \lambda_j \leq 1 \right\}, \\ \mathbb{N}_2 &= \left\{ j \in \mathbb{R}, \lambda_j > 1 \right\}. \end{aligned} \quad (83)$$

Let us assume that $j \in \mathbb{N}_1$. Let us recall that the assumption $\beta \geq 1$. By applying Lemma 12, we know that since $\lambda_j \leq 1$ then for any $\theta > 0$

$$\left| \lambda_j^\beta - \lambda_j \right| \leq \bar{C}_{1,\theta} \lambda_j^{1-\theta} (\beta - 1)^\theta. \quad (84)$$

It follows from (79) that

$$\left|\exp\left(-h\lambda_j^{\beta}\left(1+a\lambda_j\right)^{-1}\right)-\exp\left(-h\lambda_j\left(1+a\lambda_j\right)^{-1}\right)\right| \leq \frac{C_\varepsilon \bar{C}_{1,\theta}^{\varepsilon}}{a} h^{\varepsilon}\lambda_j^{-\varepsilon}\lambda_j^{(1-\theta)\varepsilon}(\beta-1)^{\theta\varepsilon} \leq \frac{C_\varepsilon \bar{C}_{1,\theta}^{\varepsilon}}{a} h^{\varepsilon}\lambda_j^{-\theta\varepsilon}(\beta-1)^{\theta\varepsilon}. \quad (85)$$

Let us assume that $j \in \mathbb{N}_2$. By applying Lemma 12, we know that since $\lambda_j > 1$, then

$$\left|\lambda_j^{\beta}-\lambda_j\right| \leq \bar{C}_{2,\theta}\lambda_j^{\beta+\theta}(\beta-1)^{\theta}. \quad (86)$$

It follows from (79) that

$$\left|\exp\left(-h\lambda_j^{\beta}\left(1+a\lambda_j\right)^{-1}\right)-\exp\left(-h\lambda_j\left(1+a\lambda_j\right)^{-1}\right)\right| \leq \frac{C_\varepsilon \bar{C}_{1,\theta}^{\varepsilon}}{a} h^{\varepsilon}\lambda_j^{-\varepsilon}\lambda_j^{(\beta+\theta)\varepsilon}(\beta-1)^{\theta\varepsilon} \leq \frac{C_\varepsilon \bar{C}_{1,\theta}^{\varepsilon}}{a} h^{\varepsilon}\lambda_j^{-\varepsilon+\beta\varepsilon+\theta\varepsilon}(\beta-1)^{\theta\varepsilon}. \quad (87)$$

Let us review that

$$f^{(\beta)}(x)=\sum_{j=1}^{\infty}\left[\frac{\int_\Omega g(x)e_j(x)dx}{\int_0^T \exp\left(-(T-s)\lambda_j^{\beta}\left(1+a\lambda_j\right)^{-1}\right)\psi(s)ds}\right]e_j(x). \quad (88)$$

By in view of Parseval's equality, we find that

$$f^{(1)}(x)=\sum_{j=1}^{\infty}\left[\frac{\int_\Omega g(x)e_j(x)dx}{\int_0^T \exp\left(-(T-s)\lambda_j\left(1+a\lambda_j\right)^{-1}\right)\psi(s)ds}\right]e_j(x). \quad (89)$$

Since two above observations, we derive that

$$\left\|f^{(\beta)}(x)-f^{(1)}(x)\right\|_{D(\mathscr{A}^r(\Omega))}^2 = \sum_{j=1}^{\infty}\left[\frac{\int_\Omega g(x)e_j(x)dx}{\int_0^T \exp\left(-(T-s)\lambda_j^{\beta}\left(1+a\lambda_j\right)^{-1}\right)\psi(s)ds} - \frac{\int_\Omega g(x)e_j(x)dx}{\int_0^T \exp\left(-(T-s)\lambda_j\left(1+a\lambda_j\right)^{-1}\right)\psi(s)ds}\right]^2\lambda_j^{2r}. \quad (90)$$

First, if $\lambda_j \leq 1$, then using the estimate (85), we get that

$$\begin{aligned}&\left|\int_0^T \exp\left(-(T-s)\lambda_j^{\beta}\left(1+a\lambda_j\right)^{-1}\right)\psi(s)\cdot ds-\int_0^T \exp\left(-(T-s)\lambda_j\left(1+a\lambda_j\right)^{-1}\right)\psi(s)ds\right| \\ &\leq \frac{C_\varepsilon \bar{C}_{1,\theta}^{\varepsilon}}{a}\lambda_j^{-\theta\varepsilon}(\beta-1)^{\theta\varepsilon}\left(\int_0^T (T-s)^{\varepsilon}ds\right)\|\psi\|_{L^\infty(0,T)} \\ &\leq M_1\|\psi\|_{L^\infty(0,T)}\lambda_j^{-\theta\varepsilon}(\beta-1)^{\theta\varepsilon},\end{aligned} \quad (91)$$

where $M_1=(C_\varepsilon \bar{C}_{1,\theta}^{\varepsilon}/a)(T^{1+\varepsilon}/1+\varepsilon)$. By a similar explanation, if $\lambda_j > 1$, then using the estimate (87), we get that

$$\begin{aligned}&\left|\int_0^T \exp\left(-(T-s)\lambda_j^{\beta}\left(1+a\lambda_j\right)^{-1}\right)\psi(s)\cdot ds-\int_0^T \exp\left(-(T-s)\lambda_j\left(1+a\lambda_j\right)^{-1}\right)\psi(s)ds\right| \\ &\leq \frac{C_\varepsilon \bar{C}_{1,\theta}^{\varepsilon}}{a}\lambda_j^{-\varepsilon+\beta\varepsilon+\theta\varepsilon}(\beta-1)^{\theta\varepsilon}\left(\int_0^T (T-s)^{\varepsilon}ds\right)\|\psi\|_{L^\infty(0,T)} \\ &\leq M_1\|\psi\|_{L^\infty(0,T)}\lambda_j^{-\varepsilon+\beta\varepsilon+\theta\varepsilon}(\beta-1)^{\theta\varepsilon}.\end{aligned} \quad (92)$$

By using Lemma 1, we find that

$$\begin{aligned}&\left[\int_0^T \exp\left(-(T-s)\lambda_j^{\beta}\left(1+a\lambda_j\right)^{-1}\right)\psi(s)ds\right]\cdot\left[\int_0^T \exp\left(-(T-s)\lambda_j\left(1+a\lambda_j\right)^{-1}\right)\psi(s)ds\right] \\ &\geq\left[1-\exp\left(-T\lambda_1^{\beta}(1+a\lambda_1)^{-1}\right)\right]^2|\psi_0|^2\left(\frac{1+a\lambda_j}{\lambda_j^{\beta}}\right)^2 \\ &= M_2^2\left(\frac{1+a\lambda_j}{\lambda_j^{\beta}}\right)^2 \geq M_2^2a^2\lambda_j^{2-2\beta},\end{aligned} \quad (93)$$

where we denote

$$M_2=\left[1-\exp\left(-T\lambda_1^{\beta}(1+a\lambda_1)^{-1}\right)\right]|\psi_0|. \quad (94)$$

From some of the above observations, we get that

$$\|f^{(\beta)}(x)-f^{(1)}(x)\|_{D(\mathscr{A}^r(\Omega))}^2 = \sum_{\lambda_j\leq 1}\left[\frac{\int_\Omega g(x)e_j(x)dx}{\int_0^T \exp\left(-(T-s)\lambda_j^{\beta}\left(1+a\lambda_j\right)^{-1}\right)\psi(s)ds} - \frac{\int_\Omega g(x)e_j(x)dx}{\int_0^T \exp\left(-(T-s)\lambda_j\left(1+a\lambda_j\right)^{-1}\right)\psi(s)ds}\right]^2\lambda_j^{2r}$$

$$+\sum_{\lambda_j>1}\left[\frac{\int_\Omega g(x)e_j(x)dx}{\int_0^T \exp\left(-(T-s)\lambda_j^\beta\left(1+a\lambda_j\right)^{-1}\right)\psi(s)ds} - \frac{\int_\Omega g(x)e_j(x)dx}{\int_0^T \exp\left(-(T-s)\lambda_j\left(1+a\lambda_j\right)^{-1}\right)\psi(s)ds}\right]^2\lambda_j^{2r}$$
$$\le\left(\frac{M_1}{M_2^2}\right)^2(\beta-1)^{2\theta\varepsilon}\|\psi\|^2_{L^\infty(0,T)}\sum_{\lambda_j\le 1}\lambda_j^{2\theta\varepsilon-4+4\beta+2r} \cdot\left(\int_\Omega g(x)e_j(x)dx\right)^2+\left(\frac{M_1}{M_2^2}\right)^2(\beta-1)^{2\theta\varepsilon}\|\psi\|^2_{L^\infty(0,T)} \cdot\sum_{\lambda_j>1}\lambda_j^{2\theta\varepsilon-4+4\beta+2r-2\varepsilon+2\beta\varepsilon}\left(\int_\Omega g(x)e_j(x)dx\right)^2. \quad (95)$$

Noting that $\beta\ge 1$, we have that

$$1\le\left(\frac{\lambda_j}{\lambda_1}\right)^{2\beta\varepsilon-2\varepsilon}. \quad (96)$$

This implies the following estimate:

$$\|f^{(\beta)}(x)-f^{(1)}(x)\|^2_{D(\mathscr{A}^r(\Omega))}\lesssim(\beta-1)^{2\theta\varepsilon}\|\psi\|^2_{L^\infty(0,T)} \cdot\sum_{j=1}^\infty\lambda_j^{2\theta\varepsilon-4+4\beta+2r-2\varepsilon+2\beta\varepsilon}\left(\int_\Omega g(x)e_j(x)dx\right)^2. \quad (97)$$

Let us choose θ and ε such that $2\theta\varepsilon+2\beta\varepsilon=4+2\varepsilon$. In order to choose such number θ and ε, we need to choose $\varepsilon>0$ if $\beta=1$ and such that

$$0<\varepsilon<\frac{2}{\beta-1},\quad \beta>1. \quad (98)$$

Let θ be such that $\theta=(2/\varepsilon)+1-\beta$. Then since (()(97)), we deduce that

$$\|f^{(\beta)}(x)-f^{(1)}(x)\|_{D(\mathscr{A}^r(\Omega))}\lesssim(\beta-1)^{\theta\varepsilon}\|\psi\|_{L^\infty(0,T)}\|g\|_{D\left(\mathscr{A}^{r+2\beta}(\Omega)\right)} =(\beta-1)^{2-(\beta-1)\varepsilon}\|\psi\|_{L^\infty(0,T)}\|g\|_{D\left(\mathscr{A}^{r+2\beta}(\Omega)\right)}. \quad (99)$$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

The authors contributed equally to this the work. The authors read and approved the final manuscript.

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References

[1] L. Jin, L. Li, and S. Fang, "The global existence and time-decay for the solutions of the fractional pseudo-parabolic equation," *Computers and Mathematics with Applications*, vol. 73, no. 10, pp. 2221–2232, 2017.

[2] G. I. Barenblatt, I. P. Zheltov, and I. Kochina, "Basic concepts in the theory of seepage of homogeneous liquids in fissured rocks [strata]," *Journal of Applied Mathematics and Mechanics*, vol. 24, no. 5, pp. 1286–1303, 1960.

[3] T. B. Benjamin, J. L. Bona, and J. J. Mahony, ""Model equations for long waves in nonlinear dispersive systems", Philosophical transactions of the Royal Society of London," *Series A, Mathematical and Physical Sciences*, vol. 272, no. 1220, pp. 47–78, 1972.

[4] R. Wang, Y. Li, and B. Wang, "Random dynamics of fractional nonclassical diffusion equations driven by colored noise," *Discrete & Continuous Dynamical Systems*, vol. 39, no. 7, pp. 4091–4126, 2019.

[5] R. Wang, L. Shi, and B. Wang, "Asymptotic behavior of fractional nonclassical diffusion equations driven by nonlinear colored noise on R^N," *Nonlinearity*, vol. 32, no. 11, pp. 4524–4556, 2019.

[6] R. Wang, Y. Li, and B. Wang, "Bi-spatial pullback attractors of fractional nonclassical diffusion equations on unbounded domains with (p, q)-growth nonlinearities," *Applied Mathematics and Optimization*, vol. 84, no. 1, pp. 425–461, 2021.

[7] N. H. Tuan and T. Caraballo, "On initial and terminal value problems for fractional nonclassical diffusion equations," *Proceedings of the American Mathematical Society*, vol. 149, no. 1, pp. 143–161, 2021.

[8] N. H. Tuan, V. V. Au, and R. Xu, "Semilinear Caputo time-fractional pseudo-parabolic equations," *Communications on Pure & Applied Analysis*, vol. 20, no. 2, pp. 583–621, 2021.

[9] M. Warma, H. Antil, and D. Verma, "Optimal control of fractional elliptic PDEs with state constraints and characterization of the dual of fractional-order Sobolev spaces," *Journal of Optimization Theory and Applications*, vol. 186, no. 1, pp. 1–23, 2020.

[10] H. Antil and M. Warma, "Optimal control of fractional semilinear PDEs," *ESAIM*, vol. 26, p. 130, 2020.

[11] H. Antil, D. Verma, and M. Warma, "External optimal control of fractional parabolic PDEs," *ESAIM*, vol. 26, p. 20, 2020.

[12] M. Warma, H. Antil, and R. Khatri, "External optimal control of nonlocal PDEs," *Inverse Problems*, vol. 35, no. 8, p. 084003, 2019.

[13] M. Warma and H. Antil, "Optimal control of the coefficient for fractional and regional fractional p-Laplace equations: approximation and convergence," *Mathematical Control and Related Fields*, vol. 9, pp. 1–38, 2019.

[14] N. H. Luc, L. D. Long, L. T. D. Hang, D. Baleanu, and N. H. Can, "Identifying the initial condition for space-fractional Sobolev equation," *Journal of Applied Analysis & Computation*, vol. 11, no. 5, pp. 2402–2422, 2021.

[15] V. V. Au, J. Hossein, Z. Hammouch, and N. H. Tuan, "On a final value problem for a nonlinear fractional pseudo-

parabolic equation," *Electronic Research Archive*, vol. 29, no. 1, pp. 1709–1734, 2021.

[16] D. H. Q. Nam, D. O'Regan, L. D. Long, T. B. Ngoc, and N. H. Tuan, "Identification of the righthand side in a bi-parabolic equation with final data," *Applicable Analysis*, vol. 101, 2020.

[17] Y. K. Ma, P. Prakash, and A. Deiveegan, "Generalized Tikhonov methods for an inverse source problem of the time- fractional diffusion equation," *Chaos, Solitons and Fractals*, vol. 108, pp. 39–48, 2018.

[18] H. W. Engl, M. Hanke, and A. Neubauer, *Regularization of Inverse Problems*, Kluwer Academic, Boston, 1996.

[19] Y. Han, X. Xiong, and X. Xue, "A fractional Landweber method for solving backward time-fractional diffusion problem," *Computers & Mathematcs with Applications*, vol. 78, no. 1, pp. 81–91, 2019.

[20] Z. S. Jiang and J. Y. Wu, "Recovering space-dependent source for a time-space fractional diffusion wave equation by fractional Landweber method," *Inverse Problems in Science and Engineering*, vol. 29, no. 7, pp. 990–1011, 2021.

[21] F. Yang, L. J. Fu, P. Fan, and X. X. Li, "Fractional Landweber iterative regularization method for identifying the unknown source of the time-fractional diffusion problem," *Acta Applicandae Mathematicae*, vol. 175, no. 1, 2021.

[22] T. T. Binh, H. K. Nashine, D. L. Le, H. L. Nguyen, and N. Can, "Identification of source term for the ill-posed Rayleigh-Stokes problem by Tikhonov regularization method," *Advances in Difference Equations*, vol. 2019, no. 1, 2019.

[23] N. H. Tuan, L. D. Long, and N. V. Thinh, "Regularized solution of an inverse source problem for a time fractional diffusion equation," *Applied Mathematical Modelling*, vol. 40, no. 19-20, pp. 8244–8264, 2016.

[24] A. B. Bakushinsky, M. Y. Kokurin, and A. Smirnova, *Iterative Methods for Ill-Posed Problems*, Inverse and Ill-Posed Problems Series, 2011.

[25] A. K. Louis, *Inverse und Schlecht Gestellte Probleme*, Springer-Verlag, 2013.

[26] N. H. Can, N. H. Luc, D. Baleanu, and Y. Zhou, "Inverse source problem for time fractional diffusion equation with Mittag-Leffler kernel," *Advances in Difference Equations*, vol. 1, 18 pages, 2020.

[27] N. H. Tuan, D. O'regan, and T. B. Ngoc, "Continuity with respect to fractional order of the time fractional diffusion-wave equation," *Evolution Equations & Control Theory*, vol. 9, no. 3, pp. 773–793, 2020.