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Research Article

Theory of Fractional Hybrid Problems in the Frame of ψ-Hilfer Fractional Operators

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In the present manuscript, we develop and extend a qualitative analysis for two classes of boundary value problems for nonlinear hybrid fractional differential equations with hybrid boundary conditions involving a ψ-Hilfer fractional order derivative introduced by Sousa and de Oliveira (2018). First, we derive the equivalent fractional integral equations to the proposed problems from some properties of the ψ-fractional calculus. Next, we establish the existence theorems in the weighted spaces via equivalent fractional integral equations with the help of Dhage's fixed-point theorem (2004). Besides, for an adequate choice of the kernel ψ, we recover most of all the preceding results on fractional hybrid equations. Finally, two examples are constructed to make our main findings effective.

1. Introduction

Recently, a lot of keen interest in the topic of fractional calculus (FC) has been shown by many researchers and investigators in view of its theoretical development and extensive applications in the applied and natural sciences. Different types of differential and integral operators of arbitrary orders have been introduced by Kilbas et al. [1]. In the same regard, Atangana and Baleanu [2] proposed a new fractional derivative (FD) based on a nonsingular and nonlocal kernel. On the advanced improvement of the FC without a singular kernel of the sinc function, the Yang-Gao-Machado-Baleanu FD was introduced in [3]. Some properties of the FD without a singular kernel were introduced by Lozada and Nieto [4].

Hilfer in [5] proposed a generalization of the Riemann-Liouville fractional derivative (RLFD) and Caputo fractional derivative (CFD) when the author deliberated fractional time evolution in physical phenomena. The author named it a generalized FD, whereas more recently, it was named the Hilfer fractional derivative (HFD). This operator carries two parameters (α, β) that may be decreased to the RLFD and CFD definitions if $\beta = 0$ and $\beta = 1$, respectively. So, such a derivative incorporates between the RLFD and CFD. Some important laws and applications of this operator are obtained in [6, 7] and the references therein.

Initial value problems (IVPs) involving the HFD were investigated by many authors, like Furati et al. [8], Gu and Trujillo [9], and Wang and Zhang [10]. Some existence and uniqueness results of IVPs for ψ-Hilfer-type coupled hybrid equations were obtained in [11]. Boundary value problems (BVPs) with the nonlocal conditions and the HFD were studied in [12]. The authors in [13] investigated the existence and stability of the solution of BVPs for ψ-Hilfer-type fractional integrodifferential equations with boundary conditions (BCs).

ψ-Fractional derivatives (ψ-FDs) have been considered in [1] to be a generalization of RLFD. Some properties of these operators have been given by Agrawal in [14]. These operators are different from the other classical operators because the kernel is shown to be linked to another function ψ. For instance, Almeida [15] gave a new generalization of CFD with some interesting properties. In addition, the authors in [16] introduced a generalized type of the Laplace transform of generalized fractional operators in the frame of both RLFD and CFD.

The new version of the HFD with respect to another function ψ has been introduced by Sousa and de Oliveira [17]. Recently, the investigation of diverse qualitative properties of solutions to several fractional differential equations (FDEs) involving generalized FDs has become the key theme of applied mathematics research. Many interesting results concerning the existence and stability of solutions by using various kinds of fixed-point techniques were formulated; e.g., Abdo et al. [18, 19] investigated various types of the Ulam-Hyers stability for ψ-Hilfer fractional problems with infinite delay and without infinite delay, respectively. The Ulam-Hyers-Rassias stability for FDEs using the ψ-Hilfer operator was discussed by Sousa and de Oliveira [20].

On the other hand, hybrid-type FDEs have attained a considerable saucepan of interest and investigations of several researchers. This category of hybrid FDEs comprises the perturbations of primitive differential equations in various manners. For instance, Dhage and Lakshmikantham [21] considered the following IVPs for hybrid DEs:

$$\begin{cases} \dfrac{d}{d\vartheta}\left(\dfrac{\varkappa(\vartheta)}{\mathbf{q}(\vartheta,\varkappa(\vartheta))}\right) = \mathbf{p}(\vartheta,\varkappa(\vartheta)), \text{a.e.}\vartheta \in \mho := [0,b], \\ \varkappa(\vartheta)|_{\vartheta=\vartheta_0} = \varkappa_0 \in \mathbb{R}, \end{cases} \tag{1}$$

where $\mathbf{p} \in C(\mho \times \mathbb{R}, \mathbb{R})$ and $\mathbf{q} \in C(\mho \times \mathbb{R}, \mathbb{R} \setminus \{0\})$. Zhao et al. [22] studied the following hybrid FDEs with RLFD:

$$\begin{cases} {}^{RL}\mathfrak{D}_{0^+}^{\theta_1}\left(\dfrac{\varkappa(\vartheta)}{\mathbf{q}(\vartheta,\varkappa(\vartheta))}\right) = \mathbf{p}(\vartheta,\varkappa(\vartheta)), \text{a.e.}\vartheta \in \mho, \\ \varkappa(\vartheta)|_{\vartheta=\vartheta_0} = \varkappa_0 \in \mathbb{R}, \end{cases} \tag{2}$$

where $\mathbf{p} \in C(\mho \times \mathbb{R}, \mathbb{R})$ and $\mathbf{q} \in C(\mho \times \mathbb{R}, \mathbb{R} \setminus \{0\})$.

On the other hand, Benchohra et al. [23] considered the following BVP for the Caputo-type FDE:

$$\begin{cases} {}^{C}\mathfrak{D}_{0^+}^{\theta_1}\varkappa(\vartheta) = \mathbf{p}(\vartheta,\varkappa(\vartheta)), \vartheta \in \mho, \\ c_1\varkappa(\vartheta)|_{\vartheta=0} + c_2\varkappa(\vartheta)|_{\vartheta=b} = d, \end{cases} \tag{3}$$

where $0 < \theta_1 < 1, c_1, c_2, d \in \mathbb{R}$ with $c_1 + c_2 \neq 0$, and $\mathbf{p} \in C(\mho \times \mathbb{R}, \mathbb{R})$.

Hilal and Kajouni [24] considered the BVP for Caputo-type hybrid FDEs with BC:

$$\begin{cases} {}^{C}\mathfrak{D}_{0^+}^{\theta_1}\left(\dfrac{\varkappa(\vartheta)}{\mathbf{q}(\vartheta,\varkappa(\vartheta))}\right) = \mathbf{p}(\vartheta,\varkappa(\vartheta)), \vartheta \in \mho, \\ c_1 \dfrac{\varkappa(\vartheta)}{\mathbf{q}(\vartheta,\varkappa(\vartheta))}\bigg|_{\vartheta=0} + c_2 \dfrac{\varkappa(\vartheta)}{\mathbf{q}(\vartheta,\varkappa(\vartheta))}\bigg|_{\vartheta=b} = d, \end{cases} \tag{4}$$

where $0 < \theta_1 < 1, c_1, c_2, d \in \mathbb{R}$ with $c_1 + c_2 \neq 0$, $\mathbf{p} \in C(\mho \times \mathbb{R}, \mathbb{R})$, and $\mathbf{q} \in C(\mho \times \mathbb{R}, \mathbb{R} \setminus \{0\})$.

However, as far as we could possibly know, no one considered the existence of solution for the hybrid-type BVPs involving HFD with respect to ψ. Here, we discuss the existence theorems of BVPs for Hilfer-type FDE with respect to ψ and BCs:

$$\begin{cases} \mathfrak{D}_{a^+}^{\theta_1,\theta_2;\psi}\varkappa(\vartheta) = \mathbf{p}(\vartheta,\varkappa(\vartheta)), \vartheta \in (a,b], \\ c_1\mathfrak{I}_{a^+}^{1-\theta;\psi}\varkappa(\vartheta)|_{\vartheta=a} + c_2\varkappa(\vartheta)|_{\vartheta=b} = d. \end{cases} \tag{5}$$

Also, we consider the Hilfer-type hybrid FDE with respect to ψ and hybrid BCs:

$$\begin{cases} \mathfrak{D}_{a^+}^{\theta_1,\theta_2;\psi}\left(\dfrac{\varkappa(\vartheta) - \mathbb{z}(\vartheta,\varkappa(\vartheta))}{\mathbf{q}(\vartheta,\varkappa(\vartheta))}\right) = \mathbf{p}(\vartheta,\varkappa(\vartheta)), \vartheta \in (a,b], \\ c_1\mathfrak{I}_{a^+}^{1-\theta;\psi}\left(\dfrac{\varkappa(\vartheta) - \mathbb{z}(\vartheta,\varkappa(\vartheta))}{\mathbf{q}(\vartheta,\varkappa(\vartheta))}\right)\bigg|_{\vartheta=a} + c_2\left(\dfrac{\varkappa(\vartheta) - \mathbb{z}(\vartheta,\varkappa(\vartheta))}{\mathbf{q}(\vartheta,\varkappa(\vartheta))}\right)\bigg|_{\vartheta=b} = d, \end{cases} \tag{6}$$

where $0 < \theta_1 < 1$ and $0 \le \theta_2 \le 1$, $\theta = \theta_1 + \theta_2(1-\theta_1), c_1, c_2, d \in \mathbb{R}$, $\mathfrak{D}_{a^+}^{\theta_1,\theta_2;\psi}$, and $\mathfrak{I}_{a^+}^{1-\theta;\psi}$ are the HFD and RLFD with respect to ψ, respectively, $\mathbf{p}, \mathbb{z} \in C(I \times \mathbb{R}, \mathbb{R})$, $\mathbf{q} \in C(I \times \mathbb{R}, \mathbb{R} \setminus \{0\})$, and $I = [a, b]$.

Observe that the considered problems (5) and (6) are the first investigations of hybrid FDEs with hybrid BCs involving ψ-HFD. Moreover, we have chosen this operator, besides the fact that it is a global operator and it generalizes more than twenty the freedom of choice of the ordinary differential operator; some of its advantages have been explained in Remark 3. So, we are sure that the obtained results will be a beneficial contribution and an extension of the current results in the literature. We will also refer here to some recent results related to the subject of our study (see [25–31]).

The rest of the work is displayed as follows. In Section 2, we give some advantageous preliminaries related to our work. In Section 3, we derive the equivalent solutions to linear problems corresponding to the proposed problems. Then, we prove the existence of solutions to given problems via Dhage's fixed-point theorem. Finally, two examples to justify reported results are offered in Section 4.

2. Preliminaries

Let us initially present some imperative definitions and primer ideas related to our work.

Let $\theta = \theta_1 + \theta_2(1-\theta_1)$ where, $0 < \theta_1 < 1$ and $0 \le \theta_2 \le 1$, and let $I = [a, b], I' = (a, b]$. Consider the functional spaces $C(I, \mathbb{R})$ and $C^{\psi}_{1-\theta}(I, \mathbb{R})$ as follows:

$$C(I, \mathbb{R}) = \{\varphi : I \longrightarrow \mathbb{R} : \varphi \text{ is continuous}\},$$
$$C^{\psi}_{1-\theta}(I, \mathbb{R}) = \left\{\varphi : I' \longrightarrow \mathbb{R} : (\psi(\vartheta) - \psi(a))^{1-\theta}\varphi(\vartheta) \in C(I, \mathbb{R})\right\}, \tag{7}$$

equipped with the norms

$$\|\varphi\|_C \coloneqq \max\{|\varphi(\vartheta)| : \vartheta \in I\},$$
$$\|\varphi\|_{C^{\psi}_{1-\theta}} = \left\|(\psi(\vartheta) - \psi(a))^{1-\theta}\varphi(\vartheta)\right\|_C. \tag{8}$$

Obviously, $(C^{\psi}_{1-\theta}(I, \mathbb{R}), \|\cdot\|_{C^{\psi}_{1-\theta}})$ is the Banach space.

In the next expressions, we will consider ψ to be an increasing function such that $\psi'(\vartheta) \ne 0$ for all $\vartheta \in I$.

Definition 1 (see [1]). Let $\theta_1 > 0$ $(\theta_1 \in \mathbb{R})$, $\varphi \in L^1(I, \mathbb{R})$, and $\psi \in C^1(I, \mathbb{R})$. Then, ψ-RL fractional integral of φ is defined by

$$\mathfrak{J}^{\theta_1;\psi}_{a^+}\varphi(\vartheta) = \frac{1}{\Gamma(\theta_1)}\int_a^{\vartheta}\psi'(s)(\psi(\vartheta) - \psi(s))^{\theta_1 - 1}\varphi(s)ds. \tag{9}$$

Definition 2 (see [17]). Let $n-1 < \theta_1 < n \in \mathbb{N}$, $0 \le \theta_2 \le 1$, and $\psi, \varphi \in C^n(I, \mathbb{R})$. Then, the ψ-HFD of φ is defined by

$$\mathfrak{D}^{\theta_1,\theta_2;\psi}_{a^+}\varphi(\vartheta) = \mathfrak{J}^{\theta_2(n-\theta_1);\psi}_{a^+}\left(\frac{1}{\psi'(\vartheta)}\frac{d}{d\vartheta}\right)^n \mathfrak{J}^{(1-\theta_2)(n-\theta_1);\psi}_{a^+}\varphi(\vartheta). \tag{10}$$

Specifically, if $0 < \theta_1 < 1$, we have

$$\mathfrak{D}^{\theta_1,\theta_2;\psi}_{a^+}\varphi(\vartheta) = \mathfrak{J}^{\theta_2(1-\theta_1);\psi}_{a^+}\left(\frac{1}{\psi'(\vartheta)}\frac{d}{d\vartheta}\right)\mathfrak{J}^{(1-\theta_2)(1-\theta_1);\psi}_{a^+}\varphi(\vartheta). \tag{11}$$

Remark 3. The operator $\mathfrak{D}^{\theta_1,\theta_2;\psi}_{a^+}$ is an interpolator of the following FDs:

(i) Classical HFD (for $\psi(\vartheta) \longrightarrow \vartheta$, see [5]), Hilfer-Hadamard FD (for $\psi(\vartheta) \longrightarrow \log \vartheta$, see [32]), and Hilfer-Katugampola FD (for $\psi(\vartheta) \longrightarrow \vartheta^{\rho}, \rho > 0$, see [33])

(ii) Standard RLFD (for $\psi(\vartheta) \longrightarrow \vartheta, \theta_2 \longrightarrow 0$, see [1]) and standard CFD (for $\psi(\vartheta) \longrightarrow \vartheta, \theta_2 \longrightarrow 1$, see [1])

(iii) Generalized RLFD (for $\theta_2 \longrightarrow 0$, see [1]) and generalized CFD (for $\theta_2 \longrightarrow 1$, see [15])

(iv) Generalized Liouville (for $\theta_2 \longrightarrow 0$, $a = 0$, see [1]) and generalized Weyl (for $\theta_2 \longrightarrow 0$, $a = -\infty$, see [34])

Lemma 4 (see [1, 17]). *Let $\theta_1, \theta_2 > 0, \eta > 0$, and $\mathscr{K}_{\psi}(\vartheta, a) \coloneqq [\psi(\vartheta) - \psi(a)]$. Then,*

$$\mathfrak{J}^{\theta_1;\psi}_{a^+}\left[\mathscr{K}^{\eta-1}_{\psi}(\vartheta, a)\right] = \frac{\Gamma(\eta)}{\Gamma(\eta+\theta_1)}\left[\mathscr{K}^{\theta_1+\eta-1}_{\psi}(\vartheta, a)\right],$$
$$\mathfrak{D}^{\theta_1,\theta_2;\psi}_{a^+}\left[\mathscr{K}^{\theta-1}_{\psi}(\vartheta, a)\right] = 0, \tag{12}$$

$$\mathfrak{J}^{\theta_1;\psi}_{a^+}\mathfrak{J}^{\theta_2;\psi}_{a^+}\varphi(\vartheta) = \mathfrak{J}^{\theta_1+\theta_2;\psi}_{a^+}\varphi(\vartheta), \textit{for } \varphi \in C(I, \mathbb{R}).$$

Lemma 5 (see [17]). *Let $0 < \theta_1 < 1$ and $0 \le \theta_2 \le 1$, where $\theta = \theta_1 + \theta_2(1-\theta_1)$. If $\mathfrak{J}^{1-\theta;\psi}_{a^+}\varphi(\vartheta) \in C(I, \mathbb{R})$, then*

$$\mathfrak{J}^{\theta_1;\psi}_{a^+}\mathfrak{D}^{\theta_1,\theta_2;\psi}_{a^+}\varphi(\vartheta) = \varphi(\vartheta) - \frac{\mathfrak{J}^{1-\theta;\psi}_{a^+}\varphi(a)}{\Gamma(\theta)}(\psi(\vartheta) - \psi(a))^{\theta-1},$$
$$\mathfrak{D}^{\theta_1,\theta_2;\psi}_{a^+}\mathfrak{J}^{\theta_1;\psi}_{a^+}\varphi(\vartheta) = \varphi(\vartheta). \tag{13}$$

Lemma 6 (see [17]). *Let $n-1 \le \theta < n$ and $\varphi \in C_{\theta}(I, \mathbb{R})$. Then, $\mathfrak{J}^{\theta_1;\psi}_{a^+}$ is bounded in $C_{\theta}(I, \mathbb{R})$. Moreover, we have*

$$\mathfrak{J}^{\theta_1;\psi}_{a^+}\varphi(a) = \lim_{\vartheta \longrightarrow a^+}\mathfrak{J}^{\theta_1;\psi}_{a^+}\varphi(\vartheta) = 0, n-1 \le \theta < \theta_1. \tag{14}$$

Theorem 7 [35]. *Let K be a nonempty, convex, closed subset of the Banach algebra $\mathscr{L}$. Let the operators $B_1, B_2 : \mathscr{L} \longrightarrow \mathscr{L}$, and $B_3 : K \longrightarrow \mathscr{L}$ such that (i) B_1 and B_2 are Lipschitzian, with Lipschitz constants κ_1 and κ_2, respectively; (ii) B_3 is continuous and compact; (iii) $\omega = B_1\varpi_1 B_3\varpi_2 + B_2\varpi_2 \in K \Rightarrow \varpi_2 \in K$ for each $\varpi_1 \in K$; and (iv) $\kappa_1 M + \kappa_2 < 1$, where $M = \|B_3(K)\|$. Then, there exists $w \in K$ such that $B_1 w B_3 w + B_2 w = w$.*

3. Main Results

In this section, we pay attention to deriving equivalent solutions to linear problems associated with problems (5) and (6); then, we prove the existence of solution to problems (5) and (6) using Dhage's fixed-point technique.

3.1. Fractional Integral Equations (FIEs). The forthcoming results give the equivalent of solution formulas for the proposed problems. For brevity, we set the following symbols:

$$\mathscr{K}^{\theta-1}_{\psi}(\vartheta, a) = (\psi(\vartheta) - \psi(a))^{\theta-1}, F^{\theta}_{\psi}(\vartheta, s) = \psi'(s)(\psi(\vartheta) - \psi(s))^{\theta-1}, \tag{15}$$

$$\frac{1}{\Gamma(\theta_1)}\int_a^{\vartheta} F_{\psi}^{\theta_1}(\vartheta,s)ds = \frac{1}{\Gamma(\theta_1)}\int_a^{\vartheta}\psi'(s)(\psi(\vartheta)-\psi(s))^{\theta_1-1}ds = \frac{(\psi(\vartheta)-\psi(a))^{\theta_1}}{\Gamma(\theta_1+1)} = \frac{\mathscr{K}_{\psi}^{\theta_1}(\vartheta,a)}{\Gamma(\theta_1+1)}, \tag{16}$$

$$\begin{aligned}\frac{1}{\Gamma(\theta_1)}&\int_a^{\vartheta} F_{\psi}^{\theta_1}(\vartheta,s)\mathscr{K}_{\psi}^{\theta-1}(s,a)ds \\ &= \frac{1}{\Gamma(\theta_1)}\int_a^{\vartheta}\psi'(s)(\psi(\vartheta)-\psi(s))^{\theta_1-1}(\psi(s)-\psi(a))^{\theta-1}ds \\ &= \mathfrak{I}_{a^+}^{\theta_1;\psi}[\psi(\vartheta)-\psi(a)]^{\theta-1} = \frac{\Gamma(\theta)}{\Gamma(\theta+\theta_1)}[\psi(\vartheta)-\psi(a)]^{\theta_1+\theta-1} \\ &= \frac{\Gamma(\theta)}{\Gamma(\theta+\theta_1)}\mathscr{K}_{\psi}^{\theta_1+\theta-1}(\vartheta,a).\end{aligned} \tag{17}$$

Lemma 8. *Let $0<\theta_1<1, 0\le\theta_2\le 1$, where $\theta=\theta_1+\theta_2-\theta_1\theta_2$, and $H: I' \longrightarrow \mathbb{R}$ is continuous. Then, the function $\varkappa \in C_{1-\theta,\psi}(I,\mathbb{R})$ is a solution of the linear fractional BVP:*

$$\begin{cases}\mathfrak{D}_{a^+}^{\theta_1,\theta_2;\psi}\varkappa(\vartheta) = H(\vartheta), \vartheta\in I', \\ c_1\mathfrak{I}_{a^+}^{1-\theta;\psi}\varkappa(\vartheta)|_{\vartheta=a} + c_2\varkappa(\vartheta)|_{\vartheta=b} = d,\end{cases} \tag{18}$$

if and only if $\varkappa$ satisfies the FIE:

$$\begin{aligned}\varkappa(\vartheta) = &\frac{\mathscr{K}_{\psi}^{\theta-1}(\vartheta,a)}{\Pi_b}\left[d - c_2\left(\mathfrak{I}_{a^+}^{\theta_1;\psi}H(s)\right)(b)\right] \\ &+ \left(\mathfrak{I}_{a^+}^{\theta_1;\psi}H(s)\right)(\vartheta), \vartheta\in I',\end{aligned} \tag{19}$$

where $\Pi_b := c_1\Gamma(\theta) + c_2\mathscr{K}_{\psi}^{\theta-1}(b,a) \neq 0$.

Proof. Let $\varkappa \in C_{1-\theta,\psi}(I,\mathbb{R})$ be a solution of (18). We need to prove that $\varkappa$ is also a solution of (19). By the definition of $C_{1-\theta,\psi}(I,\mathbb{R})$ and Lemma 6, we have $\mathfrak{I}_{a^+}^{1-\theta;\psi}\varkappa(\vartheta) \in C(I,\mathbb{R})$.

Now, by applying $\mathfrak{I}_{a^+}^{\theta_1;\psi}$ to the first equation in (18) and using Lemma 5, we can write

$$\varkappa(\vartheta) = \frac{\mathfrak{I}_{a^+}^{1-\theta;\psi}\varkappa(a^+)}{\Gamma(\theta)}(\psi(\vartheta)-\psi(a))^{\theta-1} + \left(\mathfrak{I}_{a^+}^{\theta_1;\psi}H(s)\right)(\vartheta). \tag{20}$$

Set $Y_0 := \mathfrak{I}_{a^+}^{1-\theta;\psi}\varkappa(a^+)$. Then,

$$\varkappa(\vartheta) = \frac{Y_0}{\Gamma(\theta)}(\psi(\vartheta)-\psi(a))^{\theta-1} + \left(\mathfrak{I}_{a^+}^{\theta_1;\psi}H(s)\right)(\vartheta). \tag{21}$$

Taking the limit $\vartheta \longrightarrow b$, we get

$$\varkappa(b) = \frac{Y_0}{\Gamma(\theta)}(\psi(b)-\psi(a))^{\theta-1} + \left(\mathfrak{I}_{a^+}^{\theta_1;\psi}H(s)\right)(b). \tag{22}$$

From the mixed boundary conditions $c_1Y_0 + c_2\varkappa(b) = d$, we get

$$Y_0 = \frac{d}{c_1} - \frac{c_2}{c_1}\left[\frac{Y_0}{\Gamma(\theta)}(\psi(b)-\psi(a))^{\theta-1} + \left(\mathfrak{I}_{a^+}^{\theta_1;\psi}H(s)\right)(b)\right], \tag{23}$$

which implies

$$Y_0 = \frac{\Gamma(\theta)}{c_1\Gamma(\theta) + c_2\mathscr{K}_{\psi}^{\theta-1}(b,a)}\left[d - c_2\left(\mathfrak{I}_{a^+}^{\theta_1;\psi}H(s)\right)(b)\right]. \tag{24}$$

Then, (21) becomes

$$\varkappa(\vartheta) = \frac{\mathscr{K}_{\psi}^{\theta-1}(\vartheta,a)}{\Pi_b}\left[d - c_2\left(\mathfrak{I}_{a^+}^{\theta_1;\psi}H(s)\right)(b)\right] + \left(\mathfrak{I}_{a^+}^{\theta_1;\psi}H(s)\right)(\vartheta), \tag{25}$$

which shows that formula (19) is satisfied, where

$$\left(\mathfrak{I}_{a^+}^{\theta_1;\psi}H(s)\right)(r) = \frac{1}{\Gamma(\theta_1)}\int_a^r \psi'(s)(\psi(r)-\psi(s))^{\theta_1-1}H(s)ds. \tag{26}$$

Conversely, let $\varkappa \in C_{1-\theta,\psi}(I,\mathbb{R})$ satisfy (19) which can be written as (25). In (19), taking the limit as $\vartheta \longrightarrow a$ and $\vartheta \longrightarrow b$ and then using Lemma 6, we get

$$\begin{aligned}c_1\mathfrak{I}_{a^+}^{1-\theta;\psi}\varkappa(\vartheta)|_{\vartheta=a} + c_2\varkappa(\vartheta)|_{\vartheta=b} &= \frac{c_1\Gamma(\theta)}{\Pi_b}\left[d - c_2\left(\mathfrak{I}_{a^+}^{\theta_1;\psi}H(s)\right)(b)\right] \\ &+ \frac{c_2\mathscr{K}_{\psi}^{\theta-1}(b,a)}{\Pi_b}\left[d - c_2\left(\mathfrak{I}_{a^+}^{\theta_1;\psi}H(s)\right)(b)\right] \\ &+ c_2\left(\mathfrak{I}_{a^+}^{\theta_1;\psi}H(s)\right)(b) = \left(\frac{c_1\Gamma(\theta)}{\Pi_b} + \frac{c_2\mathscr{K}_{\psi}^{\theta-1}(b,a)}{\Pi_b}\right)d \\ &- \frac{c_1c_2\Gamma(\theta)}{c_1\Gamma(\theta)+c_2\mathscr{K}_{\psi}^{\theta-1}(b,a)}\left(\mathfrak{I}_{a^+}^{\theta_1;\psi}H(s)\right)(b) \\ &+ \left(c_2 - \frac{c_2c_2\mathscr{K}_{\psi}^{\theta-1}(b,a)}{\Pi_b}\right)\left(\mathfrak{I}_{a^+}^{\theta_1;\psi}H(s)\right)(b) = d.\end{aligned} \tag{27}$$

In another direction, by applying $\mathfrak{D}_{a^+}^{\theta_1,\theta_2;\psi}$ on (19) and using Lemmas 4 and 5, we obtain

$$\begin{aligned}\mathfrak{D}_{a^+}^{\theta_1,\theta_2;\psi}\varkappa(\vartheta) &= \frac{\mathfrak{D}_{a^+}^{\theta_1,\theta_2;\psi}\mathscr{K}_{\psi}^{\theta-1}(\vartheta,a)}{\Pi_b}\left[d - c_2\left(\mathfrak{I}_{a^+}^{\theta_1;\psi}H(s)\right)(b)\right] \\ &+ \mathfrak{D}_{a^+}^{\theta_1,\theta_2;\psi}\left(\mathfrak{I}_{a^+}^{\theta_1;\psi}H(s)\right)(\vartheta) \\ &= \left(\mathfrak{D}_{a^+}^{\theta_1,\theta_2;\psi}\mathfrak{I}_{a^+}^{\theta_1;\psi}H(s)\right)(\vartheta) = H(\vartheta).\end{aligned} \tag{28}$$

This finishes the proof. ∎

Lemma 9. *Let $0<\theta_1<1, 0\le\theta_2\le 1$, where $\theta=\theta_1+\theta_2-\theta_1\theta_2$, and let $H, Z\in C(I', \mathbb{R})$ and $W\in C(I', \mathbb{R}\setminus\{0\})$. Then, the function $\varkappa\in C_{1-\theta,\psi}(I, \mathbb{R})$ is a solution of the following linear fractional hybird BVP:*

$$\begin{cases} \mathfrak{D}_{a^+}^{\theta_1,\theta_2;\psi}\left(\dfrac{\varkappa(\vartheta)-Z(\vartheta)}{W(\vartheta)}\right)=H(\vartheta), \vartheta\in I', \\ c_1\mathfrak{I}_{a^+}^{1-\theta;\psi}\left(\dfrac{\varkappa(\vartheta)-Z(\vartheta)}{W(\vartheta)}\right)\Big|_{\vartheta=a}+c_2\left(\dfrac{\varkappa(\vartheta)-Z(\vartheta)}{W(\vartheta)}\right)\Big|_{\vartheta=b}=d, \end{cases} \tag{29}$$

if and only if $\varkappa$ satisfies the FIE:

$$\varkappa(\vartheta)=W(\vartheta)\left(\frac{\mathscr{K}_\psi^{\theta-1}(\vartheta,a)}{\Pi_b}\left[d-c_2\left(\mathfrak{I}_{a^+}^{\theta_1;\psi}H(s)\right)(b)\right]+\left(\mathfrak{I}_{a^+}^{\theta_1;\psi}H(s)\right)(\vartheta)\right)+Z(\vartheta), \tag{30}$$

where Π_b is defined as in Lemma 8.

Proof. Let $\varkappa\in C_{1-\theta,\psi}(I,\mathbb{R})$ be a solution of (29). We need to prove that $\varkappa$ is also a solution of (30). By the definition of $C_{1-\theta,\psi}(I,\mathbb{R})$ and Lemma 6, we have $\mathfrak{I}_{a^+}^{1-\theta;\psi}\varkappa(\vartheta)\in C(I,\mathbb{R})$.

Now, by applying $\mathfrak{I}_{a^+}^{\theta_1;\psi}$ on the first equation of (29) and using Lemma 5, we can write

$$\frac{\varkappa(\vartheta)-Z(\vartheta)}{W(\vartheta)}=\frac{(\psi(\vartheta)-\psi(a))^{\theta-1}}{\Gamma(\theta)}\mathfrak{I}_{a^+}^{1-\theta;\psi}\left(\frac{\varkappa(a)-Z(a)}{W(a)}\right)+\left(\mathfrak{I}_{a^+}^{\theta_1;\psi}H(s)\right)(\vartheta). \tag{31}$$

Set $Y_1=\mathfrak{I}_{a^+}^{1-\theta;\psi}((\varkappa(a)-Z(a))/W(a))$; it follows that

$$\varkappa(\vartheta)=W(\vartheta)\left(\frac{Y_1}{\Gamma(\theta)}(\psi(\vartheta)-\psi(a))^{\theta-1}+\left(\mathfrak{I}_{a^+}^{\theta_1;\psi}H(s)\right)(\vartheta)\right)+Z(\vartheta). \tag{32}$$

Taking the limit $\vartheta\longrightarrow b$, we get

$$\varkappa(b)=W(b)\left(\frac{Y_1}{\Gamma(\theta)}(\psi(b)-\psi(a))^{\theta-1}+\left(\mathfrak{I}_{a^+}^{\theta_1;\psi}H(s)\right)(b)\right)+Z(b), \tag{33}$$

which implies

$$\frac{\varkappa(b)-Z(b)}{W(b)}=\frac{Y_1}{\Gamma(\theta)}(\psi(b)-\psi(a))^{\theta-1}+\left(\mathfrak{I}_{a^+}^{\theta_1;\psi}H(s)\right)(b). \tag{34}$$

To determine the constant Y_1, we use the BC of (29). It was followed by

$$Y_1=\frac{d}{c_1}-\frac{c_2}{c_1}\left[\frac{Y_1}{\Gamma(\theta)}(\psi(b)-\psi(a))^{\theta-1}+\left(\mathfrak{I}_{a^+}^{\theta_1;\psi}H(s)\right)(b)\right], \tag{35}$$

which implies

$$Y_1=\frac{\Gamma(\theta)}{c_1\Gamma(\theta)+c_2\mathscr{K}_\psi^{\theta-1}(b,a)}\left[d-c_2\left(\mathfrak{I}_{a^+}^{\theta_1;\psi}H(s)\right)(b)\right]. \tag{36}$$

Then, (32) becomes

$$\varkappa(\vartheta)=W(\vartheta)\left(\frac{\mathscr{K}_\psi^{\theta-1}(\vartheta,a)}{\Pi_b}\left[d-c_2\left(\mathfrak{I}_{a^+}^{\theta_1;\psi}H(s)\right)(b)\right]+\left(\mathfrak{I}_{a^+}^{\theta_1;\psi}H(s)\right)(\vartheta)\right)+Z(\vartheta), \tag{37}$$

which shows that formula (30) is satisfied.

Conversely, let $\varkappa\in C_{1-\theta,\psi}(I,\mathbb{R})$ satisfy (29) which can be written as (37). In (30), taking the limit as $\vartheta\longrightarrow a$ and $\vartheta\longrightarrow b$ and then using Lemma 6, we get

$$c_1\mathfrak{I}_{a^+}^{1-\theta;\psi}\left(\frac{\varkappa(\vartheta)-Z(\vartheta)}{W(\vartheta)}\right)\Big|_{\vartheta=a}+c_2\left(\frac{\varkappa(\vartheta)-Z(\vartheta)}{W(\vartheta)}\right)\Big|_{\vartheta=b}=d. \tag{38}$$

In the same context, we have from (30) that

$$\frac{\varkappa(\vartheta)-Z(\vartheta)}{W(\vartheta)}=\frac{\mathscr{K}_\psi^{\theta-1}(\vartheta,a)}{\Pi_b}\left[d-c_2\left(\mathfrak{I}_{a^+}^{\theta_1;\psi}H(s)\right)(b)\right]+\left(\mathfrak{I}_{a^+}^{\theta_1;\psi}H(s)\right)(\vartheta). \tag{39}$$

Operate $\mathfrak{D}_{a^+}^{\theta_1,\theta_2;\psi}$ on both sides of (39); then, use Lemmas 4 and 5 to get

$$\begin{aligned}\mathfrak{D}_{a^+}^{\theta_1,\theta_2;\psi}\left(\frac{\varkappa(\vartheta)-Z(\vartheta)}{W(\vartheta)}\right)&=\frac{\left(\mathfrak{D}_{a^+}^{\theta_1,\theta_2;\psi}\mathscr{K}_\psi^{\theta-1}(s,a)\right)(\vartheta)}{\Pi_b}\left[d-c_2\left(\mathfrak{I}_{a^+}^{\theta_1;\psi}H(s)\right)(b)\right]\\&\quad+\left(\mathfrak{D}_{a^+}^{\theta_1,\theta_2;\psi}\mathfrak{I}_{a^+}^{\theta_1;\psi}H(s)\right)(\vartheta)=\left(\mathfrak{D}_{a^+}^{\theta_1,\theta_2;\psi}\mathfrak{I}_{a^+}^{\theta_1;\psi}H(s)\right)(\vartheta)\\&=H(\vartheta).\end{aligned} \tag{40}$$

This finishes the proof. □

3.2. Existence Theorems. In this portion, we prove the existence theorems to problems (5) and (6) in the weighted space by means of Dhage's fixed-point technique.

By a solution of (6), we mean a function $\varkappa\in C_{1-\theta,\psi}(I,\mathbb{R})$ such that

(1) the function $(\varkappa(\vartheta)-\mathbb{z}(\vartheta,\varkappa(\vartheta)))/\mathbf{q}(\vartheta,\varkappa(\vartheta))\in C_{1-\theta,\psi}(I,\mathbb{R})$ if $[\psi(\vartheta)-\psi(a)]^{1-\theta}((\varkappa(\vartheta)-\mathbb{z}(\vartheta,\varkappa(\vartheta)))/\mathbf{q}(\vartheta,\varkappa(\vartheta)))\in C(I,\mathbb{R})$

(2) $\varkappa$ satisfies the equations in (6)

Before pursuing the main findings, we state the following assumptions:

(A_1) Let $\mathbf{q}: I'\times\mathbb{R}\longrightarrow\mathbb{R}\setminus\{0\}$, $\mathbb{z}: I'\times\mathbb{R}\longrightarrow\mathbb{R}$, and $\mathbf{p}: I'\times\mathbb{R}\longrightarrow\mathbb{R}$ be continuous functions such that $\mathbf{q}(\cdot,\varkappa(\cdot))$, $\mathbb{z}(\cdot,\varkappa(\cdot))$, $\mathbf{p}(\cdot,\varkappa(\cdot))\in C^{\psi}_{1-\theta}(I,\mathbb{R})$, for each $\varkappa\in C^{\psi}_{1-\theta}(I,\mathbb{R})$.

(A_2) There exist two positive functions $\mu_{\mathbf{q}},\mu_{\mathbb{z}}\in C(I,\mathbb{R})$ such that

$$\begin{aligned}|\mathbf{q}(\vartheta,\varkappa)-\mathbf{q}(\vartheta,\bar{\varkappa})|&\le\mu_{\mathbf{q}}|\varkappa-\bar{\varkappa}|,\\|\mathbb{z}(\vartheta,\varkappa)-\mathbb{z}(\vartheta,\bar{\varkappa})|&\le\mu_{\mathbb{z}}|\varkappa-\bar{\varkappa}|,\end{aligned}\tag{41}$$

for each $\vartheta\in I'$ and $\varkappa,\bar{\varkappa}\in\mathbb{R}$.

(A_3) There exist $\rho,\sigma\in C(I,\mathbb{R})$ such that

$$|\mathbf{p}(\vartheta,\varkappa)|\le\rho(\vartheta)+\sigma(\vartheta)\|\varkappa\|_{C^{\psi}_{1-\theta}},\ \text{for each}\ (\vartheta,\varkappa)\in I'\times\mathbb{R}.\tag{42}$$

(A_4) There exists a constant $\mu>0$ such that

$$\mu\ge\frac{\mathbf{q}_0R+\mathbb{z}_0}{1-\left(\left\|\mu_{\mathbf{q}}\right\|_C R+\|\mu_{\mathbb{z}}\|_C\right)},\ \left\|\mu_{\mathbf{q}}\right\|_C R+\|\mu_{\mathbb{z}}\|_C<1,\tag{43}$$

$$R:=\left(\frac{\mathscr{K}^{\theta-1}_{\psi}(b,a)}{\Pi_b}d+\left(\frac{\mathscr{K}^{\theta-1}_{\psi}(b,a)|c_2|}{\Pi_b}+1\right)\frac{\mathscr{K}^{\theta_1}_{\psi}(b,a)}{\Gamma(\theta_1+1)}\left(\|\rho\|_C+\|\sigma\|_C\mu\right)\right),\tag{44}$$

where $\mathbf{q}_0:=\max_{\vartheta\in I}|(\psi(\vartheta)-\psi(a))^{1-\theta}\mathbf{q}(\vartheta,0)|$ and $\mathbb{z}_0:=\max_{\vartheta\in I}|(\psi(\vartheta)-\psi(a))^{1-\theta}\mathbb{z}(\vartheta,0)|$.

As a result of Lemma 8, we present the next lemma.

Lemma 10. *Let $0<\theta_1<1, 0\le\theta_2\le1$, where $\theta=\theta_1+\theta_2-\theta_1\theta_2$ and $p: I'\times\mathbb{R}\longrightarrow\mathbb{R}$ are continuous. Then, the nonlinear fractional BVP*

$$\begin{cases}\mathfrak{D}^{\theta_1,\theta_2;\psi}_{a^+}\varkappa(\vartheta)=\mathbf{p}(\vartheta,\varkappa(\vartheta)),\ \vartheta\in I',\\ c_1\mathfrak{I}^{1-\theta;\psi}_{a^+}\varkappa(\vartheta)|_{\vartheta=a}+c_2\varkappa(\vartheta)|_{\vartheta=b}=d\end{cases}\tag{45}$$

is equivalent to

$$\begin{aligned}\varkappa(\vartheta)=&\frac{\mathscr{K}^{\theta-1}_{\psi}(\vartheta,a)}{\Pi_b}\left(d-\frac{c_2}{\Gamma(\theta_1)}\int_a^b F^{\theta_1}_{\psi}(b,s)\mathbf{p}(s,\varkappa(s))ds\right)\\&+\frac{1}{\Gamma(\theta_1)}\int_a^{\vartheta}F^{\theta_1}_{\psi}(\vartheta,s)\mathbf{p}(s,\varkappa(s))ds,\ \vartheta\in I'.\end{aligned}\tag{46}$$

As a result of Lemma 9, we present the next lemma.

Lemma 11. *Let $0<\theta_1<1$, $0\le\theta_2\le1$, where $\theta=\theta_1+\theta_2-\theta_1\theta_2$ and $p: I'\times\mathbb{R}\longrightarrow\mathbb{R}$ are continuous. Then, the nonlinear fractional hybrid BVP*

$$\begin{cases}\mathfrak{D}^{\theta_1,\theta_2;\psi}_{a^+}\left(\dfrac{\varkappa(\vartheta)-\mathbb{z}(\vartheta,\varkappa(\vartheta))}{\mathbf{q}(\vartheta,\varkappa(\vartheta))}\right)=\mathbf{p}(\vartheta,\varkappa(\vartheta)),\ \vartheta\in I',\\ c_1\mathfrak{I}^{1-\theta;\psi}_{a^+}\left(\dfrac{\varkappa(\vartheta)-\mathbb{z}(\vartheta,\varkappa(\vartheta))}{\mathbf{q}(\vartheta,\varkappa(\vartheta))}\right)\Bigg|_{\vartheta=a}+c_2\left(\dfrac{\varkappa(\vartheta)-\mathbb{z}(\vartheta,\varkappa(\vartheta))}{\mathbf{q}(\vartheta,\varkappa(\vartheta))}\right)\Bigg|_{\vartheta=b}=d\end{cases}\tag{47}$$

is equivalent to

$$\begin{aligned}\varkappa(\vartheta)=&\,\mathbb{z}(\vartheta,\varkappa(\vartheta))+\mathbf{q}(\vartheta,\varkappa(\vartheta))\\&\cdot\left[\frac{\mathscr{K}^{\theta-1}_{\psi}(\vartheta,a)}{\Pi_b}\left(d-\frac{c_2}{\Gamma(\theta_1)}\int_a^b F^{\theta_1}_{\psi}(b,s)\mathbf{p}(s,\varkappa(s))ds\right)\right.\\&\left.+\frac{1}{\Gamma(\theta_1)}\int_a^{\vartheta}F^{\theta_1}_{\psi}(\vartheta,s)\mathbf{p}(s,\varkappa(s))ds\right].\end{aligned}\tag{48}$$

Theorem 12. *Suppose that (A_1)–(A_4) hold. Then, problem (6) has at least one solution in $C^{\psi}_{1-\theta}(I,\mathbb{R})$.*

Proof. Set

$$B_{\mu}=\left\{\varkappa\in C^{\psi}_{1-\theta}(I,\mathbb{R}):\ \|\varkappa\|_{C^{\psi}_{1-\theta}}\le\mu\right\}.\tag{49}$$

Obviously, B_{μ} is a convex, closed, bounded subset of $C_{1-\theta,\psi}(I,\mathbb{R})$. By Lemma 11, problem (6) is equivalent to

$$\begin{aligned}\varkappa(\vartheta)=&\,\mathbb{z}(\vartheta,\varkappa(\vartheta))+\mathbf{q}(\vartheta,\varkappa(\vartheta))\\&\cdot\left[\frac{\mathscr{K}^{\theta-1}_{\psi}(\vartheta,a)}{\Pi_b}\left(d-\frac{c_2}{\Gamma(\theta_1)}\int_a^b F^{\theta_1}_{\psi}(b,s)\mathbf{p}(s,\varkappa(s))ds\right)\right.\\&\left.+\frac{1}{\Gamma(\theta_1)}\int_a^{\vartheta}F^{\theta_1}_{\psi}(\vartheta,s)\mathbf{p}(s,\varkappa(s))ds\right],\ \vartheta\in I'.\end{aligned}\tag{50}$$

Define the operators $A,B: C^{\psi}_{1-\theta}(I,\mathbb{R})\longrightarrow C^{\psi}_{1-\theta}(I,\mathbb{R})$ and $C: B_{\mu}\longrightarrow C^{\psi}_{1-\theta}(I,\mathbb{R})$ by

$$\begin{aligned}A\varkappa(\vartheta)&=\mathbf{q}(\vartheta,\varkappa(\vartheta)),\ \vartheta\in I',\\B\varkappa(\vartheta)&=\mathbb{z}(\vartheta,\varkappa(\vartheta)),\ \vartheta\in I',\\C\varkappa(\vartheta)&=\frac{\mathscr{K}^{\theta-1}_{\psi}(\vartheta,a)}{\Pi_b}\left(d-\frac{c_2}{\Gamma(\theta_1)}\int_a^b F^{\theta_1}_{\psi}(b,s)\mathbf{p}(s,\varkappa(s))ds\right)\\&\quad+\frac{1}{\Gamma(\theta_1)}\int_a^{\vartheta}F^{\theta_1}_{\psi}(\vartheta,s)\mathbf{p}(s,\varkappa(s))ds,\ \vartheta\in I'.\end{aligned}\tag{51}$$

Then, we can express equation (50) as follows:

$$\varkappa(\vartheta)=B\varkappa(\vartheta)+C\varkappa(\vartheta)\cdot A\varkappa(\vartheta),\ \vartheta\in I'.\tag{52}$$

Now, we will show that A, B, and C meet all the requirements for Theorem 7. This will be accomplished in a series of next steps.

Step 1. $A, B : C^{\psi}_{1-\theta}(I, \mathbb{R}) \longrightarrow C^{\psi}_{1-\theta}(I, \mathbb{R})$ are Lipschitzian on $C^{\psi}_{1-\theta}(I, \mathbb{R})$.

Let $\varkappa, \omega \in C^{\psi}_{1-\theta}(I, \mathbb{R})$ and $\vartheta \in I'$. Then, by (A_2), we have

$$\begin{aligned}\|A\varkappa - A\omega\|_{C^{\psi}_{1-\theta}} &= \max_{\vartheta\in I}\left|[\psi(\vartheta)-\psi(a)]^{1-\theta}(A\varkappa(\vartheta)-A\omega(\vartheta))\right| \\ &= \max_{\vartheta\in I}[\psi(\vartheta)-\psi(a)]^{1-\theta}|\mathbf{q}(\vartheta,\varkappa(\vartheta))-\mathbf{q}(\vartheta,\omega(\vartheta))| \\ &\le \max_{\vartheta\in I}[\psi(\vartheta)-\psi(a)]^{1-\theta}\mu_{\mathbf{q}}(\vartheta)|\varkappa(\vartheta)-\omega(\vartheta)| \\ &\le \left\|\mu_{\mathbf{q}}\right\|_{C}\|\varkappa-\omega\|_{C^{\psi}_{1-\theta}}.\end{aligned} \tag{53}$$

Therefore, A is Lipschitzian on $C^{\psi}_{1-\theta}(I, \mathbb{R})$ with the Lipschitz constant $\|\mu_{\mathbf{q}}\|_C$.

Similarly, we conclude that B is Lipschitzian on $C^{\psi}_{1-\theta}(I, \mathbb{R})$ with the Lipschitz constant $\|\mu_{\mathbb{z}}\|_C$, i.e.,

$$\|B\varkappa - B\omega\|_{C^{\psi}_{1-\theta}} \le \|\mu_{\mathbb{z}}\|_C\|\varkappa-\omega\|_{C^{\psi}_{1-\theta}}. \tag{54}$$

Step 2. $C : B_\mu \longrightarrow C^{\psi}_{1-\theta}(I, \mathbb{R})$ is completely continuous.

First, we show that $C : B_\mu \longrightarrow C^{\psi}_{1-\theta}(I, \mathbb{R})$ is continuous. Let $\{\varkappa_n\}$ be a sequence such that $\varkappa_n \longrightarrow \varkappa$ in B_μ. Then,

$$\begin{aligned}&\lim_{n\longrightarrow\infty}\left|[\psi(\vartheta)-\psi(a)]^{1-\theta}(C\varkappa_n(\vartheta)-C\varkappa(\vartheta))\right| \\ &\le \frac{1}{\Pi_b}\frac{c_2}{\Gamma(\theta_1)}\int_a^b F_{\psi}^{\theta_1}(b,s)\lim_{n\longrightarrow\infty}|\mathbf{p}(s,\varkappa_n(s))-\mathbf{p}(s,\varkappa(s))|ds \\ &\quad+\frac{\mathscr{K}_{\psi}^{1-\theta}(\vartheta,a)}{\Gamma(\theta_1)}\int_a^{\vartheta} F_{\psi}^{\theta_1}(\vartheta,s)\lim_{n\longrightarrow\infty}|\mathbf{p}(s,\varkappa_n(s))-\mathbf{p}(s,\varkappa(s))|ds \\ &= \frac{1}{\Pi_b}\frac{c_2}{\Gamma(\theta_1)}\int_a^b F_{\psi}^{\theta_1}(b,s)\mathscr{K}_{\psi}^{\theta-1}(s,a) \\ &\quad\times\lim_{n\longrightarrow\infty}\mathscr{K}_{\psi}^{1-\theta}(s,a)|\mathbf{p}(s,\varkappa_n(s))-\mathbf{p}(s,\varkappa(s))|ds \\ &\quad+\frac{\mathscr{K}_{\psi}^{1-\theta}(\vartheta,a)}{\Gamma(\theta_1)}\int_a^{\vartheta} F_{\psi}^{\theta_1}(\vartheta,s)\mathscr{K}_{\psi}^{\theta-1}(s,a) \\ &\quad\times\lim_{n\longrightarrow\infty}\mathscr{K}_{\psi}^{1-\theta}(s,a)|\mathbf{p}(s,\varkappa_n(s))-\mathbf{p}(s,\varkappa(s))|ds.\end{aligned} \tag{55}$$

Since $\mathbf{p}(\cdot,\varkappa(\cdot))$ is a continuous function with $\mathbf{p}(\cdot,\varkappa(\cdot)) \in C^{\psi}_{1-\theta}(I, \mathbb{R})$, we have

$$\begin{aligned}&\lim_{n\longrightarrow\infty}\left|[\psi(\vartheta)-\psi(a)]^{1-\theta}(C\varkappa_n(\vartheta)-C\varkappa(\vartheta))\right| \\ &\le \frac{1}{\Pi_b}\frac{c_2}{\Gamma(\theta_1)}\int_a^b F_{\psi}^{\theta_1}(b,s)\mathscr{K}_{\psi}^{\theta-1}(s,a)\lim_{n\longrightarrow\infty}\|\mathbf{p}(\cdot,\varkappa_n(\cdot))-\mathbf{p}(\cdot,\varkappa(\cdot))\|_{C^{\psi}_{1-\theta}}ds \\ &\quad+\frac{\mathscr{K}_{\psi}^{1-\theta}(\vartheta,a)}{\Gamma(\theta_1)}\int_a^{\vartheta} F_{\psi}^{\theta_1}(\vartheta,s)\mathscr{K}_{\psi}^{\theta-1}(s,a)\lim_{n\longrightarrow\infty}\|\mathbf{p}(\cdot,\varkappa_n(\cdot))-\mathbf{p}(\cdot,\varkappa(\cdot))\|_{C^{\psi}_{1-\theta}}ds.\end{aligned} \tag{56}$$

Since ψ is increasing and using (17), we obtain

$$\begin{aligned}&\lim_{n\longrightarrow\infty}\left|[\psi(\vartheta)-\psi(a)]^{1-\theta}(C\varkappa_n(\vartheta)-C\varkappa(\vartheta))\right| \\ &\le \left(\frac{c_2}{\Pi_b}+\mathscr{K}_{\psi}^{1-\theta}(b,a)\right)\frac{\Gamma(\theta)}{\Gamma(\theta+\theta_1)}\mathscr{K}_{\psi}^{\theta_1+\theta-1}(b,a) \\ &\quad\times\lim_{n\longrightarrow\infty}\|\mathbf{p}(\cdot,\varkappa_n(\cdot))-\mathbf{p}(\cdot,\varkappa(\cdot))\|_{C^{\psi}_{1-\theta}}\longrightarrow 0 \text{ as } n\longrightarrow\infty.\end{aligned} \tag{57}$$

This shows that $C : B_\mu \longrightarrow C^{\psi}_{1-\theta}(I, \mathbb{R})$ is continuous on B_μ.

Next, we show that $C(B_\mu)$ is uniformly bounded in B_μ. Indeed, for any $\varkappa \in B_\mu$, we have

$$\begin{aligned}\|C\varkappa\|_{C^{\psi}_{1-\theta}} &= \max_{\vartheta\in I}\left|[\psi(\vartheta)-\psi(a)]^{1-\theta}C\varkappa(\vartheta)\right| \le \frac{d}{\Pi_b} \\ &\quad+\left|\frac{c_2}{\Pi_b}\right|\frac{1}{\Gamma(\theta_1)}\int_a^b F_{\psi}^{\theta_1}(b,s)\max_{\vartheta\in I}|\mathbf{p}(s,\varkappa(s))|ds \\ &\quad\cdot+\frac{\mathscr{K}_{\psi}^{1-\theta}(\vartheta,a)}{\Gamma(\theta_1)}\int_a^{\vartheta} F_{\psi}^{\theta_1}(\vartheta,s)\max_{\vartheta\in I}|\mathbf{p}(s,\varkappa(s))|ds \\ &\le \frac{d}{\Pi_b}+\left|\frac{c_2}{\Pi_b}\right|\frac{1}{\Gamma(\theta_1)}\int_a^b F_{\psi}^{\theta_1}(b,s)\max_{\vartheta\in I}\left(\rho(s)+\sigma(s)\|\varkappa\|_{C^{\psi}_{1-\theta}}\right)ds \\ &\quad+\frac{\mathscr{K}_{\psi}^{1-\theta}(\vartheta,a)}{\Gamma(\theta_1)}\int_a^{\vartheta} F_{\psi}^{\theta_1}(\vartheta,s)\max_{\vartheta\in I}\left(\rho(s)+\sigma(s)\|\varkappa\|_{C^{\psi}_{1-\theta}}\right)ds \\ &\le \frac{d}{\Pi_b}+\left|\frac{c_2}{\Pi_b}\right|\frac{\mathscr{K}_{\psi}^{\theta_1}(b,a)}{\Gamma(\theta_1+1)}\left(\|\rho\|_C+\|\sigma\|_C\|\varkappa\|_{C^{\psi}_{1-\theta}}\right) \\ &\quad+\frac{\mathscr{K}_{\psi}^{1}(b,a)}{\Gamma(\theta_1+1)}\left(\|\rho\|_C+\|\sigma\|_C\|\varkappa\|_{C^{\psi}_{1-\theta}}\right)\le\frac{d}{\Pi_b} \\ &\quad+\left[\left|\frac{c_2}{\Pi_b}\right|\mathscr{K}_{\psi}^{\theta_1}(b,a)+\mathscr{K}_{\psi}^{1}(b,a)\right]\frac{(\|\rho\|_C+\|\sigma\|_C\mu)}{\Gamma(\theta_1+1)}.\end{aligned} \tag{58}$$

By (44), then $\|C\varkappa\|_{C^{\psi}_{1-\theta}} \le R$ for each $\varkappa \in B_\mu$.

Now, we prove that $C(B_\mu)$ is an equicontinuous set in $C^{\psi}_{1-\theta}(I, \mathbb{R})$.

Let $\varkappa \in B_\mu$ and $\vartheta_1, \vartheta_2 \in I'$ with $\vartheta_1 < \vartheta_2$. Then,

$$\begin{aligned}&\left|[\psi(\vartheta_2)-\psi(a)]^{1-\theta}C\varkappa(\vartheta_2)-[\psi(\vartheta_1)-\psi(a)]^{1-\theta}C\varkappa(\vartheta_1)\right| \\ &=\left|\frac{\mathscr{K}_{\psi}^{1-\theta}(\vartheta_2,a)}{\Gamma(\theta_1)}\int_a^{\vartheta_2}F_{\psi}^{\theta_1}(\vartheta_2,s)\mathbf{p}(s,\varkappa(s))ds\right. \\ &\quad\left.-\frac{\mathscr{K}_{\psi}^{1-\theta}(\vartheta_1,a)}{\Gamma(\theta_1)}\int_a^{\vartheta_1}F_{\psi}^{\theta_1}(\vartheta_1,s)\mathbf{p}(s,\varkappa(s))ds\right| \\ &=\left|\frac{\mathscr{K}_{\psi}^{1-\theta}(\vartheta_2,a)}{\Gamma(\theta_1)}\int_a^{\vartheta_2}F_{\psi}^{\theta_1}(\vartheta_2,s)\mathscr{K}_{\psi}^{\theta-1}(s,a)\left|[\psi(s)-\psi(a)]^{1-\theta}\mathbf{p}(s,\varkappa(s))\right|ds\right. \\ &\quad\left.-\frac{\mathscr{K}_{\psi}^{1-\theta}(\vartheta_1,a)}{\Gamma(\theta_1)}\int_a^{\vartheta_1}F_{\psi}^{\theta_1}(\vartheta_1,s)\mathscr{K}_{\psi}^{\theta-1}(s,a)\left|[\psi(s)-\psi(a)]^{1-\theta}\mathbf{p}(s,\varkappa(s))\right|ds\right|.\end{aligned} \tag{59}$$

Since $\mathbf{p}(\cdot,\varkappa(\cdot)) \in C^{\psi}_{1-\theta}(I, \mathbb{R})$ for any $\varkappa \in C^{\psi}_{1-\theta}(I, \mathbb{R})$ and

$[\psi(\cdot)-\psi(a)]^{1-\theta}\mathbf{p}(\cdot,\varkappa(\cdot)) \in C(I,\mathbb{R})$, there exist $\xi \in \mathbb{R}$ such that

$$\left|[\psi(s)-\psi(a)]^{1-\theta}\mathbf{p}(s,\varkappa(s))\right| \leq \xi \text{ for all } \vartheta \in I'. \tag{60}$$

Hence,

$$\begin{aligned}
&\left|[\psi(\vartheta_2)-\psi(a)]^{1-\theta}C\varkappa(\vartheta_2)-[\psi(\vartheta_1)-\psi(a)]^{1-\theta}C\varkappa(\vartheta_1)\right| \\
&\quad\leq \left|\frac{\mathscr{K}_\psi^{1-\theta}(\vartheta_2,a)}{\Gamma(\theta_1)}\xi\int_a^{\vartheta_2}F_\psi^{\theta_1}(\vartheta_2,s)\mathscr{K}_\psi^{\theta-1}(s,a)ds\right. \\
&\qquad\left.-\frac{\mathscr{K}_\psi^{1-\theta}(\vartheta_1,a)}{\Gamma(\theta_1)}\xi\int_a^{\vartheta_1}F_\psi^{\theta_1}(\vartheta_1,s)\mathscr{K}_\psi^{\theta-1}(s,a)ds\right| \\
&\quad= \left|\mathscr{K}_\psi^{1-\theta}(\vartheta_2,a)\frac{\xi\Gamma(\theta)}{\Gamma(\theta+\theta_1)}[\psi(\vartheta_2)-\psi(a)]^{\theta_1+\theta-1}\right. \\
&\qquad\left.-\mathscr{K}_\psi^{1-\theta}(\vartheta_1,a)\frac{\xi\Gamma(\theta)}{\Gamma(\theta+\theta_1)}[\psi(\vartheta_1)-\psi(a)]^{\theta_1+\theta-1}\right| \\
&\quad= \left|\frac{\xi\Gamma(\theta)}{\Gamma(\theta+\theta_1)}\left([\psi(\vartheta_2)-\psi(a)]^{\theta_1}-[\psi(\vartheta_1)-\psi(a)]^{\theta_1}\right)\right|.
\end{aligned} \tag{61}$$

The continuity of ψ shows that $|[\psi(\vartheta_2)-\psi(a)]^{1-\theta}C\varkappa(\vartheta_2)-[\psi(\vartheta_1)-\psi(a)]^{1-\theta}C\varkappa(\vartheta_1)| \longrightarrow 0$ as $|\vartheta_2-\vartheta_1| \longrightarrow 0$.

This confirms that $C(B_\mu)$ is an equicontinuous set in $C_{1-\theta}^\psi(I,\mathbb{R})$. As a result of the Arzelà-Ascoli theorem, C is completely continuous.

Step 3. $A\varkappa C\omega + B\varkappa \in B_\mu$ for $\varkappa \in C_{1-\theta}^\psi(I,\mathbb{R})$ and $\omega \in B_\mu$.

Let $\varkappa \in C_{1-\theta}^\psi(I,\mathbb{R})$, and $\omega \in B_\mu$ such that $\varkappa = A\varkappa C\omega + B\varkappa$. Then,

$$\begin{aligned}
|\varkappa(\vartheta)| &\leq |A\varkappa(\vartheta)||C\omega(\vartheta)| + |B\varkappa(\vartheta)| \leq (|\mathbf{q}(\vartheta,\varkappa(\vartheta))-\mathbf{q}(\vartheta,0)| + |\mathbf{q}(\vartheta,0)|) \\
&\quad\times\left(\frac{\mathscr{K}_\psi^{\theta-1}(\vartheta,a)}{\Pi_b}d + \frac{\mathscr{K}_\psi^{\theta-1}(\vartheta,a)}{\Pi_b}\frac{|c_2|}{\Gamma(\theta_1)}\int_a^b F_\psi^{\theta_1}(b,s)|\mathbf{p}(s,\omega(s))|ds\right. \\
&\quad\left.+\frac{1}{\Gamma(\theta_1)}\int_a^\vartheta F_\psi^{\theta_1}(\vartheta,s)|\mathbf{p}(s,\omega(s))|d\varkappa\right) + |\mathbb{z}(\vartheta,\varkappa(\vartheta))-\mathbb{z}(\vartheta,0)| \\
&\quad+|\mathbb{z}(\vartheta,0)| \leq \left(\mu_{\mathbf{q}}(\vartheta)|\varkappa(\vartheta)| + |\mathbf{q}(\vartheta,0)|\right) \\
&\quad\times\left(\frac{\mathscr{K}_\psi^{\theta-1}(\vartheta,a)}{\Pi_b}d + \frac{\mathscr{K}_\psi^{\theta-1}(\vartheta,a)}{\Pi_b}|c_2|\left(\mathfrak{I}_{a^+}^{\theta_1;\psi}\left(\rho(s)+\sigma(s)\|\omega\|_{C_{1-\theta}^\psi}\right)\right)(b)\right. \\
&\quad\left.+\left(\mathfrak{I}_{a^+}^{\theta_1;\psi}\left(\rho(s)+\sigma(s)\|\omega\|_{C_{1-\theta}^\psi}\right)\right)(\vartheta)\right) + \mu_{\mathbb{z}}(\vartheta)|\varkappa(\vartheta)| + |\mathbb{z}(\vartheta,0)| \\
&\leq \mathscr{K}_\psi^{\theta-1}(\vartheta,a)\left(\left\|\mu_{\mathbf{q}}\right\|_C\|\varkappa\|_{C_{1-\theta}^\psi} + \mathbf{q}_0\right) \\
&\quad\times\left(\frac{\mathscr{K}_\psi^{\theta-1}(\vartheta,a)}{\Pi_b}d + \frac{\mathscr{K}_\psi^{\theta-1}(\vartheta,a)}{\Pi_b}\frac{|c_2|\mathscr{K}_\psi^{\theta_1}(b,a)}{\Gamma(\theta_1+1)}\left(\|\rho\|_C + \|\sigma\|_C\|\omega\|_{C_{1-\theta}^\psi}\right)\right. \\
&\quad\left.+\left(\|\rho\|_C + \|\sigma\|_C\|\omega\|_{C_{1-\theta}^\psi}\right)\frac{\mathscr{K}_\psi^{\theta_1}(\vartheta,a)}{\Gamma(\theta_1+1)}\right) \\
&\quad+\mathscr{K}_\psi^{\theta-1}(\vartheta,a)\left(\|\mu_{\mathbb{z}}\|_C\|\varkappa\|_{C_{1-\theta}^\psi} + \mathbb{z}_0\right).
\end{aligned} \tag{62}$$

Therefore,

$$\|\varkappa\|_{C_{1-\theta}^\psi} \leq \left(\left\|\mu_{\mathbf{q}}\right\|_C\|\varkappa\|_{C_{1-\theta}^\psi} + \mathbf{q}_0\right)R + \|\mu_{\mathbb{z}}\|_C\|\varkappa\|_{C_{1-\theta}^\psi} + \mathbb{z}_0, \tag{63}$$

which implies

$$\|\varkappa\|_{C_{1-\theta}^\psi} \leq \frac{\mathbf{q}_0R + \mathbb{z}_0}{1-\left(\left\|\mu_{\mathbf{q}}\right\|_C R + \|\mu_{\mathbb{z}}\|_C\right)} \leq \mu. \tag{64}$$

Step 4. Condition (iv) in Theorem 7 holds. That is, $\kappa_1 M + \kappa_2 < 1$.

From Step 2 and (44), we have

$$\begin{aligned}
M &= \|C(B_\mu)\| = \sup_{\varkappa\in B_\mu}\left\{\sup_{\vartheta\in I}|C\varkappa(\vartheta)|\right\} \\
&\leq \left(\frac{\mathscr{K}_\psi^{\theta-1}(b,a)}{\Pi_b}d + \left(\frac{\mathscr{K}_\psi^{\theta-1}(b,a)|c_2|}{\Pi_b}+1\right)\frac{\mathscr{K}_\psi^{\theta_1}(b,a)}{\Gamma(\theta_1+1)}(\|\rho\|_C + \|\sigma\|_C\mu)\right) \\
&= R.
\end{aligned} \tag{65}$$

Hence,

$$\left\|\mu_{\mathbf{q}}\right\|_C M + \|\mu_{\mathbb{z}}\|_C \leq \left\|\mu_{\mathbf{q}}\right\|_C R + \|\mu_{\mathbb{z}}\|_C < 1, \tag{66}$$

where $\kappa_1 = \|\mu_{\mathbf{q}}\|_C$ and $\kappa_2 = \|\mu_{\mathbb{z}}\|_C$. Thus, all the assumptions of Theorem 7 are fulfilled, so the equation $\varkappa = A\varkappa C\varkappa + B\varkappa$ has a solution in B_μ. As a result, problem (6) has a solution on I. □

Theorem 13. *Suppose that (A_1)–(A_4) hold. Then, the ψ-Hilfer problem (5) has at least one solution on I.*

Proof. The proof of this theorem is quite similar to that of Theorem 12 with consideration that $\mathbb{z}(\vartheta,\varkappa(\vartheta)) \equiv 0$ and $\mathbf{q}(\vartheta,\varkappa(\vartheta)) \equiv 1$ in problem (6). Thus, we omit the details. □

Remark 14.

(1) The conditions (A_2) and (A_3) are also correct if we replace the functions $\rho, \sigma, \mu_{\mathbf{q}}, \mu_{\mathbb{z}} \in C(I,\mathbb{R})$ with constants

(2) Problem (6) reduces to problem (5) when $\mathbb{z}(\vartheta,\varkappa(\vartheta)) \equiv 0$ and $\mathbf{q}(\vartheta,\varkappa(\vartheta)) \equiv 1$. Consequently, Theorem 12 is applied to problem (5)

(3) Problem (5) reduces to problem (3) when $\psi(\vartheta) = \vartheta$, $a = 0$, and $\theta_2 = 1$ (see [23])

(4) Problem (6) reduces to problem (4) when $\psi(\vartheta) = \vartheta$, $a = 0$, $\theta_2 = 1$, and $\mathbb{z}(\vartheta,\varkappa(\vartheta)) \equiv 0$ (see [24])

(5) In particular, if $\psi(\vartheta) = \vartheta$, then the obtained results correspond to Caputo-type FDE and RL-type FDE, for $\theta_2 = 1$ and $\theta_2 = 0$, respectively

(6) The corresponding fractional problems involving the Hilfer-Katugampola type and Hilfer-Hadamard type appear as a special case of our proposed problems for $\psi(\vartheta)=\vartheta^{\rho}$, $\rho>0$, and $\psi(\vartheta)=\log\vartheta$, respectively

4. Examples

In this part, we construct two examples to explain the main results.

Example 15. Consider the ψ-Hilfer hybrid FDE with hybrid BC:

$$\begin{cases} \mathfrak{D}_{a^+}^{\theta_1,\theta_2;\psi}\left(\dfrac{\varkappa(\vartheta)-\mathbb{z}(\vartheta,\varkappa(\vartheta))}{\mathbf{q}(\vartheta,\varkappa(\vartheta))}\right)=\mathbf{p}(\vartheta,\varkappa(\vartheta)),\ \vartheta\in(0,1], \\ c_1\mathfrak{J}_{a^+}^{1-\theta;\psi}\left(\dfrac{\varkappa(\vartheta)-\mathbb{z}(\vartheta,\varkappa(\vartheta))}{\mathbf{q}(\vartheta,\varkappa(\vartheta))}\right)\Big|_{\vartheta=a}+c_2\left(\dfrac{\varkappa(\vartheta)-\mathbb{z}(\vartheta,\varkappa(\vartheta))}{\mathbf{q}(\vartheta,\varkappa(\vartheta))}\right)\Big|_{\vartheta=b}=d. \end{cases} \tag{67}$$

Define $\mathbb{z}:(0,1]\times\mathbb{R}\longrightarrow\mathbb{R}$, $\mathbf{q}:(0,1]\times\mathbb{R}\longrightarrow\mathbb{R}\setminus\{0\}$, and $\mathbf{p}:(0,1]\times\mathbb{R}\longrightarrow\mathbb{R}$ by

$$\begin{aligned} \mathbb{z}(\vartheta,\varkappa(\vartheta))&=\frac{1}{2}\cos\left(\frac{\vartheta}{3}\right)\left(\frac{\varkappa(\vartheta)}{1+\varkappa(\vartheta)}+e^{-\vartheta}\right), \\ \mathbf{q}(\vartheta,\varkappa(\vartheta))&=\left(1+\frac{\sin\vartheta}{12}\varkappa(\vartheta)\right), \\ \mathbf{p}(\vartheta,\varkappa(\vartheta))&=\frac{\psi(\vartheta)-\psi(0)}{100}\left(\frac{\varkappa(\vartheta)}{1+\varkappa(\vartheta)}+2\right). \end{aligned} \tag{68}$$

It is easy to show that for any $\varkappa,\omega\in\mathbb{R}$, we have

$$\begin{aligned} |\mathbb{z}(\vartheta,\varkappa(\vartheta))-\mathbb{z}(\vartheta,\omega(\vartheta))|&\le\frac{1}{2}\cos\left(\frac{\vartheta}{3}\right)|\varkappa(\vartheta)-\omega(\vartheta)|, \\ |\mathbf{q}(\vartheta,\varkappa(\vartheta))-\mathbf{q}(\vartheta,\omega(\vartheta))|&\le\frac{\sin\vartheta}{12}|\varkappa(\vartheta)-\omega(\vartheta)|, \end{aligned} \tag{69}$$

and for each $\varkappa\in C_{1-\theta}^{\psi}([0,1],\mathbb{R})$, we have

$$\begin{aligned} |\mathbf{p}(\vartheta,\varkappa(\vartheta))|&\le\frac{\psi(\vartheta)-\psi(0)}{100}|\varkappa(\vartheta)|+\frac{\psi(\vartheta)-\psi(0)}{50} \\ &\le\frac{[\psi(\vartheta)-\psi(0)]^{\theta}}{100}\|\varkappa\|_{1-\theta;\psi}+\frac{\psi(\vartheta)-\psi(0)}{50}. \end{aligned} \tag{70}$$

Hence, the hypotheses (A_1)–(A_3) hold with $\mu_{\mathbf{q}}(\vartheta)=\sin\vartheta/12$, $\mu_{\mathbb{z}}(\vartheta)=(1/2)\cos(\vartheta/3)$, $\rho(\vartheta)=[\psi(\vartheta)-\psi(0)]^{\theta}/100$, and $\sigma(\vartheta)=(\psi(\vartheta)-\psi(0))/50$. Then, $\|\mu_{\mathbf{q}}\|_C=1/12$, $\|\mu_{\mathbb{z}}\|_C=1/2$, $\mathbf{q}_0=\max_{\vartheta\in 0,1]}|(\psi(\vartheta)-\psi(0))^{1-\theta}|$, and $\mathbb{z}_0=\max_{\vartheta\in 0,1]}|(\psi(\vartheta)-\psi(0))^{1-\theta}\cos(\vartheta/3)e^{-\vartheta}|$. Taking $\theta_1=1/2$, $\theta_2=0$, $\theta=1/2$, $c_1=1/3$, $c_2=1/3$, $d=1$, and $\psi(\vartheta)=\vartheta$, we get $\|\rho\|_C=1/100$, $\|\sigma\|_C=1/50$, $\mathbf{q}_0=1$, and $\mathbb{z}_0=(1/e)\cos(1/3)$. From the condition (A_4), $\|\mu_{\mathbf{q}}\|_C R+\|\mu_{\mathbb{z}}\|_C<1$ when $R<6$, and $\mu\ge 1/(1-((1/12)R+(1/2)))(R+(1/e)\cos(1/3))$. Also, $\Pi_b\neq 0$; it follows from (44) that $R=(3/(1+\sqrt{\pi}))+((1/(1+\sqrt{\pi}))+1)((1/50\sqrt{\pi})+(1/25\sqrt{\pi})\mu)<6$. Hence, $\mu<(2-149\sqrt{\pi}-300\pi)/(-4-2\sqrt{\pi})$. Using the MATLAB program, μ satisfies the inequality $64.17<\mu<159.65$. Hence, all assumptions of Theorem 12 are satisfied, so problem (67) has at least one solution on $(0,1]$.

Example 16. As a special case when $\mathbb{z}(\vartheta,\varkappa(\vartheta))=0$ and $q(\vartheta,\varkappa(\vartheta))=1$, we consider the ψ-Hilfer FDE with BC:

$$\begin{cases} \mathfrak{D}_{0^+}^{1/2,\vartheta}\varkappa(\vartheta)=\dfrac{\varkappa(\vartheta)}{1+\varkappa(\vartheta)}+\dfrac{1}{100},\ \vartheta\in(0,1], \\ 100\left[\mathfrak{J}_{a^+}^{1-\theta;\psi}\varkappa(\vartheta)|_{\vartheta=0}+\varkappa(\vartheta)|_{\vartheta=1}\right]=1. \end{cases} \tag{71}$$

Comparing problem (71) with problem (5), we obtain

$$\begin{aligned} \theta_1&=\frac{1}{2}, \\ \theta_2&=1, \\ c_1=c_2&=100, \\ d&=1, \\ \psi(\vartheta)&=\vartheta, \\ \mathbf{p}(\vartheta,\varkappa(\vartheta))&=\frac{\varkappa(\vartheta)}{1+\varkappa(\vartheta)}+\frac{1}{100}. \end{aligned} \tag{72}$$

It is obvious that $\theta=1$, and in this case, the space $C_{1-\theta}^{\psi}([0,1],\mathbb{R})$ is reduced to the space of continuous functions $C([0,1],\mathbb{R})$. Hence, (A_1) holds. It is easy to show that for each $\varkappa\in\mathbb{R}$, we have

$$|\mathbf{p}(\vartheta,\varkappa(\vartheta))|\le|\varkappa(\vartheta)|+\frac{1}{100}\le\|\varkappa\|_C+\frac{1}{100}. \tag{73}$$

Thus, the hypothesis (A_3) holds with $\rho(\vartheta)=1/100$, and $\sigma(\vartheta)=1$. Then, $\|\mu_{\mathbf{q}}\|_C=1$, $\|\mu_{\mathbb{z}}\|_C=0$, $\mathbf{q}_0=1$, and $\mathbb{z}_0=0$. So, we get $\|\rho\|_C=1/100$, $\|\sigma\|_C=1$. From the condition (A_4), we have $\mu\ge R/(1-R)$. Finally, we need to show that $R<1$. Indeed, from (44), we have $R=(1/200)+(3/100\sqrt{\pi})+(3/\sqrt{\pi})\mu$. It follows that there exists $\mu>0$ with $\mu<1/600(199\sqrt{\pi}-6)$ such that $R<1$. Therefore, problem (71) can be applied to Theorem 12.

5. Concluding Remarks

We have acquired further existence results for the solution of BVPs for the ψ-Hilfer problem (6), and the ψ-Hilfer hybrid problem (5) relies on the reduction of proposed problems to FIEs. Dhage's hybrid fixed-point theorem in the Banach algebra has been applied. The reported results in the current paper are also valid for the hybrid FDEs involving RLFD and CFD, and they are also true for special cases of the function ψ. We confirm that the obtained results of this work are recent and generalize some of the previous results in the literature. More precisely, when taking different values of function ψ and parameter θ_2, the studied problems cover several problems involving classical fractional operators such as RLFD, CFD, HFD, Hilfer-Katugampola FD, and Hilfer-Hadamard FD, as mentioned in Remark 14, which have been incorporated into the operators used in our investigation. Using these investigations, other qualitative analyses of the solution such as stability and continuous dependence results can be discussed, and this is what we desire to think about in a future paper.

Data Availability

Data are available upon request.

Conflicts of Interest

The authors declare that they have no conflict regarding the publication of this paper.

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References

[1] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier B.V, Amsterdam, 2006.

[2] A. Atangana and D. Baleanu, "New fractional derivative with non-local and non-singular kernel," *Thermal Science*, vol. 20, no. 2, pp. 757–763, 2016.

[3] X. J. Yang, F. Gao, J. T. Machado, and D. Baleanu, "A new fractional derivative involving the normalized sinc function without singular kernel," *The European Physical Journal Special Topics*, vol. 226, no. 16-18, pp. 3567–3575, 2017.

[4] J. Lozada and J. J. Nieto, "Properties of a new fractional derivative without singular kernel," *Progress in Fractional Differentiation and Applications*, vol. 1, no. 1, pp. 87–92, 2015.

[5] R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific Publishing Co. Inc, River Edge, NJ, Singapore, 2000.

[6] R. Hilfer, "Fractional time evolution," in *Applications of Fractional Calculus in Physics*, vol. 87, World Sci. Publ, River Edge, NJ, 2000.

[7] R. Hilfer, "Mathematical and physical interpretations of fractional derivatives and integrals," *Handbook of Fractional Calculus with Applications*, vol. 1, pp. 47–85, 2019.

[8] K. M. Furati, M. D. Kassim, and N. Tatar, "Existence and uniqueness for a problem involving Hilfer fractional derivative," *Computers & Mathematcs with Applications*, vol. 64, no. 6, pp. 1616–1626, 2012.

[9] H. Gu and J. J. Trujillo, "Existence of mild solution for evolution equation with Hilfer fractional derivative," *Applied Mathematics and Computation*, vol. 257, pp. 344–354, 2015.

[10] J. Wang and Y. Zhang, "Nonlocal initial value problems for differential equations with Hilfer fractional derivative," *Applied Mathematics and Computation*, vol. 266, pp. 850–859, 2015.

[11] A. D. Mali, K. D. Kucche, and J. V. da Costa Sousa, "On coupled system of nonlinear ψ-Hilfer hybrid fractional differential equations," in *International Journal of Nonlinear Sciences and Numerical Simulation*, Walter de Gruyter GmbH, Berlin/Boston, 2021.

[12] S. Asawasamrit, A. Kijjathanakorn, S. K. Ntouyas, and J. Tariboon, "Nonlocal boundary value problems for Hilfer fractional differential equations," *Bulletin of the Korean Mathematical Society*, vol. 55, no. 6, pp. 1639–1657, 2018.

[13] S. T. M. Thabet, M. S. Abdo, and B. Ahmad, "The existence and Ulamâ Hyers stability results for ψ-Hilfer fractional integrodifferential equations," *Journal of Pseudo-Differential Operators and Applications*, vol. 11, no. 4, pp. 1757–1780, 2020.

[14] O. P. Agrawal, "Some generalized fractional calculus operators and their applications in integral equations," *Fractional Calculus and Applied Analysis*, vol. 15, no. 4, pp. 700–711, 2012.

[15] R. Almeida, "A Caputo fractional derivative of a function with respect to another function," *Communications in Nonlinear Science and Numerical Simulation*, vol. 44, pp. 460–481, 2017.

[16] F. Jarad and T. Abdeljawad, "Generalized fractional derivatives and Laplace transforms," *Discrete and Continuous Dynamical Systems: Series S*, vol. 13, no. 3, pp. 709–722, 2020.

[17] J. V. C. Sousa and E. C. de Oliveira, "On the ψ-Hilfer fractional derivative," *Communications in Nonlinear Science and Numerical Simulation*, vol. 60, pp. 72–91, 2018.

[18] M. S. Abdo, H. A. Wahash, and S. K. Panchal, "Ulam-Hyers-Mittag-Leffler stability for a ψ-Hilfer problem with fractional order and infinite delay," *Results in Applied Mathematics*, vol. 7, article 100115, 2020.

[19] H. A. Wahash, M. S. Abdo, and S. K. Panchal, "Fractional integrodifferential equations with nonlocal conditions and generalized Hilfer fractional derivative," *Ufa Mathematical Journal*, vol. 11, no. 1, pp. 3–21, 2019.

[20] J. V. C. Sousa and E. C. de Oliveira, "On the Ulam–Hyers–Rassias stability for nonlinear fractional differential equations using the ψ-Hilfer operator," *Journal of Fixed Point Theory and Applications*, vol. 20, p. 96, 2018.

[21] B. C. Dhage and V. Lakshmikantham, "Basic results on hybrid differential equations," *Nonlinear Analysis: Real World Applications*, vol. 4, pp. 414–424, 2010.

[22] Y. Zhao, S. Sun, Z. Han, and Q. Li, "Theory of fractional hybrid differential equations," *Computers & Mathematcs with Applications*, vol. 62, no. 3, pp. 1312–1324, 2011.

[23] M. Benchohra, S. Hamani, and S. K. Ntouyas, "Boundary value problems for differential equations with fractional order," *Surveys in Mathematics and its Applications*, vol. 3, pp. 1–12, 2008.

[24] K. Hilal and A. Kajouni, "Boundary value problem for hybrid differential equations with fractional order," *Advances in Difference Equations*, vol. 183, 19 pages, 2015.

[25] M. Abdelhakem, D. Abdelhamied, M. G. Alshehri, and M. El-Kady, "Shifted Legendre fractional pseudospectral differentiation matrices for solving fractional differential problems," *Fractals*, vol. 2021, article 2240038, 2021.

[26] M. Abdelhakem, D. Mahmoud, D. Baleanu, and M. El-kady, “Shifted ultraspherical pseudo-Galerkin method for approximating the solutions of some types of ordinary fractional problems,” *Advances in Difference Equations*, vol. 2021, 18 pages, 2021.

[27] M. Abdelhakem, H. Moussa, D. Baleanu, and M. El-Kady, “Shifted Chebyshev schemes for solving fractional optimal control problems,” *Journal of Vibration and Control*, vol. 25, pp. 2143–2150, 2019.

[28] B. P. Moghaddam, A. Dabiri, and J. A. T. Machado, “Application of variable-order fractional calculus in solid mechanics,” *Applications in Engineering, Life and Social Sciences, Part A*, vol. 7, pp. 207–224, 2019.

[29] B. P. Moghaddam and J. A. Tenreiro Machado, “Time analysis of forced variable-order fractional Van der Pol oscillator,” *The European Physical Journal Special Topics*, vol. 226, no. 16, pp. 3803–3810, 2017.

[30] S. Etemad, M. Matar, M. A. Ragusa, and S. Rezapour, “Ripled fixed points and existence study to a tripled impulsive fractional differential system via measures of noncompactness,” *Mathematics*, vol. 10, no. 1, p. 25, 2022.

[31] M. Matar, M. Ahmad, A. Zada, S. Etemad, and S. Rezapour, “On the existence and stability of two positive solutions of a hybrid differential system of arbitrary fractional order via Avery–Anderson–Henderson criterion on cones,” *Advances in Difference Equations*, vol. 2021, 23 pages, 2021.

[32] M. D. Kassim and N. E. Tatar, “Well-posedness and stability for a differential problem with Hilfer-Hadamard fractional derivative,” *Abstract and Applied Analysis*, vol. 2013, Article ID 605029, 12 pages, 2013.

[33] D. S. Oliveira and E. C. de Oliveira, “Hilfer-Katugampola fractional derivative,” *Computational and Applied Mathematics*, vol. 37, no. 3, pp. 3672–3690, 2018.

[34] R. Hilfer, “Threefold introduction to fractional derivatives, anomalous transport,” in *Foundations and Applications*, pp. 1–17, Wiley-VCH, Weinheim, Germany, 2008.

[35] B. C. Dhage, “A fixed point theorem in Banach algebras with applications to functional integral equations,” *Kyungpook National University*, vol. 44, pp. 145–155, 2004.