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Research Article

A Note on Quotient Reflective Subcategories of O-REL

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In this paper, we examine the category of ordered-RELspaces. We show that it is a normalized and geometric topological category and give the characterization of local $\overline{T}_0$, local T_0', and local T_1 ordered-RELspaces. Furthermore, we characterize explicitly several notions of T_0's and T_1 objects in **O-REL** and study their mutual relationship. Finally, it is shown that the category of T_0's (resp. T_1) ordered-RELspaces are quotient reflective subcategories of **O-REL**.

1. Introduction

Many mathematical concepts were developed to describe certain structures of topology. The concepts of uniform convergences, uniform continuity, Cartesian closedness, completeness, and total boundedness do not exist in general topology. As a remedy, several approaches have been made to define these concepts in topology by mathematicians. For example, the concepts of uniform convergence in the sense of Kent [1] and Preuss [2], of set-convergence in the sense of Wyler [3], Tozzi [4] (which scrutinize filter convergence to bounded subset and generalizes classical point-convergence and supertopologies), of nearness by Bentely [5] and Herrlich [6] (particularly containing proximities and contiguities), and that of hullness by Čech [7] and Leseberg [8] containing the concepts of b-topologies and closures, respectively. In 2018, Leseberg [9] introduced a global concept which embeds the category of the above mentioned concepts into the category of RELspaces and RELmaps as subcategories. This construct, denoted by **REL**, forms thereby a topological category [9].

Classical separation axioms are very common and important ideas in general topology, and have many applications in all fields of mathematics. With the help of T_0 reflection [10], characterizations of locally semi-simple morphisms are obtained in algebraic topology. Furthermore, lower separation axioms can be used in digital topology where they describe digital lines, and in image processing and computer graphs to construct cellular complexes [11–13]. With having the understanding of T_0 and T_1 separation properties, several mathematicians have extended this idea to arbitrary topological categories [14–18].

Classical separation axioms at some point p (locally) were generalized and have been inspected in [14], where the purpose was to describe the notion of strongly closed sets (resp., closed) in arbitrary set based topological categories [19]. Moreover, the notions of compactness [20], Hausdorffness [14], regular and normal objects [21], perfectness [20], and soberness [22] have been generalized by using the closed and strongly closed sets in some well-defined topological categories over sets [20, 23–26]. Furthermore, the notion of closedness is suitable for the formation of closure operators [27] in several well-known topological categories [28–30].

The salient objectives of this study are stated as follows:

(i) To define initial, final, discrete, and indiscrete objects in **O-REL**

(ii) To characterize local $\overline{T}_0$, local $T_0{}'$, and local T_1 objects in **O-REL** and examine their mutual relationship

(iii) To give the characterization of $\overline{T}_0$, $T_0{}'$, and T_1 objects in **O-REL** and examine their mutual relationship

(iv) To define several structures using ordered-RELspaces and discuss each of the T_0 and T_1 axioms there and examine their mutual relationship

(v) To examine the quotient-reflective properties of ordered-RELspaces.

2. Preliminaries

Recall [31, 32], a functor $\mathscr{U}:\mathscr{C}\longrightarrow$ **Set** (the category of sets and functions) is called topological if

(i) $\mathscr{U}$ is concrete

(ii) $\mathscr{U}$ consists of small fibers

(iii) Every $\mathscr{U}$-source has a unique initial lift or every $\mathscr{U}$ -sink has an unique final lift, i.e., if for every source $(f_j : X \longrightarrow (X_j,\eta_j))j\in I$ there exists an unique structure η on X such that $g:(Y,\zeta)\longrightarrow(X,\eta)$ is a morphism iff for each $j\in I$, $f_j\circ g:(Y,\zeta)\longrightarrow(X_j,\eta_j)$ is a morphism.

Moreover, a topological functor is called discrete (respectively, indiscrete) if it has a left (respectively, right) adjoint. In addition, a functor is called a normalized topological functor if constant objects, i.e., subterminals, have an unique structure, and said to be geometric functor if the discrete functor is left exact, i.e., it preserves finite limits [31, 32].

Let X be a non-empty, then $\mathscr{R}\subset \underline{P}(X\times X)$ is called a relative system for X, and it is denoted by **REL**(X). Moreover, **REL**(X) can be ordered by setting

$\bar{\mathscr{R}}<<\mathscr{R}$ iff for each $\bar{R}\in\bar{\mathscr{R}}$, there exists $R\in\mathscr{R}$ such that $R\subset\bar{R}$.

Furthermore, we denote by **sec** $\mathscr{R}:=\{\bar{R}\subset X\times X:\forall R\in\mathscr{R}, R\cap\bar{R}\neq\phi\}$ and by **stack** $\mathscr{R}=\{\bar{R}\subset X\times X:\exists R\in\mathscr{R}, R\subset\bar{R}\}$.

Definition 1 (cf. [33]). Let $X\neq\phi$, then $\beta^X\subset\underline{P}X$ is called boundedness or B-set on X, if β^X satisfies the following axioms:

(i) $\phi\in\beta^X$

(ii) $B_2\subset B_1\in\beta^X$ implies $B_2\in\beta^X$

(iii) $a\in X$ implies $\{a\}\in\beta^X$.

And for B-sets β^X and β^Y a function $g:X\longrightarrow Y$ is called bounded iff it satisfies;

$$\left\{g[B]\colon B\in\beta^X\right\}\subset\beta^Y. \tag{1}$$

By **BOUND** we denote the corresponding defined category.

Definition 2 (cf. [33]). The triple (X,β^X,r) is called RELative space (shortly RELspace) if for the boundedness β^X the function r:$\beta^X\longrightarrow\underline{P}$**REL**(**X**) satisfies the following conditions:

(i) $B\in\beta^X$ and $\bar{\mathscr{R}}<<\mathscr{R}\in r(B)$ implies $\bar{\mathscr{R}}\in r(B)$

(ii) $\{\phi\}\notin r(B)$ for $B\in\beta^X$

(iii) $\mathscr{R}\in r(\phi)$ iff $\mathscr{R}=\phi$

(iv) $a\in X$ implies $\{\{a\}\times\{a\}\}\in r(\{a\})$.

The RELspace (X,β^X,r) is called ordered-RELspace provided that the following axiom holds:

(v) $\phi\neq B_1\subset B\in\beta^X$ implies $r(B_1)\subset r(B)$.

Definition 3 (cf. [33]). Let (X,β^X,r) and (Y,β^Y,v) be two RELspaces, then a bounded function $g:X\longrightarrow Y$ is called RELative map (shortly RELmap) iff it satisfies the following condition:

$$B\in\beta^X\setminus\{\phi\}\text{ and }\mathscr{R}\in r(B)\text{ implying } g^X\mathscr{R}\in v(g[B]), \tag{2}$$

where $g^X\mathscr{R}=\{(g\times g)[R]\colon R\in\mathscr{R}\}$ with $(g\times g)[R]=\{(g\times g)(a,c)\colon (a,c)\in R\}=\{(g(a),g(c))\colon (a,c)\in R\}$. By **O-REL,** we denote the full subcategory of **REL**, whose objects are the ordered RELspaces. Note that ***O*** − ***REL*** is a bireflective subcategory of ***REL*** [34].

Example 4. Let (X,T_X) be a preuniform convergence space; then, the associated RELspace $(X,\underline{P}(X),r_{T_X})$ can be defined as follows:

$$\begin{aligned} &r_{T_X}(\phi)=\{\phi\}\text{ and for } B\in\underline{P}(X)\setminus\{\phi\},\\ &r_{T_X}(B)=\{\mathscr{R}\in REL(X)\colon \exists\mathscr{N}\in T_X, \mathscr{N}\subset \sec\mathscr{R}\}. \end{aligned} \tag{3}$$

Let **PU-REL** denotes the category, whose objects are triples $(X,\underline{P}X,r_{T_X})$ and morphisms are RELmaps. Note that **PUCONV≅PU-REL** [9], where **PUCONV** is the category of preuniform convergence spaces and uniformly continuous maps as defined in [2].

Example 5. Let (X,β^X,t) be a set-convergence space; then, the associated RELspace (X,β^X,r_t) can be defined by

$$r_t(\phi)=\{\phi\}\text{ and for } B\in\beta^X\setminus\{\phi\}, \tag{4}$$

$r_t(B) =\{\mathscr{R}\in REL(X)\colon \exists\mathscr{E}\in FIL(X)((\mathscr{E},B)\in t \text{ and } \mathscr{R}\subset \sec\mathscr{E}\otimes\mathscr{E})\}$, where $\mathscr{E}\otimes\mathscr{E}=\{R\subset X\times X:\exists E_1,E\in\mathscr{E} \mathit{such\ that}\ E_1\times E\subset R\}$ and FIL(X) is the collection of all filters defined on X.

Let **SET-REL** denotes the category, whose objects are triples (X,β^X,r_t) and morphisms are RELmaps. Note that **SETCONV≅SET-REL** [9], where **SETCONV** is the category of set-convergence spaces and morphisms are b-continuous maps as defined in [3].

Example 6. Let (x, ζ) be prenearness space; then, the associated RELspace $(X, \underline{P}(X), r_\zeta)$ can be described as

$$r_\zeta(\phi) = \{\phi\} \text{ and for } B \in \underline{P}(X) \setminus \{\phi\},$$
$$r_\zeta(B) = \{\mathscr{R} \in REL(X): \exists \mathscr{Q} \subset \underline{P}(X)(\{B\} \cup \mathscr{Q} \in \zeta \text{ and}$$
$$\mathscr{R} << \mathscr{Q} \times \mathscr{Q})\}, \text{ where } \mathscr{Q} \times \mathscr{Q} := \{D \times D : D \in \mathscr{Q}\}. \tag{5}$$

Note that **PNEAR≅PN-REL** [6, 9], where **PNEAR** is the category, whose objects are prenearness spaces and morphisms are nearness preserving maps as defined in [6], and **PN-REL** is the category of triples $(X, \underline{P}X, r_\zeta)$ and morphisms are RELmaps.

Example 7. For a B-set β^X, we put $r_b(\phi) := \{\phi\}$, and for $B \in \beta^X \setminus \{\phi\}$, we set $r_b(B) := \{\mathscr{R} \in REL(X): \exists x \in B, \mathscr{R} \subset \dot{x} \times \dot{x}\}$; hence, (X, β^X, r_b) defines a RELspace, which is diagonal, meaning that for $B \in \beta^X \setminus \{\phi\}$ and $\mathscr{R} \in s(B)$, we can find $x \in B$ such that $\forall R \in \mathscr{R}$, $(x, x) \in R$.

Let **Δ-REL** be denote the corresponding defined full subcategory of **REL**; then, **Δ-REL≅BOUND**.

Remark 8. In this context, note that **BORN**, the full subcategory of **BOUND**, whose objects are the bornological spaces, then also has evidently a corresponding counterpart in **REL**.

Example 9. Let (X, β^X, q) be b-topological space; then, the associated RELspace (X, β^X, r_q) is defined by

$$r_q(\phi) := \{\phi\} \text{ and for } B \in \beta^X \setminus \{\phi\}, r_q(B) := \left\{\mathscr{R} \in REL(X): \exists \omega \subset \beta^X, \exists a \in B(\mathscr{R} << \omega \times \omega \text{ and } a \in \cap \{q(E): E \in \omega\}\right\}. \tag{6}$$

Note that **b-TOP≅bTOP-REL** [9], where **bTOP-REL** denotes the full subcategory of **REL**, whose objects are triples (X, β^X, r_q), and **b-TOP** denotes the category of b-topological spaces and b-continuous maps as defined in [9].

3. *O − REL* as a Normalized and Geometric Topological Category

Note that the forgetful functor $\mathscr{U} : \mathscr{C} \longrightarrow$ **Set**, where $\mathscr{C} =$ ***REL*** is topological in the following sense:

Lemma 10. *Let (X_j, β^{X_j}, r_j) be a collection of RELspaces. A source $(f_i : (X, \beta_I^X, r_I^X) \longrightarrow (X_j, \beta^{X_j}, r_j))_{j \in I}$ is initial in **REL** iff*

$$\beta_I^X := \left\{B \subset X : g_j(B) \in \beta^{X_j}, \forall j \in I\right\}, \tag{7}$$

and for all $B \in \beta_I^X$,

$$r_I^X(B) := \left\{\mathscr{R} \in REL(X): g^{X_j}\mathscr{R} \in r_j\left(g_j[B]\right), \forall j \in I\right\}. \tag{8}$$

Proof. It is given in [34]. Consequently, since **O-REL** is a full and isomorphism-closed subcategory which is bireflective in **REL,** it is topological, too. □

Lemma 11. *Let (X_j, β^{X_j}, r_j) be a collection of ordered-RELspaces. A sink $(f_i : (X_j, \beta^{X_j}, r_j) \longrightarrow (X, \beta_{fin}^X, r_{fin}))_{j \in I}$ is final in **O − REL** iff*

$$\beta_{fin}^X := \left\{B \subset X : \exists j \in I, \exists B_j \in \beta_j^X \mid B \subset g_j(B_j)\right\} \cup \mathscr{D}^X, \tag{9}$$

where $\mathscr{D}^X = \{\varnothing\} \cup \{\{a\}: a \in X\}$, and for $B \in \beta_{fin}^X \setminus \{\phi\}$,

$$r_{fin}(B) := \left\{\mathscr{R} \in REL(X): \exists j \in I, \exists B_j \in \beta^{X_j}, \exists \mathscr{R}_j \in r_j[B_j] \mid \mathscr{R} << g_j^X \mathscr{R}_j\right\} \cup,$$
$$\{\mathscr{R} \in REL(X): \exists a \in B \mid (a, a) \in \cap \{R : R \in \mathscr{R}\}\} with r_{fin}(\phi) := \{\phi\}. \tag{10}$$

Proof. It is easy to observe that $(X, \beta_{fin}^X, r_{fin})$ is an ordered-RELspace and $f_i : (X_j, \beta^{X_j}, r_j)_{j \in I} \longrightarrow (X, \beta_{fin}^X, r_{fin})$ is a RELmap. Suppose that $g : (X, \beta_{fin}^X, r_{fin}) \longrightarrow (Y, \beta^Y, r_Y)$ is a mapping. We show that g is a RELmap iff $g \circ f_j$ is a RELmap. Necessity is obvious since the composition of two RELmaps is RELmap again.

Conversely, let $g \circ f_j : (X_j, \beta^{X_j}, r_j) \longrightarrow (Y, \beta^Y, r_Y)$ be a RELmap.

Then, first, we show that g is a bounded map. Let $B_i \in \beta_j^X$; it implies that $g(f_j(B_j)) = g \circ f_j(B_j) \in \beta^Y$. For our own convenience, take $f_j(B_j) = B'$, and since f_j is a RELmap, then $B' \in \beta_{fin}^X$, and consequently, g is bounded.

Now, let $B_j \in \beta^{X_j} \setminus \{\phi\}$ and $\mathscr{R}_i \in r_j(B_j)$. By the Definition 3, we have $g(f_j(B_j)) = g \circ f_j(B_j) \in r_Y(g(f_j(B_j)))$. On the other hand, f_j is a RELmap; it follows that $f_j(\mathscr{R}_j) \in r_{fin}(f_j(B_j))$. Take $f_j(\mathscr{R}_j) = \mathscr{R}'$. Then, we have $\mathscr{R}' \in r_{fin}(B')$, and subsequently, $g(\mathscr{R}') \in r_Y(g(B'))$ which shows g is a RELmap. □

Lemma 12. *Let $X \neq \phi$, and (X, β^X, r) be an ordered-RELspace.*

(i) A RELstructure (β^X, r) is discrete iff $(\beta^X, r) := (\mathscr{D}^X, r_{dis})$, where $\mathscr{D}^X = \{\varnothing\} \cup \{\{a\}: a \in X\}$ and $r_{dis}(\{a\}) = \{\mathscr{R} \in REL(X): (a, a) \in \cap \{R : R \in \mathscr{R}\}\} = \{\mathscr{R} \in REL(X): \mathscr{R} << \{\{(a, a)\}\}\}$ with $r_{dis}(\phi) := \{\phi\}$

(ii) A RELstructure (β^X, r) is indiscrete iff $(\beta^X, r) := (\underline{P}(X), r_{id})$, where $r_{id}(B) = \{\mathscr{R} \in REL(X): \{\phi\} \notin \mathscr{R}\}$ if $\beta^X \neq \phi$ with $r_{id}(\phi) := \{\phi\}$.

Proof. By applying Lemma 11, we get the desired result. □

Remark 13. The topological functor $\mathscr{U} : \mathscr{C} \longrightarrow Set$, where $\mathscr{C} =$ ***O − REL*** is normalized since an unique RELstructure

$\beta^X = \{\varnothing\}$, and $r(\varnothing) = \{\varnothing\}$ exists whenever $X = \varnothing$ and a unique RELstructure $\beta^X = \{\varnothing,\{a\}\}$, $r(\varnothing) = \{\varnothing\}$ and $r(\{a\}) = \{\varnothing,\{(a,a)\}\}$ exists whenever $X = \{a\}$. Furthermore, the topological functor $\mathscr{U} : \boldsymbol{O-REL} \longrightarrow \boldsymbol{Set}$ is geometric since the regular sub-object of a discrete RELspace is discrete, and finite product of discrete RELstructures is discrete again.

4. Local T_0 and Local T_1 Ordered-RELspaces

In this section, we define notions for T_0 and T_1 ordered-RELspaces at some point.

Let X be any set and $p \in X$. We define the *wedge product of X at p* as the two disjoint copies of X at p and denote it as $X\vee_p X$. For a point $a \in X\vee_p X$, we write it as a_1 if a belongs to the first component of the wedge product; otherwise, we write a_2 that is in the second component. Moreover, X^2 is the cartesian product of X.

Definition 14 (cf. [14]).

(i) A mapping $A_p : X\vee_p X \longrightarrow X^2$ is said to be **principal *p*-axis mapping** provided that

$$A_p(a_j) := \begin{cases} (a,p)\,; & j = 1, \\ (p,a)\,; & j = 2, \end{cases} \tag{11}$$

(ii) A mapping $S_p : X\vee_p X \longrightarrow X^2$ is said to be **skewed *p*-axis mapping** provided that

$$S_p(a_j) := := \begin{cases} (a,a)\,; & j = 1, \\ (p,a)\,; & j = 2, \end{cases} \tag{12}$$

(iii) A mapping $\nabla_p : X\vee_p X \longrightarrow X$ is said to be **fold mapping at *p*** provided that

$$\nabla_p(a_j) := a, j = 1, 2. \tag{13}$$

Assume that $\mathfrak{U} : \mathscr{C} \longrightarrow \mathbf{Set}$ is a topological functor, $X \in Obj(\mathscr{C})$ with $\mathfrak{U}X = Z$ and $p \in Z$.

Definition 15 (cf. [14]).

(i) X is $\bar{T}_0$ at p provided that the initial lift of the $\mathfrak{U}$-source $\{Z\vee_p Z \xrightarrow{A_p} \mathfrak{U}(X^2) = Z^2$ and $Z\vee_p Z \xrightarrow{\nabla_p} \mathfrak{U}DZ = Z\}$ is discrete

(ii) X is T_0' at p provided that the initial lift of the $\mathfrak{U}$-source $\{Z\vee_p Z \xrightarrow{id} \mathfrak{U}(X\vee_p X) = Z\vee_p Z$ and $Z\vee_p Z \xrightarrow{\nabla_p} \mathfrak{U}DZ = Z\}$ is discrete, where $X\vee_p X$ is the wedge product in $\mathscr{C}$, i.e., the final lift of the $\mathfrak{U}$-sink $\{\mathfrak{U}X = Z \xrightarrow{i_1,i_2} Z\vee_p Z\}$, where i_1, i_2 represent the canonical injections

(iii) X is T_1 at p provided that the initial lift of the $\mathfrak{U}$-source $\{Z\vee_p Z \xrightarrow{S_p} \mathfrak{U}(X^2) = Z^2$ and $Z\vee_p Z \xrightarrow{\nabla_p} \mathfrak{U}DZ = Z\}$ is discrete.

Remark 16.

(i) In **TOP**, $\bar{T}_0$ and T'_0 at p (respectively, T_1 at p) are equivalent to the classical T_0 at p (respectively, the classical T_1 at p), i.e., for each $a \in X$ with $a \neq p$, there exists a neighborhood N_a of "a" not containing "p" or (respectively, and); there exists a neighborhood N_p of "p" not containing "a" [35]

(ii) A topological space X is T_0 (respectively T_1) iff X is T_0 (respectively T_1) at p for each $p \in X$ [35]

(iii) Let $\mathfrak{U} : \mathscr{C} \longrightarrow \mathbf{Set}$ be a topological functor, $X \in Obj(\mathscr{C})$ and $p \in \mathfrak{U}(X)$ be a retract of X. Then, if X is $\bar{T}_0$ or T_1 at p, then X is T_0' at p but not conversely in general [36].

Theorem 17. *Let* (X, β^X, r) *be ordered-RELspace and* $p \in X$. *Then,* (X, β^X, r) *is* $\bar{T}_0$ *at p if and only if for each* $a \in X$ *with* $a \neq p$, *the following holds:*

(i) $\{a,p\} \notin \beta^X$

(ii) $\{\mathscr{R} \in REL(X)\colon \mathscr{R} << \{\{(a,p)\}\}\} \notin r(\{a\})$ *or* $\{\mathscr{R} \in REL(X)\colon \mathscr{R} << \{\{(p,a)\}\}\} \notin r(\{p\})$

(iii) $\{\mathscr{R} \in REL(X)\colon \mathscr{R} << \{\{(a,p)\}\}\} \notin r(\{p\})$ *or* $\{\mathscr{R} \in REL(X)\colon \mathscr{R} << \{\{(p,a)\}\}\} \notin r(\{a\})$

(iv) $\{\mathscr{R} \in REL(X)\colon \mathscr{R} << \{\{(a,a),(p,p)\}\}\} \notin r(\{a\})$ *or* $\{\mathscr{R} \in REL(X)\colon \mathscr{R} << \{\{(p,p),(a,a)\}\}\} \notin r(\{p\})$.

Proof. Let (X, β^X, r) be $\bar{T}_0$ at p; we show the conditions (i) to (iv) are holding:

(i) Suppose that $\{a,p\} \in \beta^X$ for all $a \in X$ with $a \neq p$. Let $U = \{a_1, a_2\} \in X\vee_p X$, then since $\nabla_p(U) = \nabla_p(\{a_1, a_2\}) = (\{\nabla_p a_1, \nabla_p a_2\}) = \{a\} \in \mathscr{D}^X$ and for $j = 1, 2$, $\pi_j A_p(U) = \{a,p\} \in \beta^X$ (by the assumption), where $\pi_j : X^2 \longrightarrow X$ for j =1,2 are projection maps. By Definitions 1 and 15 and Lemma 10, a contradiction, it follows $\{a,p\} \notin \beta^X$

(ii) Assume that $\{\mathscr{R} \in REL(X)\colon \mathscr{R} << \{\{(a,p)\}\}\} \in r(\{a\})$ and $\{\mathscr{R} \in REL(X)\colon \mathscr{R} << \{\{(p,a)\}\}\} \in r(\{p\})$. Particularly, let $\mathscr{R}_1 = \{\{(a_1,a_2)\}\} \in REL(X\vee_p X)$ and $B = \{a_1\} \in \mathscr{D}^{X\vee_p X} \setminus \{\varnothing\}$; then, $\nabla_p \mathscr{R}_1 = \nabla_p\{\{($

$a_1, a_2)\}\} = \{\{(a,a)\}\} \in r_{dis}(\{a\})$. By the assumption, $\pi_1 A_p(\mathscr{R}_1) = \{\{(\pi_1 A_p a_1, \pi_1 A_p a_2)\}\} = \{\{(a,p)\}\} \in r(\{a\})$ and $\pi_2 A_p(\mathscr{R}_1) = \{\{(\pi_2 A_p a_1, \pi_2 A_p a_2)\}\} = \{\{(p,a)\}\} \in r(\{p\})$. Since (X, β^X, r) is $\bar{T}_0$ at p, it follows that $\mathscr{R}_1 \in \bar{r}_{dis}(\{a_1\})$, where $\bar{r}_{dis}$ is the discrete structure on $X \vee_p X$.

Similarly, for $B = \{a_2\} \in \mathscr{D}^{X\vee_p X} \setminus \{\varnothing\}$, we get $\mathscr{R}_1 \in \bar{r}_{dis}(\{a_2\})$, a contradiction to the discreteness of $\bar{r}_{dis}(B)$.

Thus, $\{\mathscr{R} \in REL(X)\colon \mathscr{R} << \{\{(a,p)\}\}\} \notin r(\{a\})$ or $\{\mathscr{R} \in REL(X)\colon \mathscr{R} << \{\{(p,a)\}\}\} \notin r(\{p\})$.

(iii) Suppose that $\{\mathscr{R} \in REL(X)\colon \mathscr{R} << \{\{(a,p)\}\}\} \in r(\{p\})$ and $\{\mathscr{R} \in REL(X)\colon \mathscr{R} << \{\{(p,a)\}\}\} \in r(\{a\})$. In particular, let $\mathscr{R}_2 = \{\{(a_2, a_1)\}\} \in REL(X\vee_p X)$ and $B = \{a_1\} \in \mathscr{D}^{X\vee_p X} \setminus \{\varnothing\}$; then, $\nabla_p \mathscr{R}_1 = \nabla_p\{\{(a_2, a_1)\}\} = \{\{(a,a)\}\} \in r_{dis}(\{a\})$, and by the assumption $\pi_1 A_p(\mathscr{R}_2) = \{\{(p,a)\}\} \in r(\{a\})$ and $\pi_2 A_p(\mathscr{R}_2) = \{\{(a,p)\}\} \in r(\{p\})$. Since (X, β^X, r) is $\bar{T}_0$ at p, we get that $\mathscr{R}_2 \in \bar{r}_{dis}(\{a_1\})$, where $\bar{r}_{dis}$ is the discrete structure on $X\vee_p X$

Similarly, for $B = \{a_2\} \in \mathscr{D}^{X\vee_p X} \setminus \{\varnothing\}$, we get $\mathscr{R}_2 \in \bar{r}_{dis}(\{a_2\})$, a contradiction.

Therefore, $\{\mathscr{R} \in REL(X)\colon \mathscr{R} << \{\{(a,p)\}\}\} \notin r(\{p\})$ or $\{\mathscr{R} \in REL(X)\colon \mathscr{R} << \{\{(p,a)\}\}\} \notin r(\{a\})$.

(iv) Assume that $\{\mathscr{R} \in REL(X)\colon \mathscr{R} << \{\{(a,a),(p,p)\}\}\} \in r(\{a\})$ and $\{\mathscr{R} \in REL(X)\colon \mathscr{R} << \{\{(p,p),(a,a)\}\}\} \in r(\{p\})$. Let $\mathscr{R}_3 = \{\{(a_1,a_1),(a_2,a_2)\}\} \in REL(X\vee_p X)$ and $B = \{a_1\} \in \mathscr{D}^{X\vee_p X} \setminus \{\varnothing\}$; then, $\nabla_p \mathscr{R}_3 = \nabla_p\{\{(a_1,a_1),(a_2,a_2)\}\} = \{\{(a,a)\}\} \in r_{dis}(\{a\})$, $\pi_1 A_p(\mathscr{R}_3) = \{\{(a,a),(p,p)\}\} \in r(\{a\})$, $\pi_2 A_p(\mathscr{R}_3) = \{\{(p,p),(a,a)\}\} \in r(\{p\})$ (by the assumption). Since (X, β^X, r) is $\bar{T}_0$ at p, it follows that $\mathscr{R}_3 \in \bar{r}_{dis}(\{a_1\})$, where $\bar{r}_{dis}$ is the discrete structure on $X\vee_p X$.

Similarly, for $B = \{a_2\} \in \mathscr{D}^{X\vee_p X} \setminus \{\varnothing\}$, we get $\mathscr{R}_3 \in \bar{r}_{dis}(\{a_2\})$, a contradiction.

Hence, $\{\mathscr{R} \in REL(X)\colon \mathscr{R} << \{\{(a,a),(p,p)\}\}\} \notin r(\{a\})$ or $\{\mathscr{R} \in REL(X)\colon \mathscr{R} << \{\{(p,p),(a,a)\}\}\} \notin r(\{p\})$.

Conversely, suppose (i) to (iv) are holding.

Let $(\beta^{X\vee_p X}, \bar{r})$ be the initial structure induced by $A_p : X\vee_p X \longrightarrow (X^2, \beta^{X^2}, r^2)$ and $\nabla_p : X\vee_p X \longrightarrow (X, \mathscr{D}^X, r_{dis})$, where (B^{X^2}, r^2) is the product RELstructure on X^2 and $(\mathscr{D}^X, r_{dis})$ the discrete RELstructure on X.

We show that $(\beta^{X\vee_p X}, \bar{r})$ is the discrete REL structure on $X\vee_p X$, i.e., we show that $\beta^{X\vee_p X} = \mathscr{D}^{X\vee_p X} = \{\{\varnothing\} \cup \{a_j\}\,;\, j = 1,2 \text{ and } a_j \in X\vee_p X\}$ and for $B \in \mathscr{D}^{X\vee_p X}$, $\bar{r}(B) = \{\bar{\mathscr{R}} \in REL(X\vee_p X)\colon \bar{\mathscr{R}} << \{\{(a_j,a_j)\}\}\,;\, j = 1,2\}$.

Let $U \in \beta^{X\vee_p X}$ and $\nabla_p U \in \mathscr{D}^X$; if $\nabla_p U = \varnothing$, then $U = \varnothing$. Suppose $\nabla_p U \neq \varnothing$. Then, we have $\nabla_p U = \{a\}$ for some $a \in X$, and if $a = p$, then $U = \{p\}$; let $a \neq p$; then, it further implies that $U = \{a_1\}$ or $U = \{a_2\}$ and $U = \{a_1, a_2\}$. By the assumption, $\pi_j A_p U = \pi_j A_p\{a_1, a_2\} = \{a, p\} \notin \beta^X$ (for j = 1,2). Thus, $U = \{a_1\}$ and $U = \{a_2\}$; subsequently, $\beta^{X\vee_p X} = \mathscr{D}^{X\vee_p X}$.

Now, $B \in \mathscr{D}^{X\vee_p X} \setminus \{\varnothing\}$ implies $B = \{a_1\}$ and $B = \{a_2\}$, and by Lemma 10, $\bar{r}(B) = \{\bar{\mathscr{R}} \in REL(X\vee_p X)\colon \pi_j A_p(\bar{\mathscr{R}}) \in r(\pi_j A_p(B)) \text{and} \nabla_p(\bar{\mathscr{R}}) \in r_{dis}(\nabla_p B)), where j = 1,2\}$.

Suppose $B = \{a_1\}$, then
$\bar{r}(\{a_1\}) = \{\bar{\mathscr{R}} \in REL(X\vee_p X)\colon \pi_j A_p(\bar{\mathscr{R}}) \in r(\pi_j A_p(\{a_1\}))$ and $\nabla_p \bar{\mathscr{R}} \in r_{dis}(\nabla_p\{a_1\}), where j = 1,2\}$; it follows that $\bar{r}(\{a_1\}) = \{\bar{\mathscr{R}} \in REL(X\vee_p X)\colon \pi_1 A_p(\bar{\mathscr{R}}) \in r(\{a\}) \text{and} \pi_2 A_p(\bar{\mathscr{R}}) \in r(\{p\}) \text{and} \nabla_p(\bar{\mathscr{R}}) \in r_{dis}(\{a\})\}$.

Since $\nabla_p(\bar{\mathscr{R}}) \in r_{dis}(\{a\}) = \{\mathscr{R} \in REL(X)\colon \mathscr{R} << \{\{(a,a)\}\}\}$, we have the following possibilities of $\mathscr{R}$:

$\{\bar{\mathscr{R}} \in REL(X\vee_p X)\colon \bar{\mathscr{R}} << \{\{(a_1,a_1)\}\}\}$,
$\{\bar{\mathscr{R}} \in REL(X\vee_p X)\colon \bar{\mathscr{R}} << \{\{(a_2,a_2)\}\}\}$,
$\{\bar{\mathscr{R}} \in REL(X\vee_p X)\colon \bar{\mathscr{R}} << \{\{(a_1,a_2)\}\}\}$,
$\{\bar{\mathscr{R}} \in REL(X\vee_p X)\colon \bar{\mathscr{R}} << \{\{(a_2,a_1)\}\}\}$,
$\{\bar{\mathscr{R}} \in REL(X\vee_p X)\colon \bar{\mathscr{R}} \ll \{\{(a_1,a_1),(a_2,a_2)\}\}\}$.

Case (i). Suppose $\{\bar{\mathscr{R}} \in REL(X\vee_p X)\colon \bar{\mathscr{R}} << \{\{(a_1,a_1)\}\}\}$. It follows that for all $\bar{R} \in \bar{\mathscr{R}}$ such that $\{(a_1,a_1)\} \subseteq \bar{R}$, and $\pi_1 A_p\{(a_1,a_1)\} \subseteq \pi_1 A_p \bar{R}$, $\pi_1 A_p \bar{\mathscr{R}} << \pi_1 A_p\{\{(a_1,a_1)\}\} = \{\{(\pi_1 A_p a_1, \pi_2 A_p a_1)\}\} = \{\{(a,a)\}\}$. By Definition 2, $\pi_1 A_p \bar{\mathscr{R}} << \{\{(a,a)\}\} \in r(\{a\})$. Similarly, $\pi_2 A_p \bar{\mathscr{R}} << \{\{(p,p)\}\} \in r(\{p\})$. Therefore, $\{\bar{\mathscr{R}} \in REL(X\vee_p X)\colon \mathscr{R} << \{\{(a_1,a_1)\}\}\}$ holds

Case (ii). $\{\bar{\mathscr{R}} \in REL(X\vee_p X)\colon \bar{\mathscr{R}} << \{\{(a_2,a_2)\}\}\}$ holds. The proof is similar to Case (i)

Case (iii). Let $\{\bar{\mathscr{R}} \in REL(X\vee_p X)\colon \bar{\mathscr{R}} << \{\{(a_1,a_2)\}\}\}$. It follows that for all $\bar{R} \in \bar{\mathscr{R}}$ such that $\{(a_1,a_2)\} \subseteq \bar{R}$, and $\pi_1 A_p\{(a_1,a_2)\} \subseteq \pi_1 A_p \bar{R}$, $\pi_1 A_p \bar{\mathscr{R}} << \pi_1 A_p\{\{(a_1,a_2)\}\} = \{\{(a,p)\}\}$. By the assumption, we get $\pi_1 A_p \bar{\mathscr{R}} << \{\{(a,p)\}\} \notin r(\{a\})$. Similarly, $\pi_2 A_p \bar{\mathscr{R}} << \{\{(p,a)\}\} \notin r(\{p\})$. Thus, $\{\bar{\mathscr{R}} \in REL(X\vee_p X)\colon \bar{\mathscr{R}} << \{\{(a_1,a_2)\}\}\}$ cannot be possible

Case (iv). Similar to Case (iii), we conclude that $\{\bar{\mathscr{R}} \in REL(X\vee_p X)\colon \bar{\mathscr{R}} << \{\{(a_2,a_1)\}\}\}$ is not possible

Case (v). If $\{\bar{\mathscr{R}} \in REL(X\vee_p X)\colon \bar{\mathscr{R}} << \{\{(a_1,a_1),(a_2,a_2)\}\}\}$. It follows that for all $\bar{R} \in \bar{\mathscr{R}}$ such that $\{(a_1,a_1),(a_2,a_2)\} \subseteq \bar{R}$, and $\pi_1 A_p\{(a_1,a_1),(a_2,a_2)\} \subseteq \pi_1 A_p \bar{R}$, for all $\bar{R} \in \bar{\mathscr{R}}$ implying $\pi_1 A_p \bar{\mathscr{R}} << \pi_1 A_p\{\{(a_1,a_1),(a_2,a_2)\}\} = \{\{(a,a),(p,p)\}\}$. By the assumption, $\pi_1 A_p \bar{\mathscr{R}} << \{\{(a,a),(p,p)\}\} \notin r(\{a\})$. Similarly, $\pi_2 A_p \bar{\mathscr{R}} << \{\{(p,p),(a,a)\}\} \notin r(\{p\})$. Hence, $\{\bar{\mathscr{R}} \in REL(X\vee_p X)\colon \bar{\mathscr{R}} << \{\{(a_1,a_1),(a_2,a_2)\}\}\}$ is not possible.

□

Similarly, if $B=\{a_2\}$, only Case (i) and Case (ii) are holding. By Lemma 12, $\bar{r}(B)=\{\bar{\mathscr{R}}\in REL(X\vee_p X)\colon \bar{\mathscr{R}}<<\{\{(a_j,a_j)\}\};j=1,2\}$ is discrete.

Therefore, by Definition 15, (X,β^X,r) is $\bar{T}_0$ at p.

Theorem 18. *Let* (X,β^X,r) *be an ordered-RELspace and* $p\in X$.

(X,β^X,r) *is* T_1 *at p if and only if for any* $a\in X$ *with* $a\neq p$, *the following holds:*

(i) $\{a,p\}\notin\beta^X$

(ii) $\{\mathscr{R}\in REL(X)\colon \mathscr{R}<<\{\{(a,p)\}\}\}\notin r(\{a\})$ *and* $\{\mathscr{R}\in REL(X)\colon \mathscr{R}<<\{\{(p,a)\}\}\}\notin r(\{p\})$

(iii) $\{\mathscr{R}\in REL(X)\colon \mathscr{R}<<\{\{(a,p)\}\}\}\notin r(\{p\})$ *and* $\{\mathscr{R}\in REL(X)\colon \mathscr{R}<<\{\{(p,a)\}\}\}\notin r(\{a\})$

(iv) $\{\mathscr{R}\in REL(X)\colon \mathscr{R}<<\{\{(a,a),(p,p)\}\}\}\notin r(\{a\})$ *and* $\{\mathscr{R}\in REL(X)\colon \mathscr{R}<<\{\{(p,p),(a,a)\}\}\}\notin r(\{p\})$.

Proof. By following the same technique used in Theorem 17, and replacing the mapping A_p by the mapping S_p, we get the proof. □

Theorem 19. *All ordered-RELspaces are* T'_0 *at p.*

Proof. Let (X,β^X,r) be ordered-RELspace and $p\in X$. By Definition 15, we show that for each $U\in\beta^{X\vee_p X}$, $U\subset i_k(V)$ (where $k=1,2$) for some $V\in\beta^X$ and $\nabla_p U\in\mathscr{D}^X$. $\nabla_p U=\phi$ implying $U=\phi$. Suppose $\nabla_p U\neq\phi$, it implies that $\nabla_p U=\{a\}$ for some $a\in X$. If $a=p$, then $\nabla_p U=\{p\}$ implying $U=\{p\}$.

Suppose $a\neq p$, it follows that $U=\{a_1\}$, $\{a_2\}$ or $\{a_1,a_2\}$. If $U=\{a_1,a_2\}$, then $\{a_1,a_2\}\subset i_1(V)$ for some $V\in\beta^X$ which shows that a_2 should be in the first component of the wedge product $X\vee_p X$, a contradiction. In similar manner, $\{a_1,a_2\}\subset i_2(V)$ for some $V\in\beta^X$. Hence, U=$\{a_1,a_2\}$. Thus, we must have $U=\{a_j\}$ for $j=1,2$ only and consequently, $\beta^{X\vee_p X}=\mathscr{D}^{X\vee_p X}$, the discrete RELstructure on $X\vee_p X$.

Now, for $B\in\mathscr{D}^{X\vee_p X}\setminus\{\phi\}$, by Lemma 10, $\bar{r}(B)=\{\bar{\mathscr{R}}\in REL(X\vee_p X)\colon \bar{\mathscr{R}}<<i_1(s)$ for some $s\in r(B)$, $\bar{\mathscr{R}}<<i_2(s)$ for some $s\in r(B)$ and $\nabla_p(\bar{\mathscr{R}})\in r_{dis}(\nabla_p B)\}$. Since $\nabla_p(\bar{\mathscr{R}})\in r_{dis}(\{B\})=\{\mathscr{R}\in REL(X)\colon \mathscr{R}<<\{\{(a_j,a_j)\}\} where j=1,2\}$, we have the following possibilities of $\mathscr{R}$:

$\{\bar{\mathscr{R}}\in REL(X\vee_p X)\colon \bar{\mathscr{R}}<<\{\{(a_1,a_1)\}\}\}$,
$\{\bar{\mathscr{R}}\in REL(X\vee_p X)\colon \bar{\mathscr{R}}<<\{\{(a_2,a_2)\}\}\}$,
$\{\bar{\mathscr{R}}\in REL(X\vee_p X)\colon \bar{\mathscr{R}}<<\{\{(a_1,a_2)\}\}\}$,
$\{\bar{\mathscr{R}}\in REL(X\vee_p X)\colon \bar{\mathscr{R}}<<\{\{(a_2,a_1)\}\}\}$,
$\{\bar{\mathscr{R}}\in REL(X\vee_p X)\colon \bar{\mathscr{R}}<<\{\{(a_1,a_1),(a_2,a_2)\}\}\}$.

In particular, for $\{\bar{\mathscr{R}}\in REL(X\vee_p X)\colon \bar{\mathscr{R}}<<\{\{(a_1,a_2)\}\}\}$. It follows that, for all $\bar{R}\in\bar{\mathscr{R}}$ such that $\{(a_1,a_2)\}\subset\bar{R}$, and (for k=1,2), $i_k\{(a_1,a_2)\}\subset i_K\bar{R}$ implying $i_k\bar{\mathscr{R}}<<i_k\{\{(a_1,a_2)\}\}$. It follows a_2 (respectively, a_1) in the first (respectively, second) component of the wedge product $X\vee_p X$, a contradiction. Similarly, for $\{\bar{\mathscr{R}}\in REL(X\vee_p X)\colon \bar{\mathscr{R}}<<\{\{(a_2,a_1)\}\}\}$ and $\{\bar{\mathscr{R}}\in REL(X\vee_p X)\colon \bar{\mathscr{R}}<<\{\{(a_1,a_1),(a_2,a_2)\}\}\}$, we get a contradiction.

Therefore, $\bar{r}(B)=\{\bar{\mathscr{R}}\in REL(X\vee_p X)\colon \mathscr{R}<<\{\{(a_j,a_j)\}\};j=1,2\}$. Consequently, by Definition 15(i) and Lemma 10, (X,β^X,r) is T'_0 at p. □

5. T_0 and T_1 Ordered-RELspaces

In this section, we define generically notions of T_0 and T_1 in ordered-RELspaces.

The characterization of T_0 objects in categorical topology has been an important idea in a topological universe. Therefore, several attempts has been made such as in 1971 Brümmer [15], in 1973 Marny [18], in 1974 Hoffman [17], in 1977 Harvey [16], and in 1991 Baran [14] to discuss various approaches to generalize classical T_0 object and examined the relationship between different forms of generalized T_0 objects. One of the main purposes of generalization is to define Hausdorff objects in arbitrary topological categories. In 1991, Baran [14, 37] also generalizes the classical T_1 objects of topology to topological categories [14, 37]. In abstract topological categories [21], T_1 objects are used to define T_3, T_4, normal objects, regular, and completely regular. To characterize separation axioms, Baran's approach was to use initial and final lifts and discreteness.

In 1991, Baran [14] used the generic element method of topos theory introduced by Johnstone [38], to define generic separation axioms, due to the fact that points does not make sense in topos theory. In general, the wedge product $X\vee_p X$ at p can be replaced by $X^2\vee_\Delta X^2$ at diagonal Δ. Any element $(a,b)\in X^2\vee_\Delta X^2$ is written as $(a,b)_1$ (resp., $(a,b)_1$) if it lies in the first (resp., second) component of $X^2\vee_\Delta X^2$. Clearly, $(a,b)_1=(a,b)_2$, if and only if $a=b$.

Definition 20 (cf. [14]).

(i) A mapping $A: X^2\vee_\Delta X^2\longrightarrow X^3$ is called **principal axis mapping** provided that

$$A\left((a,b)_j\right) := \begin{cases}(a,b,a); & j=1,\\ (a,a,b); & j=2,\end{cases} \tag{14}$$

(ii) A mapping $S: X^2\vee_\Delta X^2\longrightarrow X^3$ is called **skewed axis mapping** provided that

$$S\left((a,b)_j\right) := \begin{cases}(a,b,b); & j=1,\\ (a,a,b); & j=2,\end{cases} \tag{15}$$

(iii) A mapping $\nabla: X^2\vee_\Delta X^2\longrightarrow X^2$ is called **fold mapping** provided that

$$\nabla\left((a,b)_j\right) := (a,b), j = 1,2. \quad (16)$$

Any element $(a,b) \in X^2 \vee_\Delta X^2$ is written as $(a,b)_1$ (resp., $(a,b)_1$) if it lies in the first (resp., second) component of $X^2 \vee_\Delta X^2$. Clearly, $(a,b)_1 = (a,b)_2$ if and only if $a = b$.

Now, we replace the point p by any generic point δ and define the following separation axioms.

Definition 21. Let $\mathfrak{U} : \mathscr{C} \longrightarrow$ **Set** be a topological functor, $X \in Obj(\mathscr{C})$ with $\mathfrak{U}X = Z$.

(i) X is $\bar{T}_0$ provided that the initial lift of the $\mathfrak{U}$-source $\{Z^2 \vee_\Delta Z^2 \xrightarrow{A} \mathfrak{U}(X^3) = Z^3$ and $Z^2 \vee_\Delta Z^2 \xrightarrow{\nabla} \mathfrak{U}D(Z^2) = Z^2\}$ is discrete [14]

(ii) X is T'_0 provided that the initial lift of the $\mathfrak{U}$-source $\{Z^2 \vee_\Delta Z^2 \xrightarrow{id} \mathfrak{U}(Z^2 \vee_\Delta Z^2)' = Z^2 \vee_\Delta Z^2$ and $Z^2 \vee_\Delta Z^2 \xrightarrow{\nabla} \mathfrak{U}D(Z^2) = Z^2\}$ is discrete, where $(Z^2 \vee_\Delta Z^2)'$ is the final lift of the $\mathfrak{U}$-sink $\{\mathfrak{U}(X^2) = Z^2 \xrightarrow{i_1, i_2} Z^2 \vee_\Delta Z^2\}$ [14, 39]

(iii) X is called T_0 provided that X doesn't contain an indiscrete subspace with at least two points [18, 40]

(iv) X is T_1 provided that the initial lift of the $\mathfrak{U}$-source $\{Z^2 \vee_\Delta Z^2 \xrightarrow{S} \mathfrak{U}(X^3) = Z^3$ and $Z^2 \vee_\Delta Z^2 \xrightarrow{\nabla} \mathfrak{U}D(Z^2) = Z^2\}$ is discrete [14].

Remark 22.

(i) In **TOP**, all the properties of being T_0, $\bar{T}_0$ and T_0' (respectively, T_1) are equivalent to those classical ones which are T_0 (respectively, T_1), i.e., for each $a, b \in X$ with $a \neq b$, there exists a neighbourhood N_a of "a" not containing "b" or (respectively and), there exists a neighbourhood N_b of "b" not containing "a" [14, 18, 40]

(ii) In any topological category, $\bar{T}_0$ implies is T'_0 but not conversely in general. Also, each of the $\bar{T}_0$ and T'_0 has no relation to a T_0 [39]

(iii) Let $\mathfrak{U} : \mathscr{C} \longrightarrow$ **Set** be a topological functor, $X \in Obj(\mathscr{C})$ and $p \in \mathfrak{U}(X)$ be a retract of X. Then, if X is $\bar{T}_0$ (respectively T_1), then X is $\bar{T}_0$ at p (respectively T_1 at p) but not conversely in general [36].

Theorem 23. *Let (X, β^X, r) be an ordered-RELspace.*

(X, β^X, r) is $\bar{T}_0$ iff for each $a, b \in X$ with $a \neq b$, the following holds:

(i) $\{a,b\} \notin \beta^X$

(ii) $\{\mathscr{R} \in REL(X)\colon \mathscr{R} << \{\{(a,b)\}\}\} \notin r(\{a\}$ *or* $\{\mathscr{R} \in REL(X)\colon \mathscr{R} << \{\{(b,a)\}\}\} \notin r(\{b\})$

(iii) $\{\mathscr{R} \in REL(X)\colon \mathscr{R} << \{\{(a,b)\}\}\} \notin r(\{b\})$ *or* $\{\mathscr{R} \in REL(X)\colon \mathscr{R} << \{\{(b,a)\}\}\} \notin r(\{a\})$

(iv) $\{\mathscr{R} \in REL(X)\colon \mathscr{R} << \{\{(a,a),(b,b)\}\}\} \notin r(\{a\})$ *or* $\{\mathscr{R} \in REL(X)\colon \mathscr{R} << \{\{(b,b),(a,a)\}\}\} \notin r(\{b\})$.

Proof. Suppose (X, β^X, r) is $\bar{T}_0$, we show that conditions (i) to (iv) are holding.

(i) Suppose that $\{a,b\} \in \beta^X$ for each a,b∈ X, $a \neq b$. Let $U = \{((a,b)_1, (a,b)_2)\} \in X^2 \vee_\Delta X^2$. Note that $\nabla(U) = \nabla\{((a,b)_1,(a,b)_2)\} = \{(a,b)\} \in \mathscr{D}^{X^2}$ and $\pi_1 A(U) = \{a\} \in \beta^X$. By the assumption, $\pi_k A(U) = \pi_k A\{((a,b)_1,(a,b)_2)\} = \{a,b\} \in \beta^X$, where $\pi_k : X^3 \longrightarrow X^2$ (for k=2,3) are projection maps. By Definitions 1 and 15 and Lemma 10, it leads to a contradiction, it follows that $\{a,b\} \notin \beta^X$

(ii) Suppose that $\{\mathscr{R} \in REL(X)\colon \mathscr{R} << \{\{(a,b)\}\}\} \in r(\{a\})$ and $\{\mathscr{R} \in REL(X)\colon \mathscr{R} << \{\{(b,a)\}\}\} \in r(\{b\})$. Let $\mathscr{R}_1 = \{\{((a,b)_1,(a,b)_2)\}\} \in REL(X^2 \vee_\Delta X^2)$ and $B = \{(a,b)_1\} \in \mathscr{D}^{X^2 \vee_\Delta X^2} \setminus \{\phi\}$, then $\nabla(\mathscr{R}_1) = \nabla\{\{((a,b)_1,(a,b)_2)\}\} = \{\{(\nabla(a,b)_1, \nabla(a,b)_2)\}\} = \{\{(a,b)\}\} \in r^2_{dis}(\{(a,b)\})$. By Definition 2, $\pi_1 A\{\{((a,b)_1,(a,b)_2)\}\} = \{\{(\pi_1 A(a,b)_1, \pi_1 A(a,b)_2)\}\} = \{\{(a,a)\}\} \in r(\{a\})$ and by the assumption, $\pi_2 A\{\{((a,b)_1,(a,b)_2)\}\} = \{\{(b,a)\}\} \in r(\{b\})$ and $\pi_3 A\{\{((a,b)_1,(a,b)_2)\}\} = \{\{(a,b)\}\} \in r(\{a\})$. Since (X, β^X, r) is $\bar{T}_0$, we conclude $\mathscr{R}_1 \in \bar{r}^2_{dis}(\{(a,b)_1\})$, where $\bar{r}^2_{dis}$ is the discrete structure on $X^2 \vee_\Delta X^2$

Similarly, for $B = \{(a,b)_2\} \in \mathscr{D}^{X^2 \vee_\Delta X^2} \setminus \{\phi\}$, we get $\mathscr{R}_1 \in \bar{r}^2_{dis}(\{(a,b)_2\})$, a contradiction.

Therefore, $\{\mathscr{R} \in REL(X)\colon \mathscr{R} << \{\{(a,b)\}\}\} \notin r(\{a\}$ or $\{\mathscr{R} \in REL(X)\colon \mathscr{R} << \{\{(b,a)\}\}\} \notin r(\{b\})$.

(iii) Suppose that $\{\mathscr{R} \in REL(X)\colon \mathscr{R} << \{\{(a,b)\}\}\} \in r(\{b\})$ and $\{\mathscr{R} \in REL(X)\colon \mathscr{R} << \{\{(b,a)\}\}\} \in r(\{a\})$. In particular, let $\mathscr{R}_2 = \{\{((a,b)_2,(a,b)_1)\}\} \in REL(X^2 \vee_\Delta X^2)$ and $B = \{(a,b)_1\} \in \mathscr{D}^{X^2 \vee_\Delta X^2} \setminus \{\phi\}$, then $\nabla(\mathscr{R}_2) = \nabla\{\{((a,b)_2,(a,b)_1)\}\} = \{\{(a,b)\}\} \in r^2_{dis}(\{(a,b)\})$. By Definition 2, $\pi_1 A\{\{((a,b)_2,(a,b)_1)\}\} = \{\{(a,a)\}\} \in r(\{a\})$ and by the assumption, $\pi_2 A\{\{((a,b)_2,(a,b)_1)\}\} = \{\{(\pi_2 A(a,b)_2, \pi_2 A(a,b)_1)\}\} = \{\{(a,b)\}\} \in r(\{b\})$ and $\pi_3 A\{\{((a,b)_2,(a,b)_1)\}\} = \{\{(\pi_3 A(a,b)_2, \pi_3 A(a,b)_1)\}\} = \{\{(b,a)\}\} \in r(\{a\})$. Since (X, β^X, r) is $\bar{T}_0$ it follows that $\mathscr{R}_2 \in \bar{r}^2_{dis}(\{(a,b)_2\})$, where $\bar{r}^2_{dis}$ is the discrete structure on $X^2 \vee_\Delta X^2$

Similarly, for $B = \{(a,b)_2\} \in \mathscr{D}^{X^2 \vee_\Delta X^2} \setminus \{\phi\}$, we get $\mathscr{R}_2 \in \bar{r}^2_{dis}(\{(a,b)_1\})$, a contradiction.

Thus, $\{\mathscr{R} \in REL(X)\colon \mathscr{R} << \{\{(a,b)\}\}\} \notin r(\{b\})$ or $\{\mathscr{R} \in REL(X)\colon \mathscr{R} << \{\{(b,a)\}\}\} \notin r(\{a\})$.

(iv) Suppose that $\{\mathscr{R} \in REL(X)\colon \mathscr{R} << \{\{(a,a),(b,b)\}\}\} \in r(\{a\})$ and $\{\mathscr{R} \in REL(X)\colon \mathscr{R} << \{\{(b,b),(a,a)\}\}$

$\} \in r(\{b\})$. Let $\mathscr{R}_3 = \{(((a,b)_1, (a,b)_1), ((a,b)_2, (a,b)_2)\}\} \in REL(X^2 \vee_\Delta X^2)$, and $B = \{(a,b)_1\} \in \mathscr{D}^{X^2 \vee_\Delta X^2} \setminus \{\phi\}$; then, $\nabla \mathscr{R}_3 = \nabla\{\{(((a,b)_1, (a,b)_1), ((a,b)_2, (a,b)_2)\}\} = \{\{(a,b)\}\} \in r^2_{dis}(\{(a,b)\})$, and by Definition 2, $\pi_1 A\{\{(((a,b)_1, (a,b)_1), ((a,b)_2, (a,b)_2)\}\} = \{\{(a,a)\}\} \in r(\{a\})$. By the assumption, $\pi_2 A\{\{(((a,b)_1, (a,b)_1), ((a,b)_2, (a,b)_2)\}\} = \{\{(b,b), (a,a)\}\} \in r(\{b\})$ and $\pi_3 A\{\{(((a,b)_1, (a,b)_1), ((a,b)_2, (a,b)_2)\}\} = \{\{(a,a), (b,b)\}\} \in r(\{a\})$. Since (X, β^X, r) is $\bar{T}_0$, we conclude $\mathscr{R}_3 \in \bar{r}^2_{dis}(\{(a,b)_1\})$, where $\bar{r}^2_{dis}$ is the discrete structure on $X^2 \vee_\Delta X^2$.

Similarly, for $B = \{(a,b)_2\} \in \mathscr{D}^{X^2 \vee_\Delta X^2} \setminus \{\phi\}$, we get $\mathscr{R}_3 \in \bar{r}^2_{dis}(\{(a,b)_2\})$, a contradiction to the discreteness of $\bar{r}^2_{dis}(B)$.

Hence, $\{\mathscr{R} \in REL(X)\colon \mathscr{R} << \{\{(a,a), (b,b)\}\}\} \notin r(\{a\})$ or $\{\mathscr{R} \in REL(X)\colon \mathscr{R} << \{\{(b,b), (a,a)\}\}\} \notin r(\{b\})$. □

Conversely, suppose (i) to (iv) are holding.

Let $(\beta^{X^2 \vee_\Delta X^2}, \bar{r}^2)$ be the initial structure induced by $A : X^2 \vee_\Delta X^2 \longrightarrow (X^3, \beta^{X^3}, r^3)$ and $\nabla : X^2 \vee_\Delta X^2 \longrightarrow (X^2, \mathscr{D}^{X^2}, r^2_{dis})$, where (β^{X^3}, r^3) is the product RELstructure on X^3 and $(\mathscr{D}^{X^2}, r^2_{dis})$ the discrete RELstructure on X^2.

We show that $(\beta^{X^2 \vee_\Delta X^2}, \bar{r}^2)$ is the discrete RELstructure on $X^2 \vee_\Delta X^2$, i.e, $\beta^{X^2 \vee_\Delta X^2} = \mathscr{D}^{X^2 \vee_\Delta X^2} = \{\{\phi\} \cup \{(a,b)_j\}\colon (a,b)_j \in X^2 \vee_\Delta X^2 for j = 1, 2\}$ and for $B \in \mathscr{D}^{X^2 \vee_\Delta X^2} \setminus \{\phi\}, \bar{r}^2(B) = \{\bar{\mathscr{R}} \in REL(X^2 \vee_\Delta X^2)\colon \bar{\mathscr{R}} << \{\{(a,b)_j, (a,b)_j, j = 1, 2\}\}$.

Let $U \in \beta^{X^2 \vee_\Delta X^2}$ and $\nabla U \in B^{X^2}$. If $\nabla U = \phi$, then $U = \phi$. Suppose $\nabla U \neq \phi$, then it follows that $\nabla U = \{(a,b)\}$ for some $(a,b) \in X^2$. If $a = b$, then $U = \{(b,b)\}$. Next, let $a \neq b$; then, we have $U = \{(a,b)_1\}$ or $U = \{(a,b)_2\}$ or $U = \{(a,b)_1, (a,b)_2\}$ and $\pi_1 AU = \pi_1 A\{(a,b)_1, (a,b)_2\} = \{\pi_1 A(a,b)_1, \pi_1 A(a,b)_2\} = \{a, a\}$, and by the assumption, we get $\pi_k A\{(a,b)_1, (a,b)_2\} = \{a, b\} \notin \beta^X$, (for k=2,3). Thus, $U = \{(a,b)_1\}$ or $U = \{(a,b)_2\}$, and subsequently, $\beta^{X^2 \vee_\Delta X^2} = \mathscr{D}^{X^2 \vee_\Delta X^2}$.

Now, $B \in \beta^{X^2 \vee_\Delta X^2} \setminus \{\phi\}$ implies $B = \{(a,b)_1\}$ and $B = \{(a,b)_2\}$, and by Lemma 10, $\bar{r}^2(B) = \{\bar{\mathscr{R}} \in REL(X^2 \vee_\Delta X^2)\colon \pi_j A\bar{\mathscr{R}} \in r(\pi_j A(B))and\nabla \bar{\mathscr{R}} \in r^2_{dis}(\nabla B))$, where j = 1, 2, 3$\}$.

Suppose $B = U = \{(a,b)_1\}$, thensince $\nabla \mathscr{R} \in r^2_{dis}(\{a,b\}) = \{\mathscr{R} \in REL(X^2)\colon \mathscr{R} << \{\{((a,b), (a,b))\}\}\}$, we have the following possibilities:

$\{\bar{\mathscr{R}} \in REL(X^2 \vee_\Delta X^2)\colon \bar{\mathscr{R}} << \{\{(a,b)_1, (a,b)_1\}\}\}$,
$\{\bar{\mathscr{R}} \in REL(X^2 \vee_\Delta X^2)\colon \bar{\mathscr{R}} << \{\{(a,b)_2, (a,b)_2\}\}\}$,
$\{\bar{\mathscr{R}} \in REL(X^2 \vee_\Delta X^2)\colon \bar{\mathscr{R}} << \{\{(a,b)_1, (a,b)_2\}\}\}$,
$\{\bar{\mathscr{R}} \in REL(X^2 \vee_\Delta X^2)\colon \bar{\mathscr{R}} << \{\{(a,b)_2, (a,b)_1\}\}\}$,
$\{\bar{\mathscr{R}} \in REL(X^2 \vee_\Delta X^2)\colon \bar{\mathscr{R}} << \{\{(((a,b)_1, (a,b)_1), ((a,b)_2, (a,b)_2)\}\}\}$.

Case (i). If $\{\bar{\mathscr{R}} \in REL(X^2 \vee_\Delta X^2)\colon \bar{\mathscr{R}} << \{\{(a,b)_1, (a,b)_1\}\}\}$. It follows that for all $\bar{R} \in \bar{\mathscr{R}}\{(a,b)_1, (a,b)_1\} \subseteq \bar{R}$ and $\pi_1 A\{(a,b)_1, (a,b)_1\} \subseteq \pi_1 A\bar{R}$, $\pi_1 A\bar{\mathscr{R}} << \pi_1 A\{\{(a,b)_1, (a,b)_1\}\} = \{\{\pi_1 A(a,b)_1, \pi_1 A(a,b)_1\}\} = \{\{(a,a)\}\}$, and by the Definition 2, we get $\pi_1 A\bar{\mathscr{R}} << \{\{(a,a)\}\} \in r(\{a\})$.

In a similar way, $\pi_2 A\bar{\mathscr{R}} << \{\{(b,b)\}\} \in r(\{b\})$ and $\pi_3 A\bar{\mathscr{R}} << \{\{(a,a)\}\} \in r(\{a\})$.

Thus, $\{\bar{\mathscr{R}} \in REL(X^2 \vee_\Delta X^2)\colon \bar{\mathscr{R}} << \{\{(a,b)_1, (a,b)_1\}\}\}$ holds

Case (ii). $\{\bar{\mathscr{R}} \in REL(X^2 \vee_\Delta X^2)\colon \bar{\mathscr{R}} << \{\{(a,b)_2, (a,b)_2\}\}\}$ holds. The proof is similar to Case (i)

Case (iii). $\{\bar{\mathscr{R}} \in REL(X^2 \vee_\Delta X^2)\colon \bar{\mathscr{R}} << \{\{(a,b)_1, (a,b)_2\}\}\}$. It follows that, for all $\bar{R} \in \bar{\mathscr{R}}$ such that $\{(a,b)_1, (a,b)_2\} \subseteq \bar{R}$. And $\pi_1 A\{(a,b)_1, (a,b)_2\} \subseteq \pi_1 A\bar{R}$, $\pi_1 A\bar{\mathscr{R}} << \pi_1 A\{\{(a,b)_1, (a,b)_2\}\} = \{\{(a,a)\}\}$, and by Definition 2 $\pi_1 A\bar{\mathscr{R}} << \{\{(a,a)\}\} \in r(\{a\})$

Similarly, by the assumption $\pi_2 A\bar{\mathscr{R}} << \{\{(b,a)\}\} \notin r(\{b\})$ and $\pi_3 A\bar{\mathscr{R}} << \{\{(a,b)\}\} \notin r(\{a\})$.

Therefore, $\{\bar{\mathscr{R}} \in REL(X^2 \vee_\Delta X^2)\colon \bar{\mathscr{R}} << \{\{(a,b)_1, (a,b)_2\}\}\}$ is not possible.

Case (iv). Similar to Case (iii), we conclude that $\{\bar{\mathscr{R}} \in REL(X^2 \vee_\Delta X^2)\colon \bar{\mathscr{R}} << \{\{(a,b)_2, (a,b)_1\}\}\}$ is not possible.

Case (v). If $\{\bar{\mathscr{R}} \in REL(X^2 \vee_\Delta X^2)\colon \bar{\mathscr{R}} << \{\{(((a,b)_1, (a,b)_1), ((a,b)_2, (a,b)_2)\}\}\}$. It follows that, for all $\bar{R} \in \bar{\mathscr{R}}$ such that $\{(((a,b)_1, (a,b)_1), ((a,b)_2, (a,b)_2)\} \subseteq \bar{R}$ and $\pi_1 A\{(((a,b)_1, (a,b)_1), ((a,b)_2, (a,b)_2)\} \subseteq \pi_1 A\bar{\mathscr{R}}$ implies $\pi_1 A\bar{\mathscr{R}} << \pi_1 A\{\{(((a,b)_1, (a,b)_1), ((a,b)_2, (a,b)_2)\}\} = \{\{(a,a)\}\}$. By Definition 2, $\pi_1 A\bar{\mathscr{R}} << \{\{(a,a)\}\} \in r(\{a\})$.

Similarly, by the assumption, $\pi_2 A\bar{\mathscr{R}} << \{\{((b,b), (a,a))\}\} \notin r(\{b\})$ and $\pi_3 A\bar{\mathscr{R}} << \{\{((a,a), (b,b))\}\} \notin r(\{b\})$.

Hence, $\{\bar{\mathscr{R}} \in REL(X^2 \vee_\Delta X^2)\colon \bar{\mathscr{R}} << \{\{(((a,b)_1, (a,b)_1), ((a,b)_2, (a,b)_2)\}\}\}$ is not possible.

Similarly, if B= $\{(a,b)_2\}$ only Case (i) and Case (ii) are holding. By Lemma 12, $\{\bar{\mathscr{R}} \in REL(X^2 \vee_\Delta X^2)\colon \bar{\mathscr{R}} << \{\{((a,b)_j, (a,b)_j), j = 1, 2\}\}\}$ is discrete. Therefore, by Definition 21 (i), (X, β^X, r) is $\bar{T}_0$.

Theorem 24. *Let (X, β^X, r) be an ordered-RELspace.*

(X, β^X, r) is T_0 iff for each $a, b \in X$ with $a \neq b$, each of the following conditions are satisfied:

(i) $\{a, b\} \notin \beta^X$

(ii) $\{\mathscr{R} \in REL(X)\colon \mathscr{R} << \{\{(a,b)\}\}\} \notin r(\{a\})$ or $\{\mathscr{R} \in REL(X)\colon \mathscr{R} << \{\{(a,b)\}\}\} \notin r(\{b\})$

(iii) $\{\mathscr{R} \in REL(X)\colon \mathscr{R} << \{\{(b,a)\}\}\} \notin r(\{a\})$ or $\{\mathscr{R} \in REL(X)\colon \mathscr{R} << \{\{(b,a)\}\}\} \notin r(\{b\})$

(iv) $\{\mathscr{R} \in REL(X)\colon \mathscr{R} << \{\{(a,a)\}\}\} \notin r(\{b\})$ or $\{\mathscr{R} \in REL(X)\colon \mathscr{R} << \{\{(b,b)\}\}\} \notin r(\{a\})$.

Proof. Let (X, β^X, r) be T_0, $\{a, b\} \in \beta^X$ and $\{\mathscr{R} \in REL(X)\colon \mathscr{R} << \{\{(a,b)\}\}\} \in r(\{a\})$ and $\{\mathscr{R} \in REL(X)\colon \mathscr{R} << \{\{(a,b)\}\}\} \in r(\{b\})$ and $\{\mathscr{R} \in REL(X)\colon \mathscr{R} << \{\{(b,a)\}\}\} \in r(\{a\})$ and $\{\mathscr{R} \in REL(X)\colon \mathscr{R} << \{\{(b,a)\}\}\} \in r(\{b\})$ and $\{\mathscr{R} \in REL(X)\colon \mathscr{R} << \{\{(a,a)\}\}\} \in r(\{a\})$ and $\{\mathscr{R} \in REL(X)\colon \mathscr{R} << \{\{(b,b)\}\}\} \in r(\{b\})$.

Let $U = \{a, b\}$. Note that (U, β^U, r_U) is the subspace of (X, β^X, r), where (β^U, r_U) is the initial lift of the ordered-RELsystem induced by the inclusion map $i : S \longrightarrow U$ and for any $S \subset U$, $S \in \beta^U$, whenever $i(S) = S \in \beta^U$ and for any $\mathscr{R} \in REL(U)$, $\mathscr{R} \in r(S)$, whenever $i(\mathscr{R}) = \mathscr{R} \in r(B)$.

By the assumption, $i(U) = U = \{a, b\} \in \beta^U$ and by Definition 1, we get $\beta^U = \underline{P}U$.

Now, for any $\mathscr{R} \in REL(U)$ let $\mathscr{R} = \{\{(a, a)\}\} \in REL(U)$. By Definition 2, $i(\{\{(a, a)\}\}) = \{\{(a, a)\}\} \in r(\{a\})$. By the assumption, $\mathscr{R} = \{\{(a, a)\}\} \in r(\{b\})$ implying that $\{\mathscr{R} \in REL(X)$: $\mathscr{R} << \{\{(a, a)\}\}\} \in r(\{a\})$ and $\{\mathscr{R} \in REL(X)$: $\mathscr{R} << \{\{(a, a)\}\}\} \in r(\{b\})$.

Similarly, for $\mathscr{R} = \{\{(b, b)\}\} \in REL(U)$, it follows that $\{\mathscr{R} \in REL(X)$: $\mathscr{R} << \{\{(b, b)\}\}\} \in r(\{a\})$ and $\{\mathscr{R} \in REL(X)$: $\mathscr{R} << \{\{(b, b)\}\}\} \in r(\{b\})$.

Now, if $\mathscr{R} = \{\{(a, b)\}\} \in REL(U)$ then by the assumption, $\{\mathscr{R} \in REL(X)$: $\mathscr{R} << \{\{(a, b)\}\}\} \in r(\{a\})$ and $\{\mathscr{R} \in REL(X)$: $\mathscr{R} << \{\{(a, b)\}\}\} \in r(\{b\})$.

And for $\mathscr{R} = \{\{(b, a)\}\} \in REL(U)$ then by the assumption, $\{\mathscr{R} \in REL(X)$: $\mathscr{R} << \{\{(b, a)\}\}\} \in r(\{a\})$ and $\{\mathscr{R} \in REL(X)$: $\mathscr{R} << \{\{(b, a)\}\}\} \in r(\{b\})$.

Therefore, $r_U = \{\mathscr{R} \in REL(U)$: $\{\phi\} \in \mathscr{R}\}$ and $(\beta^U, r_U) = (\underline{P}(U), r_{id})$, which is a contradiction by Lemma 12. Thus $(i) - (iv)$ are holding.

Conversely, suppose that for all $a, b \in X$ with $a \neq b$, conditions $(i) - (iv)$ are holding. We show that the initial structure (β^U, r_U) is not an indiscrete ordered-RELstructure on U. Let $U = \{a, b\} \subset X$. By the assumption, $\{a, b\} \notin \beta^X$ and $\{\mathscr{R} \in REL(X)$: $\mathscr{R} << \{\{(a, b)\}\}\} \notin r(\{a\})$ or $\{\mathscr{R} \in REL(X)$: $\mathscr{R} << \{\{(a, b)\}\}\} \notin r(\{b\})$ and $\{\mathscr{R} \in REL(X)$: $\mathscr{R} << \{\{(b, a)\}\}\} \notin r(\{a\})$ or $\{\mathscr{R} \in REL(X)$: $\mathscr{R} << \{\{(b, a)\}\}\} \notin r(\{b\})$ and $\{\mathscr{R} \in REL(X)$: $\mathscr{R} << \{\{(a, a)\}\}\} \notin r(\{b\})$ or $\{\mathscr{R} \in REL(X)$: $\mathscr{R} << \{\{(b, b)\}\}\} \notin r(\{a\})$. Thus, (U, β^U, r) is not an indiscrete ordered-RELsubspace of (X, β^X, r). Hence, by Definition 21 (iii), (X, β^X, r) is T_0. □

Theorem 25. *Let (X, β^X, r) be an ordered-RELspace. Then, (X, β^X, r) is T_1 iff for all $a, b \in X$ with $a \neq b$, the following holds:*

(i) $\{a, b\} \notin \beta^X$

(ii) $\{\mathscr{R} \in REL(X)$: $\mathscr{R} << \{\{(a, b)\}\}\} \notin r(\{a\})$ *and* $\{\mathscr{R} \in REL(X)$: $\mathscr{R} << \{\{(b, a)\}\}\} \notin r(\{b\})$

(iii) $\{\mathscr{R} \in REL(X)$: $\mathscr{R} << \{\{(a, b)\}\}\} \notin r(\{b\})$ *and* $\{\mathscr{R} \in REL(X)$: $\mathscr{R} << \{\{(b, a)\}\}\} \notin r(\{a\})$

(iv) $\{\mathscr{R} \in REL(X)$: $\mathscr{R} << \{\{(a, a), (b, b)\}\}\} \notin r(\{a\})$ *and* $\{\mathscr{R} \in REL(X)$: $\mathscr{R} << \{\{(b, b), (a, a)\}\}\} \notin r(\{b\})$.

Proof. Similarly, using Theorem 23, and replacing mapping A by the mapping S, we obtain the proof. □

Theorem 26. *All ordered-RELspaces are ${T'}_0$.*

Proof. Let (X, β^X, r) be an ordered-RELspace. By Definition 21, we show that for any $U \in \beta^{X^2 \vee_\Delta X^2}$, $U \subset i_k(V)$ (where $k = 1, 2$) for some $V \in \beta^{X^2}$ and $\nabla U \in \mathscr{D}^{X^2}$. If $\nabla U = \phi$ implies $U = \phi$. Suppose $\nabla U \neq \phi$, hence $\nabla U = \{(a, b)\}$ for some $(a, b) \in X^2$.

Suppose $a \neq b$, it follows that $U = \{(a, b)_1\}$or$\{(a, b)_2\}$or $\{(a, b)_1, (a, b)_2\}$. If $U = \{(a, b)_1, (a, b)_2\}$, then $\{(a, b)_1, (a, b)_2\} \subset i_1(V)$ for some $V \in \beta^{X^2}$, which shows that $(a, b)_2$ must be in the first component of $X^2 \vee_\Delta X^2$, a contradiction. Similarly, $\{(a, b)_1, (a, b)_2\} \subset i_2(V)$, for $V \in \beta^{X^2}$. Hence, $U = \{\{(a, b)_j\}\}$ for $j = 1, 2$. Consequently, $\beta^{X^2 \vee_\Delta X^2} = \mathscr{D}^{X^2 \vee_\Delta X^2}$, the discrete ordered-RELstructure on $X^2 \vee_\Delta X^2$.

Now, for $B \in \mathscr{D}^{X^2 \vee_\Delta X^2} \setminus \{\phi\}$, and by Lemma 10, $\bar{r}^2(B) = \{\bar{\mathscr{R}} \in REL(X^2 \vee_\Delta X^2)$: $\bar{\mathscr{R}} << i_1(s)$ for some $s \in r(B)$, $\bar{\mathscr{R}} << i_2(s)$ for some $s \in r(B)$ and $\nabla(\bar{\mathscr{R}}) \in r^2_{dis}(\nabla B)\}$. But $\nabla(\bar{\mathscr{R}}) \in r^2_{dis}(\nabla B)$ gives the following possibilities:

$\{\bar{\mathscr{R}} \in REL(X^2 \vee_\Delta X^2)$: $\bar{\mathscr{R}} << \{\{(a, b)_1, (a, b)_1\}\}\}$,
$\{\bar{\mathscr{R}} \in REL(X^2 \vee_\Delta X^2)$: $\bar{\mathscr{R}} << \{\{(a, b)_2, (a, b)_2\}\}\}$,
$\{\bar{\mathscr{R}} \in REL(X^2 \vee_\Delta X^2)$: $\bar{\mathscr{R}} << \{\{(a, b)_1, (a, b)_2\}\}\}$,
$\{\bar{\mathscr{R}} \in REL(X^2 \vee_\Delta X^2)$: $\bar{\mathscr{R}} << \{\{(a, b)_2, (a, b)_1\}\}\}$,
$\{\bar{\mathscr{R}} \in REL(X^2 \vee_\Delta X^2)$: $\bar{\mathscr{R}} << \{\{(((a, b)_1, (a, b)_1), ((a, b)_2, (a, b)_2)\}\}\}$.

In particular, for $\{\bar{\mathscr{R}} \in REL(X^2 \vee_\Delta X^2)$: $\bar{\mathscr{R}} << \{\{(a, b)_1, (a, b)_2\}\}\}$. Then, it follows, for all $\bar{R} \in \bar{\mathscr{R}}$ $\{(a, b)_1, (a, b)_2\} \subset \bar{R}$, and consequently, $i_k\{(a, b)_1, (a, b)_2\} \subset \bar{R}$ (for k=1,2). As a result, $(a, b)_2$ (respectively, $(a, b)_1$) is in the first (respectively, second) component of the wedge product $X^2 \vee_\Delta X^2$ which leads to a contradiction. Similarly, for $\{\bar{\mathscr{R}} \in REL(X^2 \vee_\Delta X^2)$: $\bar{\mathscr{R}} << \{\{(a, b)_2, (a, b)_1\}\}\}$ and $\{\bar{\mathscr{R}} \in REL(X^2 \vee_\Delta X^2)$: $\bar{\mathscr{R}} << \{\{(((a, b)_1, (a, b)_1), ((a, b)_2, (a, b)_2)\}\}\}$, we get a contradiction.

Hence, $\bar{r}^2(B) = \{\bar{\mathscr{R}} \in REL(X^2 \vee_\Delta X^2)$: $\bar{\mathscr{R}} << \{\{((a, b)_j, (a, b)_j)\}\ ; j = 1, 2\}\}$. Thus, by Lemma 10 and Definition 21, (X, β^X, r) is ${T'}_0$. □

Remark 27. Let X be an ordered-RELspace.

(i) By Theorems 17 and 23, X is $\bar{T}_0$ iff X is $\bar{T}_0$ at p, for each $p \in X$

(ii) By Theorems 18 and 25, X is T_1 iff X is T_1 at p, for each $p \in X$

(iii) By Theorems 19 and 26, X is ${T'}_0$ iff X is ${T'}_0$ at p, for each $p \in X$

(iv) By Theorems 23–26, $T_1 \Longrightarrow \bar{T}_0 \Longrightarrow T_0 \Longrightarrow {T'}_0$ but the converse does not hold in general.

Corollary 28. *Let $(X, \underline{P}(X), r)$ be in* **PU − REL**. *Then the following statements are equivalent.*

(i) $(X, \underline{P}(X), r)$ *is* $\bar{T}_0$

(ii) $(X, \underline{P}(X), r)$ *is* $\bar{T}_0$**PUCONV**, *where* $\bar{T}_0$**PUCONV** *is the category of* $\bar{T}_0$ *pre-uniform convergence spaces and uniformly continuous maps*

(iii) *For each* $a, b \in X$ *with* $a=b$, *and for all* $B \in \underline{P}(X)$, $\{\mathscr{R} \in REL(X)$: $\mathscr{R}<<\{\{(a,b)\}\}\} \notin r(B)$ *or* $\{\mathscr{R} \in REL(X)$: $\mathscr{R}<<\{\{(b,a)\}\}\} \notin r(B)$, *and* $\{\mathscr{R} \in REL(X)$: $\mathscr{R}<<\{\{(a,a),(b,b)\}\}\} \notin r(B)$.

Proof. By applying Example 4, Theorem 23, and Theorem 3.1.10 of [41]. □

Corollary 29. *Let* $(X, \underline{P}(X), r)$ *be in* **PU – REL**. *Then, the following statements are equivalent:*

(i) $(X, \underline{P}(X), r)$ *is* T_1

(ii) $(X, \underline{P}(X), r)$ *is* T_1**PUCONV**, *where* T_1**PUCONV** *is the category of* T_1 *pre-uniform convergence spaces and uniformly continuous maps*

(iii) *For all* $a, b \in X$ *with* $a=b$, *and for all* $B \in \underline{P}(X)$, $\{\mathscr{R} \in REL(X)$: $\mathscr{R}<<\{\{(a,b)\}\}\} \notin r(B)$ *and* $\{\mathscr{R} \in REL(X)$: $\mathscr{R}<<\{\{(b,a)\}\}\} \notin r(B)$, *and* $\{\mathscr{R} \in REL(X)$: $\mathscr{R}<<\{\{(a,a),(b,b)\}\}\} \notin r(B)$.

Proof. This follows from Example 4, Theorem 25, and Theorem 3.2.4 of [41]. □

6. Quotient-Reflective Subcategories of the Category of Ordered-RELspaces

Definition 30 (cf. [42]). Given a topological functor $\mathfrak{U} : \mathscr{C} \longrightarrow$ **Set**, and a full and isomorphism-closed subcategory $\mathscr{H}$ of $\mathscr{C}$, we say that $\mathscr{H}$ is

(i) Epireflective in $\mathscr{C}$ and closed if and only if $\mathscr{H}$ is closed under the formation of products and extremal subobjects (i.e., subspaces)

(ii) Quotient-reflective in $\mathscr{C}$ if and only if $\mathscr{H}$ is epireflective and is closed under finer structures (i.e., if $A \in \mathscr{H}$, $B \in \mathscr{C}$, $\mathfrak{U}(A) = \mathfrak{U}(B)$, and $id : A \longrightarrow B$ is a $\mathscr{C}$-morphism, then $B \in \mathscr{H}$).

Theorem 31.

(i) *Any* $\bar{T}_0$***O-REL***, T_0***O-REL*** *and* T_1***O-REL*** *is a quotient-reflective subcategory of* ***O-REL***

(ii) T_0'***O-REL*** *is a normalized topological construct*

Proof. (i) Suppose $\mathscr{C} = \bar{\mathbf{T}}_\mathbf{0}\boldsymbol{O} - \boldsymbol{REL}$ and $(X, \beta^X, r) \in \mathscr{C}$. It can be easily verified that $\mathscr{C}$ is a full and isomorphism-closed subcategory of **O-REL** and closed under finer structures. It remains to show that X is closed under extremal subobjects and closed under the formation of products.

Let $A \subset X$ and (β^A, r_A) denotes the sub **O-REL** structure on A, induced by the inclusion map $i : A \longrightarrow X$. We show that (A, β^A, r_A) is $\bar{\mathbf{T}}_\mathbf{0}$**O-REL** space. Suppose that for any a,b $\in$ A with $a \neq b$, $\{a,b\} \in \beta^A$, then by the inclusion map $i(\{a,b\}) = \{i(a), i(b)\} = \{a,b\} \in \beta^X$, a contradiction by Theorem 23.Thus, $\{a,b\} \notin \beta^A$.

Now, suppose $\{\mathscr{R} \in REL(A)$: $\mathscr{R}<<\{\{(a,b)\}\}\} \in r_A(\{a\})$ and $\{\mathscr{R} \in REL(A)$: $\mathscr{R}<<\{\{(b,a)\}\}\} \in r_A(\{b\})$. It follows that, for all $R \in \mathscr{R}$ such that $\{(a,b)\} \subset R$, and by the inclusion map $i\{(a,b)\} \subset i(R)$ implying $\{(a,b)\} \subset R$. It follows that $\{\mathscr{R} \in REL(X)$: $\mathscr{R}<<\{\{(a,b)\}\}\} \in r_X(\{a\})$, a contradiction by Theorem 23. Similarly, by the same argument $\{\mathscr{R} \in REL(A)$: $\mathscr{R}<<\{\{(b,a)\}\}\} \in r_X(\{b\})$, a contradiction. Therefore, $\{\mathscr{R} \in REL(A)\mathscr{R}<<\{\{(a,b)\}\}\} \notin r_A(\{a\})$ or $\{\mathscr{R} \in REL(A)$: $\mathscr{R}<<\{\{(b,a)\}\}\} \notin r_A(\{b\})$.

In similar way, $\{\mathscr{R} \in REL(A)$: $\mathscr{R}<<\{\{(a,b)\}\}\} \notin r_A(\{b\})$ or $\{\mathscr{R} \in REL(A)$: $\mathscr{R}<<\{\{(b,a)\}\}\} \notin r_A(\{a\})$, and $\{\mathscr{R} \in REL(A)$: $\mathscr{R}<<\{\{(a,a),(b,b)\}\}\} \notin r_A(\{a\})$ or $\{\mathscr{R} \in REL(A)$: $\mathscr{R}<<\{\{(b,b),(a,a)\}\}\} \notin r_A(\{b\})$. Hence, X is closed under extremal subobjects.

Next, suppose that $X = \prod_{k \in I} X_k$, where (β^{X_k}, r_{X_k}) are the $\bar{\mathbf{T}}_\mathbf{0}$**O-REL** structures on X_k induced by projection map $\pi_k : X_k \longrightarrow X$ for all $k \in I$, i.e., $(X_k, \beta^{X_k}, r_{X_k}) \in \mathscr{C}$. We show that (X, β^X, r_X) is a $\bar{\mathbf{T}}_\mathbf{0}$**O-REL** space. Let $\{a,b\} \in \beta^X$ for any $a, b \in X$ with $a \neq b$. Then, $\pi_k(\{a,b\}) = \{\pi_k(a), \pi_k(b)\} = \{a_k, b_k\} \in \beta^{X_k}$, a contradiction by Theorem 23. Thus, $\{a,b\} \notin \beta^X$.

Now, suppose $\{\mathscr{R} \in REL(X)$: $\mathscr{R}<<\{\{(a,b)\}\}\} \in r_X(\{a\})$ and $\{\mathscr{R} \in REL(X)$: $\mathscr{R}<<\{\{(b,a)\}\}\} \in r_X(\{b\})$. It follows $R \in \mathscr{R}$ implies $\{(a,b)\} \subset R$. Then, there is $k \in I$ for which $a_k \neq b_k \in X_k$, and $\pi_k\{(a,b)\} \subset \pi_k R$ implying $\{(\pi_k a, \pi_k b)\} = \{(a_k, b_k)\} \subset \pi_k R$. It follows that $\{\mathscr{R} \in REL(X)$: $\pi_k\mathscr{R}<<\{\{(a_k,b_k)\}\}\} \in r_{X_k}(\{a_k\})$, a contradiction by Theorem 23. By the same process, $\{\mathscr{R} \in REL(X)$: $\pi_k(\mathscr{R})<<\{\{(b_k,a_k)\}\}\} \in r_{X_k}(\{b_k\})$, a contradiction. Hence, $\{\mathscr{R} \in REL(X)$: $\mathscr{R}<<\{\{(a,b)\}\}\} \notin r_X(\{a\}$ or $\{\mathscr{R} \in REL(X)$: $\mathscr{R}<<\{\{(b,a)\}\}\} \notin r_X(\{b\})$. In similar way, $\{\mathscr{R} \in REL(X)$: $\mathscr{R}<<\{\{(a,b)\}\}\} \notin r_X(\{b\})$ or $\{\mathscr{R} \in REL(X)$: $\mathscr{R}<<\{\{(b,a)\}\}\} \notin r_X(\{a\})$, and $\{\mathscr{R} \in REL(X)$: $\mathscr{R}<<\{\{(a,a),(b,b)\}\}\} \notin r_X(\{a\})$ or $\{\mathscr{R} \in REL(X)$: $\mathscr{R}<<\{\{(b,b),(a,a)\}\}\} \notin r_X(\{b\})$. Hence, X is closed under the formation of products.

Therefore, the category $\bar{\boldsymbol{T}}_\mathbf{0}$**O-REL** is a quotient-reflective subcategory of **O-REL**.

Analogous to the above argument, setting $\mathscr{C} = \boldsymbol{T}_\mathbf{0}$ **O – REL** or $\boldsymbol{T}_\mathbf{1}$**O – REL**, the proof can be easily followed by using Theorem 24 or Theorem 25, respectively.

(ii) By the Theorem 26 and Remark 13, $\boldsymbol{T}_\mathbf{0}'$**O-REL** and **O-REL** are isomorphic categories and thus $\boldsymbol{T}_\mathbf{0}'$**O-REL** is normalized □

Data Availability

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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