Research Article

A Note on Quotient Reflective Subcategories of O-REL

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In this paper, we examine the category of ordered-REL spaces. We show that it is a normalized and geometric topological category and give the characterization of local $T_0$, local $T_0'$, and local $T_1$ ordered-REL spaces. Furthermore, we characterize explicitly several notions of $T_0$'s and $T_1$ objects in O-REL and study their mutual relationship. Finally, it is shown that the category of $T_0$'s (resp. $T_1$) ordered-REL spaces are quotient reflective subcategories of O-REL.

1. Introduction

Many mathematical concepts were developed to describe certain structures of topology. The concepts of uniform convergences, uniform continuity, Cartesian closedness, completeness, and total boundedness do not exist in general topology. As a remedy, several approaches have been made to define these concepts in topology by mathematicians. For example, the concepts of uniform convergence in the sense of Kent [1] and Preuss [2], of set-convergence in the sense of Wyler [3], Tozzi [4] (which scrutinize filter convergence to bounded subset and generalizes classical point-convergence and supertopologies), of nearness by Bentely [5] and Herrlich [6] (particularly containing proximities and contiguities), and that of hullness by Čech [7] and Leseberg [8] containing the concepts of b-topologies and closures, respectively.

In this paper, we examine the category of ordered-REL spaces. We show that it is a normalized and geometric topological category and give the characterization of local $T_0$, local $T_0'$, and local $T_1$ ordered-REL spaces. Furthermore, we characterize explicitly several notions of $T_0$'s and $T_1$ objects in O-REL and study their mutual relationship. Finally, it is shown that the category of $T_0$'s (resp. $T_1$) ordered-REL spaces are quotient reflective subcategories of O-REL.

1.1. Initial, Final, Discrete, and Indiscrete Objects

The salient objectives of this study are stated as follows:

(i) To define initial, final, discrete, and indiscrete objects in O-REL

(ii) To characterize local $T_0$, local $T_0'$, and local $T_1$ objects in O-REL and examine their mutual relationship

(iii) To give the characterization of $T_0$, $T_0'$, and $T_1$ objects in O-REL and examine their mutual relationship
(iv) To define several structures using ordered-RELspaces and discuss each of the $T_0$ and $T_1$ axioms there and examine their mutual relationship.

(v) To examine the quotient-reflective properties of ordered-RELspaces.

2. Preliminaries

Recall [31, 32], a functor $\mathcal{U} : \mathcal{C} \to \mathbf{Set}$ (the category of sets and functions) is called topological if

(i) $\mathcal{U}$ is concrete

(ii) $\mathcal{U}$ consists of small fibers

(iii) Every $\mathcal{U}$-source has a unique initial lift or every $\mathcal{U}$-sink has an unique final lift, i.e., if for every source $(f_j : X \to (X_j, \eta_j))_{j \in I}$ there exists an unique structure $\eta$ on $X$ such that $g : (Y, \zeta) \to (X, \eta)$ is a morphism iff for each $j \in I$, $f_j \circ g : (Y, \zeta) \to (X, \eta_j)$ is a morphism.

Moreover, a topological functor is called discrete (respectively, indiscrete) if it has a left (respectively, right) adjoint. In addition, a functor is called a normalized topological functor if constant objects, i.e., subterminal objects, have an unique structure, and said to be geometric functor if the discrete functor is left exact, i.e., it preserves finite limits [31, 32].

Let $X$ be a non-empty, then $\mathcal{R} \subset \mathcal{P}(X \times X)$ is called a relative system for $X$, and it is denoted by $\text{REL}(X)$. Moreover, $\text{REL}(X)$ can be ordered by setting $\mathcal{R} \ll \mathcal{R}$ iff for each $R \in \mathcal{R}$, there exists $R' \in \mathcal{R}$ such that $R \subseteq R'$.

Furthermore, we denote by $\text{sec} \ \mathcal{R} = \{ R \subset X \times X : \forall R \in \mathcal{R}, R \cap R' \setminus \phi \}$ and by $\text{stack} \ \mathcal{R} = \{ R \subset X \times X : \exists R \in \mathcal{R}, R \subseteq R' \}$.

Definition 1 (cf. [33]). Let $X \neq \phi$, then $\beta^X \subset \mathcal{P}(X)$ is called boundedness or $B$-set on $X$, if $\beta^X$ satisfies the following axioms:

(i) $\phi \in \beta^X$

(ii) $B \subseteq B_1 \in \beta^X$ implies $B_2 \in \beta^X$

(iii) $a \in X$ implies $\{ a \} \in \beta^X$.

And for $B$-sets $\beta^X$ and $\beta^Y$ a function $g : X \to Y$ is called bounded iff it satisfies:

$$\{ g[B] : B \in \beta^X \} \subset \beta^Y. \quad (1)$$

By $\text{BOUND}$ we denote the corresponding defined category.

Definition 2 (cf. [33]). The triple $(X, \beta^X, r)$ is called RELative space (shortly RELspace) if for the boundedness $\beta^X$ the function $r : \beta^X \to \text{PREL}(X)$ satisfies the following conditions:

(i) $B \in \beta^X$ and $\mathcal{R} \ll \mathcal{R} \in r(B)$ implies $\mathcal{R} \in r(B)$

(ii) $\{ \phi \} \notin r(B)$ for $B \in \beta^X$

(iii) $\mathcal{R} \in r(\phi)$ iff $\mathcal{R} = \phi$

(iv) $a \in X$ implies $\{ \{ a \} \times \{ a \} \} \in r(\{ a \})$.

The RELspace $(X, \beta^X, r)$ is called ordered-RELspace provided that the following axiom holds:

(v) $\phi \neq B_1 \subset B \in \beta^X$ implies $r(B_1) \subset r(B)$.

Definition 3 (cf. [33]). Let $(X, \beta^X, r)$ and $(Y, \beta^Y, r)$ be two RELspaces, then a bounded function $g : X \to Y$ is called RELative map (shortly RELmap) iff it satisfies the following condition:

$$B \in \beta^X \setminus \{ \phi \} \text{ and } \mathcal{R} \in r(B) \text{ implying } g[B] \subseteq \mathcal{R} \in v(g[B]), \quad (2)$$

where $g[B] \mathcal{R} = \{(g \times g)[R] : R \in \mathcal{R}\}$ with $(g \times g)[R] = \{(g \times g)(a, c) : (a, c) \in R\} = \{(g(a), g(c)) : (a, c) \in R\}$. By $\text{O-REL}$, we denote the full subcategory of REL, whose objects are the ordered RELspaces. Note that $\text{O-REL}$ is a bifree left subcategory of $\text{REL}$ [34].

Example 4. Let $(X, T_X)$ be a preuniform convergence space; then, the associated RELspace $(X, P_X(X), r_{T_X})$ can be defined as follows:

$$r_{T_X}(\phi) = \{ \phi \} \text{ and for } B \in P_X(X) \setminus \{ \phi \},$$

$$r_{T_X}(B) = \{ \mathcal{R} \in \text{REL}(X) : \forall X' \in T_X, X' \subset \text{sec} \ \mathcal{R} \}. \quad (3)$$

Let $\text{PU-REL}$ denotes the category, whose objects are triples $(X, P_X(X), r_{T_X})$ and morphisms are RELmaps. Note that $\text{PUCONV} \equiv \text{PU-REL}$ [9], where $\text{PUCONV}$ is the category of preuniform convergence spaces and uniformly continuous maps as defined in [2].

Example 5. Let $(X, \beta^X, t)$ be a set-convergence space; then, the associated RELspace $(X, \beta^X, r_t)$ can be defined by

$$r_t(\phi) = \{ \phi \} \text{ and for } B \in \beta^X \setminus \{ \phi \},$$

$$r_t(B) = \{ \mathcal{R} \in \text{REL}(X) : \exists X' \in \text{FIL}(X)((X', B) \in t \text{ and } \mathcal{R} \subset \text{sec} \ X' \otimes X') \},$$

where $X' \otimes X' = \{ R \subset X \times X : \exists E, E \in X' \text{ such that } E_1 \times E \subseteq R \}$ and $\text{FIL}(X)$ is the collection of all filters defined on $X$.

Let $\text{SET-REL}$ denotes the category, whose objects are triples $(X, \beta^X, r_t)$ and morphisms are RELmaps. Note that $\text{SETCONV} \equiv \text{SET-REL}$ [9], where $\text{SETCONV}$ is the category of set-convergence spaces and morphisms are b-continuous maps as defined in [3].
Example 6. Let \((x, \zeta)\) be prenearness space; then, the associated RELspace \((X, P(X), r_\zeta)\) can be described as

\[
r_\zeta(\emptyset) = \{ \emptyset \} \quad \text{and for } B \in P(X) \setminus \{ \emptyset \},
\]

\[
r_\zeta(B) = \{ \mathcal{S} \in \text{REL}(X) : \exists \mathcal{O} \subseteq P(X)(\{ B \} \cup \mathcal{O} \in \zeta \text{ and } \mathcal{S} \ll \mathcal{O} \times \mathcal{O}) \}, \quad \text{where } \mathcal{O} \times \mathcal{O} = \{ D \times D : D \in \mathcal{O} \}.
\]  
(5)

Note that PNEAR\(\equiv\)PN-REL [6, 9], where PNEAR is the category, whose objects are prenearness spaces and morphisms are nearness preserving maps as defined in [6], and PN-REL is the category of triples \((X, P(X), r_\zeta)\) and morphisms are RELmaps.

Example 7. For a B-set \(B^X\), we put \(r_\emptyset(\emptyset) = \{ \emptyset \}\), and for \(B \in B^X \setminus \{ \emptyset \}\), we set \(r_\emptyset(B) = \{ \mathcal{S} \in \text{REL}(X) : \exists a \in B, \mathcal{S} \ll a \times a \}\); hence, \((X, B^X, r_\emptyset)\) defines a RELspace, which is diagonal, meaning that for \(B \in B^X \setminus \{ \emptyset \}\) and \(\mathcal{S} \in \text{REL}(B)\), we can see \(a \in B\) such that \(VR \in \mathcal{S}, (x, x) \in R\).

Let \(\Delta-\text{REL}\) be the corresponding defined full subcategory of REL; then, \(\Delta-\text{REL}\equiv\text{BOUND}^\Delta\).

Remark 8. In this context, note that BORN, the full subcategory of BOUND, whose objects are the bornological spaces, then also has evidently a corresponding counterpart in REL.

Example 9. Let \((X, B^X, q)\) be b-topological space; then, the associated RELspace \((X, B^X, r_q)\) is defined by

\[
r_q(\emptyset) = \{ \emptyset \} \quad \text{and for } B \in B^X \setminus \{ \emptyset \},
\]

\[
r_q(B) = \{ \mathcal{S} \in \text{REL}(X) : \exists a \in B, \mathcal{S} \ll a \times a \} = \{ \mathcal{S} \in \text{REL}(X) : \exists a \in B, \mathcal{S} \ll a \times a \}.
\]  
(6)

Note that b-TOP\(\equiv\)bTOP-REL [9], where bTOP-REL denotes the full subcategory of REL, whose objects are triples \((X, B^X, r_q)\), and b-TOP denotes the category of b-topological spaces and b-continuous maps as defined in [9].

3. \(O-\text{REL}\) as a Normalized and Geometric Topological Category

Note that the forgetful functor \(U : \mathcal{C} \rightarrow \text{Set}\), where \(\mathcal{C} = \text{REL}\) is topological in the following sense:

Lemma 10. Let \((X_j, B^X_j, r_j)\) be a collection of RELspaces. A source \((f_j : (X_j, B^X_j, r_j))_{j \in I}\) is initial in REL iff

\[
B_j^X = \{ B \subseteq X : g_j(B) \in B^X, \forall j \in I \}.
\]  
(7)

and for all \(B \in B_j^X\),

\[
r_j(X) = \{ \mathcal{S} \in \text{REL}(X) : g_j(\mathcal{S}) \in r_j(g_j[B]), \forall j \in I \}.
\]  
(8)

Proof. It is given in [34]. Consequently, since O-REL is a full and isomorphism-closed subcategory which is bireflective in REL, it is topological, too.

Lemma 11. Let \((X_j, B^X_j, r_j)\) be a collection of ordered-RELspaces. A sink \((f_j : (X_j, B^X_j, r_j) \rightarrow (X, B^X_j, r_j))_{j \in I}\) is final in O-REL iff

\[
B^X_j = \{ B_j \subseteq X : \exists j \in I, \exists B_j \in B^X_j \mid B \in g_j(B_j) \} \cup \Delta, \tag{9}
\]

where \(\Delta = \{ \emptyset \} \cup \{ \{ a \} : a \in X \}\), and for \(B \in B^X_j \setminus \{ \emptyset \}\),

\[
r_j(B) = \{ \mathcal{S} \in \text{REL}(X) : \exists j \in I, \exists B_j \in B^X_j, \exists \mathcal{R} \in \mathcal{S} \mid \mathcal{S} \ll g_j(\mathcal{R}) \}, \quad \{ \mathcal{S} \in \text{REL}(X) : \exists a \in B \mid (a, a) \in \{ R : R \in \mathcal{S} \} \text{ with } r_j(\emptyset) = \{ \emptyset \}. \tag{10}
\]

Proof. It is easy to observe that \((X, B^X_j, r_j)\) is an ordered-RELspace and \(f_j : (X_j, B^X_j, r_j) \rightarrow (X, B^X_j, r_j)\) is a RELmap. Suppose that \(g : (X, B^X_j, r_j) \rightarrow (Y, B^Y_j, r_y)\) is a mapping. We show that \(g\) is a RELmap iff \(g \circ f_j\) is a RELmap.

Conversely, let \(g \circ f_j : (X_j, B^X_j, r_j) \rightarrow (Y, B^Y_j, r_y)\) be a RELmap.

Then, first, we show that \(g\) is a bounded map. Let \(B \subseteq B^X_j\); it implies that \(g(f_j(B)) \subseteq Y\). For our own convenience, take \(f_j(B) = B'\), and since \(f_j\) is a RELmap, then \(B' \subseteq B^Y_j\), and consequently, \(g\) is bounded.

Now, let \(B \subseteq B^X_j\) and \(\mathcal{R} \in r_j(B)\). By the Definition 3, we have \(g(f_j(B)) = g \circ f_j(B) \in r_y(g(f_j(B)))\). On the other hand, \(f_j\) is a RELmap; it follows that \(f_j(\mathcal{R}) \subseteq r_j(f_j(B))\). Take \(f_j(\mathcal{R}) = B'\). Then, we have \(\mathcal{R} \in r_j(B')\), and subsequently, \(g(\mathcal{R}) \in r_y(g(B'))\) which shows \(g\) is a RELmap.

Lemma 12. Let \(X \neq \emptyset\), and \((X, B^X, r)\) be an ordered-RELspace.

(i) A RELstructure \((B^X, r)\) is discrete iff \((B^X, r) = (\Delta, r_{dis})\), where \(\Delta = \{ \emptyset \} \cup \{ \{ a \} : a \in X \}\) and \(r_{dis}(\{ a \}) = \{ \mathcal{R} \in \text{REL}(X) : (a, a) \in \{ R : R \in \mathcal{R} \} \mid \mathcal{R} \ll \{ (a, a) \} \} \) with \(r_{dis}(\emptyset) = \{ \emptyset \}.

(ii) A RELstructure \((B^X, r)\) is indiscrete iff \((B^X, r) = (\mathcal{P}(X), r_{id})\), where \(r_{id}(B) = \{ \mathcal{R} \in \text{REL}(X) : \emptyset \in \mathcal{R} \} \) if \(B^X \neq \emptyset\) with \(r_{id}(\emptyset) = \{ \emptyset \}.

Proof. By applying Lemma 11, we get the desired result.

Remark 13. The topological functor \(U : \mathcal{C} \rightarrow \text{Set}\), where \(\mathcal{C} = O-REL\) is normalized since an unique RELstructure
$\beta^X = \{\emptyset\}$, and $r(\emptyset) = \{\emptyset\}$ exists whenever $X = \emptyset$ and a unique RELstructure $\beta^X = \{\emptyset, \{a\}\}$, $r(\emptyset) = \{\emptyset\}$ and $r(\{a\}) = \{\emptyset, \{a, a\}\}$ exists whenever $X = \{a\}$. Furthermore, the topological functor $\mathcal{U} : \mathcal{O} - \text{REL} \rightarrow \text{Set}$ is geometric since the regular sub-object of a discrete RELspace is discrete, and finite product of discrete RELstructures is discrete again.

4. Local $T_0$ and Local $T_1$ Ordered-RELspaces

In this section, we define notions for $T_0$ and $T_1$ ordered-RELspaces at some point.

Let $X$ be any set and $p \in X$. We define the wedge product of $X$ at $p$ as the two disjoint copies of $X$ at $p$ and denote it as $X \cup_p X$. For a point $a \in X \cup_p X$, we write it as $a_1$ if $a$ belongs to the first component of the wedge product; otherwise, we write $a_2$ that is in the second component. Moreover, $X^2$ is the cartesian product of $X$.

Definition 14 (cf. [14]).

(i) A mapping $A_p : X \cup_p X \rightarrow X^2$ is said to be principal $p$-axis mapping provided that

$$A_p(a_j) = \begin{cases} (a, p) ; & j = 1, \\ (p, a) ; & j = 2, \end{cases}$$

(ii) A mapping $S_p : X \cup_p X \rightarrow X^2$ is said to be skewed $p$-axis mapping provided that

$$S_p(a_j) = \begin{cases} (a, a) ; & j = 1, \\ (p, a) ; & j = 2, \end{cases}$$

(iii) A mapping $\nabla_p : X \cup_p X \rightarrow X$ is said to be fold mapping at $p$ provided that

$$\nabla_p(a_j) = a, j = 1, 2.$$

Assume that $\mathcal{U} : \mathcal{C} \rightarrow \text{Set}$ is a topological functor, $X \in \text{Obj}(\mathcal{C})$ with $\mathcal{U}X = Z$ and $p \in Z$.

Definition 15 (cf. [14]).

(i) $X$ is $T_0$ at $p$ provided that the initial lift of the $\mathcal{U}$-source $\{Z \rightarrow \mathcal{U}(X^2) \rightarrow Z^2 \text{ and } Z \rightarrow \mathcal{U}Z \text{ at } p\}$ is discrete

(ii) $X$ is $T_1$ at $p$ provided that the initial lift of the $\mathcal{U}$-source $\{Z \rightarrow \mathcal{U}(X \cup_p X) \rightarrow Z \text{ and } Z \rightarrow \mathcal{U}Z \text{ at } p\}$ is discrete, where $X \cup_p X$ is the wedge product in $\mathcal{C}$, i.e., the final lift of the $\mathcal{U}$-sink $\{UX = Z \rightarrow i_1, i_2 \rightarrow Z \}$, where $i_1, i_2$ represent the canonical injections.

(iii) $X$ is $T_1$ at $p$ provided that the initial lift of the $\mathcal{U}$-source $\{Z \rightarrow \mathcal{U}(X^2) \rightarrow Z^2 \text{ and } Z \rightarrow \mathcal{U}Z \text{ at } p\}$ is discrete.

Remark 16.

(i) In $\text{TOP}$, $T_0$, and $T_1$ at $p$ (respectively, $T_1$ at $p$) are equivalent to the classical $T_0$ at $p$ (respectively, the classical $T_1$ at $p$), i.e., for each $a \in X$ with $a \neq p$, there exists a neighborhood $N_a$ of $a$ not containing $p$ or (respectively, and); there exists a neighborhood $N_p$ of $p$ not containing $a$ [35]

(ii) A topological space $X$ is $T_0$ (respectively $T_1$) if $X$ is $T_0$ (respectively $T_1$) at $p$ for each $p \in X$ [35]

(iii) Let $\mathcal{U} : \mathcal{C} \rightarrow \text{Set}$ be a topological functor, $X \in \text{Obj}(\mathcal{C})$ and $p \in \mathcal{U}(X)$ be a retract of $X$. Then, if $X$ is $T_0$ or $T_1$ at $p$, then $X$ is $T_0$ at $p$ but not conversely in general [36].

Theorem 17. Let $(X, \beta^X, r)$ be ordered-RELspace and $p \in X$. Then, $(X, \beta^X, r)$ is $T_0$ at $p$ if and only if for each $a \in X$ with $a \neq p$, the following holds:

(i) $\{a, p\} \notin \beta^X$

(ii) $\{a, p\} \in \beta^X$

(iii) $\{a, p\} \notin \beta^X$

(iv) $\{a, p\} \in \beta^X$

Proof. Let $(X, \beta^X, r)$ be $T_0$ at $p$; we show the conditions (i) to (iv) are holding:

(i) Suppose that $\{a, p\} \notin \beta^X$ for all $a \in X$ with $a \neq p$. Let $U = \{a_1, a_2\} \in X \cup_p X$, then since $\nabla_p(U) = \nabla_p(\{a_1, a_2\}) = \{\nabla_p a_1, \nabla_p a_2\} = \{a\} \in \beta^X$ and for $j = 1, 2$, $\nabla_p a_j(U) = \{a, p\} \in \beta^X$ (by the assumption), where $\nabla_p : X^2 \rightarrow X$ is the projection maps. By Definitions 1 and 15 and Lemma 10, a contradiction, it follows $\{a, p\} \notin \beta^X$

(ii) Assume that $\{a, p\} \in \beta^X$ for all $a \in X$ with $a \neq p$. Let $U = \{a_1, a_2\} \in X \cup_p X$, then since $\nabla_p(U) = \nabla_p(\{a_1, a_2\}) = \{a_1, a_2\} \in \beta^X$ and for $j = 1, 2$, $\nabla_p a_j(U) = \{a, p\} \in \beta^X$ (by the assumption), where $\nabla_p : X^2 \rightarrow X$ is the projection maps. By Definitions 1 and 15 and Lemma 10, a contradiction, it follows $\{a, p\} \notin \beta^X$
Similarly, for \( B = \{ a \} \in \mathcal{D}^{X,V}_\partial \setminus \emptyset \), we get \( \mathcal{R} \in \bar{r}_{\partial \mathcal{R}}(\{ a \}) \), a contradiction to the discreteness of \( \bar{r}_{\partial \mathcal{R}}(B) \).

Hence, \( \mathcal{R} \in \mathcal{R} \). Therefore, \( \mathcal{R} \in \mathcal{R} \) and \( \mathcal{R} \in \mathcal{R} \).

Similarly, for \( B = \{ a \} \in \mathcal{D}^{X,V}_\partial \setminus \emptyset \), we get \( \mathcal{R} \subseteq \bar{r}_{\partial \mathcal{R}}(\{ a \}) \), a contradiction to the discreteness of \( \bar{r}_{\partial \mathcal{R}}(B) \).

(iv) Assume that \( \mathcal{R} \in \mathcal{R} \) and \( \mathcal{R} \in \mathcal{R} \). Therefore, \( \mathcal{R} \in \mathcal{R} \). Conversely, suppose (i) to (iv) are holding.

Let \( \beta^{X,V} \) be the initial structure induced by \( A_p : X \) and \( \mathcal{R} \subseteq \mathcal{R} \).

We show that \( \beta^{X,V} \) is the discrete RELStructure on \( X_{\partial \mathcal{R}} \).
Similarly, if $B = \{a_2\}$, only Case (i) and Case (ii) are holding. By Lemma 12, $\bar{r}(B) = \{\bar{R} \in \text{REL}(X \vee \rho X) : \bar{R} \subset \{(a, a_1)\} \}$ is discrete. Therefore, by Definition 15, $(X, \beta^X, r)$ is $T_0$ at $p$.

**Theorem 18.** Let $(X, \beta^X, r)$ be an ordered-RELspace and $p \in X$.

$(X, \beta^X, r)$ is $T_1$ at $p$ if and only if for any $a \in X$ with $a \neq p$, the following holds:

(i) $\{a, p\} \notin \beta^X$

(ii) $\{\bar{R} \in \text{REL}(X) : \bar{R} \subset \{(a, p)\}\} \subset \{p\}$ and $\{\bar{R} \in \text{REL}(X) : \bar{R} \subset \{(a, p)\}\} \not\subset \{p\}$

(iii) $\{\bar{R} \in \text{REL}(X) : \bar{R} \subset \{(a, p)\}\} \not\subset \{p\}$ and $\{\bar{R} \in \text{REL}(X) : \bar{R} \subset \{(a, p)\}\} \not\subset \{p\}$

(iv) $\{\bar{R} \in \text{REL}(X) : \bar{R} \subset \{(a, p)\}\} \not\subset \{p\}$ and $\{\bar{R} \in \text{REL}(X) : \bar{R} \subset \{(a, p)\}\} \not\subset \{p\}$.

**Proof.** By following the same technique used in Theorem 17, and replacing the mapping $A_p$ by the mapping $s_p$, we get the proof.

**Theorem 19.** All ordered-RELspaces are $T'_0$ at $p$.

**Proof.** Let $(X, \beta^X, r)$ be ordered-RELspace and $p \in X$. By Definition 15, we show that for each $U \in \beta^{X^p}$, $U \subseteq i_k(V)$ (where $k = 1, 2$) for some $V \in \beta^X$ and $\bigvee_p U = X$. If $U = \emptyset$, it implies that $U = \emptyset$ for some $a \in X$. If $a = p$, then $\bigvee_p U = \emptyset$ implying $U = \emptyset$.

Suppose $a \neq p$, it follows that $U = a_1$, $a_2$ or $a_1, a_2$. If $U = a_1, a_2$, then $a_1, a_2 \subseteq i_k(V)$ for some $V \in \beta^X$. Hence, $U = a_1, a_2$. Thus, we must have $U = a_1$ for $j = 1, 2$ only and consequently, $\beta^{X^p} = X^{X^p}$, the discrete RELstructure on $X^p$.

Now, for $B \in \beta^{X^p} \setminus \{\emptyset\}$, by Lemma 10, $\bar{r}(B) = \{\bar{R} \in \text{REL}(X \vee \rho X) : \bar{R} \subset \{(a_1, a_2)\}\} \setminus \{p\}$ if it lies in the first (resp., second) component of the wedge product $X \vee \rho X$, a contradiction. In similar manner, $\bar{r}(B) \subset \{p\}$.

**5. $T_0$ and $T_1$ Ordered-RELspaces**

In this section, we define generically notions of $T_0$ and $T_1$ in ordered-RELspaces.

The characterization of $T_0$ objects in categorical topology has been an important idea in a topological universe. Therefore, several attempts has been made such as in 1971 Brümmer [15], in 1973 Marny [18], in 1974 Hoffman [17], in 1977 Harvey [16], and in 1991 Baran [14] to discuss various approaches to generalize classical $T_0$ object and examined the relationship between different forms of generalized $T_0$ objects. One of the main purposes of generalization is to define Hausdorff objects in arbitrary topological categories. In 1991, Baran [14, 37] also generalizes the classical $T_1$ objects of topology to topological categories [14, 37]. In abstract topological categories [21], $T_1$ objects are used to define $T_1$, $T_4$, normal objects, regular, and completely regular. To characterize separation axioms, Baran’s approach was to use initial and final lifts and discreteness.

In 1991, Baran [14] used the generic element method of topos theory introduced by Johnstone [38], to define generic separation axioms, due to the fact that points does not make sense in topos theory. In general, the wedge product $X \vee X$ at $p$ can be replaced by $X^2 \vee X^2$ at diagonal $\Delta$. Any element $(a, b) \in X^2 \vee X^2$ is written as $(a, b)_1$ (resp., $(a, b)_2$) if it lies in the first (resp., second) component of $X^2 \vee X^2$. Clearly, $(a, b)_1 = (a, b)_2$, if and only if $a = b$.

**Definition 20** (cf. [14]).

1. A mapping $A : X^{2} \vee X^2 \rightarrow X^3$ is called **principal axis mapping** provided that

$$A((a, b)_j) = \begin{cases} (a, b, a) ; & j = 1, \\ (a, a, b) ; & j = 2. \end{cases}$$ (14)

2. A mapping $S : X^2 \vee X^2 \rightarrow X^3$ is called **skewed axis mapping** provided that

$$S((a, b)_j) = \begin{cases} (a, b, b) ; & j = 1, \\ (a, a, b) ; & j = 2. \end{cases}$$ (15)

3. A mapping $V : X^2 \vee X^2 \rightarrow X^2$ is called **fold mapping** provided that...
\[ \forall (a, b)_j = (a, b), j = 1, 2. \]  

Any element \((a, b) \in X^2 \gamma_X X^2\) is written as \((a, b)_j\) (resp., \((a, b)_i\)) if it lies in the first (resp., second) component of \(X^2 \gamma_X X^2\). Clearly, \((a, b)_j = (a, b)_i\) if and only if \(a = b\).

Now, we replace the point \(p\) by any generic point \(\delta\) and define the following separation axioms.

**Definition 21.** Let \(U : \mathcal{C} \to \text{Set}\) be a topological functor, \(X \in \text{Obj}(\mathcal{C})\) with \(UX = Z\).

(i) \(X\) is \(\tilde{T}_0\) provided that the initial lift of the \(U\)-source \(\{Z^2 \gamma_X Z_2 \xrightarrow{\Delta} \text{U}(X^3) = Z_3\}\) is discrete \([14]\)

(ii) \(X\) is \(T'_0\) provided that the initial lift of the \(U\)-source \(\{Z^2 \gamma_X Z_2 \xrightarrow{id} \text{U}(Z^2) = Z_2\}\) is discrete, where \((Z^2 \gamma_X Z_2)\) is the final lift of the \(U\)-sink \(\{U(X^2) = Z^2 \xrightarrow{\gamma} \tilde{\text{U}}(Z^2) = Z_2^2\}\) \([14, 39]\).

(iii) \(X\) is called \(T_0\) provided that \(X\) doesn’t contain an indiscrete subspace with at least two points \([18, 40]\).

(iv) \(X\) is \(T_1\) provided that the initial lift of the \(U\)-source \(\{Z^2 \gamma_X Z_2 \xrightarrow{S} \text{U}(X^3) = Z_3\}\) is discrete \([14]\).

**Remark 22.**

(i) In \(\text{TOP}\), all the properties of being \(T_0\), \(\tilde{T}_0\) and \(T'_0\) (respectively, \(T_1\)) are equivalent to those classical ones which are \(T_0\) (respectively, \(T_1\)), i.e., for each \(a, b \in X\) with \(a \neq b\), there exists a neighbourhood \(N_a\) of \(a\) not containing \(b\) (or respectively) and, there exists a neighbourhood \(N_b\) of \(b\) not containing \(a\) \([14, 18, 40]\). (ii) In any topological category, \(\tilde{T}_0\) implies \(T'_0\) but not conversely in general. Also, each of the \(\tilde{T}_0\) and \(T'_0\) has no relation to a \(T_0\) \([39]\).

(iii) Let \(U : \mathcal{C} \to \text{Set}\) be a topological functor, \(X \in \text{Obj}(\mathcal{C})\) and \(p \in \text{U}(X)\) be a retract of \(X\). Then, if \(X\) is \(T_0\) (respectively \(T_1\)), then \(X\) is \(\tilde{T}_0\) at \(p\) (respectively \(T_1\) at \(p\)) but not conversely in general \([36]\).

**Theorem 23.** Let \((X, \beta^X, r)\) be an ordered-RELspace. \((X, \beta^X, r)\) is \(\tilde{T}_0\) iff for each \(a, b \in X\) with \(a \neq b\), the following holds:

(i) \(\{a, b\} \notin \beta^X\)

(ii) \(\{R \in \text{REL}(X) : R \ll \{\{(a, b)\}\} \notin \{R \in \text{REL}(X) : R \ll \{\{(b, a)\}\} \notin \{\}

(iii) \(\{a, b\} \notin \beta^X\)

(iv) \(\{R \in \text{REL}(X) : R \ll \{\{(a, b, b)\}\} \notin \{R \in \text{REL}(X) : R \ll \{\{(b, a, a)\}\} \notin \{\}

Proof. Suppose \((X, \beta^X, r)\) is \(\tilde{T}_0\), we show that conditions (i) to (iv) are holding.

(i) Suppose that \(\{a, b\} \in \beta^X\) for each \(a, b \in X\), \(a \neq b\). Let \(U = \{(a, b), (a, b)\} \in X^2 \gamma_X X^2\). Note that \(\forall (U) = \forall \{(a, b), (a, b)\} \in X^2 \gamma_X X^2\) and \(\pi_1(A(U)) = \{a\} \in \beta^X\). By the assumption, \(\pi_1(A(U)) = \pi_1(A((a, b), (a, b))) = \{a, b\} \in \beta^X\), where \(\pi_1 : X^3 \to X^2\) (for \(k = 2, 3\)) are projection maps. By Definitions 1 and 15 and Lemma 10, it leads to a contradiction, it follows that \(\{a, b\} \notin \beta^X\).

(ii) Suppose that \(\{R \in \text{REL}(X) : R \ll \{\{(a, b)\}\} \notin \{R \in \text{REL}(X) : R \ll \{\{(b, a)\}\} \notin \{\}

(iii) Suppose that \(\{R \in \text{REL}(X) : R \ll \{\{(a, b)\}\} \notin \{R \in \text{REL}(X) : R \ll \{\{(b, a)\}\} \notin \{\}

(iv) Suppose that \(\{R \in \text{REL}(X) : R \ll \{\{(a, b, b)\}\} \notin \{R \in \text{REL}(X) : R \ll \{\{(b, a, a)\}\} \notin \{\}

Similarly, for \(B = \{(a, b, b)\} \in X^2 \gamma_X X^2\), \(\forall (B) \), and \(\forall (B) \), we get \(R_1 \notin \beta^X\). Therefore, \(\{R \in \text{REL}(X) : R \ll \{\{(a, b)\}\} \notin \{R \in \text{REL}(X) : R \ll \{\{(a, b)\}\} \notin \{\}

Similarly, for \(B = \{(a, b, b)\} \in X^2 \gamma_X X^2\), \(\forall (B) \), and \(\forall (B) \), we get \(R_2 \notin \beta^X\). Thus, \(\{R \in \text{REL}(X) : R \ll \{\{(a, b)\}\} \notin \{R \in \text{REL}(X) : R \ll \{\{(a, b)\}\} \notin \{\}

(iv) Suppose that \(\{R \in \text{REL}(X) : R \ll \{\{(a, b, b)\}\} \notin \{R \in \text{REL}(X) : R \ll \{\{(b, a, a)\}\} \notin \{\}

Similarly, for \(B = \{(a, b, b)\} \in X^2 \gamma_X X^2\), \(\forall (B) \), and \(\forall (B) \), we get \(R_2 \notin \beta^X\). Thus, \(\{R \in \text{REL}(X) : R \ll \{\{(a, b)\}\} \notin \{R \in \text{REL}(X) : R \ll \{\{(a, b)\}\} \notin \{\}

Similarly, for \(B = \{(a, b, b)\} \in X^2 \gamma_X X^2\), \(\forall (B) \), and \(\forall (B) \), we get \(R_2 \notin \beta^X\). Thus, \(\{R \in \text{REL}(X) : R \ll \{\{(a, b)\}\} \notin \{R \in \text{REL}(X) : R \ll \{\{(a, b)\}\} \notin \{\}

Similarly, for \( B = \{(a, b)_2\} \in X_2 \times \Delta \times X_2 \setminus \{\phi\} \), we get \( r_d\in \Delta \) is not a contradiction to the discreteness of \( r_d\) of \( B \).

Conversely, suppose (i) to (iv) are holding.

Let \( (\beta_2X, X_2, r_d) \) be the initial structure induced by \( A : X_2 \times \Delta \times X_2 \to (X_2, \beta_2X, r_d) \) and \( \Delta \times X_2 \times X_2 \to (X_2, \Delta \times X_2, r_{d_2}) \), where \( (\beta_2X, r_d) \) is the product RELstructure on \( X_2 \) and \( (\Delta \times X_2, r_{d_2}) \) is the discrete RELstructure on \( X_2 \times \Delta \times X_2 \).

We show that \( (\beta_2X, X_2, r_d) \) is the discrete RELstructure on \( X_2 \times \Delta \times X_2 \), i.e., \( \beta_2X = X_2 \times \Delta \times X_2 \) for \( i, j \in \{1, 2\} \) and for \( B \in X_2 \times \Delta \times X_2 \setminus \{\phi\} \), \( r_d(B) = 0 \).

Let \( U \in \beta_2X \times \Delta \times X_2 \) and \( \forall V \in B \). If \( U = \phi \), then \( U = \phi \). Suppose \( U \neq \phi \), then it follows that \( \forall V \neq \phi \) for some \( (a, b) \in X_2 \). If \( a = b \), then \( U = \{(a, b)\} \).

Case (i). If \( \{a, b\} \notin \beta_2X \times \Delta \times X_2 \times X_2 \), then \( U = \{(a, b)\} \in B \).

Thus, \( \{ \mathcal{A} \in \text{REL}(X_2 \times \Delta \times X_2) : \mathcal{A} \notin \{(a, b)_1, (a, b)_2\} \} \) holds.

Case (ii). \( \{ \mathcal{A} \in \text{REL}(X_2 \times \Delta \times X_2) : \mathcal{A} \notin \{(a, b)_2, (a, b)_2\} \} \) holds. The proof is similar to Case (i).

Case (iii). \( \{ \mathcal{A} \in \text{REL}(X_2 \times \Delta \times X_2) : \mathcal{A} \notin \{(a, b)_1, (a, b)_2\} \} \) holds. Therefore, \( \{ \mathcal{A} \in \text{REL}(X_2 \times \Delta \times X_2) : \mathcal{A} \notin \{(a, b)_1, (a, b)_2\} \} \) is not possible.

Case (iv). Similar to Case (iii), we conclude that \( \{ \mathcal{A} \in \text{REL}(X_2 \times \Delta \times X_2) : \mathcal{A} \notin \{(a, b)_1, (a, b)_2\} \} \) is not possible.
Let $U = \{a, b\}$. Note that $(U, \beta^U, r_U)$ is the subspace of $(X, \beta^X, r)$, where $(\beta^X, r_U)$ is the initial lift of the ordered-RELsystem induced by the inclusion map $i: S \to U$ and for any $S \subset U$, $S \subset \beta^U$, whenever $i(S) = S \subset \beta^U$ and for any $R \in REL(U)$, $R \subset r(S)$, whenever $i(R) = R \subset r(B)$.

By the assumption, $i(U) = U = \{a, b\} \subset \beta^X$ and by Definition 1, we get $\beta^U = PU$.

Now, for any $R \in REL(U)$ let $R = \{\{a, a\}\} \in REL(U)$. By Definition 2, $i(\{a, a\}) \not\in r\{a\}$. By the assumption, $R = \{\{a, a\}\} \not\subset r\{b\}$ implying $R \subset R EL(X)$ if $(a, a) \not\subset r\{b\}$ and $R \subset R EL(X)$ if $(a, a) \not\subset r\{b\}$.

Similarly, for $R = \{\{b, b\}\} \in REL(U)$, it follows that $R \subset R EL(X)$ if $(b, b) \not\subset r\{a\}$ and $R \subset R EL(X)$ if $(b, b) \not\subset r\{a\}$.

Now, if $R = \{\{a, b\}\} \subset REL(U)$ then by the assumption, $R \subset R EL(X)$ if $(a, b) \not\subset r\{b\}$ and $R \subset R EL(X)$ if $(a, b) \not\subset r\{b\}$.

And for $R = \{\{b, a\}\} \subset REL(U)$ then by the assumption, $R \subset R EL(X)$ if $(b, a) \not\subset r\{a\}$ and $R \subset R EL(X)$ if $(b, a) \not\subset r\{a\}$.

Therefore, $r_U = \{R \in REL(U): \emptyset \subset R \}$ and $(\beta^U, r_U) = (P(U), r_d)$, which is a contradiction by Lemma 12. Thus $(i) - (iv)$ are holding.

Conversely, suppose that for all $a, b \in X$ with $a \neq b$, conditions $(i) - (iv)$ are holding. We show that the initial structure $(\beta^U, r_U)$ is not an indiscrete ordered-RELstructure on $U$. Let $U = \{a, b\} \subset X$. By the assumption, $\{a, b\} \notin \beta^X$ and $\{R \in REL(X): \emptyset \not\subset r\{a\}\} \not\subset r\{a\}$ or $\{R \in REL(X): \emptyset \not\subset r\{a\}\} \not\subset r\{a\}$ and $\{R \in REL(X): \emptyset \not\subset r\{a\}\} \not\subset r\{a\}$ or $\{R \in REL(X): \emptyset \not\subset r\{a\}\} \not\subset r\{a\}$ and $\{R \in REL(X): \emptyset \not\subset r\{a\}\} \not\subset r\{a\}$.

Thus, $(U, \beta^U, r)$ is not an indiscrete ordered-RELsubspace of $(X, \beta^X, r)$. Hence, by Definition 21 (iii), $(X, \beta^X, r)$ is $T_0$.

**Theorem 25.** Let $(X, \beta^X, r)$ be an ordered-RELspace. Then, $(X, \beta^X, r)$ is $T_1$ iff for all $a, b \in X$ with $a \neq b$, the following holds:

(i) $\{a, b\} \notin \beta^X$

(ii) $\{R \in REL(X): \emptyset \not\subset r\{a\}\} \not\subset r\{a\}$ and $\{R \in REL(X): \emptyset \not\subset r\{a\}\} \not\subset r\{a\}$

(iii) $\{R \in REL(X): \emptyset \not\subset r\{a\}\} \not\subset r\{a\}$ and $\{R \in REL(X): \emptyset \not\subset r\{a\}\} \not\subset r\{a\}$

(iv) $\{R \in REL(X): \emptyset \not\subset r\{a\}\} \not\subset r\{a\}$ and $\{R \in REL(X): \emptyset \not\subset r\{a\}\} \not\subset r\{a\}$.

**Proof.** Similarly, using Theorem 23, and replacing mapping $A$ by the mapping $S$, we obtain the proof.

**Theorem 26.** All ordered-RELspaces are $T'_{0}$.
(iii) For each \( a, b \in X \) with \( a = b \), and for all \( B \in \mathcal{P}(X) \), \( \{ R \in \text{REL}(X) : R \not< \{(a, b)\} \} \not= r(B) \) or \( \{ R \in \text{REL}(X) : R \not< \{(a, b)\} \} \not= r(B) \). and \( \{ R \in \text{REL}(X) : R \not< \{(a, b)\} \} \not= r(B) \).

Proof. By applying Example 4, Theorem 23, and Theorem 3.1.10 of [41].

\[ \blacksquare \]

Corollary 29. Let \( (X, \mathcal{P}(X), r) \) be in \( \text{PU - REL} \). Then, the following statements are equivalent:

(i) \( (X, \mathcal{P}(X), r) \) is \( T_1 \)

(ii) \( (X, \mathcal{P}(X), r) \) is \( T_1 \text{PUCONV} \), where \( T_1 \text{PUCONV} \) is the category of \( T_1 \) pre-uniform convergence spaces and uniformly continuous maps

(iii) For all \( a, b \in X \) with \( a = b \), and for all \( B \in \mathcal{P}(X) \), \( \{ R \in \text{REL}(X) : R \not< \{(a, b)\} \} \not= r(B) \) and \( \{ R \in \text{REL}(X) : R \not< \{(a, b)\} \} \not= r(B) \).

Proof. This follows from Example 4, Theorem 25, and Theorem 3.2.4 of [41].

\[ \blacksquare \]

6. Quotient-Reflective Subcategories of the Categorized-RELSpaces

Definition 30 (cf. [42]). Given a topological functor \( U : \mathcal{C} \longrightarrow \text{Set} \), a full and isomorphism-closed subcategory \( \mathcal{H} \) of \( \mathcal{C} \), we say that \( \mathcal{H} \) is

(i) Epireflective in \( \mathcal{C} \) and closed if and only if \( \mathcal{H} \) is closed under the formation of products and extremal subobjects (i.e., subspaces)

(ii) Quotient-reflective in \( \mathcal{C} \) if and only if \( \mathcal{H} \) is epireflective and is closed under finer structures (i.e., if \( A \in \mathcal{H}, B \in \mathcal{C}, U(A) = U(B) \), and \( id : A \longrightarrow B \) is a \( \mathcal{C} \)-morphism, then \( B \in \mathcal{H} \)).

Theorem 31.

(i) Any \( T_0\text{O-REL} \), \( T_0\text{O-REL} \) and \( T_0\text{O-REL} \) is a quotient-reflective subcategory of \( \text{O-REL} \)

(ii) \( T_0\text{O-REL} \) is a normalized topological construct

Proof. (i) Suppose \( \mathcal{C} = T_0 \text{O-REL} \) and \( (X, \beta^X, r) \in \mathcal{C} \). It can be easily verified that \( \mathcal{C} \) is a full and isomorphism-closed subcategory of \( \text{O-REL} \) and closed under finer structures. It remains to show that \( X \) is closed under extremal subobjects and closed under the formation of products.

Let \( A \subset X \) and \( (\beta^A, r_A) \) denotes the sub \( \text{O-REL} \) structure on \( A \), induced by the inclusion map \( i : A \longrightarrow X \). We show that \( (A, \beta^A, r_A) \) is \( T_0\text{O-REL} \) space. Suppose that for any \( a, b \in A \) with \( a \neq b \), \( \{a, b\} \in \beta^A \), then by the inclusion map \( i : (a, b) = \{i(a), i(b)\} = \{a, b\} \in \beta^X \), a contradiction by Theorem 23. Thus, \( \{a, b\} \not\in \beta^A \).

Now, suppose \( \{ R \in \text{REL}(X) : R \not< \{(a, b)\} \} \not= r_A(\{a\}) \) and \( \{ R \in \text{REL}(X) : R \not< \{(a, b)\} \} \not= r_A(\{b\}) \). It follows that, for all \( R \in \mathcal{C} \) such that \( \{a, b\} \in R \), and by the inclusion map \( \{(a, b)\} \in i(R) \) implying \( \{(a, b)\} \in R \). It follows that \( \{ R \in \text{REL}(X) : R \not< \{(a, b)\} \} \not= r_A(\{a\}) \), a contradiction by Theorem 23. Similarly, by the same argument \( \{ R \in \text{REL}(X) : R \not< \{(a, b)\} \} \not= r_A(\{b\}) \), a contradiction by Theorem 23. Therefore, \( \{ R \in \text{REL}(A) : R \not< \{(a, b)\} \} \not= r_A(\{a\}) \) or \( \{ R \in \text{REL}(A) : R \not< \{(a, b)\} \} \not= r_A(\{b\}) \). Hence, \( X \) is closed under extremal subobjects.

Next, suppose that \( X = \prod_{k \in \mathcal{I}} X_k \), where \( (\beta^{X_k}, r_{X_k}) \) are the \( T_0\text{O-REL} \) structures on \( X_k \) induced by projection map \( \pi_k : X_k \longrightarrow X \) for all \( k \in \mathcal{I} \), i.e., \( (X_k, \beta^{X_k}, r_{X_k}) \in \mathcal{C} \). We show that \( (X, \beta^X, r_X) \) is a \( T_0\text{O-REL} \) space. Let \( \{a, b\} \in \beta^X \) for any \( a, b \in X \) with \( a \neq b \). Then, \( \pi_k(\{a, b\}) = \{\pi_k(a), \pi_k(b)\} \neq \{a, b\} \), a contradiction by Theorem 23. Thus, \( \{a, b\} \not\in \beta^X \).

Now, suppose \( \{ R \in \text{REL}(X) : R \not< \{(a, b)\} \} \not= r_X(\{a\}) \) and \( \{ R \in \text{REL}(X) : R \not< \{(a, b)\} \} \not= r_X(\{b\}) \). It follows \( R \in \mathcal{C} \) implies \( \{a, b\} \not\subset R \). Then, there is \( k \in \mathcal{I} \) for which \( a_k \neq b_k \in X_k \), and \( \pi_k(\{a, b\}) \not\subset \pi_k R \) implying \( \{\pi_k(a), \pi_k(b)\} \neq \{a_k, b_k\} \subset \pi_k R \). It follows that \( \{ R \in \text{REL}(X) : R \not< \{(a, b)\} \} \not= r_X(\{a\}) \), a contradiction by Theorem 23. By the same process, \( \{ R \in \text{REL}(X) : R \not< \{(a, b)\} \} \not= r_X(\{b\}) \), a contradiction. Hence, \( \{ R \in \text{REL}(X) : R \not< \{(a, b)\} \} \not= r_X(\{a\}) \) or \( \{ R \in \text{REL}(X) : R \not< \{(a, b)\} \} \not= r_X(\{b\}) \). In similar way, \( \{ R \in \text{REL}(X) : R \not< \{(a, b)\} \} \not= r_X(\{a\}) \) or \( \{ R \in \text{REL}(X) : R \not< \{(a, b)\} \} \not= r_X(\{b\}) \). Hence, \( X \) is closed under the formation of products.

Therefore, the category \( T_0\text{O-REL} \) is a quotient-reflective subcategory of \( \text{O-REL} \).

Analogous to the above argument, setting \( \mathcal{C} = T_0 \text{O-REL} \) or \( T_0\text{O-REL} \), the proof can be easily followed by using Theorem 24 or Theorem 25, respectively.

(ii) By the Theorem 26 and Remark 13, \( T_0\text{O-REL} \) and \( \text{O-REL} \) are isomorphic categories and thus \( T_0\text{O-REL} \) is normalized

\[ \blacksquare \]

Data Availability

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.
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