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Research Article

Some Inequalities of Hermite–Hadamard Type for MT-h-Convex Functions via Classical and Generalized Fractional Integrals

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Convexity plays a vital role in pure and applied mathematics specially in optimization theory, but the classical convexity is not enough to fulfil the needs of modern mathematics; hence, it is important to study generalized notion of convexity. Fraction integral operators also become an important tool for solving problems of model physical and engineering processes that are found to be best described by fractional differential equations. The aim of this paper is to study MT-h-convex functions via fractional integral operators. We establish several Hermite–Hadamard-type inequalities for MT-h-convex function via classical and generalized fractional integrals. We also obtain special means related to our results and present some error estimates for the trapezoidal formulas.

1. Introduction

One of the most important notions in mathematics is convex functions which are very important for both pure and applied mathematicians. Convex functions are helpful in solving problems of optimization theory and many other problems of applied nature.

Definition 1. Let $J \subseteq \mathfrak{R}$ and J^o be interior of J, a mapping $\psi : J \longrightarrow \mathfrak{R}$ said to be convex on J, if the following inequality holds for all $c, d \in J$ and $\lambda \in [0, 1]$,

$$\psi(\lambda c + (1 - \lambda d)) \leq \lambda\psi(c) + (1 - \lambda)\psi(d). \tag{1}$$

The mapping ψ is said to be concave if $-\psi$ is convex.

The theory of inequalities got the attention of many researchers, and the new inequalities are always appreciable not only in real analysis but also the researchers working in applied sciences use inequalities as a very effective tool for analyzing different practical problems and to study various properties of solution of different equations [1]. Jensen-type inequalities, Hardy-type inequalities [2], Gagliardo-Nirenberg-type inequalities [3], Grüss-type inequalities [4], Ostrowski-type inequality [5, 6], etc. are extensively studied in the literature. The most famous inequality in literature is known as Hermite–Hadamard inequality, which has fundamental role in convex analysis. The Hermite–Hadamard-type inequalities for different classes of convex function can be found in [7, 8] and references therein.

Let $\psi : J \subseteq \mathfrak{R} \longrightarrow \mathfrak{R}$ be a convex mapping defined on the interval J of real numbers and $c, d \in J$. Then, the Hermite–Hadamard inequality is as follows:

$$\psi\left(\frac{c+d}{2}\right) \leq \frac{1}{c-d}\int_c^d \psi(\lambda)d\lambda \leq \frac{\psi(c)+\psi(d)}{2}, \tag{2}$$

with $c < d$ and $c, d \in J$. If ψ is concave, then both the inequalities reverse their direction.

Hermite–Hadamard inequality is the most important inequality so far in inequality theory, and several extensions of this inequality are given by researchers in recent years [9]. Our motivation is to establish a generalized version of Hermite–Hadamard-type inequality for MT-h-convex functions. It is worthy to mention here that several results of literature can be obtained from our established results as a particular case by taking suitable values of involved parameters.

Since classical notion of convexity is not enough for solving today's problems, so this notion has been generalized by several researchers to meet the needs of modern mathematics. Now, we present some generalized notions of convexity.

Definition 2 (see [10]). Let $h : I \longrightarrow \mathfrak{R}$ be a nonnegative mapping, $h \neq 0$. The mapping $\psi : J \longrightarrow \mathfrak{R}$ is said to be h-convex, if ψ is nonnegative and for all $c, d \in J$, $\lambda \in (0, 1)$, the following inequality holds:

$$\psi(\lambda c + (1-\lambda)d) \leq h(\lambda)\psi(c) + h(1-\lambda)\psi(d). \tag{3}$$

The mapping is said to be h-concave if inequality (3) is reversed.

Definition 3 (see [11, 12]). A mapping $\psi : J \subseteq \mathfrak{R} \longrightarrow \mathfrak{R}$ is said to be MT-convex on J if it is nonnegative and satisfies the following inequality:

$$\psi(\lambda c + (1-\lambda)d) \leq \frac{\sqrt{\lambda}}{2\sqrt{1-\lambda}}\psi(c) + \frac{\sqrt{1-\lambda}}{2\sqrt{\lambda}}\psi(d). \tag{4}$$

Motivated by the above two notions, we introduce the following notion of MT-h-convex function.

Definition 4. A mapping $\psi : J \subseteq \mathfrak{R} \longrightarrow \mathfrak{R}$ is said to be MT-h-convex on J, if for all $c, d \in J$ and $h(\lambda) \in (0, 1)$, it is nonnegative and satisfies the following inequality:

$$\psi(\lambda c + (1-\lambda)d) \leq \frac{\sqrt{h(\lambda)}}{2\sqrt{h(1-\lambda)}}\psi(c) + \frac{\sqrt{h(1-\lambda)}}{2\sqrt{h(\lambda)}}. \tag{5}$$

Fraction integral operators also become an important tool for solving problems of model physical and engineering processes that are found to be best described by fractional differential equations. Fractional calculus creates a diversity in inequality theory of convex analysis [13, 14].

The following generalized fractional integral operators were introduced by Ertugral and Sariaya [15] as follows:

Let $\rho : [0,\infty) \longrightarrow [0,\infty)$ be such that

$$\int_0^1 \frac{\rho(\lambda)}{\lambda} d\lambda < \infty, \tag{6}$$

then the left-hand-sided and right-hand-sided generalized fractional integral operators are defined as:

$${}_{c}^{+}J_{\rho}\psi(\nu) = \int_c^{\nu} \frac{\rho(\nu-\lambda)}{\nu-\lambda}\psi(\lambda), \nu > c, \tag{7}$$

$${}_{d}^{-}J_{\rho}\psi(\nu) = \int_{\nu}^{d} \frac{\rho(\lambda-\nu)}{\lambda-\nu}\psi(\lambda), \nu < d, \tag{8}$$

respectively.

The following remark justifies the generality of the above fractional integral operators.

Remark 5. (1) If $\rho(\lambda) = \lambda$, then (7) and (8) convert to usual Riemann fractional integral, respectively

$$\begin{aligned} J_c^{+}\psi(\nu) &= \int_c^{\nu} \psi(\lambda) d\lambda, \nu > c, \\ J_d^{-}\psi(\nu) &= \int_{\nu}^{d} \psi(\lambda) d\lambda, \nu < d. \end{aligned} \tag{9}$$

(2) If $\rho(\lambda) = \lambda^{\eta}/\Gamma(\eta)$, then (7) and (8) reduce to the Riemann-Liouville integral [14, 16]

$$\begin{aligned} J_{c^+}^{\eta}\psi(\nu) &= \frac{1}{\Gamma(\eta)}\int_c^{\nu} (\nu-\lambda)^{\eta-1}\psi(\lambda) d\lambda, \nu > c, \\ J_{d^-}^{\eta}\psi(\nu) &= \frac{1}{\Gamma(\eta)}\int_{\nu}^{d} (\lambda-\nu)^{\eta-1}\psi(\lambda) d\lambda, \nu < d. \end{aligned} \tag{10}$$

Here, $\Gamma(\eta) = \int_0^{\infty} t^{-m} m^{\eta-1} dm$ and $J_{c^+}^{0}\psi(\nu) = J_{d^-}^{0}\psi(\nu) = \psi(\nu)$.

Note that, for $\eta = 1$, the Riemann-Liouville integral converts to the classical integrals. For the other interesting special cases of (7) and (8), we refer to the readers [17, 18].

The aim of this paper is to establish Hermite–Hadamard-type inequalities for the proposed notion of MT-h-convex function in the setting of classical and generalized fractional integral operators. As applications of our results, we present special means related with our results. We also establish some error estimates for the trapezoidal formula.

2. Hermite–Hadamard-Type Inequalities via Classical Fractional Integral

First, we present the following identity that has been obtained in [19] which will play a crucial role in the proof of our main results.

Lemma 6. *Let $\psi : J \subset \mathfrak{R} \longrightarrow \mathfrak{R}$ be a differentiable function on J^o and $c, d \in J$ with $c < d$. If $\psi' \in L_1[c, d]$, then the* *following inequality holds:*

$$\frac{(d-\nu)\psi(d)+(\nu-c)\psi(c)}{d-c}-\frac{1}{d-c}\int_c^d\psi(s)ds = \frac{(\nu-c)^2}{d-c}\int_0^1(1-\lambda)\psi'(\lambda\nu+(1-\lambda)c)d\lambda + \frac{(d-\nu)^2}{d-c}\int_0^1(1-\lambda)\psi'(\lambda\nu+(1-\lambda)d)d\lambda, \quad (11)$$

for each $\nu\in[c,d]$.

Theorem 7. *Let* $\psi: J\subset\Re^+\longrightarrow\Re$ *be a differentiable function on* J^o *such that* $\psi'\in L_1[c,d]$ *where* $c,d\in J$. *If* $|\psi'|$ *is MT-h-convex function on* $[c,d]$ *and* $|\psi'|\le Q$ *where* $\nu\in[c,d]$ *, then we have*

$$\left|\frac{(d-\nu)\psi(d)+(\nu-c)\psi(c)}{d-c}-\frac{1}{d-c}\int_c^d\psi(s)ds\right| \le Q\frac{(\nu-c)^2+(d-\nu)^2}{d-c}\times\left(\frac{1}{2}(S_1+S_2)\right), \quad (12)$$

where

$$S_1=\int_0^1(1-\lambda)\frac{\sqrt{h(\lambda)}}{\sqrt{h(1-\lambda)}}d\lambda, S_2=\int_0^1(1-\lambda)\frac{\sqrt{h(1-\lambda)}}{\sqrt{h(\lambda)}}d\lambda, \quad (13)$$

with $h(\lambda)/h(1-\lambda)<\infty$ *is finite.*

Proof. From Lemma 6, we have

$$\frac{(d-\nu)\psi(d)+(\nu-c)\psi(c)}{d-c}-\frac{1}{d-c}\int_c^d\psi(s)ds = \frac{(\nu-c)^2}{d-c}\int_0^1(1-\lambda)\psi'(\lambda\nu+(1-\lambda)c)d\lambda + \frac{(d-\nu)^2}{d-c}\int_0^1(1-\lambda)\psi'(\lambda\nu+(1-\lambda)d)d\lambda. \quad (14)$$

Applying mode on both sides,

$$\left|\frac{(d-\nu)\psi(d)+(\nu-c)\psi(c)}{d-c}-\frac{1}{d-c}\int_c^d\psi(s)ds\right| \le \frac{(\nu-c)^2}{d-c}\int_0^1(1-\lambda)\left|\psi'(\lambda\nu+(1-\lambda)c)\right|d\lambda + \frac{(d-\nu)^2}{d-c}\int_0^1(1-\lambda)\left|\psi'(\lambda\nu+(1-\lambda)d)\right|d\lambda. \quad (15)$$

Employing MT-h-convexity of $|\psi'|$, we have

$$\le\frac{(\nu-c)^2}{d-c}\int_0^1(1-\lambda)\left[\frac{\sqrt{h(\lambda)}}{2\sqrt{h(1-\lambda)}}|\psi'(\nu)| + \frac{\sqrt{h(1-\lambda)}}{2\sqrt{h(\lambda)}}|\psi'(c)|\right]d\lambda + \frac{(d-\nu)^2}{d-c}\int_0^1(1-\lambda) \cdot\left[\frac{\sqrt{h(\lambda)}}{2\sqrt{h(1-\lambda)}}|\psi'(\nu)|+\frac{\sqrt{h(1-\lambda)}}{2\sqrt{h(\lambda)}}|\psi'(d)|\right]d\lambda. \quad (16)$$

Since $|\psi'(\nu)|\le Q$, so

$$\begin{aligned}&\le Q\frac{(\nu-c)^2}{d-c}\left[\int_0^1(1-\lambda)\frac{\sqrt{h(\lambda)}}{2\sqrt{h(1-\lambda)}}d\lambda + \int_0^1(1-\lambda)\frac{\sqrt{h(1-\lambda)}}{2\sqrt{h(\lambda)}}d\lambda\right]+Q\frac{(d-\nu)^2}{d-c}\\&\cdot\left[\int_0^1(1-\lambda)\frac{\sqrt{h(\lambda)}}{2\sqrt{h(1-\lambda)}}d\lambda+\int_0^1(1-\lambda)\frac{\sqrt{h(1-\lambda)}}{2\sqrt{h(\lambda)}}d\lambda\right]\\&\le Q\frac{(\nu-c)^2}{d-c}\left(\frac{1}{2}(S_1+S_2)\right)+Q\frac{(d-\nu)^2}{d-c}\left(\frac{1}{2}(S_1+S_2)\right)\\&\le Q\frac{(\nu-c)^2+(d-\nu)^2}{d-c}\times\left(\frac{1}{2}(S_1+S_2)\right).\end{aligned} \quad (17)$$

The proof is completed. □

Corollary 8. *In Theorem 7, if we substitute* $h(\lambda)=\lambda$, *we get [20] Theorem 7.*

Remark 9. If we take $\nu=(c+d)/2$ in Theorem 7, then we obtain

$$\left|\frac{(d-\nu)\psi(d)+(\nu-c)\psi(c)}{d-c}-\frac{1}{d-c}\int_c^d\psi(s)ds\right| \le\frac{1}{4}(d-c)Q\times(S_1+S_2). \quad (18)$$

Theorem 10. *Let* $\psi: J\subset\Re^+\longrightarrow\Re$ *be a differentiable function on* J^o *such that* $\psi'\in L_1[c,d]$ *where* $c,d\in J$. *If* $|\psi'|^q$ *is MT-h-convex mapping on* $[c,d]$ *and* $q>1$, $(1/p)+(1/q)=1$ *also* $|\psi'|\le Q$ *and* $\nu\in[c,d]$, *then we have*

$$\frac{(d-\nu)\psi(d)+(\nu-c)\psi(c)}{d-c}-\frac{1}{d-c}\int_c^d\psi(s)ds \le\frac{(\nu-c)^2+(d-\nu)^2}{d-c}\times\left(\frac{1}{p+1}\right)^{1/p} \times\left(Q^q\int_0^1\frac{\sqrt{h(\lambda)}}{\sqrt{h(1-\lambda)}}d\lambda\right)^{1/q}. \quad (19)$$

Proof. Assume that $p > 1$ and using Lemma 6, we have

$$\left|\frac{(d-\nu)\psi(d)+(\nu-c)\psi(c)}{d-c}-\frac{1}{d-c}\int_c^d \psi(s)ds\right| \leq \frac{(\nu-c)^2}{d-c}\int_0^1(1-\lambda)\left|\psi'(\lambda\nu+(1-\lambda)c)\right|d\lambda + \frac{(d-\nu)^2}{d-c}\int_0^1(1-\lambda)\left|\psi'(\lambda\nu+(1-\lambda)d)\right|d\lambda. \quad (20)$$

Using Hölder inequality,

$$\leq \frac{(\nu-c)^2}{d-c}\left(\int_0^1(1-\lambda)^p d\lambda\right)^{1/p}\left(\int_0^1\left|\psi'(\lambda\nu+(1-\lambda)c)\right|^q d\lambda\right)^{1/q} + \frac{(d-\nu)^2}{d-c}\left(\int_0^1(1-\lambda)^p d\lambda\right)^{1/p}\left(\int_0^1\left|\psi'(\lambda\nu+(1-\lambda)d)\right|^q d\lambda\right)^{1/q}. \quad (21)$$

Since $|\psi'|^q$ is MT-h-convex mapping and $|\psi(\nu)| \leq Q$, so we have

$$\leq \frac{(\nu-c)^2}{d-c}\times\left(\frac{1}{p+1}\right)^{1/p}\times\left(\int_0^1\left[\frac{\sqrt{h(\lambda)}}{2\sqrt{h(1-\lambda)}}\left|\psi'(\nu)\right|^q + \frac{\sqrt{h(1-\lambda)}}{2\sqrt{h(\lambda)}}+\left|\psi'(c)\right|^q\right]\right)^{1/q} + \frac{(d-\nu)^2}{d-c} \times\left(\frac{1}{p+1}\right)^{1/p}\times\left(\int_0^1\left[\frac{\sqrt{h(\lambda)}}{2\sqrt{h(1-\lambda)}}\left|\psi'(\nu)\right|^q + \frac{\sqrt{h(1-\lambda)}}{2\sqrt{h(\lambda)}}+\left|\psi'(d)\right|^q\right]\right)\frac{1}{q}. \quad (22)$$

Some small calculations yield that

$$\int_0^1\frac{\sqrt{h(\lambda)}}{h(1-\lambda)}d\lambda = \int_0^1\frac{\sqrt{h(1-\lambda)}}{h(\lambda)}d\lambda. \quad (23)$$

This implies that

$$\leq \frac{(\nu-c)^2}{d-c}\times\left(\frac{1}{p+1}\right)^{1/p}\times\left(Q^q\int_0^1\frac{\sqrt{h(\lambda)}}{\sqrt{h(1-\lambda)}}d\lambda\right)^{1/q} + \frac{(d-\nu)^2}{d-c}\times\left(\frac{1}{p+1}\right)^{1/p}\times\left(Q^q\int_0^1\frac{\sqrt{h(\lambda)}}{\sqrt{h(1-\lambda)}}d\lambda\right)^{1/q} = \frac{(\nu-c)^2+(d-\nu)^2}{d-c}\times\left(\frac{1}{p+1}\right)^{1/p}\times\left(Q^q\int_0^1\frac{\sqrt{h(\lambda)}}{\sqrt{h(1-\lambda)}}d\lambda\right)^{1/q}. \quad (24)$$

The proof is completed. □

Corollary 11. *In Theorem 10, if we substitute $h(\lambda)=\lambda$, we get [20] Theorem 2.4.*

Remark 12. If we take $\nu = (c+d)/2$ in Theorem 10, then we obtain

$$\left|\frac{(d-\nu)\psi(d)+(\nu-c)\psi(c)}{d-c}-\frac{1}{d-c}\int_c^d \psi(s)ds\right| \leq \left(\frac{(d-c)}{2}\right)\times\left(\frac{1}{p+1}\right)^{1/p}\times\left(Q^q\int_0^1\frac{\sqrt{h(\lambda)}}{\sqrt{h(1-\lambda)}}d\lambda\right)^{1/q}. \quad (25)$$

Theorem 13. *Let $\psi : J \subset \mathfrak{R}^+ \longrightarrow \mathfrak{R}$ be a differentiable function on J^o such that $\psi' \in L_1[c,d]$ with $c<d$ where $c,d \in J$. If $|\psi'|^q$ is MT-h-convex function on $[c,d]$ with $q>1$ and $|\psi'(\nu)| \leq Q$ where $\nu \in [c,d]$, then we have*

$$\left|\frac{(d-\nu)\psi(d)+(\nu-c)\psi(c)}{d-c}-\frac{1}{d-c}\int_c^d \psi(s)ds\right| \leq \frac{(\nu-c)^2+(d-\nu)^2}{d-c}\times\left(\frac{1}{2}\right)^{1-(1/q)}\left(\frac{1}{2}Q^q S_1+\frac{1}{2}Q^q S_2\right)^{1/q}, \quad (26)$$

where

$$S_1 = \int_0^1(1-\lambda)\frac{\sqrt{h(\lambda)}}{\sqrt{h(1-\lambda)}}d\lambda, \quad S_2 = \int_0^1(1-\lambda)\frac{\sqrt{h(1-\lambda)}}{\sqrt{h(\lambda)}}d\lambda, \quad (27)$$

with $h(\lambda)/h(1-\lambda)<\infty$ is finite.

Proof. Using Lemma 6, Hölder inequality, and MT-h-convexity of $|\psi'|^q$, we have

$$\left|\frac{(d-\nu)\psi(d)+(\nu-c)\psi(c)}{d-c}-\frac{1}{d-c}\int_c^d \psi(s)ds\right| \leq \frac{(\nu-c)^2}{d-c}\int_0^1(1-\lambda)\left|\psi'(\lambda\nu+(1-\lambda)c)\right|d\lambda + \frac{(d-\nu)^2}{d-c}\int_0^1(1-\lambda)\left|\psi'(\lambda\nu+(1-\lambda)d)\right|d\lambda$$
$$\leq \frac{(\nu-c)^2}{d-c}\left(\int_0^1(1-\lambda)d\lambda\right)^{1-(1/q)} \cdot\left(\int_0^1(1-\lambda)\left|\psi'(\lambda\nu+(1-\lambda)c)\right|^q d\lambda\right)^{1/q} + \frac{(d-\nu)^2}{d-c}\left(\int_0^1(1-\lambda)d\lambda\right)^{1-(1/q)} \cdot\left(\int_0^1(1-\lambda)\left|\psi'(\lambda\nu+(1-\lambda)d)\right|^q d\lambda\right)^{1/q}$$

$$\leq \frac{(\nu-c)^2}{d-c} \times \left(\frac{1}{2}\right)^{1-(1/q)} \left(\int_0^1 (1-\lambda)\frac{\sqrt{h(\lambda)}}{2\sqrt{h(1-\lambda)}}\left|\psi'(\nu)\right|^q \cdot d\lambda + \int_0^1 (1-\lambda)\frac{\sqrt{h(\lambda)}}{2\sqrt{h(1-\lambda)}}\left|\psi'(c)\right|^q d\lambda\right)^{1/q}$$
$$+ \frac{(d-\nu)^2}{d-c} \times \left(\frac{1}{2}\right)^{1-(1/q)} \times \left(\int_0^1 (1-\lambda)\frac{\sqrt{h(\lambda)}}{2\sqrt{h(1-\lambda)}} \cdot \left|\psi'(\nu)\right|^q d\lambda + \int_0^1 (1-\lambda)\frac{\sqrt{h(\lambda)}}{2\sqrt{h(1-\lambda)}}\left|\psi'(c)\right|^q d\lambda\right)^{1/q}$$
$$\leq \frac{(\nu-c)^2}{d-c} \times \left(\frac{1}{2}\right)^{1-(1/q)} \left(\frac{1}{2}Q^q S_1 + \frac{1}{Q^q} S_2\right)^{1/q} + \frac{(d-\nu)^2}{d-c} \times \left(\frac{1}{2}\right)^{1-(1/q)} \left(\frac{1}{2}Q^q S_1 + \frac{1}{Q^q} S_2\right)^{1/q}$$
$$\leq \frac{(\nu-c)^2 + (d-\nu)^2}{d-c} \times \left(\frac{1}{2}\right)^{1-(1/q)} \left(\frac{1}{2}Q^q S_1 + \frac{1}{2}Q^q S_2\right)^{1/q}. \tag{28}$$

The proof is completed. □

Corollary 14. *In Theorem 13, if we substitute $h(\lambda) = \lambda$, we get [20] Theorem 2.6.*

Remark 15. If we take $\nu = (c+d)/2$ in Theorem 13, then we obtain

$$\left|\frac{(d-\nu)\psi(d) + (\nu-c)\psi(c)}{d-c} - \frac{1}{d-c}\int_c^d \psi(s)ds\right| \leq \frac{(d-c)}{2} \times \left(\frac{1}{2}\right)^{1-(1/q)} \left(\frac{1}{2}Q^q S_1 + \frac{1}{2}Q^q S_2\right)^{1/q}. \tag{29}$$

3. Hermite–Hadamard-Type Inequalities via Generalized Fractional Operators

To establish Hermite–Hadamard-type inequalities via generalized fractional operators for MT-h-convex function, we need the following identity [21].

Lemma 16. *Let $\psi : [c,d] \longrightarrow \Re$ be differentiable function on (c,d) with $c < d$ such that $\psi \in L^1[c,d]$. Then, for each $t \in (0,1)$, we have*

$$\frac{(1-t)^\eta \Omega(1) + t^\eta \nabla(1)}{d-c}\psi(\nu) - \frac{1}{d-c}\left[(1-t)^\eta \left(w^- I_\rho \psi(c)\right) + t^\eta \left(\nu^+ I_\rho \psi(d)\right)\right] = (1-t)^{\eta+1}\int_0^1 \Omega(\lambda)\psi'(\lambda\nu + (1-\lambda)c)d\lambda - t^{\eta+1}\int_0^1 \nabla(\lambda)\psi'(\lambda\nu + (1-\lambda)d)d\lambda, \tag{30}$$

where $\nu = tc + (1-t)d$, and

$$\Omega(\lambda) = \int_0^\lambda \frac{\rho((\nu-c)s)}{s} ds < \infty, \quad \nabla(\lambda) = \int_0^\lambda \frac{\rho((d-\nu)s)}{s} ds < \infty. \tag{31}$$

Theorem 17. *Let $\psi : [c,d] \longrightarrow \Re$ be a differentiable function on (c,d) and $\psi' \in L^1[c,d]$ with $0 \leq c < d$ and $\eta > 0$. Then, the following inequality holds for each $t \in (0,1)$. If $|\psi'|$ is MT-h-convex on $[c,d]$,*

$$\left|\frac{(1-t)^\eta \Omega(1) + t^\eta \nabla(1)}{d-c}\psi(\nu) - \frac{1}{d-c}\left[(1-t)^\eta \left(\nu^- I_\rho \psi(c)\right) + t^\eta \left(\nu^+ I_\rho \psi(d)\right)\right]\right| \leq \frac{(1-t)^{\eta+1}}{2}\left[M_1|\psi'(\nu)| + N_1|\psi'(c)|\right] + \frac{t^{\eta+1}}{2}\left[M_1|\psi'(\nu)| + N_1|\psi'(d)|\right], \tag{32}$$

where the constants M_1, M_2, N_1, and N_2 are

$$M_1 = \int_0^1 \sqrt{\frac{h(\lambda)}{h(1-\lambda)}}|\Omega(\lambda)|d\lambda, \quad M_2 = \int_0^1 \sqrt{\frac{h(\lambda)}{h(1-\lambda)}}|\nabla(\lambda)|d\lambda, \quad N_1 = \int_0^1 \sqrt{\frac{h(1-\lambda)}{h(\lambda)}}|\Omega(\lambda)|d\lambda, \quad N_2 = \int_0^1 \sqrt{\frac{h(1-\lambda)}{h(\lambda)}}|\nabla(\lambda)|d\lambda. \tag{33}$$

Proof. From Lemma 16, we have

$$\frac{(1-t)^\eta \Omega(1) + t^\eta \nabla(1)}{d-c}\psi(\nu) - \frac{1}{d-c}\left[(1-t)^\eta \left(\nu^- I_\rho \psi(c)\right) + t^\eta \left(\nu^+ I_\rho \psi(d)\right)\right] = (1-t)^{\eta+1}\int_0^1 \Omega(\lambda)\psi'(\lambda\nu + (1-\lambda)c)d\lambda - t^{\eta+1}\int_0^1 \nabla(\lambda)\psi'(\lambda\nu + (1-\lambda)d)d\lambda. \tag{34}$$

Using mode property on both sides, we obtain

$$\begin{aligned}&\left|\frac{(1-t)^{\eta}\Omega(1)+t^{\eta}\nabla(1)}{d-c}\psi(\nu)\right.\\&\left.\quad-\frac{1}{d-c}\left[(1-t)^{\eta}\left(\nu^{-}I_{\rho}\psi(c)\right)+t^{\eta}\left(\nu^{+}I_{\rho}\psi(d)\right)\right]\right|\\&\leq(1-t)^{\eta+1}\int_0^1|\Omega(\lambda)|\left|\psi'(\lambda\nu+(1-\lambda)c)\right|d\lambda\\&\quad+t^{\eta+1}\int_0^1|\nabla(\lambda)|\left|\psi'(\lambda\nu+(1-\lambda)d)\right|d\lambda.\end{aligned}\tag{35}$$

Since$|\psi'|$ is MT-h-convex, so we have

$$\begin{aligned}&\leq(1-t)^{\eta+1}\int_0^1|\Omega(\lambda)|\left[\frac{\sqrt{h(\lambda)}}{2\sqrt{h(1-\lambda)}}\left|\psi'(\nu)\right|\frac{\sqrt{h(1-\lambda)}}{2\sqrt{h(\lambda)}}\left|\psi'(c)\right|\right]\\&\quad+t^{\eta+1}\int_0^1|\nabla(\lambda)|\left[\frac{\sqrt{h(\lambda)}}{2\sqrt{h(1-\lambda)}}\left|\psi'(\nu)\right|+\frac{\sqrt{h(1-\lambda)}}{2\sqrt{h(\lambda)}}\left|\psi'(d)\right|\right]d\lambda.\end{aligned}\tag{36}$$

After simplification, we obtain desired result

$$\begin{aligned}&=\frac{(1-t)^{\eta+1}}{2}\left[M_1\left|\psi'(\nu)\right|+N_1\left|\psi'(c)\right|\right]\\&\quad+\frac{t^{\eta+1}}{2}\left[M_1\left|\psi'(\nu)\right|+N_1\left|\psi'(d)\right|\right].\end{aligned}\tag{37}$$

□

Corollary 18. *In Theorem 17, if we substitute $h(\lambda)=\lambda$, we get [21] Theorem 2.1.*

Theorem 19. *Let $\psi:[c,d]\longrightarrow\mathfrak{R}$ be a differentiable function over (c,d) and $\psi'\varepsilon L^1[c,d]$ with $0\leq c<d$ and $\eta>0$. If $|\psi'|^q$ is MT-h-convex with $q>1$. Then for each $t\in(0,1)$, the following inequality holds:*

$$\begin{aligned}&\left|\frac{(1-t)^{\eta}\Omega(1)+t^{\eta}\nabla(1)}{d-c}\psi(\nu)\right.\\&\left.\quad-\frac{1}{d-c}\left[(1-t)^{\eta}\left(\nu^{-}I_{\rho}\psi(c)\right)+t^{\eta}\left(\nu^{+}I_{\rho}\psi(d)\right)\right]\right|\\&\leq(1-t)^{\eta+1}\left(\int_0^1\left|\Omega(\lambda)^p d\lambda\right.\right)^{1/p}\\&\quad\cdot\left(\frac{1}{2}\left(\left|\psi'(\nu)\right|^q+\left|\psi'(c)\right|^q\right)\int_0^1\frac{\sqrt{h(\lambda)}}{\sqrt{h(1-\lambda)}}d\lambda\right)^{1/q}\\&\quad+t^{\eta+1}\left(\int_0^1\left|\Omega(\lambda)^p d\lambda\right.\right)^{1/p}\\&\quad\cdot\left(\frac{1}{2}\left(\left|\psi'(\nu)\right|^q+\left|\psi'(d)\right|^q\right)\int_0^1\frac{\sqrt{h(\lambda)}}{\sqrt{h(1-\lambda)}}d\lambda\right)^{1/q},\end{aligned}\tag{38}$$

where $(1/p)+(1/q)=1$ and $p>1$.

Proof. By using Lemma 16, Hölder's inequality, and MT-h-convexity of $|\psi'|^q$, we get

$$\begin{aligned}&\left|\frac{(1-t)^{\eta}\Omega(1)+t^{\eta}\nabla(1)}{d-c}\psi(\nu)-\frac{1}{d-c}\right|\\&\quad\cdot\left[(1-t)^{\eta}\left(\nu^{-}I_{\rho}\psi(c)\right)+t^{\eta}\left(\nu^{+}I_{\rho}\psi(d)\right)\right]\\&\leq(1-t)^{\eta+1}\int_0^1|\Omega(\lambda)|\left|\psi'(\lambda\nu+(1-\lambda)c)\right|d\lambda\\&\quad+t^{\eta+1}\int_0^1|\nabla(\lambda)|\left|\psi'(\lambda\nu+(1-\lambda)d)\right|d\lambda\\&\leq(1-t)^{\eta+1}\left(\int_0^1\left|\Omega(\lambda)^p d\lambda\right.\right)^{1/p}\left(\int_0^1\left|\psi'(\lambda\nu+(1-\lambda)c)\right|^q d\lambda\right)^{1/q}\\&\quad+t^{\eta+1}\left(\int_0^1|\nabla(\lambda)|^p d\lambda\right)^{1/p}\left(\int_0^1\left|\psi'(\lambda\nu+(1-\lambda)d)\right|^q d\lambda\right)^{1/q}\\&\leq(1-t)^{\eta+1}\left(\int_0^1\left|\Omega(\lambda)^p d\lambda\right.\right)^{1/p}\left(\int_0^1\left[\frac{\sqrt{h(\lambda)}}{2\sqrt{h(1-\lambda)}}\left|\psi'(\nu)\right|^q\right.\right.\\&\quad\left.\left.+\frac{\sqrt{h(1-\lambda)}}{2\sqrt{h(\lambda)}}\left|\psi'(c)\right|^q\right]d\lambda\right)^{1/q}+t^{\eta+1}\left(\int_0^1\left|\nabla(\lambda)^p d\lambda\right.\right)^{1/p}\\&\quad\cdot\left(\int_0^1\left[\frac{\sqrt{h(\lambda)}}{2\sqrt{h(1-\lambda)}}\left|\psi'(\nu)\right|^q+\frac{\sqrt{h(1-\lambda)}}{2\sqrt{h(\lambda)}}\left|\psi'(d)\right|^q\right]d\lambda\right)^{1/q}.\end{aligned}\tag{39}$$

Since,

$$\int_0^1\frac{\sqrt{h(\lambda)}}{\sqrt{h(1-\lambda)}}d\lambda=\int_0^1\frac{\sqrt{h(1-\lambda)}}{\sqrt{h(\lambda)}}d\lambda,\tag{40}$$

this implies that

$$\begin{aligned}&\leq(1-t)^{\eta+1}\left(\int_0^1\left|\Omega(\lambda)^p d\lambda\right.\right)^{1/p}\left(\frac{1}{2}\left(\left|\psi'(\nu)\right|^q+\left|\psi'(c)\right|^q\right)\right.\\&\quad\left.\cdot\int_0^1\frac{\sqrt{h(\lambda)}}{\sqrt{h(1-\lambda)}}d\lambda\right)^{1/q}+t^{\eta+1}\left(\int_0^1\left|\Omega(\lambda)^p d\lambda\right.\right)^{1/p}\\&\quad\cdot\left(\frac{1}{2}\left(\left|\psi'(\nu)\right|^q+\left|\psi'(d)\right|^q\right)\int_0^1\frac{\sqrt{h(\lambda)}}{\sqrt{h(1-\lambda)}}d\lambda\right)^{1/q}.\end{aligned}\tag{41}$$

The proof is completed. □

Corollary 20. *If we substitute $h(\lambda)=\lambda$ in Theorem 19, we obtain [21] Theorem 7.*

Theorem 21. *Let $\psi:[c,d]\longrightarrow\mathfrak{R}$ be a differentiable function on (c,d) and $\psi'\in L^1[c,d]$ with $0\leq c<d$ and $\eta>0$. If function*

$|\psi'|^q$ *is MT-h-convex on* $[c, d]$ *for* $q > 1$, *then for each* $t\varepsilon(0, 1)$, *we have*

$$\begin{aligned}&\left|\frac{(1-t)^{\eta}\Omega(1)+t^{\mu}\nabla(1)}{d-c}\psi(\nu)\right.\\&\left.\quad-\frac{1}{d-c}\left[(1-t)^{\eta}\left(\nu^{-}I_{\rho}\psi(c)\right)+t^{\eta}\left(\nu^{+}I_{\rho}\psi(\eta)\right)\right]\right|\\&\leq\left(\frac{1}{2}\right)^{1/q}(1-t)^{\eta+1}\left[\int_0^1|\Omega(\lambda)|d\lambda\right]^{1-(1/q)}\\&\quad\cdot\left(M_1|\psi'(\nu)|^q+N_1|\psi'(c)|^q\right)^{1/q}\\&\quad+\left(\frac{1}{2}\right)^{1/q}t^{\eta+1}\left[\int_0^1|\nabla(\lambda)|d\lambda\right]^{1-(1/q)}\\&\quad\cdot\left(M_2|\psi'(\nu)|^q+N_2|\psi'(d)|^q\right)^{1/q}.\end{aligned}\tag{42}$$

Proof. By using Lemma 16, power mean integral inequality, and MT-h-convexity of $|\psi'|^q$, we have

$$\begin{aligned}&\left|\frac{(1-t)^{\eta}\Omega(1)+t^{\mu}\nabla(1)}{d-c}\psi(\nu)\right.\\&\left.\quad-\frac{1}{d-c}\left[(1-t)^{\eta}\left(\nu^{-}I_{\rho}\psi(c)\right)+t^{\eta}\left(\nu^{+}I_{\rho}\psi(\eta)\right)\right]\right|\\&\leq(1-t)^{\eta+1}\int_0^1|\Omega(\lambda)||\psi'(\lambda\nu+(1-\lambda)c)|d\lambda\\&\quad+t^{\eta+1}\int_0^1|\nabla(\lambda)||\psi'(\lambda\nu+(1-\lambda)d)|d\lambda\\&\leq(1-t)^{\eta+1}\left(\int_0^1|\Omega(\lambda)|d\lambda\right)^{1-(1/q)}\left(\int_0^1|\Omega(\lambda)||\psi'(\lambda\nu+(1-\lambda)c|^q d\lambda\right)^{1/q}\\&\quad+t^{\eta+1}\left(\int_0^1\nabla(\lambda)d\lambda\right)^{1-(1/q)}\left(\int_0^1|\nabla(\lambda)||\psi'(\lambda\nu+(1-\lambda)d|^q d\lambda\right)^{1/q}\\&\leq(1-t)^{\eta+1}\left(\int_0^1|\Omega(\lambda)|d\lambda\right)^{1-(1/q)}\left(\int_0^1|\Omega(\lambda)|\left[\frac{\sqrt{h(\lambda)}}{2\sqrt{h(1-\lambda)}}|\psi'(\nu)|^q\right.\right.\\&\quad\left.\left.+\frac{\sqrt{h(1-\lambda)}}{2\sqrt{h(\lambda)}}|\psi'(\nu)|^q\right]d\lambda\right)^{1/q}+t^{\eta+1}\left(\int_0^1|\nabla(\lambda)|d\lambda\right)^{1-(1/q)}\\&\quad\times\left(\int_0^1|\nabla(\lambda)|\left[\frac{\sqrt{h(\lambda)}}{2\sqrt{h(1-\lambda)}}|\psi'(\nu)|^q+\frac{\sqrt{h(1-\lambda)}}{2\sqrt{h(\lambda)}}|\psi'(d)|^q\right]d\lambda\right)^{1/q}\\&\leq\left(\frac{1}{2}\right)^{1/q}(1-t)^{\eta+1}\left[\int_0^1|\Omega(\lambda)|d\lambda\right]^{1-(1/q)}\left(M_1|\psi'(\nu)|^q+N_1|\psi'(c)|^q\right)^{1/q}\\&\quad+\left(\frac{1}{2}\right)^{1/q}t^{\eta+1}\left[\int_0^1|\nabla(\lambda)|d\lambda\right]^{1-(1/q)}\left(M_2|\psi'(\nu)|^q+N_2|\psi'(d)|^q\right)^{1/q}.\end{aligned}\tag{43}$$

The proof is completed. □

Corollary 22. *If we substitute identity function* $h(\lambda) = \lambda$ *in Theorem 21, we obtain [21] Theorem 2.3.*

4. Application to Special Means

In this section, we present applications of our results in special means. Firstly, we give definitions of special means.

(1) The arithmetic mean

$$A = A(c, d) = \frac{c+d}{2}\ ;\ \mathrm{c, d} \in \mathfrak{R}.\tag{44}$$

(2) The logarithmic mean

$$L(c, d) = \frac{d-c}{\ln|d| - \ln|c|}\ ;\ |c| \neq |d|, cd \neq 0, \mathrm{c, d} \in \mathfrak{R}.\tag{45}$$

(3) The generalized logarithmic mean

$$L_n(c, d) = \left[\frac{d^{n+1} - c^{n+1}}{(d-c)(n+1)}\right]^{1/n}\ ;\ n \in \mathbb{Z} - 1, 0, c, d \in \mathfrak{R}, c \neq d.\tag{46}$$

Now, using the results of §2, we give some applications to special means of real numbers.

Proposition 23. *Assume that* $c, d \in \mathfrak{R}$, $0 < c < d$ *and* $n \in \mathbb{Z}$, $|n| \geq 2$. *Then,* $\forall\ q \geq 1$ *following inequality holds:*

$$\begin{aligned}&|A(c^n, d^n) - L_n^n(c, d)|\\&\leq\left(\frac{d-c}{2}\right)\times\left(\frac{1}{p+1}\right)^{1/p}\times\left(Q^q\int_0^1\frac{\sqrt{h(\lambda)}}{\sqrt{h(1-\lambda)}}d\lambda\right)^{1/q}\\&|A(c^n, d^n) - L_n^n(c, d)|\\&\quad\leq\frac{(d-c)}{2}\times\left(\frac{1}{2}\right)^{1-(1/q)}\left(\frac{1}{2}Q^q S_1+\frac{1}{2}Q^q S_2\right)^{1/q}.\end{aligned}\tag{47}$$

Proof. For $\psi(\nu) = \nu^n$ the statement follows by Remark 12 and 15 where $\nu \in \mathfrak{R}$, $n \in \mathbb{Z}$, $|n| \geq 2$. □

Proposition 24. *Assume that* $c, d \in \mathfrak{R}$, $0 < a < d$. *Then,* $\forall\ q \geq 1$, *we have*

$$\begin{aligned}&\left|A\left(c^{-1}, d^{-1}\right) - L^{-1}(c, d)\right|\\&\leq\left(\frac{d-c}{2}\right)\times\left(\frac{1}{p+1}\right)^{1/p}\times\left(Q^q\int_0^1\frac{\sqrt{h(\lambda)}}{\sqrt{h(1-\lambda)}}d\lambda\right)^{1/q}\\&\left|A\left(c^{-1}, d^{-1}\right) - L^{-1}(c, d)\right|\\&\quad\leq\frac{(d-c)}{2}\times\left(\frac{1}{2}\right)^{1-(1/q)}\left(\frac{1}{2}Q^q S_1+\frac{1}{2}Q^q S_2\right)^{1/q}.\end{aligned}\tag{48}$$

Proof. The statement follows by Remark 12 and 15 for $\psi(\nu) = 1/\nu$. □

5. Estimates of Error for Trapezoidal Formula

Assume that f is a division $c = v_0 < v_1 < \cdots < v_{n-1} < v_n = d$ of interval $[c, d]$ and consider the quadrature formula

$$\int_0^1 \psi(v)d(v) = T_r(\psi, f) + E_r(\psi, f), \tag{49}$$

where

$$T_r(\psi, f) = \sum_{k=0}^{n-1} \frac{\psi(v_k)\psi(v_{k+1})}{2}(v_{k+1} - v_k), \tag{50}$$

for the trapezoidal version $E_r(\psi, f)$ donates the associated approximation error.

Proposition 25. *Let $\psi : J \subseteq \Re \longrightarrow \Re$ be a differentiable function on J^o such that $\psi' \in L_1[c, d]$, where $c, d \in J$ with $c < d$ and $|\psi'|^q$ is MT-h-convex on $[c, d]$. Then, in 24, for every division f of $[c, d]$ and $|\psi'(v)| \leq Q, v \in [c, d]$, the trapezoidal error estimate satisfies*

$$|E_r(\psi, f)| \leq \left(\frac{1}{2}\right) \times \left(\frac{1}{p+1}\right)^{1/p} \times \left(Q^q \int_0^1 \frac{\sqrt{h(\lambda)}}{\sqrt{h(1-\lambda)}} d\lambda\right)^{1/q} \sum_{k=0}^{n-1} (v_{k+1} - v_k)^2, \tag{51}$$

where $p > 1$, $(1/p) + (1/q) = 1$.

Proof. On applying Remark 12 on the subinterval $[v_k, v_{k+1}]$ $(k = 0, 1, 2, \cdots, n-1)$ of the division, we have

$$\begin{aligned} \left| \int_c^d \psi(v)dv - T_r(\psi, f) \right| &= \left| \sum_{k=0}^{n-1} \left\{ \int_{v_k}^{v_k+1} \psi(v)dv - \frac{\psi(v_k) + \psi(v_{k+1})}{2}(v_{k+1} - v_k) \right\} \right| \\ &\leq \sum_{k=0}^{n-1} \left| \int_{v_k}^{v_k+1} \psi(v)dv - \frac{\psi(v_k) + \psi(v_{k+1})}{2}(v_{k+1} - v_k) \right| \\ &\leq \left(\frac{1}{2}\right) \times \left(\frac{1}{p+1}\right)^{1/p} \times \left(Q^q \int_0^1 \frac{\sqrt{h(\lambda)}}{\sqrt{h(1-\lambda)}} d\lambda\right)^{1/q} \\ &\quad \cdot \sum_{k=0}^{n-1} (v_{k+1} - v_k)^2. \end{aligned} \tag{52}$$

With this, the proof is completed. □

Proposition 26. *Let $\psi : J \subseteq \Re \longrightarrow \Re$ be a differentiable function on J^o such that $\psi' \in L_1[c, d]$, where $c, d \in J$ with $c < d$ and $|\psi'|^q$ is MT-h-convex on $[c, d]$ where $q \geq 1$, for every division f of $[c, d]$ and $|\psi'(v)| \leq Q, v \in [c, d]$, the trapezoidal error estimate satisfies*

$$|E_r(\psi, f)| \leq \left(\frac{1}{2}\right)^{2-(1/q)} \left(\frac{1}{2}Q^q S_1 + \frac{1}{2}Q^q S_2\right)^{1/q} \sum_{k=0}^{n-1} (v_{k+1} - v_k)^2. \tag{53}$$

Proof. By using Remark 15, the proof comes the same as Proposition 25. □

6. Conclusion

Convexity and fractional integral operators are the most important notions to deal with the problems of today's world. In the present paper, we introduced a more general notion of convexity, called as MT-h-convexity. The classical and generalized fractional integral operators are used to establish the most famous and most studied Hermite–Hadamard-type inequalities for the proposed class of convex functions. Applications of presented results to special means are also given. Corollaries and remarks presented in this paper justify the generality of our results. It is interesting to establish Hermite–Hadamard-type inequalities for the other variants of fractional integral operators, like Caputo fractional integral operators and Atangana fractional integral operators.

Data Availability

All data required for this research is included within this paper.

Conflicts of Interest

The authors do not have any conflict of interests.

Authors' Contributions

Hengxiao Qi designed the paper and proved the main results of this paper; Waqas Nazeer analyzed the results, wrote the final version of the paper, and relate this paper with the existing results to justify that the results of this paper are more generalized; Fatima Abbas proposed the problem and supervised this work; and Wenbo Liao wrote the first version of this paper.

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References

[1] B. P. Moghaddam, Z. S. Mostaghim, A. A. Pantelous, and J. T. Machado, "An integro quadratic spline-based scheme for solving nonlinear fractional stochastic differential equations with

constant time delay," *Communications in Nonlinear Science and Numerical Simulation*, vol. 92, article 105475, 2021.

[2] P. Ciatti, M. Cowling, and F. Ricci, "Hardy and uncertainty inequalities on stratified Lie groups," *Advances in Mathematics*, vol. 277, pp. 365–387, 2015.

[3] Y. Sawano and H. Wadade, "*On the Gagliardo-Nirenberg type inequality in the critical Sobolev-Morrey space*," *Journal of Fourier Analysis and Applications*, vol. 19, no. 1, pp. 20–47, 2013.

[4] Z. Dahmani, L. Tabharit, and S. Taf, "New generalizations of Gruss inequality using Riemann-Liouville fractional integrals," *Bulletin of Mathematical Analysis and Applications*, vol. 2, pp. 93–99, 2010.

[5] B. Gavrea and I. Gavrea, "On some Ostrowski type inequalities," *General Mathematics*, vol. 18, pp. 33–44, 2010.

[6] W. J. Lin, "Some Ostrowski type inequalities via Riemann-Liouville fractional integrals for h-convex functions," *Journal of Computational Analysis and Applications*, vol. 16, pp. 998–1004.1, 2014.

[7] M. Z. Sarikaya and H. Yildrim, "On Hermite-Hadamard type inequalities for Riemann-Liouville fractional integrals," *Miskolc Mathematical Notes*, vol. 17, no. 2, pp. 1049–1059, 2016.

[8] P. O. Mohammed, "Inequalities of type Hermite-Hadamard for fractional integrals via differentiable convex functions," *Turkish Journal of Analysis and Number Theory*, vol. 4, no. 5, pp. 135–139, 2016.

[9] Y. Wu, H. P. Yin, and B. N. Guo, "Generalizations of Hermite–Hadamard type integral inequalities for convex functions," *Axioms*, vol. 10, no. 3, p. 136, 2021.

[10] S. Varošanec, "On h-convexity," *Journal of Mathematical Analysis and Applications*, vol. 326, pp. 303–311, 2007.

[11] Y. M. Chu, M. A. Khan, T. U. Khan, and T. Ali, "Generalizations of Hermite–Hadamard type inequalities for MT-convex functions," *Journal of Nonlinear Sciences and Applications*, vol. 9, no. 6, pp. 4305–4316, 2016.

[12] M. Tung and H. Yildirimon, "MT-convexity," 2012, https://arxiv.org/pdf/1205.5453.

[13] A. Fernandez and P. Mohammed, "Hermite-Hadamard inequalities in fractional calculus defined using Mittag-Leffler kernels," *Mathematical Methods in the Applied Sciences*, vol. 44, pp. 8414–8431, 2020.

[14] I. Podlubny, *Fractional Differential Equations*, Academic Press, Inc., San Diego, CA, 1999.

[15] M. Z. Sarikaya and F. Ertugral, *On the Generalized Hermite-Hadamard Inequalities*2017, http://www.researchgate.net/publication/321760443.

[16] S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional Integral and DerivativesTheory and Application*, Gordon and Breach, Yverdon, Switzerland, 1993.

[17] S. Mubeen and G. M. Habibullah, "*k -fractional integrals and application*," *International Journal of Contemporary Mathematical Sciences*, vol. 7, no. 2, pp. 89–94, 2012.

[18] U. N. Katugampola, "New approach to a generalized fractional integral," *Applied Mathematics and Computation*, vol. 218, no. 3, pp. 860–865, 2011.

[19] H. Kavurmaci, M. Avci, and M. E. Özdemir, "New inequalities of Hermite-Hadamard type for convex functions with applications," *Journal of Inequalities and Applications*, vol. 2011, no. 1, p. 11, 2011.

[20] W. Liu, W. Wen, and J. Park, "Hermite-Hadamard type inequalities for MT-convex functions via classical integrals and fractional integrals," *Journal of Nonlinear Sciences and Applications*, vol. 9, no. 3, pp. 766–777, 2016.

[21] J. Han, P. O. Mohammed, and H. Zeng, "Generalized fractional integral inequalities of Hermite-Hadamard-type for a convex function," *Open Mathematics*, vol. 18, pp. 794–806, 2020.