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Research Article

On System of Mixed Fractional Hybrid Differential Equations

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In this article, we find the necessary conditions for the existence and uniqueness of solutions to a system of hybrid equations that contain mixed fractional derivatives (Caputo and Riemann-Liouville). We also verify the stability of these solutions using the Ulam-Hyers (U-H) technique. Finally, this study ends with applied examples that show how to proceed and verify the conditions of our theoretical results.

1. Introduction

Although the concept of fractional calculus was established 300 years ago, interest in this type of derivative appeared for a short period. So that it is no secret to anyone that the most important use of fractional derivatives is to find analytical solutions to differential equations if possible, or by using numerical analysis methods to find an approximation to these solutions. In this study, we will focus on the idea of studying theories that investigate the existence of a solution to a system of hybrid fractional equations that contain mixed fractional derivatives with boundary conditions attached to them.

As mentioned before, fractional calculus as a concept is not very recent. It is worth mentioning here the great names who have given a lot to this science, such as A.V. Letnikov, J. Hadamard, J. Liouville, B. Riemann M., and Caputo L. worked in this field. These names must be mentioned by way of example. To get acquainted with some of the names of scientists who have made great contributions to fractional calculus in the modern world, we ask the reader to look at [1].

Fractional derivatives have played a very important role in mathematical modeling in many diverse applied sciences, see [2, 3]. For example, the authors in [4] employed the fractional derivative of the Psi-Caputo type in modeling the logistic population equation, through which they were able to show that the model with the fractional derivative led to a better approximation of the variables than the classical model. In addition, the authors in [5] employed the fractional derivative of the Psi-Caputo type and used the kernel Rayleigh, to improve the model again in modeling the logistic population equation.

As a final example, the authors in [6] employed the fractional derivatives of the Caputo and Caputo-Fabrizio type by modeling the equation that gives the relationship between atmospheric pressure and altitude, and they were also able to show that the fractional equation gave less error in estimating atmospheric pressure at a certain altitude. There are many scientific papers in the literature that prove the superiority of fractional derivatives over classical ones.

There are a large number of manuscripts published in the literature that investigate the issue of the existence of a solution to fractional differential equations, whether they are sequential equations of type or nonsequential equations [7–14].

In 2012, the authors in [7] studied a nonlinear three-point boundary value problem of sequential fractional differential equations. Green's function of the associated problem involving the classical gamma function is obtained. Existence results are obtained using Banach's contraction mapping principle and Krasnoselskii's fixed point theorem.

$$\begin{cases} {}^{C}D^{q}(D+\lambda)\chi(\tau) = w(\tau, \chi(\tau)), \tau \in [0,1] q \in (1,2], \\ \chi(0) = 0, \chi'(0) = 0, \chi(1) = \delta\chi(\eta), \eta \in (0,1). \end{cases} \quad (1)$$

Here, D is the ordinary derivative, $\psi : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}, \lambda \in \mathbb{R}+, \delta$ is a real number such that $\delta \neq ((\lambda + e^{-\lambda} - 1) / (\lambda\eta + e^{-\lambda\eta} - 1))$.

In 2019, Ahmad et al. [15] developed the existence theory for a new kind of nonlocal three-point boundary value problems for differential equations involving both Caputo and Riemann–Liouville fractional derivatives. The existence of solutions for the multivalued problem concerning the upper semicontinuous and Lipschitz cases is proved by applying nonlinear alternative for Kakutani maps and Covitz and Nadler fixed point theorem.

$$\begin{cases} \left({}^{C}D^{q}_{1-}\,{}^{RL}D^{p}_{0+}\right)\omega(\tau) = \vartheta(\tau, \omega(\tau)), 1 < q \leq 2, 0 < p \leq 1, \\ \omega(0) = \omega'(0) = 0, \omega(1) = \delta\omega(\zeta), \end{cases} \quad (2)$$

where $\varphi : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}, \delta \in \mathbb{R}, \zeta \in (0,1)$.

It is known that fractional calculus and FDEs are used in different fields such as physics, signal and image processing, control theory, robotics, economics, biology, and metallurgy, see for example [16, 17] and references therein. On the other hand, recently, many researchers have paid much attention to hybrid differential equations of fractional order. This is because of the development and new advanced applications of fractional calculus. The fractional hybrid modeling is of great significance in different engineering fields, and it can be a unique idea for future combined research between various applied sciences, for example, see [18] in which fractional hybrid modeling of a thermostat is simulated, for some recent results on hybrid.

For FDEs, we refer to [19, 20]. Freshly, some authors have studied different characteristics of hybrid FDEs including the existence of solutions, see for some detail [21–29], and some go further and studied Hyers-Ulam stability for FDEs by different mathematical theories, see for some detail [26].

Zhao et al. [29] investigated the existence result for the fractional hybrid differential equations with Riemann–Liouville fractional derivatives given by

$$\begin{cases} \left({}^{RL}D^{r}_{0+}\right)\left(\dfrac{x(t)}{f(t, x(t))}\right) = g(t, x(t)), t \in [0, T], r \in (0,1), \\ x(0) = 0, \end{cases} \quad (3)$$

where ${}^{RL}D^{r}$ is Riemann–Liouville fractional derivative, $f : [0, T] \times \mathbb{R} \longrightarrow \mathbb{R}/\{0\}, g : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ are assumed to be continuous.

Hilal and Kajouni [30] studied the Caputo hybrid BVP of the form

$$\begin{cases} \left({}^{C}D^{r}_{0+}\right)\left(\dfrac{x(t)}{f(t, x(t))}\right) = g(t, x(t)), t \in [0, L], r \in (0,1), \\ a_1 \dfrac{x(0)}{f(0, x(0))} + a_2 \dfrac{x(L)}{f(L, x(L))} = d, \end{cases} \quad (4)$$

in which $f : [0, T] \times \mathbb{R} \longrightarrow \mathbb{R}/\{0\}$, $g : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ are assumed to be continuous and $a_1 + a_2 \neq 0$.

In [31], the authors have considered the following coupled hybrid system. A new generalization of Darbo's theorem associated with measures of noncompactness is the main tool in their approach:

$$\begin{cases} \left({}^{C}D^{q}\right)\left(\dfrac{\chi(\tau)}{\hbar(\tau, \chi(\tau), \vartheta(\tau))}\right) = \psi(\tau, \chi(\tau), \vartheta(\tau)), \tau \in [0,1]\ 0 < q \leq 1, \\ \left({}^{RL}D^{p}\right)\left(\dfrac{\vartheta(t)}{\bar{\lambda}(\tau, \chi(\tau), \vartheta(\tau))}\right) = \varphi(\tau, \chi(\tau), \vartheta(\tau)), 1 < p \leq 2, \end{cases} \quad (5)$$

supplemented with nonlocal hybrid boundary conditions.

Inspired by the aforementioned studies, the following sequential hybrid BVP is considered for investigating the existence of the solution and for the stability of its solution via the U-H sense

$$\begin{cases} \left({}^{C}D^{q}_{1-}\,{}^{RL}D^{r}_{0+}\right)\left(\dfrac{\chi(\tau)}{\hbar(\tau, \chi(\tau), \vartheta(\tau))}\right) = \psi(\tau, \chi(\tau), \vartheta(\tau)), & \tau \in [0,1]\ 1 < q \leq 2, 0 < r \leq 1, \\ \left({}^{C}D^{q}_{1-}RLD^{p}_{0+}\right)\left(\dfrac{\vartheta(t)}{\bar{\lambda}(\tau, \chi(\tau), \vartheta(\tau))}\right) = \varphi(\tau, \chi(\tau), \vartheta(\tau)), & 0 < p \leq 1, \\ \chi(0) = \chi'(0) = 0, & \chi(1) = \delta\chi(\zeta), \delta \in \mathbb{R}, \zeta \in (0,1), \\ \vartheta(0) = \vartheta'(0) = 0, & \vartheta(1) = \varepsilon\vartheta(\xi), \varepsilon \in \mathbb{R}, \xi \in (0,1). \end{cases} \quad (6)$$

After this introductory section of this work, the manuscript is organized as the following hierarchical structure: Section 2 delivers the basic elements of fractional calculus definitions, Section 3 introduces the main results of the work, Section 4 introduces the (U-H) stability result for our problem, and the last section is arranged for a numerical example to support the theoretical results.

2. Preliminaries

In this part, we present some basic elements and definitions needed to find solutions to the main mathematical problem presented in this study.

Definition 1 (see [3]). The Riemann-Liouville (RL) fractional integral is defined by

$$\left({}^{RL}I_{0^+}^{\delta}\vartheta\right)(\varpi) := \frac{1}{\Gamma(\delta)}\int_0^{\varpi}(\varpi-t)^{\delta-1}\vartheta(t)dt, \varpi>0, \operatorname{Re}(\delta)>0. \tag{7}$$

Definition 2 (see [3]). The Caputo fractional derivative of order ν of a function $\vartheta : \mathbb{R}^+ \longrightarrow \mathbb{R}$ is given by

$$\left({}^{C}D_{0^+}^{\nu}\vartheta\right)(\tau) = \int_0^{\tau}\frac{(\tau-z)^{p-\nu-1}\vartheta^{(p)}(z)}{\Gamma(p-\nu)}dz, p-1<\nu<p, p=[\nu]+1. \tag{8}$$

Theorem 3 (see [3], Banach's contraction mapping principle). *Let (S,d) be a complete metric space; $H : S \longrightarrow S$ is a contraction then*

(i) H has a unique fixed point $s \in S$; that, is $H(s)=s$

(ii) $\forall s_0 \in S$, we have $\lim_{n\longrightarrow\infty} H^n(u_0)=u$

Theorem 4 (see [3], nonlinear alternative of Leray-Schauder type). *Assume that V is an open subset of a Banach space U, $0 \in V$, and $F : \bar{V} \longrightarrow U$ be a contraction such that $F(\bar{V})$ is bounded then either*

(i) F has a fixed point in $\bar{V}$, or

(ii) $\exists\mu \in (0,1)$ and $v \in \partial V$ such that $v=\mu F(v)$ holds

Theorem 5 (see [2], Arzela-Ascoli theorem). *$F \subset C(U,\mathbb{R})$ is compact if and only if it is closed, bounded, and equicontinuous.*

3. Main Results

Lemma 6. *If $h \in C([0,1],\mathbb{R})$, and*

$$\begin{cases} \left({}^{C}D_{1-}^{q}\,{}^{RL}D_{0+}^{r}\right)\left(\dfrac{\chi(\tau)}{\hbar(\tau,\chi(\tau),\vartheta(\tau))}\right)=w(\tau) & ,\tau\in[0,1], 1<q\le 2, 0<r\le 1, \\ \chi(0)=\chi'(0)=0, & \chi(1)=\delta\chi(\zeta), \delta\in\mathbb{R}, \zeta\in(0,1), \end{cases} \tag{9}$$

then the solution to the problem mentioned above is given by

$$\begin{aligned}\chi(\tau) &= \hbar(\tau,\chi(\tau),\vartheta(\tau)) \\ &\quad\times\left(\frac{1}{\Gamma(r)}\int_0^{\tau}(\tau-z)^{r-1}I_{1-}^{q}w(z)dz+\frac{\tau^{r+1}}{\left(1-\delta\zeta^{r+1}\right)\Gamma(r)}\right. \\ &\quad\left.\cdot\left[\delta\int_0^{\zeta}(\zeta-z)^{r-1}I_{1-}^{q}w(z)dz-\int_0^1(1-z)^{r-1}I_{1-}^{q}w(z)dz\right]\right).\end{aligned} \tag{10}$$

Proof. Taking ${}^{RL}I_{1-}^{q}$ to $({}^{C}D_{1-}^{q}\,{}^{RL}D_{0+}^{r})(\chi(\tau)/\hbar(\tau,\chi(\tau),\vartheta(\tau))) = w(\tau)$, then take ${}^{RL}I_{0+}^{r}$ to the resulting equation, we get

$$\begin{aligned}\frac{\chi(\tau)}{\hbar(\tau,\chi(\tau),\vartheta(\tau))} &= {}^{RL}I_{0+}^{r}\left({}^{RL}I_{1-}^{q}w(\tau)+a_0+a_1t\right) \\ &\quad + a_2\tau^{r-1} = {}^{RL}I_{0+}^{r}\,{}^{RL}I_{1-}^{q}w(\tau)+a_0\frac{\tau^{r}}{\Gamma(r+1)} \\ &\quad + a_1\frac{\tau^{r+1}}{\Gamma(r+2)}+a_2\tau^{r-1}.\end{aligned} \tag{11}$$

Substitution of $\chi(0)=0$ and $\chi'(0)=0$ in Equation (11) gives $a_2=0$ and $a_0=0$, respectively, and consequently, Equation (6) becomes

$$\frac{\chi(\tau)}{\hbar(\tau,\chi(\tau),\vartheta(\tau))}={}^{RL}I_{0+}^{r}\,{}^{RL}I_{1-}^{q}w(\tau)+a_1\frac{\tau^{r+1}}{\Gamma(r+2)}. \tag{12}$$

Use of the condition $\chi(1)=\delta\chi(\zeta)$ in Equation (12) yields

$$a_1=\frac{\Gamma(r+2)}{1-\delta\zeta^{r+1}}\left(\delta^{RL}I_{0+}^{r}\,{}^{RL}I_{1-}^{q}w(\zeta)-{}^{RL}I_{0+}^{r}\,{}^{RL}I_{1-}^{q}w(1)\right). \tag{13}$$

Inserting a_1 in Equation (12) gives

$$\frac{\chi(\tau)}{\hbar(\tau,\chi(\tau),\vartheta(\tau))}={}^{RL}I_{0+}^{r}\,{}^{RL}I_{1-}^{q}w(\tau)+\frac{\Gamma(r+2)}{1-\delta\zeta^{r+1}}\cdot\left(\delta^{RL}I_{0+}^{r}\,{}^{RL}I_{1-}^{q}w(\zeta)-{}^{RL}I_{0+}^{r}\,{}^{RL}I_{1-}^{q}w(1)\right)\frac{\tau^{r+1}}{\Gamma(r+2)}. \tag{14}$$

Alternatively, we have

$$\chi(\tau)=\hbar(\tau,\chi(\tau),\vartheta(\tau))\times\left({}^{RL}I_{0+}^{r}\,{}^{RL}I_{1-}^{q}w(\tau)+\frac{\Gamma(r+2)}{1-\delta\zeta^{r+1}}\left(\delta^{RL}I_{0+}^{r}\,{}^{RL}I_{1-}^{q}w(\zeta)-{}^{RL}I_{0+}^{r}\,{}^{RL}I_{1-}^{q}w(1)\right)\frac{\tau^{r+1}}{\Gamma(r+2)}\right),\ \tau\in[0,1]. \tag{15}$$

Equation (15) is equivalent to Equation (10), which makes the proof done. □

Denote the Banach space by $C=C[0,1]$ with the norm $\|h\|=\sup_{0\le t\le 1}|h(t)|$. Then, the product space $(C\times C,\|(\chi,\vartheta)\|)$ with the norm $\|(\chi,\vartheta)\|=\|\chi\|+\|\vartheta\|,\ \forall(x,y)\in C\times C$ is indeed a Banach space too. We define an operator $\Upsilon: C\times C\longrightarrow C\times C$ as

$$\Upsilon(\chi,\vartheta)(\tau)=\begin{pmatrix}\Upsilon_1(\chi,\vartheta)(\tau)\\ \Upsilon_2(\chi,\vartheta)(\tau)\end{pmatrix}, \tag{16}$$

where

$$\begin{aligned}\Upsilon_1(\chi,\vartheta)(\tau)&=\hbar(\tau,\chi(\tau),\vartheta(\tau))\times\left(\frac{1}{\Gamma(r)}\int_0^\tau(\tau-z)^{r-1RL}I_{1-}^{q}\psi(z,\chi(z),\vartheta(z))dz\right.\\&\quad+\frac{\tau^{r+1}}{\left(1-\delta\zeta^{r+1}\right)\Gamma(r)}\times\left[\delta\int_0^\zeta(\zeta-z)^{r-1RL}I_{1-}^{q}\psi(z,\chi(z),\vartheta(z))dz\right.\\&\quad\left.\left.-\int_0^1(1-z)^{r-1RL}I_{1-}^{q}\psi(z,\chi(z),\vartheta(z))dz\right]\right),\ \Upsilon_2(\chi,\vartheta)(\tau)\\&=\bar{\lambda}(\tau,\chi(\tau),\vartheta(\tau))\times\left(\frac{1}{\Gamma(p)}\int_0^\tau(\tau-z)^{p-1RL}I_{1-}^{q}\varphi(z,\chi(z),\vartheta(z))dz\right.\\&\quad+\frac{\tau^{p+1}}{\left(1-\varepsilon\xi^{p+1}\right)\Gamma(p)}\times\left[\varepsilon\int_0^\xi(\xi-z)^{p-1RL}I_{1-}^{q}\varphi(z,\chi(z),\vartheta(z))dz\right.\\&\quad\left.\left.-\int_0^1(1-z)^{p-1RL}I_{1-}^{q}\varphi(z,\chi(z),\vartheta(z))dz\right]\right).\end{aligned} \tag{17}$$

To construct the necessary conditions for the results of uniqueness and existence of the problem (6), let us consider the following hypotheses.

($C1$) Let the functions f and g are assumed to be continuous and bounded; that is, $\exists\lambda_f,\lambda_g>0$ such that

$$|\hbar(\tau,\chi,\vartheta)|\le\lambda_\hbar,\ \text{and}\ |\bar{\lambda}(\tau,\chi,\vartheta)|\le\lambda_{\bar{\lambda}},\forall(\tau,\chi,\vartheta)\in[0,1]\times\mathbb{R}^2. \tag{18}$$

($C2$) Let the functions ψ and φ are assumed to be continuous, and $\exists\upsilon_i,\ell_i>0,(i=1,2)$ such that

$$|\psi(\tau,\chi_1,\vartheta_1)-\psi(\tau,\chi_2,\vartheta_2)|\le\upsilon_1|\chi_1-\chi_2|+\upsilon_2|\vartheta_1-\vartheta_2|,$$

$$|\varphi(\tau,\chi_1,\vartheta_1)-\varphi(\tau,\chi_2,\vartheta_2)|\le\ell_1|\chi_1-\chi_2|+\ell_2|\vartheta_1-\vartheta_2|,\forall\tau\in[0,1],\chi_i,\vartheta_i\in\mathbb{R},(i=1,2). \tag{19}$$

($C3$) There is positive constants ω_0, θ_0, and $\omega_i,\theta_i\ge 0$ ($i=1,2$) such that

$$|\psi(\tau,\chi,\vartheta)|\le\omega_0+\omega_1|\chi|+\omega_2|\vartheta|, \tag{20}$$

$$|\varphi(\tau,\chi,\vartheta)|\le\theta_0+\theta_1|\chi|+\theta_2|\vartheta|,\forall\tau\in[0,1],\chi_i,\vartheta_i\in\mathbb{R},(i=1,2). \tag{21}$$

($C4$) Let $S\subset C\times C$ be a bounded set, then $\exists\kappa_i>0,(i=1,2)$ such that $|\psi(\tau,\chi(\tau),\vartheta(\tau))|\le\kappa_1$,and

$$|\varphi(\tau,\chi(\tau),\vartheta(\tau))|\le\kappa_2,\forall(\chi,\vartheta)\in S. \tag{22}$$

Observe that

$$\begin{aligned}
&\frac{1}{\Gamma(r)\Gamma(q)}\int_0^\tau(\tau-z)^{r-1}\int_s^1(u-z)^{q-1}dudz \\
&\quad=\int_0^\tau\frac{(\tau-z)^{r-1}}{\Gamma(r)}\int_z^1\frac{(u-z)^{q-1}}{\Gamma(q)}dudz \\
&\quad=\int_0^\tau\frac{(\tau-z)^{r-1}}{\Gamma(r)}\frac{(u-z)^{q}}{\Gamma(q+1)}\Bigg|_{u=z}^{u=1}dz \\
&\quad=\int_0^\tau\frac{(\tau-z)^{r-1}}{\Gamma(r)}\frac{(u-z)^{q}}{\Gamma(q+1)}\Bigg|_{u=z}^{u=1}dz=\int_0^\tau\frac{(\tau-z)^{r-1}}{\Gamma(r)}\frac{(1-z)^{q}}{\Gamma(q+1)}dz \\
&\quad\le\int_0^\tau\frac{(\tau-z)^{r-1}}{\Gamma(r)}\frac{1}{\Gamma(q+1)}dz,\ ((1-s)^q\le 1,1<q\le 2) \\
&\quad=\frac{\tau^r}{\Gamma(r+1)\Gamma(q+1)}.
\end{aligned}\tag{23}$$

To facilitate the calculations below, let us say

$$\begin{aligned}
\Lambda_1&=\sup_{0\le\tau\le1}\left\{\left\{\frac{1}{\Gamma(r)\Gamma(q)}\int_0^\tau(\tau-z)^{r-1}\int_z^1(u-z)^{q-1}dudz\right.\right. \\
&\quad+\frac{\tau^{r+1}}{\left|1-\delta\zeta^{r+1}\right|\Gamma(r)\Gamma(q)}\times\left[|\delta|\int_0^\zeta(\zeta-z)^{r-1}\int_z^1(u-z)^{q-1}dudz\right. \\
&\quad\left.\left.-\int_0^1(1-z)^{r-1}\int_z^1(u-z)^{q-1}dudz\right]\right\} \\
&\le\frac{1}{\Gamma(q+1)\Gamma(r+1)}\left[1+\frac{|\delta|\zeta^r}{\left|1-\delta\zeta^{r+1}\right|}\right],
\end{aligned}$$

$$\begin{aligned}
\Lambda_2&=\sup_{0\le t\le1}\left\{\frac{1}{\Gamma(p)\Gamma(q)}\int_0^\tau(\tau-z)^{p-1}\int_z^1(u-z)^{q-1}dudz\right. \\
&\quad+\frac{\tau^{p+1}}{\Gamma(p)\left|1-\varepsilon\xi^{p+1}\right|\Gamma(q)}\times\left[|\varepsilon|\int_0^\xi(\xi-z)^{p-1}\int_z^1(u-z)^{q-1}dudz\right. \\
&\quad\left.\left.-\int_0^1(1-z)^{p-1}\int_s^1(u-z)^{q-1}dudz\right]\right\} \\
&\le\frac{1}{\Gamma(p+1)\Gamma(q+1)}\left[1+\frac{|\varepsilon|\xi^p}{\left|1-\varepsilon\xi^{p+1}\right|}\right].
\end{aligned}\tag{24}$$

Theorem 7. *If both (C1) and (C2) are satisfied, and assume that $[\lambda_\hbar\Lambda_1(\upsilon_1+\upsilon_2)+\lambda_{\bar{\lambda}}\Lambda_2(\ell_1+\ell_2)]<1$. Then, the system in Equation (6) has a unique solution.*

Proof. Define a closed ball $\overline{\mathfrak{B}_\gamma}=\{(\chi,\vartheta)\in C\times C:\|(\chi,\vartheta)\|\le\gamma\}$ with $\gamma\ge(\lambda_\hbar\Lambda_1N_\psi+\lambda_{\bar{\lambda}}\Lambda_2N_\varphi)/(1-(\lambda_\hbar\Lambda_1(\upsilon_1+\upsilon_2)+\lambda_{\bar{\lambda}}\Lambda_2(\ell_1+\ell_2)))$, where $N_\psi=\sup_{0\le\tau\le T}|\psi(\tau,0,0)|$, $N_\varphi=\sup_{0\le\tau\le T}|\varphi(\tau,0,0)|$.

Observe that $|\psi(\tau,\chi,\vartheta)|=|\psi(\tau,\chi,\vartheta)-\psi(\tau,0,0)+\psi(\tau,0,0)|\le\upsilon_1\|\chi\|+\upsilon_2\|\vartheta\|+N_\psi\le(\upsilon_1+\upsilon_2)\gamma+N_\psi$.

First, we show that $\mathfrak{J}\bar{\mathfrak{B}}_\gamma\subset\overline{\mathfrak{B}_\gamma}$. For any $(\chi,\vartheta)\in\overline{\mathfrak{B}_\gamma}$, $\tau\in[0,1]$, we have

$$\begin{aligned}
|\Upsilon_1(\chi,\vartheta)(\tau)|&=\Bigg|\hbar(\tau,\chi(\tau),\vartheta(\tau))\times\left(\frac{1}{\Gamma(r)}\int_0^\tau(\tau-z)^{r-1RL}I_{1-}^q\psi(z,\chi(z),\vartheta(z))dz\right. \\
&\quad+\frac{\tau^{r+1}}{\left(1-\delta\zeta^{r+1}\right)\Gamma(r)}\times\left[\delta\int_0^\zeta(\zeta-z)^{r-1RL}I_{1-}^q\psi(z,\chi(z),\vartheta(z))dz\right. \\
&\quad\left.\left.\cdot\int_0^1(1-z)^{r-1RL}I_{1-}^q\psi(z,\chi(z),\vartheta(z))dz\right]\right)\Bigg| \\
&\le\lambda_\hbar\sup_{0\le\tau\le1}\left\{\frac{1}{\Gamma(r)\Gamma(q)}\int_0^\tau(\tau-z)^{r-1}\right. \\
&\quad\cdot\int_z^1(u-z)^{q-1}|\psi(z,\chi(z),\vartheta(z))|dudz+\frac{\tau^{r+1}}{\left|1-\delta\zeta^{r+1}\right|\Gamma(r)\Gamma(q)} \\
&\quad\cdot\left[|\delta|\int_0^\zeta(\zeta-z)^{r-1}\int_z^1(u-z)^{q-1}|\psi(z,\chi(z),\vartheta(z))|dudz\right. \\
&\quad\left.\left.-\int_0^1(1-z)^{r-1}\int_z^1(u-z)^{q-1}|\psi(z,\chi(z),\vartheta(z))|dudz\right]\right\} \\
&\le\lambda_\hbar\left[(\upsilon_1+\upsilon_2)\gamma+N_\psi\right]\sup_{0\le\tau\le1}\left\{\int_0^\tau(\tau-z)^{r-1}\int_z^1(u-z)^{q-1}dudz\right. \\
&\quad+\frac{\tau^{r+1}}{\left|1-\delta\zeta^{r+1}\right|\Gamma(r)\Gamma(q)}\times\left[|\delta|\int_0^\zeta(\zeta-z)^{r-1}\int_z^1(u-z)^{q-1}dudz\right. \\
&\quad\left.\left.-\int_0^1(1-z)^{r-1}\int_z^1(u-z)^{q-1}|\psi(z,\chi(z),\vartheta(z))|dudz\right]\right\} \\
&\le\lambda_\hbar\Lambda_1\left[(\upsilon_1+\upsilon_2)\gamma+N_\psi\right],
\end{aligned}\tag{25}$$

similar to what was done above, we get

$$\|\Upsilon_2(\chi,\vartheta)\|\le\lambda_{\bar{\lambda}}\Lambda_2\left[(\ell_1+\ell_2)\gamma+N_\varphi\right].\tag{26}$$

From Equation (25) and Equation (26), we deduce that $\|\Upsilon(\chi,\vartheta)\|\le\gamma$.

Next, for $(\chi_1,\vartheta_1),(\chi_2,\vartheta_2)\in C\times C,\forall\tau\in[0,1]$, we have

$$\begin{aligned}
&|\Upsilon_1(\chi_1,\vartheta_1)(\tau)-\Upsilon_1(\chi_2,\vartheta_2)(\tau)| \\
&\quad\le\lambda_\hbar\sup_{0\le\tau\le1}\left\{\frac{1}{\Gamma(r)\Gamma(q)}\int_0^\tau(\tau-z)^{r-1}\int_z^1(u-z)^{q-1}|\psi(z,\chi_1(z),\vartheta_1(z))\right. \\
&\qquad-\psi(z,\chi_2(z),\vartheta_2(z))|dudz+\frac{\tau^{r+1}}{\left|1-\delta\zeta^{r+1}\right|\Gamma(r)\Gamma(q)} \\
&\qquad\times\left[|\delta|\int_0^\zeta(\zeta-z)^{r-1}\int_z^1(u-z)^{q-1}|\psi(z,\chi_1(z),\vartheta_1(z))\right. \\
&\qquad-\psi(z,\chi_2(z),\vartheta_2(z))|dudz-\int_0^1(1-z)^{r-1} \\
&\qquad\left.\left.\times\int_z^1(u-z)^{q-1}|\psi(z,\chi_1(z),\vartheta_1(z))-\psi(z,\chi_2(z),\vartheta_2(z))|dudz\right]\right\} \\
&\quad\le\lambda_\hbar(\upsilon_1\|\chi_1-\chi_2\|+\upsilon_2\|\vartheta_1-\vartheta_2\|)\sup_{0\le\tau\le1}\left\{\frac{1}{\Gamma(r)\Gamma(q)}\int_0^\tau(\tau-z)^{r-1}\right. \\
&\qquad\times\int_z^1(u-z)^{q-1}dudz+\frac{\tau^{r+1}}{\left|1-\delta\zeta^{r+1}\right|\Gamma(r)\Gamma(q)}\times\left[|\delta|\int_0^\zeta(\zeta-z)^{r-1}\right. \\
&\qquad\left.\left.\times\int_z^1(u-z)^{q-1}dudz-\int_0^1(1-z)^{r-1}\int_z^1(u-z)^{q-1}dudz\right]\right\} \\
&\quad\le\lambda_\hbar\Lambda_1(\upsilon_1\|\chi_1-\chi_2\|+\upsilon_2\|\vartheta_1-\vartheta_2\|)\le\lambda_\hbar\Lambda_1(\upsilon_1+\upsilon_2)(\|\chi_1-\chi_2\|+\|\vartheta_1-\vartheta_2\|).
\end{aligned}\tag{27}$$

Similarly, we can find

$$\| \Upsilon_2(\chi_1, \vartheta_1) - \Upsilon_2(\chi_2, \vartheta_2)\| \le \lambda_{\bar{\lambda}} \Lambda_2(\ell_1 + \ell_2)(\|\chi_1 - \chi_2\| + \|\vartheta_1 - \vartheta_2\|). \tag{28}$$

Combining Equation (27) and Equation (28) yields

$$\| \Upsilon(\chi_1, \vartheta_1) - \Upsilon(\chi_2, \vartheta_2)\| \le [\lambda_\hbar \Lambda_1(\upsilon_1 + \upsilon_2) + \lambda_{\bar{\lambda}} \Lambda_2(\ell_1 + \ell_2)] \times (\|\chi_1 - \chi_2\| + \|\vartheta_1 - \vartheta_2\|). \tag{29}$$

Equation (29) becomes $\| \Upsilon(\chi_1, \vartheta_1) - \Upsilon(\chi_2, \vartheta_2)\| \le (\|\chi_1 - \chi_2\| + \|\vartheta_1 - \vartheta_2\|)$. That is, Υ is a contraction; consequently, Banach fixed point theorem applies; thus, the uniqueness of solutions for Equation (6) holds on $[0, 1]$. □

Theorem 8. *If (C1), (C3), and (C4) are satisfied, and if $(\lambda_\hbar \Lambda_1\omega_1 + \lambda_{\bar{\lambda}} \Lambda_2\theta_1) < 1$ and $(\lambda_\hbar \Lambda_1\omega_2 + \lambda_{\bar{\lambda}} \Lambda_2\theta_2) < 1$, then Equation (6) has at least one solution.*

Proof. In the first step, we verify that the operator $\Upsilon : C \times C \longrightarrow C \times C$ is completely continuous; obviously, the operator is continuous as a result that $\hbar$, $\bar{\lambda}$, ψ, and φ are all assumed to be continuous.

With the aid of $(C4)$, $\forall(\chi, \vartheta) \in S$, we have

$$\begin{aligned} |\Upsilon_1(\chi, \vartheta)(\tau)| \le \lambda_\hbar \sup_{0\le\tau\le1} \Bigg\{ & \frac{1}{\Gamma(r)\Gamma(q)} \int_0^\tau (\tau - z)^{r-1} \cdot \int_z^1 (u - z)^{q-1} |\psi(z, \chi(z), \vartheta(z))| dudz \\ & + \frac{\tau^{r+1}}{\left|1 - \delta\zeta^{r+1}\right| \Gamma(r)\Gamma(q)} \times \left[|\delta| \int_0^\zeta (\zeta - z)^{r-1} \cdot \int_z^1 (u - z)^{q-1} |\psi(z, \chi(z), \vartheta(z))| dudz \right. \\ & \left. - \int_0^1 (1 - z)^{r-1} \int_z^1 (u - z)^{q-1} |\psi(z, \chi(z), \vartheta(z))| dudz \right] \Bigg\} \\ \le \lambda_\hbar \Lambda_1 \kappa_1. & \end{aligned} \tag{30}$$

Similarly,

$$\| \Upsilon_2(\chi, \vartheta)\| \le \lambda_{\bar{\lambda}} \Lambda_2 \kappa_2. \tag{31}$$

Combining the inequalities (30) and (31) yields $\| \Upsilon(\chi, \vartheta)\| \le \lambda_\hbar \Lambda_1 \kappa_1 + \lambda_{\bar{\lambda}} \Lambda_2 \kappa_2$, implying that Υ is uniformly bounded.

Next, to verify the equicontinuity for the operator Υ, we let $\tau_1, \tau_2 \in [0, 1]$, $(\tau_1 < \tau_2)$ then

$$\begin{aligned} & |\Upsilon_1(\chi, \vartheta)(\tau_2) - \Upsilon_1(\chi, \vartheta)(\tau_1)| \\ & \le \lambda_\hbar \sup_{0\le\tau\le1} \Bigg\{ \frac{1}{\Gamma(r)\Gamma(q)} \int_0^{\tau_1} \left((\tau_1 - z)^{r-1} - (\tau_2 - z)^{r-1} \right) \\ & \quad \times \int_z^1 (u - z)^{q-1} |\psi(z, \chi(z), \vartheta(z))| dudz \\ & \quad - \frac{1}{\Gamma(r)\Gamma(q)} \int_{\tau_1}^{\tau_2} (\tau_2 - z)^{r-1} \int_z^1 (u - z)^{q-1} |\psi(z, \chi(z), \vartheta(z))| dudz \\ & \quad + \frac{\left|\tau_2^{r+1} - \tau_1^{r+1}\right|}{\left|1 - \delta\zeta^{r+1}\right| \Gamma(r)\Gamma(q)} \times \left[|\delta| \int_0^\zeta (\zeta - z)^{r-1} \right. \\ & \quad \times \int_z^1 (u - z)^{q-1} |\psi(z, \chi(z), \vartheta(z))| dudz - \int_0^1 (1 - z)^{r-1} \\ & \quad \left. \times \int_z^1 (u - z)^{q-1} |\psi(z, \chi(z), \vartheta(z))| dudz \right] \Bigg\}, \end{aligned} \tag{32}$$

$$\begin{aligned} & |\Upsilon_1(\chi, \vartheta)(\tau_2) - \Upsilon_1(\chi, \vartheta)(\tau_1)| \le \lambda_\hbar \kappa_1 \\ & \quad \times \left| \frac{1}{\Gamma(r)\Gamma(q)} \int_0^{\tau_1} \left((\tau_1 - z)^{r-1} - (\tau_2 - z)^{r-1} \right) \int_z^1 (u - z)^{q-1} dudz \right. \\ & \quad \left. - \frac{1}{\Gamma(r)\Gamma(q)} \int_{\tau_1}^{\tau_2} (\tau_2 - z)^{r-1} \int_z^1 (u - z)^{q-1} dudz \right| \\ & \quad + \frac{\left|\tau_2^{r+1} - \tau_1^{r+1}\right|}{\left|1 - \delta\zeta^{r+1}\right| \Gamma(r)\Gamma(q)} \times \left[|\delta| \int_0^\zeta (\zeta - z)^{r-1} \int_z^1 (u - z)^{q-1} dudz \right. \\ & \quad \left. - \int_0^1 (1 - z)^{r-1} \int_z^1 (u - z)^{q-1} dudz \right], \end{aligned} \tag{33}$$

$$\begin{aligned} & |\Upsilon_2(\chi, \vartheta)(\tau_2) - \Upsilon_2(\chi, \vartheta)(\tau_1)| \le \lambda_{\bar{\lambda}} \kappa_2 \\ & \quad \times \left| \frac{1}{\Gamma(p)\Gamma(q)} \int_0^{\tau_1} \left((\tau_1 - z)^{p-1} - (\tau_2 - z)^{p-1} \right) \int_s^1 (u - s)^{q-1} duds \right. \\ & \quad \left. - \frac{1}{\Gamma(p)\Gamma(q)} \int_{t_1}^{t_2} (t_2 - s)^{p-1} \int_s^1 (u - s)^{q-1} duds \right| \\ & \quad + \frac{\left|\tau_2^{p+1} - \tau_1^{p+1}\right|}{\left|1 - \varepsilon\xi^{p+1}\right| \Gamma(p)\Gamma(q)} \times \left[|\varepsilon| \int_0^\xi (\xi - z)^{p-1} \int_z^1 (u - z)^{q-1} dudz \right. \\ & \quad \left. - \int_0^1 (1 - z)^{p-1} \int_z^1 (u - z)^{q-1} dudz \right]. \end{aligned} \tag{34}$$

The R.H.S for both inequalities (33) and (34) tend to zero as $\tau_1 \longrightarrow \tau_2$, and they are both independent on (χ, ϑ). So, operator $\Upsilon(\chi, \vartheta)$ is equicontinuous and yields; $\Upsilon(\chi, \vartheta)$ is completely continuous.

Finally, we establish the bounded set given by $\Omega = \{(x, y) \in C \times C | (x, y) = \beta\mathfrak{F}(x, y), \beta \in [0, 1]\}$; then, $\forall \tau \in [0, 1]$; the equation $(\chi, \vartheta) = \beta\Upsilon(\chi, \vartheta)$ gives

$$\chi(\tau) = \beta\Upsilon_1(\chi, \vartheta)(\tau), \quad \vartheta(\tau) = \beta\Upsilon_2(\chi, \vartheta)(\tau). \tag{35}$$

Using the hypothesis ($C3$), we get

$$\begin{aligned} \|\chi\| &\le \lambda_{\hbar}\Lambda_1(\omega_0 + \omega_1\|\chi\| + \omega_2\|\vartheta\|), \\ \|\vartheta\| &\le \lambda_{\eth}\Lambda_2(\theta_0 + \theta_1\|\chi\| + \theta_2\|\vartheta\|). \end{aligned} \tag{36}$$

Consequently, we have

$$\begin{aligned} \|\chi\| + \|\vartheta\| \le{}& (\lambda_{\hbar}\Lambda_1\omega_0 + \lambda_{\eth}\Lambda_2\theta_0) + (\lambda_{\hbar}\Lambda_1\omega_1 + \lambda_{\eth}\Lambda_2\theta_1)\|\chi\| \\ &+ (\lambda_{\hbar}\Lambda_1\omega_2 + \lambda_{\eth}\Lambda_2\theta_2)\|\vartheta\|. \end{aligned} \tag{37}$$

Inequality (37) can be written as follows:

$$\|(\chi, \vartheta)\| \le \frac{(\lambda_{\hbar}\Lambda_1\omega_0 + \lambda_{\eth}\Lambda_2\theta_0)}{\Lambda_0}, \tag{38}$$

where $\Lambda_0 = \min\{1 - (\lambda_{\hbar}\Lambda_1\omega_1 + \lambda_{\eth}\Lambda_2\theta_1), 1 - (\lambda_{\hbar}\Lambda_1\omega_2 + \lambda_{\eth}\Lambda_2\theta_2)\}$.

Inequality (38) shows that Ω is bounded. Hence, *Leray-Schauder alternative* applies, implying the existence of the solution for Equation (6). □

4. Stability

In this part, we address the issue of stability of solutions to the system of equations defined by Equation (6) via U-H definition.

Definition 9. The system of the coupled sequential fractional differential BVPs Equation (6) is stable in U–H sense if a real number $c = \max(c_1, c_2) > 0$ exists so that, for any $\varepsilon = \max(\varepsilon_1, \varepsilon_2) > 0$ and for any $(\bar{\chi}, \bar{\vartheta}) \in C \times C$ satisfying

$$\begin{cases} \left| \left({}^{C}D_{1-}^{q}\,{}^{RL}D_{0+}^{r}\right)\left(\dfrac{\bar{\chi}(\tau)}{\hbar(\tau, \bar{\chi}(\tau), \bar{\vartheta}(\tau))}\right) - \psi(\tau, \bar{\chi}(\tau), \bar{\vartheta}(\tau)) \right| < \varepsilon_1, \tau \in [0, 1], \\ \left| \left({}^{C}D_{1-}^{q}\,{}^{RL}D_{0+}^{p}\right)\left(\dfrac{\bar{\vartheta}(\tau)}{\eth(\tau, \bar{\chi}(\tau), \bar{\vartheta}(\tau))}\right) - \varphi(\tau, \bar{\chi}(\tau), \bar{\vartheta}(\tau)) \right| < \varepsilon_2, \end{cases} \tag{39}$$

there exists a unique solution $(\chi, \vartheta) \in C \times C$ of (6) with

$$\left\|(\chi, \vartheta) - (\bar{\chi}, \bar{\vartheta})\right\| < c\varepsilon. \tag{40}$$

It is clear that $(\bar{\chi}, \bar{\vartheta}) \in C \times C$ satisfies the inequalities (39) if there exists a function $(h_1, h_2) \in C \times C$ (which depends on $(\bar{\chi}, \bar{\vartheta})$), such that

(i) $|h_1(\tau)| < \varepsilon_1$ and $|h_2(\tau)| < \varepsilon_2$, $\tau \in [0, 1]$

(ii) For $\tau \in [0, 1]$

$$\begin{cases} \left({}^{C}D_{1-}^{q}\,{}^{RL}D_{0+}^{r}\right)\left(\dfrac{\bar{\chi}(\tau)}{\hbar(\tau, \bar{\chi}(\tau), \bar{\vartheta}(\tau))}\right) = \psi(\tau, \bar{\chi}(\tau), \bar{\vartheta}(\tau)) + h_1(\tau), \\ \left({}^{C}D_{1-}^{q}\,{}^{RL}D_{0+}^{p}\right)\left(\dfrac{\bar{\vartheta}(\tau)}{\eth(\tau, \bar{\chi}(\tau), \bar{\vartheta}(\tau))}\right) = \varphi(\tau, \bar{\chi}(\tau), \bar{\vartheta}(\tau)) + h_2(\tau). \end{cases} \tag{41}$$

Theorem 10. *Suppose that (C2) is fulfilled. Moreover*

$$\begin{aligned} &\lambda_{\hbar}\Lambda_1(\upsilon_1 + \upsilon_2) < 1, \lambda_{\eth}\Lambda_2(\ell_1 + \ell_2) < 1, \\ &\Delta = (1 - \lambda_{\hbar}\Lambda_1(\upsilon_1 + \upsilon_2))(1 - \lambda_{\eth}\Lambda_2(\ell_1 + \ell_2)) \\ &\quad - \lambda_{\hbar}\Lambda_1(\upsilon_1 + \upsilon_2)\lambda_{\eth}\Lambda_2(\ell_1 + \ell_2) > 0. \end{aligned} \tag{42}$$

Then, the system of coupled sequential fractional differential BVPs (6) is U–H stable.

Proof. Assume that for $\varepsilon_1, \varepsilon_2 > 0$ a couple $(\bar{\chi}, \bar{\vartheta}) \in C \times C$ satisfies the inequalities (39). Introduce the following operator

$$\begin{aligned} K_1(\tau; h) &= \frac{1}{\Gamma(r)}\int_0^{\tau}(\tau - z)^{r-1}\,{}^{RL}I_{1-}^{q}h(z)dz, \\ K_2(\tau; h) &= \frac{1}{\Gamma(p)}\int_0^{\tau}(\tau - u)^{p-1}\,{}^{RL}I_{1-}^{q}h(u)du. \end{aligned} \tag{43}$$

Then

$$\begin{aligned} \bar{\chi}(\tau) ={}& \hbar(\tau, \bar{\chi}(\tau), \bar{\vartheta}(\tau))\Big(K_1(\tau; \psi(\cdot, \bar{\chi}(\cdot), \bar{\vartheta}(\cdot))) \\ &+ \frac{\tau^{r+1}}{(1 - \delta\zeta^{r+1})}[\delta K_1(\zeta; \psi(\cdot, \bar{\chi}(\cdot), \bar{\vartheta}(\cdot))) \\ &- K_1(1; \psi(\cdot, \bar{\chi}(\cdot), \bar{\vartheta}(\cdot)))]\Big) + \hbar(\tau, \bar{\chi}(\tau), \bar{\vartheta}(\tau)) \\ &\cdot \left(K_1(\tau; h_1) + \frac{\tau^{r+1}}{(1 - \delta\zeta^{r+1})}[\delta K_1(\zeta; h_1) - K_1(1; h_1)]\right), \end{aligned} \tag{44}$$

$$\begin{aligned} \bar{\vartheta}(\tau) ={}& \eth(\tau, \bar{\chi}(\tau), \bar{\vartheta}(\tau))\Big(K_2(\tau; \varphi(\cdot, \bar{\chi}(\cdot), \bar{\vartheta}(\cdot))) \\ &+ \frac{\tau^{r+1}}{(1 - \delta\zeta^{r+1})}[\delta K_2(\zeta; \varphi(\cdot, \bar{\chi}(\cdot), \bar{\vartheta}(\cdot))) - K_2(1; \varphi(\cdot, \bar{\chi}(\cdot), \bar{\vartheta}(\cdot)))]\Big) \\ &+ \eth(\tau, \bar{\chi}(\tau), \bar{\vartheta}(\tau))\left(K_2(\tau; h_2) + \frac{\tau^{r+1}}{(1 - \delta\zeta^{r+1})}[\delta K_2(\zeta; h_2) - K_2(1; h_2)]\right). \end{aligned} \tag{45}$$

From Equation (44) and Equation (45), we obtain

$$\bar{\chi}(\tau)-\hbar\left(\tau,\bar{\chi}(\tau),\bar{\vartheta}(\tau)\right)\left(K_1\left(\tau;\psi\left(\frac{\ddot{A}}{n},\bar{\chi}\left(\frac{\ddot{A}}{n}\right),\bar{\vartheta}\left(\frac{\ddot{A}}{n}\right)\right)\right)\right.$$

$$\left.+\frac{\tau^{r+1}}{\left(1-\delta\zeta^{r+1}\right)}\left[\delta K_1\left(\zeta;\psi\left(\frac{\ddot{A}}{n},\bar{\chi}\left(\frac{\ddot{A}}{n}\right),\bar{\vartheta}\left(\frac{\ddot{A}}{n}\right)\right)\right)-K_1\left(1;\psi\left(\frac{\ddot{A}}{n},\bar{\chi}\left(\frac{\ddot{A}}{n}\right),\bar{\vartheta}\left(\frac{\ddot{A}}{n}\right)\right)\right)\right]\right)$$

$$\leq\hbar\left(\tau,\bar{\chi}(\tau),\bar{\vartheta}(\tau)\right)\left(K_1(\tau;h_1)+\frac{\tau^{r+1}}{\left(1-\delta\zeta^{r+1}\right)}[\delta K_1(\zeta;h_1)-K_1(1;h_1)]\right),\quad(46)$$

$$\bar{\vartheta}(\tau)-\bar{\lambda}\left(\tau,\bar{\chi}(\tau),\bar{\vartheta}(\tau)\right)\left(\left(K_2\left(\tau;\varphi\left(\frac{\ddot{A}}{n},\bar{\chi}\left(\frac{\ddot{A}}{n}\right),\bar{\vartheta}\left(\frac{\ddot{A}}{n}\right)\right)\right)\right.\right.$$

$$\left.+\frac{\tau^{r+1}}{\left(1-\delta\zeta^{r+1}\right)}\left[\delta K_2\left(\zeta;\varphi\left(\frac{\ddot{A}}{n},\bar{\chi}\left(\frac{\ddot{A}}{n}\right),\bar{\vartheta}\left(\frac{\ddot{A}}{n}\right)\right)\right)-K_2\left(1;\varphi\left(\frac{\ddot{A}}{n},\bar{\chi}\left(\frac{\ddot{A}}{n}\right),\bar{\vartheta}\left(\frac{\ddot{A}}{n}\right)\right)\right)\right]\right)$$

$$\leq\bar{\lambda}\left(\tau,\bar{\chi}(\tau),\bar{\vartheta}(\tau)\right)\left(K_2(\tau;h_2)+\frac{\tau^{r+1}}{\left(1-\delta\zeta^{r+1}\right)}[\delta K_2(\zeta;h_2)-K_2(1;h_2)]\right)\quad(47)$$

From Equation (27) and Equation (28), we obtain

$$\begin{aligned}&\left|\bar{\chi}(\tau)-\hbar\left(\tau,\bar{\chi}(\tau),\bar{\vartheta}(\tau)\right)\left(K_1\left(\tau;\psi\left(\cdot,\bar{\chi}(\cdot),\bar{\vartheta}(\cdot)\right)\right)\right.\right.\\&\quad\left.\left.+\frac{\tau^{r+1}}{\left(1-\delta\zeta^{r+1}\right)}\left[\delta K_1\left(\zeta;\psi\left(\cdot,\bar{\chi}(\cdot),\bar{\vartheta}(\cdot)\right)\right)-K_1\left(1;\psi\left(\cdot,\bar{\chi}(\cdot),\bar{\vartheta}(\cdot)\right)\right)\right]\right)\right|\\&\quad\leq\lambda_\hbar\Lambda_1\|h_1\|\leq\lambda_\hbar\Lambda_1\varepsilon_1,\end{aligned}\quad(48)$$

$$\begin{aligned}&\bar{\vartheta}(\tau)-\bar{\lambda}\left(\tau,\bar{\chi}(\tau),\bar{\vartheta}(\tau)\right)\left(\left(K_2\left(\tau;\varphi\left(\cdot,\bar{\chi}(\cdot),\bar{\vartheta}(\cdot)\right)\right)\right)\right.\\&\quad\left.+\frac{\tau^{r+1}}{\left(1-\delta\zeta^{r+1}\right)}\left[\delta K_2\left(\zeta;\varphi\left(\cdot,\bar{\chi}(\cdot),\bar{\vartheta}(\cdot)\right)\right)-K_2\left(1;\varphi\left(\cdot,\bar{\chi}(\cdot),\bar{\vartheta}(\cdot)\right)\right)\right]\right)\\&\quad\leq\lambda_{\bar{\lambda}}\Lambda_2\|h_2\|\leq\lambda_{\bar{\lambda}}\Lambda_2\varepsilon_2.\end{aligned}\quad(49)$$

Let $(\chi,\vartheta)\in C\times C$ be a solution of Equation (6). Thanks to Lemma 6, it is equivalent to the following integral equations:

$$\begin{aligned}\chi(\tau)&=\hbar(\tau,\chi(\tau),\vartheta(\tau))(K_1(\tau;\psi(\cdot,\chi(\cdot),\vartheta(\cdot)))\\&\quad\left.+\frac{\tau^{r+1}}{\left(1-\delta\zeta^{r+1}\right)}[\delta K_1(\zeta;\psi(\cdot,\chi(\cdot),\vartheta(\cdot)))-K_1(1;\psi(\cdot,\chi(\cdot),\vartheta(\cdot)))]\right),\vartheta(t)\\&=\bar{\lambda}(\tau,\chi(\tau),\vartheta(\tau))(K_2(t;\varphi(\cdot,\chi(\cdot),\vartheta(\cdot)))\\&\quad\left.+\frac{\tau^{r+1}}{\left(1-\delta\zeta^{r+1}\right)}[\delta K_2(\zeta;\varphi(\cdot,\chi(\cdot),\vartheta(\cdot)))-K_2(1;\varphi(\cdot,\chi(\cdot),\vartheta(\cdot)))]\right).\end{aligned}\quad(50)$$

By the same arguments in Theorem 7, we get

$$\begin{aligned}|\chi(\tau)-\bar{\chi}(\tau)|&=\left|\Upsilon_1(\chi,\vartheta)(\tau)-\Upsilon_1(\bar{\chi},\bar{\vartheta})(\tau)-\hbar\left(\tau,\bar{\chi}(\tau),\bar{\vartheta}(\tau)\right)\right.\\&\quad\left.\cdot\left(K_1(\tau;h_1)+\frac{\tau^{r+1}}{\left(1-\delta\zeta^{r+1}\right)}[\delta K_1(\zeta;h_1)-K_1(1;h_1)]\right)\right|\\&\leq\left|\Upsilon_1(\chi,\vartheta)(\tau)-\Upsilon_1(\bar{\chi},\bar{\vartheta})(\tau)\right|+\left|\hbar\left(\tau,\bar{\chi}(\tau),\bar{\vartheta}(\tau)\right)\right.\\&\quad\left.\cdot\left(K_1(\tau;h_1)+\frac{\tau^{r+1}}{\left(1-\delta\zeta^{r+1}\right)}[\delta K_1(\zeta;h_1)-K_1(1;h_1)]\right)\right|\\&\leq\lambda_\hbar\Lambda_1(v_1+v_2)\left(\|\chi-\bar{\chi}\|+\left\|\vartheta-\bar{\vartheta}\right\|\right)+\lambda_\hbar\Lambda_1\varepsilon_1,\end{aligned}\quad(51)$$

$$\begin{aligned}\left|\vartheta(\tau)-\bar{\vartheta}(\tau)\right|&=\left|\Upsilon_2(\chi,\vartheta)(\tau)-\Upsilon_2(\bar{\chi},\bar{\vartheta})(\tau)-\bar{\lambda}\left(\tau,\bar{\chi}(\tau),\bar{\vartheta}(\tau)\right)\right.\\&\quad\left.\cdot\left(K_2(\tau;h_2)+\frac{\tau^{r+1}}{\left(1-\delta\zeta^{r+1}\right)}[\delta K_2(\zeta;h_2)-K_2(1;h_2)]\right)\right|\\&\leq\left|\Upsilon_2(\chi,\vartheta)(\tau)-\Upsilon_2(\bar{\chi},\bar{\vartheta})(\tau)\right|+\left|\bar{\lambda}\left(\tau,\bar{\chi}(\tau),\bar{\vartheta}(\tau)\right)\right.\\&\quad\left.\cdot\left(K_2(\tau;h_2)+\frac{\tau^{r+1}}{\left(1-\delta\zeta^{r+1}\right)}[\delta K_2(\zeta;h_2)-K_2(1;h_2)]\right)\right|\\&\leq\lambda_{\bar{\lambda}}\Lambda_2(\ell_1+\ell_2)\left(\|\chi-\bar{\chi}\|+\left\|\vartheta-\bar{\vartheta}\right\|\right)+\lambda_{\bar{\lambda}}\Lambda_2\varepsilon_2.\end{aligned}\quad(52)$$

It follows that

$$\begin{aligned}&\|\chi-\bar{\chi}\|-\lambda_\hbar\Lambda_1(v_1+v_2)\left(\|\chi-\bar{\chi}\|+\left\|\vartheta-\bar{\vartheta}\right\|\right)\leq\lambda_\hbar\Lambda_1\varepsilon_1,\\&\left\|\vartheta-\bar{\vartheta}\right\|-\lambda_g\Lambda_2(\ell_1+\ell_2)\left(\|\chi-\bar{\chi}\|+\left\|\vartheta-\bar{\vartheta}\right\|\right)\leq\lambda_{\bar{\lambda}}\Lambda_2\varepsilon_2.\end{aligned}\quad(53)$$

Representing these inequalities as matrices, we get

$$\begin{pmatrix}1-\lambda_\hbar\Lambda_1(v_1+v_2)&-\lambda_\hbar\Lambda_1(v_1+v_2)\\1-\lambda_{\bar{\lambda}}\Lambda_2(\ell_1+\ell_2)&-\lambda_{\bar{\lambda}}\Lambda_2(\ell_1+\ell_2)\end{pmatrix}\begin{pmatrix}\|\chi-\bar{\chi}\|\\\left\|\vartheta-\bar{\vartheta}\right\|\end{pmatrix}\leq\begin{pmatrix}\lambda_\hbar\Lambda_1\varepsilon_1\\\lambda_{\bar{\lambda}}\Lambda_2\varepsilon_2\end{pmatrix}.\quad(54)$$

Solving the above inequality, we get

$$\begin{aligned}\|\chi-\bar{\chi}\|&\leq\frac{1-\lambda_\hbar\Lambda_1(v_1+v_2)}{\Delta}\lambda_\hbar\Lambda_1\varepsilon_1+\frac{\lambda_\hbar\Lambda_1(v_1+v_2)}{\Delta}\lambda_{\bar{\lambda}}\Lambda_2\varepsilon_2,\\\left\|\vartheta-\bar{\vartheta}\right\|&\leq\frac{\lambda_{\bar{\lambda}}\Lambda_2(\ell_1+\ell_2)}{\Delta}\lambda_\hbar\Lambda_1\varepsilon_1+\frac{1-\lambda_{\bar{\lambda}}\Lambda_2(\ell_1+\ell_2)}{\Delta}\lambda_{\bar{\lambda}}\Lambda_2\varepsilon_2,\end{aligned}\quad(55)$$

where $\Delta=(1-\lambda_\hbar\Lambda_1(v_1+v_2))(1-\lambda_{\bar{\lambda}}\Lambda_2(\ell_1+\ell_2))-\lambda\lambda_\hbar\Lambda_1(v_1+v_2)\lambda_{\bar{\lambda}}\Lambda_2(\ell_1+\ell_2)\neq0$.

Thus

$$\begin{aligned}\|\chi-\bar{\chi}\|+\left\|\vartheta-\bar{\vartheta}\right\|&\leq\left(\frac{1-\lambda_\hbar\Lambda_1(v_1+v_2)}{\Delta}+\frac{\lambda_{\bar{\lambda}}\Lambda_2(\ell_1+\ell_2)}{\Delta}\right)\lambda_\hbar\Lambda_1\varepsilon_1\\&\quad+\left(\frac{1-\lambda_{\bar{\lambda}}\Lambda_2(\ell_1+\ell_2)}{\Delta}+\frac{\lambda_\hbar\Lambda_1(v_1+v_2)}{\Delta}\right)\lambda_{\bar{\lambda}}\Lambda_2\varepsilon_2.\end{aligned}\quad(56)$$

For $\varepsilon = \max(\varepsilon_1, \varepsilon_2)$ and

$$c = \frac{(1 - \lambda_\hbar \Lambda_1(\upsilon_1 + \upsilon_2) + \lambda_{\bar{\lambda}} \Lambda_2(\ell_1 + \ell_2))\lambda_\hbar \Lambda_{11} + (1 - \lambda_{\bar{\lambda}} \Lambda_2(\ell_1 + \ell_2) + \lambda_\hbar \Lambda_1(\upsilon_1 + \upsilon_2))\lambda_{\bar{\lambda}} \Lambda_2}{\Delta}, \tag{57}$$

we get

$$\left\|(\chi, \bar{\chi}) - (\vartheta, \bar{\vartheta})\right\| \le \|\chi - \bar{\chi}\| + \left\|\vartheta - \bar{\vartheta}\right\| \le c\varepsilon. \tag{58}$$

Therefore, with the aid of Definition 9, the solution of the problem Equation (6) is U–H stable. □

5. Example

In this part, we present an applied example to support the theoretical results we reached in the previous part, consider the following system:

$$\begin{cases} \left({}^C D_{1-}^{7/4} {}^{RL} D_{0+}^{1/4}\right)\left(\dfrac{\chi(\tau)}{1/2|\sin \chi(\tau)| + 7/5}\right) = 3e^\tau + \dfrac{1}{10\sqrt{\tau^3 + 15}} \dfrac{|\chi|}{1 + |\chi|} + \dfrac{1}{40} \tan^{-1}\vartheta, \tau \in [0, 1], \\ \left({}^C D_{1-}^{7/4} {}^{RL} D_{0+}^{3/4}\right)\left(\dfrac{\vartheta(\tau)}{1/3|\cos \vartheta(\tau)| + 1}\right) = 2e^{-3\tau} \sin \tau + \dfrac{1}{20}\left(\tan^{-1}\vartheta + \tan^{-1}\chi\right), \\ \chi(0) = \chi'(0) = 0, \quad \chi(1) = \chi\left(\dfrac{1}{2}\right), \\ \vartheta(0) = \vartheta'(0) = 0, \quad \vartheta(1) = \vartheta\left(\dfrac{1}{3}\right). \end{cases} \tag{59}$$

Here,

$$\begin{gathered} 1 = \delta = \varepsilon, \\ p = \frac{7}{4}, q = \frac{5}{4}, \hbar(\tau, \chi, \vartheta) = \frac{1}{2}|\sin \chi(\tau)| + \frac{7}{5}, \bar{\lambda}(\tau, \chi, \vartheta) = \frac{1}{3}|\cos \vartheta(\tau)| + 1, \\ \psi(\tau, \chi(\tau), \vartheta(\tau)) = 3e^\tau + \frac{1}{10\sqrt{\tau^3 + 15}} \frac{|\chi|}{1 + |\chi|} + \frac{1}{40} \tan^{-1}\vartheta, \varphi(\tau, \chi(\tau), \vartheta(\tau)) \\ = 2e^{-3\tau} \sin \tau + \frac{1}{20}\left(\tan^{-1}\vartheta + \tan^{-1}\chi\right). \end{gathered} \tag{60}$$

Observe that

$$\begin{gathered} |\psi(\tau, \chi_1, \vartheta_1) - \psi(\tau, \chi_2, \vartheta_2)| \le \frac{1}{40}|\chi_2 - \chi_1| + \frac{1}{40}|\vartheta_2 - \vartheta_1|, \\ |\varphi(\tau, \chi_1, \vartheta_1) - \varphi(\tau, \chi_2, \vartheta_2)| \le \frac{1}{20}|\chi_2 - \chi_1| + \frac{1}{20}|\vartheta_2 - \vartheta_1|, \\ [\lambda_\hbar \Lambda_1(\upsilon_1 + \upsilon_2) + \lambda_{\bar{\lambda}} \Lambda_2(\ell_1 + \ell_2)] \le 0.330291 < 1. \end{gathered} \tag{61}$$

Thus, the boundary value problem Equation (59) satisfies all the conditions of Theorem 7; consequently, the uniqueness of solution of Equation (59) is satisfied on $[0, 1]$.

In order to explain Theorem 7, it is clear that $(C1)$ is satisfied as follows:

$$\begin{aligned} |\hbar(\tau, \chi, \vartheta)| &\le \frac{1}{2}|\sin \chi(\tau)| + \frac{7}{5} \le 2 = \lambda_\hbar, \\ |\bar{\lambda}(\tau, \chi, \vartheta)| &\le \frac{1}{3}|\cos \vartheta(\tau)| + 1 \le \frac{3}{2} = \lambda_{\bar{\lambda}}, \end{aligned} \tag{62}$$

Also, one can easily show that $(C3)$ holds, taking into account that $\tau \in [0, 1]$, then

$$\begin{aligned} |\psi(\tau, \chi, \vartheta)| &= \left|3e^\tau + \frac{1}{10\sqrt{\tau^3 + 15}} \frac{|\chi|}{1 + |\chi|} + \frac{1}{40} \tan^{-1}\vartheta\right| \\ &\le 3e + \frac{1}{40}|\chi| + \frac{1}{40}|\vartheta|, \\ |\varphi(\tau, \chi, \vartheta)| &= \left|2e^{-3\tau} \sin \tau + \frac{1}{20}\left(\tan^{-1}\vartheta + \tan^{-1}\chi\right)\right| \\ &\le 2 + \frac{1}{20}|\chi| + \frac{1}{20}|\vartheta|. \end{aligned} \tag{63}$$

Also, $(C4)$ satisfied with

$$|\psi(\tau, \chi, \vartheta)| \le 3e + \frac{1 + 2\pi}{40}, |\varphi(\tau, \chi, \vartheta)| \le 2 + \frac{\pi}{5}. \tag{64}$$

Finally, easy calculations with the data above give $(\lambda_\hbar \Lambda_1 \omega_1 + \lambda_{\bar{\lambda}} \Lambda_2 \theta_1) = 0.203275 < 1$ and $(\lambda_\hbar \Lambda_1 \omega_2 + \lambda_{\bar{\lambda}} \Lambda_2 \theta_2) =$

0.203 < 1; all conditions of Theorem 8 hold; that is, the problem (59) has at least one solution in $[0, 1]$.

6. Conclusion

We have studied a coupled hybrid FDEs consisting of mixed fractional derivatives such as Caputo and Riemann-Liouville fractional derivatives and nonlocal boundary conditions. Existence/uniqueness results are established via a nonlinear alternative of the Leray-Schauder and Banach fixed point theorem. We also studied the Ulam-Hyers stability of these couple of hybrid FDEs. The obtained result is well illustrated by a numerical example. The result obtained in this paper is new and significantly contributes to the existing literature on the topic.

One possible direction in which to extend the results of this paper is toward different kinds of mixed fractional differential and mixed conformable fractional differential systems of higher order. Another challenge is to find out if similar results can be derived in the case of constant/variable delays in linear/nonlinear terms.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

Authors' Contributions

M.A and N. M contributed to each part of this work equally and read and approved the final version of the manuscript.

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