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Research Article

Novel Optical Soliton Waves in Metamaterials with Parabolic Law of Nonlinearity via the IEFM and ISEM

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Here, the miscellaneous soliton solutions of the generalized nonlinear Schrödinger equation are considered that describe the model of few-cycle pulse propagation in metamaterials with parabolic law of nonlinearity. The novel analytical wave solutions to the mentioned nonlinear equation in the sense of the nonlinear ordinary differential transform equation are obtained. The techniques are the improved exp $(-\Gamma(\varpi))$ function method and the improved simple equation method. The nonlinear ordinary transform to concern the generalized Schrodinger equation to convert it for a solvable integer-order differential equation is used. After the successful implementation of the presented methods, the exact solitary wave solutions in the form of trigonometric, rational, and hyperbolic functions are obtained. Hence, the presented methods are relatable and efficient to solve nonlinear problems in mathematical physics.

1. Introduction

In nonlinear engineering, physics, and mathematics, solving the nonlinear models would still be a useful approach for comprehending the complicated systems. The usefulness of explicit traveling wave solutions for nonlinear partial differential equations (NLPDEs) is noteworthy in the current situation [1]. NLPDEs have recently become quite popular in various topics such as optical fiber, fractional dynamics, fluid mechanics, control theory, chemical kinematics, signal transmission control theory, plasma physics, earthquakes, relativistic, solid-state physics, chemical physics, geochemistry, ecosystem, biomechanics, biophysics, gas dynamics, and so forth [2–10].

Furthermore, the scientific community has taken an interest in extraction for dynamic wave propagation recognized by many other types of nonlinear evolution equations (NLEEs). It is operating several sorts of population models in atmospheric engineering and is also used to represent wave propagation and heat movement in physics [11, 12].

Many academics have focused on finding the optimum nonlinear differential equation solution. In the last decade, the development of the computer algebraic systems such as Mathematica and Maple has been utilized for obtaining the exact solutions to the NLPDEs that help a lot to identify solutions to the traveling wave of NLPDEs, and various strategies have been created and established [13].

There are numerous effective approaches available, notably the extended tanh-coth method and extended rational sinh-cosh method [14], the improved $\tan(\phi/2)$ method [15], the Riccati equation expansion approach [16], the csch-function method [17], the multiple exp-function scheme [18], the exp-function method [19], the machine learning method [20], the bilinear transformation [21], the molecular dynamic simulation method [22], the generalized Darboux transformation scheme [23], the binary Darboux transformation method [24], a partial least square structural equation modeling [25], the modified simple equation (MSE) method [26], the modified Kudryashov method [27], the Jacobi elliptic function method [28], a CFD-based simulation approach by using nanofluids [29], the modified auxiliary expansion method [30], the Lie symmetry analysis approach [31], the sine-Gordon expansion method [32], the direct algebraic method [33], understanding by a design model as a useful tool for a meaningful and permanent learning [34], the generalized $\exp(-\phi(\xi))$-expansion method [35, 36], and residual power series method [37] which have been created and effectively developed in order to attain the exact solution to NLPDEs. Mixed Morrey spaces and their applications were studied for partial differential equations of parabolic type [38] and boundedness of fractional integral operator Morrey-type and anisotropic spaces to the KdV equation [39] by Benia and Scapellato.

The nonlinear dynamics that describes the propagation of pulses in optical metamaterials is modeled by the nonlinear Schrödinger equation [40] which is shown as follows:

$$\begin{aligned} iu_t + au_{xx} &+ \left(b_1|u|^{-4} + b_2|u|^2 + b_3|u|^4\right)u \\ &= i\left[\alpha u_x + \beta\left(|u|^2u\right)_x + \nu\left(|u|^2\right)_x u\right] + \theta_1\left(|u|^2u\right)_{xx} \\ &\quad + \theta_2|u|^2u_{xx} + \theta_3 u^2 u^*_{xx}, \end{aligned} \tag{1}$$

and has been noted by the numerous researchers of authors. Zhou et al. investigated the optical metamaterials with parabolic law nonlinearity and spatiotemporal dispersion [41]. Ekici and coauthors determined the exact solutions for the nonlinear negative-indexed materials with anticubic nonlinearity [42]. Also, Foroutan and coworkers probed soliton perturbation in optical metamaterials, with anticubic nonlinearity, by implementing the improved $\tan(\phi/2)$-expansion method [43]. Take the optical metamaterials modeled by the nonlinear Schrödinger equation with power law nonlinearity [44] which reads as

$$\begin{aligned} iu_t + au_{xx} &+ b\left(|u|^2 + \sigma_1|u|^4\right)u \\ &= i\left[\alpha_1 u_x + \lambda_1\left(|u|^2u\right)_x + \nu\left(|u|^2\right)_x u\right] \\ &\quad + \theta_1\left(|u|^2u\right)_{xx} + \theta_2|u|^2u_{xx} + \theta_3 u^2 u^*_{xx}. \end{aligned} \tag{2}$$

In [45], Li and coauthors found new types of solitary wave solutions for the higher-order nonlinear Schrödinger equation describing propagation of femtosecond light pulses in an optical fiber. Biswas et al. [46] probed the bright and dark soliton solutions for equation (2). Moreover, Yakadaa and coworkers [47] implemented the new extended direct algebraic method to determine more general and new soliton solutions for equation (2). In equation (2), $u = u(x, t)$ and a shows the group velocity dispersion, b stands for the self-modulation term, and σ_1 is the coefficient of quintic nonlinearity. And constant λ_1 shows the self-steepening effect term, α_1 shows the intermodal dispersion, ν is the nonlinear dispersion, and the constants θ_k for $k = 1, 2, 3$ are the perturbation terms that appear in the context of metamaterials [43, 46]. The authors of [48–50] have worked on the Kundu-Eckhaus, Vakhnenko-Parkes, and unsteady Korteweg-de Vries equations, respectively, and obtained the exact solutions.

The main goal of this research is to find a well-formed exact solution for the NLPDEs that are broad-ranging and well-known, such as soliton, kinks, bells, and other sorts of solutions. The solutions of the generalized nonlinear Schrödinger equation were solved using the improved exp $(-\Gamma(\varpi))$ function method and improved simple equation method. So, solving this proposed equation utilizing the mentioned methods with the nonlinear ordinary differential transform equation is completely new. These methods are utilized in this article to obtain some more recent and broad results that are simple to implement, as well as versatile, flexible, configurable, extensible, and faster to simulate. On the other hand, we introduced three new graphical styles, lislpointplot3D, density, and vector plot, which also have an important impact on this type of research. Additionally, we visualize a three-dimensional plotline and density and two-dimensional plots. Those eight types of graphical description help us to describe the physical sketch a lot more explicitly.

The remainder of the article will be organized in the following manner: the governing equation and the base configuration are introduced in Section 2; the execution of the methodology about the improved exp $(-\Gamma(\varpi))$ function method has been depicted in Section 3. In Section 4, we used the second mentioned method to construct the specific solution for the proposed equation. Section 5 contains a short discussion, as well as a graphical description and classification, which are part of the visual explanation. The final thoughts are presented in the last section.

2. The Governing Equation

Pulse propagation in a one-core fiber has distinction from continuous wave propagation. In a conventional one-core fiber, pulse propagation has been investigated extensively by solving the model equation. To determine soliton waves of equation (2), we commence the following wave transformations [46, 51] as

$$u(x, t) = V(\xi(x, t)) \exp\left(i\phi(x, t)\right), \tag{3}$$

where

$$\xi(x, t) = x + \nu t. \tag{4}$$

Here, $V(\xi(x, t))$ is an amplitude component of the wave profile, while $\phi(x, t)$ is a phase component of the profile where it is given as

$$\phi(x, t) = -kx + \omega t + \phi_0. \tag{5}$$

The parameters k, ω, and ϕ_0 are the wave number, frequency, and phase constant, respectively, while ν represents the velocity of soliton. Substitute equation (3) and its derivatives into equation (2), and decompose it into real and imaginary parts. The real part and imaginary part equation for the component is given below, for

$$\begin{aligned}&\left(\omega + \alpha_1 k + ak^2\right)V - aV'' + \left(k(\lambda_1 - k(\theta_1 + \theta_2 + \theta_3)) - b\right)V^3 \\ &\quad - b\sigma_1 V^5 - 6\theta_1 V\left(V'\right)^2 - (3\theta_1 + \theta_2 + \theta_3)V^2V'' = 0,\end{aligned} \tag{6}$$

$$[\nu - (\alpha_1 + 2ak)]V = [3\lambda_1 + 2\nu - 6k\theta_1 - 2k\theta_2 + 2k\theta_3]V^2V'. \tag{7}$$

Then, we obtain

$$\nu = \alpha_1 + 2ak. \tag{8}$$

3. Description of the First Proposed Method

In this section, we seek to utilize a suitable traveling wave transformation for the NLPDEs to translate it into nonlinear-order ordinary differential equations. Furthermore, we study the special ODE and then determine the obtained solutions. Therefore, consider the improved $\exp(-\Gamma(\varpi))$ function method. Let the NLPDEs

$$\mathcal{S}(U, D_xU, D_tU, D_{xx}U, D_{tt}U, \cdots) = 0. \tag{9}$$

Let

$$U = U(\varpi), \varpi = x + \nu t. \tag{10}$$

Substitute equation (10) into equation (9):

$$\mathcal{W}\left(U, U', U'', U''', \cdots\right) = 0. \tag{11}$$

Let the solution of equation (11) be

$$U(\varpi) = \frac{\sum_{i=0}^{P_1} H_i\left(e^{-\Gamma(\varpi)}\right)^i}{\sum_{i=0}^{P_2} N_i\left(e^{-\Gamma(\varpi)}\right)^i}. \tag{12}$$

Let Γ satisfy

$$\Gamma' = \rho_1 e^{\Gamma(\varpi)} + e^{-\Gamma(\varpi)} + \rho_2. \tag{13}$$

Put equation (12) with equation (11) into equation (14). Then, the solutions are presented as

Product 1: with $\rho_1 \neq 0$ and $\rho_2^2 - 4\rho_1 > 0$, then we have

$$\Gamma(\varpi) = \ln\left(-\frac{\sqrt{\rho_2^2 - 4\rho_1}}{2\rho_1}\tanh\left(\frac{\sqrt{\rho_2^2 - 4\rho_1}}{2}(\varpi + \varpi_0)\right) - \frac{\rho_2}{2\rho_1}\right), \tag{14}$$

where ϖ_0 is the integral constant.

Product 2: with $\rho_1 \neq 0$ and $\rho_2^2 - 4\rho_1 < 0$, then we have

$$\Gamma(\varpi) = \ln\left(\frac{\sqrt{-\rho_2^2 + 4\rho_1}}{2\rho_1}\tan\left(\frac{\sqrt{-\rho_2^2 + 4\rho_1}}{2}(\varpi + \varpi_0)\right) - \frac{\rho_2}{2\rho_1}\right). \tag{15}$$

Product 3: with $\rho_1 = 0$, $\rho_2 \neq 0$, and $\lambda^2 - 4\mu > 0$, then we get

$$\Gamma(\varpi) = -\ln\left(\frac{\rho_2}{\exp(\rho_2(\varpi + \varpi_0)) - 1}\right). \tag{16}$$

Product 4: with $\rho_1 \neq 0$, $\rho_2 \neq 0$, and $\rho_2^2 - 4\rho_1 = 0$, then we get

$$\Gamma(\varpi) = \ln\left(-\frac{2\lambda(\varpi + \varpi_0) + 4}{\rho_2^2(\varpi + \varpi_0)}\right). \tag{17}$$

Product 5: with $\rho_1 = 0$, $\rho_2 = 0$, and $\rho_2^2 - 4\rho_1 = 0$, then we get

$$\Omega(\eta) = \ln(\varpi + \varpi_0), \tag{18}$$

where $H_i (0 \le i \le P_1)$, $N_i (0 \le i \le P_2)$, and ρ_1 and ρ_2 are also the values to be explored later.

Consider $G = G(\varpi) = e^{-\Gamma(\varpi)}$, and we have

$$\begin{aligned} V(\varpi) &\simeq \delta F^{N-M}, \\ V'(\varpi) &\simeq \delta G^{P_1 - P_2 - 1}G' \\ &= -\delta\left(\rho_1 G^{P_1 - P_2 - 1} + \rho_2 G^{P_1 - P_2} + G^{P_1 - P_2 + 1}\right) \\ &\simeq -\delta G^{P_1 - P_2 + 1}, \\ V''(\varpi)) &\simeq \delta G^{P_1 - P_2 - 1}, \\ (V(\varpi))^4 &\simeq \delta^4 G^{5P_1 - 5P_2}, \end{aligned} \tag{19}$$

where $\delta = H_{P_1}/N_{P_2}$. Balancing V^2V'' with V^5 in equation (6) concludes

$$G^{3P_1 - 3P_2 + 2} \simeq (V(\varpi))^2 V''(\varpi) = (V(\varpi))^5 \simeq G^{5P_1 - 5P_2}. \tag{20}$$

We can determine values of N and M as follows:

$$3P_1 - 3P_2 + 2 = 5P_1 - 5P_2 \Longrightarrow P_1 = P_2 + 1. \tag{21}$$

3.1. Case I: $P_1 = 1, P_2 = 0$. The IEFM allows us to recruit the substitutions:

$$\nu(\varpi) = \frac{H_0 + H_1 G(\varpi)}{N_0}. \tag{22}$$

Plugging (35) along with (12) into equation (6), we get

$$\left\{\begin{array}{l} -bH_1{}^5\sigma_1 - 2\,H_1{}^3N_0{}^2\Theta_2 - 6\,H_1{}^3N_0{}^2\theta_1 = 0 - 5\,bH_0H_1{}^4\sigma_1 - 3\,H_1{}^3N_0{}^2\,\Theta_2\rho_2 - 12\,H_1{}^3N_0{}^2\rho_2\theta\,1 - 4\,H_0H_1{}^2N_0{}^2\Theta_2 \\ -\,6\,H_0H_1{}^2N_0{}^2\theta_1 = 0 - k^2H_1{}^3N_0{}^2\Theta_1 - H_1{}^3N_02\Theta_2\rho_2{}^2 - 6\,H_1{}^3N_0{}^2\rho_2{}^2\theta_1 - 10\,bH_0{}^2H_1{}^3\sigma_1 + kH_1{}^3N_0{}^2\lambda_1 \\ -\,6\,H_0H_1{}^2N_0{}^2\,\Theta_2\rho_2 - 12\,H_0H_1{}^2N_0{}^2\rho_2\,\theta_1 - 2\,H_1{}^3N_0{}^2\Theta_2\rho_1 - 12\,H_1{}^3N_0{}^2\rho_1\theta_1 - 2\,aH_1N_04 - bH_1{}^3N_0{}^2 \\ -\,2\,H_0{}^2H_1N_0{}^2\,\Theta_2 = 0 - 3\,k^2H_0H_1{}^2N_0{}^2\Theta_1 - 2\,H_0H1^2N_0{}^2\Theta_2\rho_2{}^2 - 6\,H_0H_1{}^2N_0{}^2\rho_2{}^2\theta_1 - H_1{}^3N_02\Theta_2\rho_1\rho_2 \\ -\,12\,H_1{}^3N_0{}^2\rho_1\rho_2\theta_1 - 3\,aH_1N_0{}^4\rho_2 - 10\,b\,H_0{}^3H_1{}^2\sigma_1 + 3\,kH_0H_1{}^2N_0{}^2\lambda_1 - 3\,H_0{}^2H_1N_0{}^2\Theta_2\,\rho_2 - 4\,H_0H_1{}^2N_0{}^2\Theta_2\rho_1 \\ -\,12\,H_0H_1{}^2N_0{}^2\rho_1\theta_1 - 3\,bH_0\,H_1{}^2N_0{}^2 = 0ak^2H_1N_0{}^4 - 3\,k^2H_0{}^2H_1N_0{}^2\Theta_1 - H_0{}^2H_1N_0{}^2\Theta_2\rho_2{}^2 - 2\,H_0H_1{}^2N_0{}^2\Theta_2\rho_1\rho_2 \\ -\,12\,H_0H_1{}^2N_0{}^2\rho_1\rho_2\theta_1 - 6\,H_1{}^3N_02\rho_1{}^2\theta_1 + -aH_1N_0{}^4\rho_2{}^2 - 2\,aH_1N_0{}^4\rho_1 - 5\,bH_0{}^4H_1\sigma_1 + 3\,kH_0{}^2H_1N_0{}^2\,\lambda_1 + kH_1N_0{}^4\alpha_1 \\ -\,2\,H_0{}^2H_1\,N_0{}^2\Theta_2\rho_1 - 3\,bH_0{}^2H_1N_0{}^2 + \omega\,H_1N_0{}^4 = 0 \\ ak^2H_0N_0{}^4 - aH_1N_0{}^4\rho_1\rho_2 - k^2H_0{}^3N_0{}^2\Theta_1 - H_0{}^2H_1N0^2\Theta_2\rho_1\rho_2 - 6\,H_0H_1{}^2N_0{}^2\rho_1{}^2\theta_1 - bH_0{}^5\sigma_1 + kH_0{}^3N_0{}^2\lambda_1 \\ +\,kH_0N_0{}^4\alpha_1 - bH_0{}^3N_0{}^2 + \omega\,H_0N_0{}^4 = 0. \end{array}\right. . \tag{23}$$

Solving the above nonlinear system, we get the following.

Product I:

$$\begin{gathered} \rho_1 = \frac{1}{2}\,\frac{N_0{}^2\left(-k^2\Theta_1\Theta_2 - 3\,k^2\Theta_1\theta_1 + ab\sigma_1 + 3\,k\lambda_1\theta_1 - 3\,b\theta_1\right) - \Theta_2\left(bH_0{}^2\sigma_1 - kN_0{}^2\lambda_1 + bN_0{}^2\right) - 6\,bH_0{}^2\sigma_1\theta_1}{N_0{}^2(\Theta_2 + 3\,\theta_1)(\Theta_2 + 6\,\theta_1)}, \\ k = k, H_0 = H_0, H_1 = \frac{\sqrt{-2\,b\sigma_1(\Theta_2 + 3\,\theta_1)}N_0}{b\sigma_1}, N_0 = N_0, \rho_2 = 2\,\frac{H_0}{H_1}, \\ \rho_2^2 - 4\rho_1 = -2\,\frac{-k^2\Theta_1\Theta_2 - 3\,k^2\Theta_1\theta_1 + ab\sigma_1 + k\Theta_2\lambda_1 + 3\,k\lambda_1\theta_1 - b\Theta_2 - 3\,b\theta_1}{(\Theta_2 + 3\,\theta_1)(\Theta_2 + 6\,\theta_1)},\ \omega = \frac{\Lambda_1 + \Lambda_2 + \Lambda_3}{(\Theta_2 + 6\,\theta_1)^2 b\sigma_1}, \\ \Lambda_1 = -3\,k^4\theta_1\Theta_1{}^2\Theta_2 - 9\,k^4\theta_1{}^2\Theta_1{}^2 - k^2\Theta_1\Theta_2(ab\sigma_1 - 6\,k\lambda_1\theta_1 + 6\,b\theta_1) - bk\Theta_2{}^2\sigma_1(ak + \alpha_1) - 18\,k^2\Theta_1\theta_1{}^2(-k\lambda_1 + b), \\ \Lambda_2 = -\Theta_2\left(12\,abk^2\sigma_1\theta_1 - abk\lambda_1\sigma_1 + 12\,bk\alpha_1\sigma_1\theta_1 + 3\,k^2\lambda_1{}^2\theta_1 + ab^2\sigma_1 - 6\,bk\lambda_1\theta_1 + 3\,b^2\theta_1\right), \\ \Lambda_3 = -36\,abk^2\sigma_1\theta_1{}^2 + a^2b^2\sigma_1{}^2 - 36\,bk\alpha_1\sigma_1\theta_1{}^2 - 9\,k^2\lambda_1{}^2\theta_1{}^2 + 18\,bk\lambda_1\theta_1{}^2 - 9\,b^2\theta_1{}^2. \end{gathered} \tag{24}$$

Through (35) and (36), then we find the below cases:
Product IA:

By using (14), we have

$$q(x,t) = \left\{\frac{H_0}{N_0} + \frac{\sqrt{-2\,b\sigma_1(\Theta_2 + 3\,\theta_1)}/b\sigma_1}{\sqrt{\rho_2^2 - 4\rho_1}/2\rho_1\ \tanh\left(\left(\sqrt{\rho_2^2 - 4\rho_1}/2\right)(\varpi + \varpi_0)\right) + \rho_2/2\rho_1}\right\} e^{i(-kx+\omega t)}, \tag{25}$$

where H_0, N_0, ϖ_0 are free amounts. This solution is valid for the following case:

$$\left(-k^2\Theta_1\Theta_2 - 3k^2\Theta_1\theta_1 + ab\sigma_1 + k\Theta_2\lambda_1 + 3k\lambda_1\theta_1 - b\Theta_2 - 3b\theta_1\right)(9\theta_1 + \theta_2 + \theta_3)(6\theta_1 + \theta_2 + \theta_3) > 0. \tag{26}$$

Product IB:

By utilizing (15), we have

$$q_2(x,t) = \left\{\frac{H_0}{N_0} - \frac{\sqrt{-2b\sigma_1(\Theta_2 + 3\theta_1)}/b\sigma_1}{\sqrt{-\rho_2^2 + 4\rho_1}/2\rho_1 \tan\left(\left(\sqrt{-\rho_2^2 + 4\rho_1}/2\right)(\varpi + \varpi_0)\right) - \rho_2/2\rho_1}\right\} e^{i(-kx+\omega t)}, \tag{27}$$

where H_0, N_0, ϖ_0 are free amounts. This solution is valid for the following case:

$$\left(-k^2\Theta_1\Theta_2 - 3k^2\Theta_1\theta_1 + ab\sigma_1 + k\Theta_2\lambda_1 + 3k\lambda_1\theta_1 - b\Theta_2 - 3b\theta_1\right)(9\theta_1 + \theta_2 + \theta_3)(6\theta_1 + \theta_2 + \theta_3) < 0. \tag{28}$$

Product IC:

By using (16), we get

$$q_3(x,t) = \left\{\frac{H_0}{N_0} + \frac{\sqrt{-2b\sigma_1(\Theta_2 + 3\theta_1)}/b\sigma_1}{\rho_2/(\exp(\rho_2(x + (\alpha_1 + 2ak)t + \varpi_0)) - 1)}\right\} e^{i(-kx+\omega t)}, \tag{29}$$

in which

$$H_0 = \frac{N_0\sqrt{\left(-k^2\Theta_1\Theta_2 - 3k^2\Theta_1\theta_1 + ab\sigma_1 + k\Theta_2\lambda_1 + 3k\lambda_1\theta_1 - b\Theta_2 - 3b\theta_1\right)b\sigma_1(\Theta_2 + 6\theta_1)}}{-k^2\Theta_1\Theta_2 - 3k^2\Theta_1\theta_1 + ab\sigma_1 + k\Theta_2\lambda_1 + 3k\lambda_1\theta_1 - b\Theta_2 - 3b\theta_1}, \tag{30}$$

where H_0, ϖ_0 are free values. This solution is valid for the following case:

$$(9\theta_1 + \theta_2 + \theta_3) > 0,\ (6\theta_1 + \theta_2 + \theta_3) > 0,\ H_0\left(-k^2\Theta_1\Theta_2 - 3k^2\Theta_1\theta_1 + ab\sigma_1 + k\Theta_2\lambda_1 + 3k\lambda_1\theta_1 - b\Theta_2 - 3b\theta_1\right) > 0. \tag{31}$$

Product ID:

By utilizing (17), we conclude

$$q_4(x,t) = \left\{\frac{H_0}{N_0} - \frac{\sqrt{-2b\sigma_1(\Theta_2 + 3\theta_1)}/b\sigma_1}{\rho_2^2(x + (\alpha_1 + 2ak)t + \varpi_0))/(2\rho_2(x + (\alpha_1 + 2ak)t + \varpi_0)) + 4)}\right\} e^{i(-kx+\omega t)}, \tag{32}$$

in which

$$k = \frac{\Theta_2\lambda_1 + 3\lambda_1\theta_1 + \sqrt{4ab\Theta_1\Theta_2\sigma_1 + 12ab\Theta_1\sigma_1\theta_1 - 4b\Theta_1{\Theta_2}^2 - 24b\Theta_1\Theta_2\theta_1 - 36b\Theta_1{\theta_1}^2 + {\Theta_2}^2{\lambda_1}^2 + 6\Theta_2{\lambda_1}^2\theta_1 + 9{\lambda_1}^2{\theta_1}^2}}{2\Theta_1(\Theta_2 + 3\theta_1)}, \tag{33}$$

where H_0, ϖ_0 are free amounts. This solution is valid for the following case:

$$\rho_2, \rho_1 \neq 0, 4\,ab\Theta_1\Theta_2\sigma_1 + 12\,ab\Theta_1\sigma_1\theta_1 - 4\,b\Theta_1{\Theta_2}^2 - 24\,b\Theta_1\Theta_2\theta_1 - 36\,b\Theta_1{\theta_1}^2 + {\Theta_2}^2{\lambda_1}^2 + 6\,\Theta_2{\lambda_1}^2\theta_1 + 9\,{\lambda_1}^2{\theta_1}^2 > 0. \tag{34}$$

3.2. Case II: $P_1 = 2, P_2 = 1$. By using (21) and $P_1 = P_2 + 1$, we get $P_1 = 2, P_2 = 1$. Therefore, we get

$$v(\varpi) = \frac{H_0 + H_1 F(\varpi) + H_2 F(\varpi)^2}{N_0 + N_1 F(\varpi)}, \tag{35}$$

where the values H_0, H_1, H_2, N_0, N_1 will be found. To get to the exact solutions, put (35) along with (12) into equation (6), and equating all the coefficients of powers of $F^k(\varpi), k = 0:10$ to be zero, one obtains a system of algebraic equations. Solving the related system concludes the following.

Set I:

$$\begin{aligned}
&b = -2\frac{{N_1}^2(\Theta_2 + 3\,\theta_1)}{{H_2}^2\sigma_1}, H_0 = \frac{N_0(H_1N_1 - H_2N_0)}{{N_1}^2}, H_1 = H_1, H_2 = H_2, N_0 = N_0,\ N_1 = N_1,\\
&\rho_2 = 2\frac{H_1N_1 - H_2N_0}{H_2N_1},\ \rho_1 = -\frac{1}{2}\ \frac{\sigma_1 k{H_2}^2(k\Theta_1 - \lambda_1) - 2\,\sigma_1(\Theta_2 + 6\,\theta_1)(H_1N_1 - H_2N_0)^2 + 2\,{N_1}^4(a\sigma_1 - \Theta_2 - 3\,\theta_1)}{{H_2}^2{N_1}^2\sigma_1(\Theta_2 + 6\,\theta_1)},\\
&\rho_2^2 - 4\rho_1 = 2\ \frac{2\,a{N_1}^4\sigma_1 + k^2{H_2}^2\Theta_1\sigma_1 - k{H_2}^2\lambda_1\sigma_1 - 2\,{N_1}^4\Theta_2 - 6\,{N_1}^4\theta_1}{{H_2}^2{N_1}^2\sigma_1(\Theta_2 + 6\,\theta_1)}, k = k,\ \omega = \frac{\Lambda_1 + \Lambda_2 + \Lambda_3}{2{N_1}^2{\sigma_1}^2(\Theta_2 + 6\,\theta_1)^2{H_2}^2},\\
&\Lambda_1 = -3\,k^2{H_2}^4{\sigma_1}^2\theta_1(k\Theta_1 - \lambda_1)^2 + 2\,ak^2{H_2}^2{N_1}^2{\sigma_1}^2\left(\Theta_1\Theta_2 + {\Theta_2}^2 + 12\,\Theta_2\theta_1 + 36\,{\theta_1}^2\right),\\
&\Lambda_2 = -2\,k{H_2}^2{N_1}^2\sigma_1\left(a\Theta_2\lambda_1\sigma_1 - 6\,k\Theta_1\Theta_2\theta_1 - 18\,k\Theta_1{\theta_1}^2 - {\Theta_2}^2\alpha_1\sigma_1 - 12\,\Theta_2\alpha_1\sigma_1\theta_1 - 36\,\alpha_1\sigma_1{\theta_1}^2\right),\\
&\Lambda_3 = 4\,{N_1}^2(\Theta_2 + 3\,\theta_1)\left(a^2{N_1}^2{\sigma_1}^2 - 3\,k{H_2}^2\lambda_1\sigma_1\theta_1 - a{N_1}^2\Theta_2\sigma_1 - 3\,{N_1}^2\Theta_2\theta_1 - 9\,{N_1}^2{\theta_1}^2\right),
\end{aligned} \tag{36}$$

where H_0, N_0, and N_1 are arbitrary amounts. From the first set, (36) into (35), we have the following:

Product IA:

By utilizing (14), we gain

$$q_1(x,t) = \left\{\frac{(N_0(H_1N_1 - H_2N_0)/{N_1}^2) - \left(H_1/\left(\left(\sqrt{\rho_2^2 - 4\rho_1}/2\rho_1\right)\tanh\left(\sqrt{\rho_2^2 - 4\rho_1}/2(\varpi + \varpi_0)\right) + \rho_2/2\rho_1\right)\right) + H_2/\left(\left(\sqrt{\rho_2^2 - 4\rho_1}/2\rho_1\right)\tanh\left(\sqrt{\rho_2^2 - 4\rho_1}/2(\varpi + \varpi_0)\right) + (\rho_2/2\rho_1)\right)^2}{N_0 - \left(N_1/\left(\sqrt{\rho_2^2 - 4\rho_1}/2\rho_1\right)\tanh\left(\sqrt{\rho_2^2 - 4\rho_1}/2(\varpi + \varpi_0)\right) + (\rho_2/2\rho_1)\right)}\right\} \times e^{i\left(-kx + \frac{\Lambda_1 + \Lambda_2 + \Lambda_3}{2{N_1}^2{\sigma_1}^2(\Theta_2 + 6\theta_1)^2{H_2}^2}t\right)}, \tag{37}$$

where H_0, H_1, ϖ_0 are free amounts. This solution is valid for the following case:

$$\left(2\,a{N_1}^4\sigma_1 + k^2{H_2}^2\Theta_1\sigma_1 - k{H_2}^2\lambda_1\sigma_1 - 2\,{N_1}^4\Theta_2 - 6\,{N_1}^4\theta_1\right)\sigma_1(9\theta_1 + \theta_2 + \theta_3) > 0. \tag{38}$$

Product IB:

By utilizing (15), we get

$$q_2(x,t) = \left\{\frac{(N_0(H_1N_1 - H_2N_0)/{N_1}^2) + H_1/\left(\left(\sqrt{-\rho_2^2 + 4\rho_1}/2\rho_1\right)\tan\left(\sqrt{-\rho_2^2 + 4\rho_1}/2(\varpi + \varpi_0)\right) - (\rho_2/2\rho_1)\right) + H_2/\left(\left(\sqrt{-\rho_2^2 + 4\rho_1}/2\rho_1\right)\tan\left(\sqrt{-\rho_2^2 + 4\rho_1}/2(\varpi + \varpi_0)\right) - (\rho_2/2\rho_1)\right)^2}{N_0 + \left(N_1/\left(\left(\sqrt{-\rho_2^2 + 4\rho_1}/2\rho_1\right)\tan\left(\left(\sqrt{-\rho_2^2 + 4\rho_1}/2\right)(\varpi + \varpi_0)\right) - (\rho_2/2\rho_1)\right)\right)}\right\} \times e^{i\left(-kx + \frac{\Lambda_1 + \Lambda_2 + \Lambda_3}{2{N_1}^2{\sigma_1}^2(\Theta_2 + 6\theta_1)^2{H_2}^2}t\right)}, \tag{39}$$

where H_0, H_1, ϖ_0 are free amounts. This solution is valid for the following case:

$$\left(2\,aN_1{}^4\sigma_1 + k^2 H_2{}^2\Theta_1\sigma_1 - kH_2{}^2\lambda_1\sigma_1 - 2\,N_1{}^4\Theta_2 - 6\,N_1{}^4\theta_1\right)\sigma_1(9\theta_1 + \theta_2 + \theta_3) < 0. \tag{40}$$

Product IC:

By using (16), we gain

$$q_3(x,t) = \left\{\frac{\left(N_0(H_1N_1 - H_2N_0)/N_1{}^2\right) + \left(H_1/(\rho_2/(\exp(\rho_2(x + (\alpha_1 + 2ak)t + \varpi_0)) - 1))\right) + \left(H_2/(\rho_2/(\exp(\rho_2(x + (\alpha_1 + 2ak)t + \varpi_0)) - 1))^2\right)}{N_0 + \left(N_1/(\rho_2/(\exp(\rho_2(x + (\alpha_1 + 2ak)t + \varpi_0)) - 1))\right)}\right\} e^{i\left(-kx + \frac{\Lambda_1 + \Lambda_2 + \Lambda_3}{2N_1{}^2\sigma_1{}^2(\Theta_2 + 6\theta_1)^2H_2{}^2}t\right)}, \tag{41}$$

in which

$$\begin{aligned} k &= \frac{\lambda_1\sigma_1H_2 + \sqrt{\Pi}}{2\Theta_1\sigma_1H_2}, \\ \Pi &= -8\,aN_1{}^4\Theta_1\sigma_1{}^2 + 8\,H_1{}^2N_1{}^2\Theta_1\Theta_2\sigma_1{}^2 \\ &\quad + 48\,H_1{}^2N_1{}^2\Theta_1\sigma_1{}^2\theta_1 - 16\,H_1H_2N_0N_1\Theta_1\Theta_2\sigma_1{}^2 \\ &\quad - 96\,H_1H_2N_0N_1\Theta_1\sigma_1{}^2\theta_1 + 8\,H_2{}^2N_0{}^2\Theta_1\Theta_2\sigma_1{}^2 \\ &\quad + 48\,H_2{}^2N_0{}^2\Theta_1\sigma_1{}^2\theta_1 + 8\,N_1{}^4\Theta_1\Theta_2\sigma_1 \\ &\quad + 24\,N_1{}^4\Theta_1\sigma_1\theta_1 + \lambda_1{}^2\sigma_1{}^2H_2{}^2, \end{aligned} \tag{42}$$

where H_0, ϖ_0 are free arbitraries. This solution is valid for the following case:

$$\begin{aligned} &-8\,N_1{}^4\Theta_1\sigma_1(a\sigma_1 - \Theta_2 - 3\,\theta_1) \\ &\quad + 8\,\Theta_1\sigma_1{}^2(H_1N_1 - H_2N_0)^2(\Theta_2 + 6\,\theta_1) \\ &\quad + \lambda_1{}^2\sigma_1{}^2H_2{}^2 > 0. \end{aligned} \tag{43}$$

Product ID:
By utilizing (17), we obtain

$$\begin{aligned} q_4(x,t) = &\left\{\frac{\left(N_0(H_1N_1 - H_2N_0)/N_1{}^2\right) + \left(H_1/\left(\rho_2^2(x + (\alpha_1 + 2ak)t + \varpi_0)\right)/(2\rho_2(x + (\alpha_1 + 2ak)t + \varpi_0)) + 4)\right) + \left(H_2/\left(\rho_2^2(x + (\alpha_1 + 2ak)t + \varpi_0)\right)/(2\rho_2(x + (\alpha_1 + 2ak)t + \varpi_0)) + 4)^2\right)}{N_0 + \left(N_1/\left(\rho_2^2(x + (\alpha_1 + 2ak)t + \varpi_0)\right)/(2\rho_2(x + (\alpha_1 + 2ak)t + \varpi_0)) + 4)\right)}\right\} \\ &\cdot e^{i\left(-kx + \frac{\Lambda_1 + \Lambda_2 + \Lambda_3}{2N_1{}^2\sigma_1{}^2(\Theta_2 + 6\theta_1)^2H_2{}^2}t\right)}, \end{aligned} \tag{44}$$

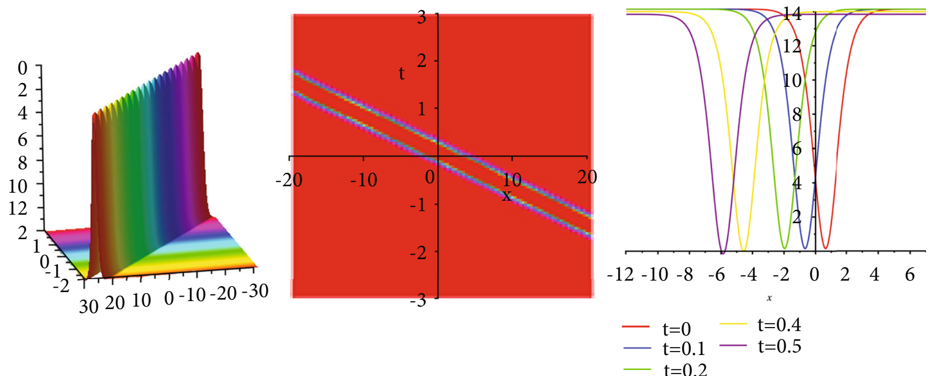
in which

$$k = \frac{1}{2}\,\frac{\lambda_1\sigma_1H_2 + \sqrt{-8\,aN_1{}^4\Theta_1\sigma_1{}^2 + 8\,N_1{}^4\Theta_1\Theta_2\sigma_1 + 24\,N_1{}^4\Theta_1\sigma_1\theta_1 + \lambda_1{}^2\sigma_1{}^2H_2{}^2}}{\Theta_1\sigma_1H_2}, \tag{45}$$

where H_0, ϖ_0 are free arbitraries. This solution is valid for the following case:

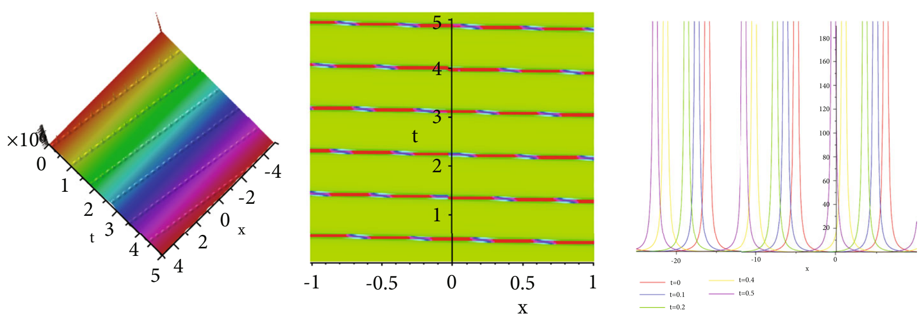
$$\begin{aligned} &-8\,aN_1{}^4\Theta_1\sigma_1{}^2 + 8\,N_1{}^4\Theta_1\Theta_2\sigma_1 \\ &\quad + 24\,N_1{}^4\Theta_1\sigma_1\theta_1 + \lambda_1{}^2\sigma_1{}^2H_2{}^2 > 0. \end{aligned} \tag{46}$$

Inserting $\theta_1 = \theta_2 = \theta_3 = N_0 = N_1 = 1, H_1 = 3, b = 5, \alpha_1 = 1$ $, a = 2, k = 3, b = 1, \sigma_1 = -1, \lambda_1 = 3$, the solution $u(x,t)$ given by (37) shows the soliton wave solution. Moreover, the solution finds to the soliton solution as suggested in Figures 1(a)–1(c). Likewise, by taking $\theta_1 = \theta_2 = \theta_3 = N_0 = N_1 = 1, H_1 = 3, b = 5, \alpha_1 = 1, a = 2, k = 3, b = 1, \sigma_1 = -1, \lambda_1 = 10$, the solution $u(x,t)$ given by (38) denotes the periodic wave solution. Moreover, the solution transforms to the periodic solution as mentioned in Figures 2(a)–2(c). Also,



(a) (b) (c)

FIGURE 1: Graph of (37) by taking $\theta_1 = 1, \theta_2 = 1, \theta_3 = 1, N_0 = 1, N_1 = 1, H_1 = 3, b = 5, \alpha_1 = 1, a = 2, k = 3, b = 1, \sigma_1 = -1, \lambda_1 = 3$: (a) 3D graph, (b) density plot, and (c) 2D plot.



(a) (b) (c)

FIGURE 2: Graph of (38) by taking $\theta_1 = 1, \theta_2 = 1, \theta_3 = 1, N_0 = 1, N_1 = 1, H_1 = 3, b = 5, \alpha_1 = 1, a = 2, k = 3, b = 1, \sigma_1 = -1, \lambda_1 = 10$: (a) 3D graph, (b) density plot, and (c) 2D plot.

by plugging $\theta_1 = \theta_2 = \theta_3 = N_0 = N_1 = 1, H_1 = 3, b = 5, \alpha_1 = 1, a = 2, k = 3, b = 1, \sigma_1 = -1, \lambda_1 = 10$, the solution $u(x, t)$ given by (38) shows the kink-singular wave solution. Moreover, the solution finds to the soliton solution as found in Figures 3(a)–3(c). Finally, by putting $\theta_1 = \theta_2 = \theta_3 = N_0 = N_1 = 1, H_1 = 3, b = 5, \alpha_1 = 1, a = 2, k = 3, b = 1, \sigma_1 = -1, \lambda_1 = 3$, the solution $u(x, t)$ given by (38) finds the cupson-singular wave solution. Moreover, the solution finds to the rational-singular solution as offered in Figures 4(a)–4(c).

Set II:

$$\begin{aligned}
&H_0 = -\frac{aN_1^{\,2}}{H_2(2\Theta_2 - 3\theta_1)}, H_1 = 0, H_2 = \frac{\sqrt{-2b\sigma_1(\Theta_2 + 3\theta_1)}N_1}{b\sigma_1}, N_0 = 0, N_1 = N_1,\\
&\rho_2 = 0, \rho_1 = -\frac{aN_1^{\,2}}{(2\Theta_2 - 3\theta_1)H_2^{\,2}}, \rho_2^2 - 4\rho_1 = \frac{4aN_1^{\,2}}{(2\Theta_2 - 3\theta_1)H_2^{\,2}},\\
&k = \frac{2H_2\Theta_2\lambda_1\sigma_1 - 3H_2\lambda_1\sigma_1\theta_1 + \sqrt{4H_2^{\,2}\Theta_2^{\,2}\lambda_1^{\,2}\sigma_1^{\,2} - 12H_2^{\,2}\Theta_2\lambda_1^{\,2}\sigma_1^{\,2}\theta_1 + 9H_2^{\,2}\lambda_1^{\,2}\sigma_1^{\,2}\theta_1^{\,2} - 8\Lambda_1\Theta_1\Theta_2\sigma_1 + 12\Lambda_1\Theta_1\sigma_1\theta_1}}{2\Theta_1\sigma_1(2\Theta_2 - 3\theta_1)},\\
&\Lambda_1 = 8aN_1^{\,2}\Theta_2\sigma_1 + 18aN_1^{\,2}\sigma_1\theta_1 - 4N_1^{\,2}\Theta_2^{\,2} - 6N_1^{\,2}\Theta_2\theta_1 + 18N_1^{\,2}\theta_1^{\,2},\\
&\omega = -\frac{\Lambda_2 + \Lambda_3}{H_2^{\,2}\Theta_1\sigma_1(2\Theta_2 - 3\theta_1)}, \Lambda_2 = 2\Theta_2\left(kH_2^{\,2}\Theta_1\alpha_1\sigma_1 + 2aN_1^{\,2}\Theta_2\right) - \Theta_1\sigma_1\left(3kH_2^{\,2}\alpha_1\theta_1 + 4a^2N_1^{\,2}\right),\\
&\Lambda_3 = -2a\Theta_2\left(-kH_2^{\,2}\lambda_1\sigma_1 + 4aN_1^{\,2}\sigma_1 - 3B_1^{\,2}\theta_1\right) - 3a\theta_1\left(kH_2^{\,2}\lambda_1\sigma_1 + 6aN_1^{\,2}\sigma_1 + 6N_1^{\,2}\theta_1\right),
\end{aligned} \tag{47}$$

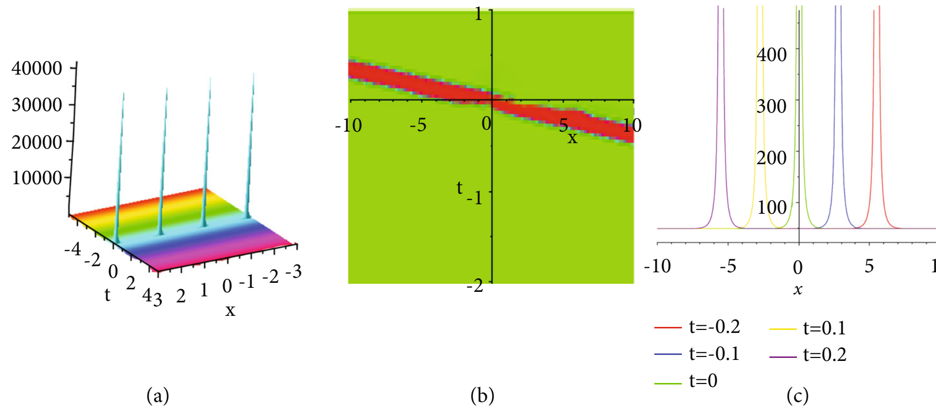


FIGURE 3: Graph of (39) by taking $\theta_1=1, \theta_2=1, \theta_3=1, N_0=1, N_1=1, H_1=3, b=5, \alpha_1=1, a=2, k=3, b=1, \sigma_1=-1, \lambda_1=10$: (a) 3D graph, (b) density plot, and (c) 2D plot.

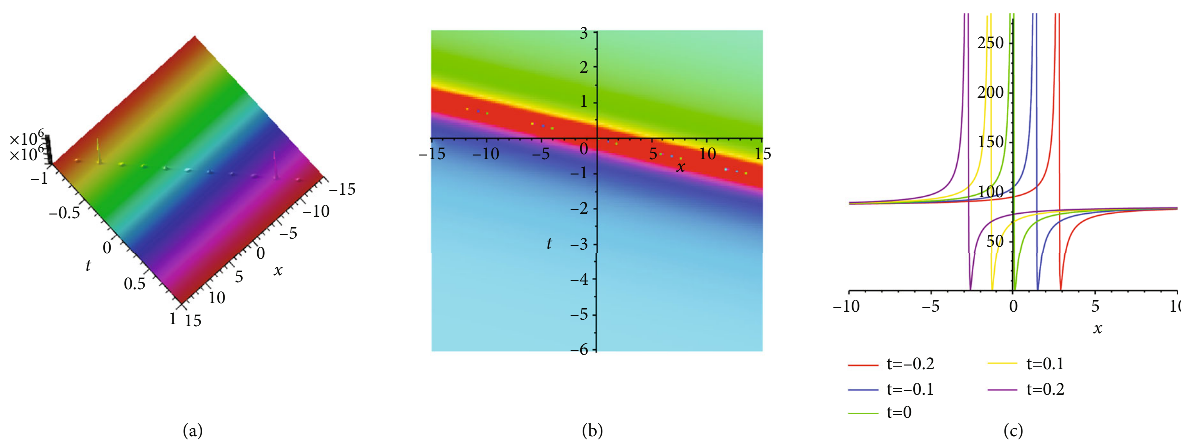


FIGURE 4: Graph of (60) by taking $\theta_1=1, \theta_2=1, \theta_3=1, N_0=1, N_1=1, H_1=3, b=5, \alpha_1=1, a=2, k=3, b=1, \sigma_1=-1, \lambda_1=10$: (a) 3D graph, (b) density plot, and (c) 2D plot.

where H_0, N_0, and N_1 are arbitrary amounts. Imposing the solution set (36) into (35), we get the following.

Product IIA:

By using (14), one gains

$$q_5(x,t)=\left\{\frac{\sqrt{-2b\sigma_1(\Theta_2+3\theta_1)}}{b\sigma_1}\sqrt{\frac{aN_1^2}{(2\Theta_2-3\theta_1)H_2^2}}\coth\left(\sqrt{\frac{aN_1^2}{(2\Theta_2-3\theta_1)H_2^2}}(x+(\alpha_1+2ak)t+\varpi_0)\right)-\frac{aN_1}{\sqrt{(aN_1^2/(2\Theta_2-3\theta_1)H_2^2)}H_2(2\Theta_2-3\theta_1)}\tanh\left(\sqrt{\frac{aN_1^2}{(2\Theta_2-3\theta_1)H_2^2}}(x+(\alpha_1+2ak)t+\varpi_0)\right)\right\}\cdot e^{i\left(-kx-\frac{\Lambda_2+\Lambda_3}{H_2^2\Theta_1\sigma_1(2\Theta_2-3\theta_1)}t\right)}, \quad (48)$$

where N_0, N_1, ϖ_0 are free values. This solution is valid for the following case:

$$a(3\theta_1+2\theta_2+2\theta_3)>0. \quad (49)$$

Product IIB:

By utilizing (15), we get

$$q_6(x,t)=\left\{\frac{\sqrt{-2b\sigma_1(\Theta_2+3\theta_1)}}{b\sigma_1}\sqrt{-\frac{aN_1^2}{(2\Theta_2-3\theta_1)H_2^2}}\cot\left(\sqrt{-\frac{aN_1^2}{(2\Theta_2-3\theta_1)H_2^2}}(x+(\alpha_1+2ak)t+\varpi_0)\right)-\frac{aN_1}{\sqrt{-(aN_1^2/(2\Theta_2-3\theta_1)H_2^2)}H_2(2\Theta_2-3\theta_1)}\tan\left(\sqrt{-\frac{aN_1^2}{(2\Theta_2-3\theta_1)H_2^2}}(x+(\alpha_1+2ak)t+\varpi_0)\right)\right\}\cdot e^{i\left(-kx-\frac{\Lambda_2+\Lambda_3}{H_2^2\Theta_1\sigma_1(2\Theta_2-3\theta_1)}t\right)}, \quad (50)$$

where N_0, N_1, ϖ_0 are free values. This solution is valid for the following case:

$$a(3\theta_1 + 2\theta_2 + 2\theta_3) < 0. \quad (51)$$

Set III:

$$\begin{aligned}
H_1 &= 2\frac{\sqrt{-2\,\sigma_1(\Theta_2 + 6\,\theta_1)\left(k^2H_2{}^2\Theta_1\sigma_1 - kH_2{}^2\lambda_1\sigma_1 + 2\,aN_1{}^2\sigma_1 - 8\,A_0H_2\Theta_2\sigma_1 - 48\,H_0H_2\sigma_1\theta_1 - 2\,N_1{}^2\Theta_2 - 6\,N_1{}^2\theta_1\right)}}{4\sigma_1(\Theta_2 + 6\,\theta_1)},\\
H_0 &= H_0, H_2 = \frac{\sqrt{-2\,b\sigma_1(\Theta_2 + 3\,\theta_1)}N_1}{b\sigma_1}, N_0 = \frac{H_1N_1}{2A_2}, N_1 = N_1, k = k,\\
\rho_2 &= \frac{H_1}{H_2}, \rho_1 = -\frac{k^2H_2{}^2\Theta_1\sigma_1 - kH_2{}^2\lambda_1\sigma_1 + 2\,aN_1{}^2\sigma_1 - 4\,H_0H_2\Theta_2\sigma_1 - 24\,H_0H_2\sigma_1\theta_1 - 2\,N_1{}^2\Theta_2 - 6\,N_1{}^2\theta_1}{4H_2{}^2\sigma_1(\Theta_2 + 6\,\theta_1)},\\
\rho_2^2 - 4\rho_1 &= \frac{k^2H_2{}^2\Theta_1\sigma_1 - kH_2{}^2\lambda_1\sigma_1 + 2\,aN_1{}^2\sigma_1 - 2\,N_1{}^2\Theta_2 - 6\,N_1{}^2\theta_1}{2H_2{}^2\sigma_1(\Theta_2 + 6\,\theta_1)}, \ \omega = -\frac{\Lambda_1 + \Lambda_2 + \Lambda_3}{2N_1^2\sigma_1^2(\Theta_2 + 6\theta_1)^2H_2^2},\\
\Lambda_1 &= -3\,k^4H_2{}^4\sigma_1{}^2\theta_1\Theta_1{}^2 + 72\,ak^2H_2{}^2N_1{}^2\sigma_1{}^2\theta_1{}^2 - 3\,k^2H_2{}^4\lambda_1{}^2\sigma_1{}^2\theta_1 + 72\,kH_2{}^2N_1{}^2\alpha_1\sigma_1{}^2\theta_1{}^2 + 12\,a^2N_1{}^4\sigma_1{}^2\theta_1\\
&\quad - 36\,kH_2{}^2N_1{}^2\lambda_1\sigma_1\theta_1{}^2 - 108\,N_1{}^4\theta_1{}^3,\\
\Lambda_2 &= 2\,N_1{}^2\Theta_2\left(12\,ak^2H_2{}^2\sigma_1{}^2\theta_1 - akH_2{}^2\lambda_1\sigma_1{}^2 + 12\,kH_2{}^2\alpha_1\sigma_1{}^2\theta_1 + 2\,a^2N_1{}^2\sigma_1{}^2 - 6\,kH_2{}^2\lambda_1\sigma_1\theta_1 - 6\,aN_1{}^2\sigma_1\theta_1 - 36\,N_1{}^2\theta_1{}^2\right),\\
\Lambda_3 &= 6\,k^2H_2{}^2\Theta_1\sigma_1\theta_1\left(kH_2{}^2\lambda_1\sigma_1 + 6\,N_1{}^2\theta_1\right) + 2\,N_1{}^2\Theta_2{}^2\left(ak^2H_2{}^2\sigma_1{}^2 + kH_2{}^2\alpha_1\sigma_1{}^2 - 2\,aN_1{}^2\sigma_1 - 6\,N_1{}^2\theta_1\right)\\
&\quad + 2\,k^2H_2{}^2N_1{}^2\Theta_1\Theta_2\sigma_1(a\sigma_1 + 6\,\theta_1),
\end{aligned} \quad (52)$$

where H_0, N_0, and N_1 are free values. Imposing the solution set (36) into (35), one gains the following.

Product IIIA:
By using (14), we gain

$$q_7(x,t) = \left\{\frac{H_0 - \left(H_1/\left(\left(\sqrt{\rho_2^2 - 4\rho_1}/2\rho_1\right)\tanh\left(\left(\sqrt{\rho_2^2 - 4\rho_1}/2\right)(\varpi + \varpi_0)\right) + (\rho_2/2\rho_1)\right)\right) + \left(\left(\sqrt{-2\,b\sigma_1(\Theta_2 + 3\,\theta_1)}N_1/b\sigma_1\right)/\left(\left(\sqrt{\rho_2^2 - 4\rho_1}/2\rho_1\right)\tanh\left(\left(\sqrt{\rho_2^2 - 4\rho_1}/2\right)(\varpi + \varpi_0)\right) + (\rho_2/2\rho_1)\right)^2\right)}{(H_1N_1/2H_2) - \left(N_1/\left(\sqrt{\rho_2^2 - 4\rho_1}/2\rho_1\right)\tanh\left(\left(\sqrt{\rho_2^2 - 4\rho_1}/2\right)(\varpi + \varpi_0)\right) + (\rho_2/2\rho_1)\right)}\right\} \times e^{i\left(-kx - \frac{\Lambda_1 + \Lambda_2 + \Lambda_3}{2N_1^2\sigma_1^2(\Theta_2 + 6\theta_1)^2H_2^2}t\right)}, \quad (53)$$

where N_0, N_1, ϖ_0 are free amounts. This solution is valid for the following case:

$$\left(k^2H_2{}^2\Theta_1\sigma_1 - kH_2{}^2\lambda_1\sigma_1 + 2\,aN_1{}^2\sigma_1 - 2\,N_1{}^2\Theta_2 - 6\,N_1{}^2\theta_1\right)\sigma_1(9\theta_1 + \theta_2 + \theta_3) > 0. \quad (54)$$

Product IIIB:

Via utilizing (15), we get

$$q_8(x,t) = \left\{\frac{H_0 + \left(H_1/\left(\left(\sqrt{\rho_2^2 - 4\rho_1}/2\rho_1\right)\tan\left(\left(\sqrt{\rho_2^2 - 4\rho_1}/2\right)(\varpi + \varpi_0)\right) - (\rho_2/2\rho_1)\right)\right) + \left(\sqrt{-2\,b\sigma_1(\Theta_2 + 3\,\theta_1)}N_1/b\sigma_1\right)/\left(\left(\sqrt{\rho_2^2 - 4\rho_1}/2\rho_1\right)\tan\left(\left(\sqrt{\rho_2^2 - 4\rho_1}/2\right)(\varpi + \varpi_0)\right) - (\rho_2/2\rho_1)\right)^2}{(H_1N_1/2H_2) + \left(N_1/\left(\left(\sqrt{\rho_2^2 - 4\rho_1}/2\rho_1\right)\tan\left(\left(\sqrt{\rho_2^2 - 4\rho_1}/2\right)(\varpi + \varpi_0)\right) - (\rho_2/2\rho_1)\right)\right)}\right\} \times e^{i\left(-kx - \frac{\Lambda_1 + \Lambda_2 + \Lambda_3}{2N_1^2\sigma_1^2(\Theta_2 + 6\theta_1)^2H_2^2}t\right)}, \quad (55)$$

where N_0, N_1, ϖ_0 are free values. This solution is valid for the following case:

$$\left(k^2 H_2{}^2 \Theta_1 \sigma_1 - k H_2{}^2 \lambda_1 \sigma_1 + 2\,a N_1{}^2 \sigma_1 - 2\,N_1{}^2 \Theta_2 - 6\,N_1{}^2 \theta_1\right) \sigma_1 (9\theta_1 + \theta_2 + \theta_3) < 0. \tag{56}$$

Product IIIC:

By the help of (16), we get

$$q_9(x,t) = \left\{ \frac{H_0 + (H_1/(\rho_2/(\exp(\rho_2(x+(\alpha_1+2ak)t+\varpi_0))-1))) + \left(\left(\sqrt{-2\,b\sigma_1(\Theta_2+3\,\theta_1)}N_1/b\sigma_1\right)/(\rho_2/(\exp(\rho_2(x+(\alpha_1+2ak)t+\varpi_0))-1))^2\right)}{(H_1 N_1/2H_2) + (N_1/(\rho_2/\exp(\rho_2(x+(\alpha_1+2ak)t+\varpi_0))-1))} \right\} \cdot e^{i\left(-kx - \frac{\Lambda_1+\Lambda_2+\Lambda_3}{2B_1^2\sigma_1^2(\Theta_2+6\theta_1)^2 H_2^2} t\right)}, \tag{57}$$

in which

$$k = \frac{H_2{}^2 \lambda_1 \sigma_1 + \sqrt{-H_2{}^2 \sigma_1 \left(8\,a N_1{}^2 \Theta_1 \sigma_1 - 16\,H_0 H_2 \Theta_1 \Theta_2 \sigma_1 - 96\,H_0 H_2 \Theta_1 \sigma_1 \theta_1 - H_2{}^2 \lambda_1{}^2 \sigma_1 - 8\,N_1{}^2 \Theta_1 \Theta_2 - 24\,N_1{}^2 \Theta_1 \theta_1\right)}}{H_2{}^2 \Theta_1 \sigma_1}, \tag{58}$$

where H_0, ϖ_0 are free amounts. This solution is valid for the following case:

$$\frac{H_0 N_1 \sqrt{-2\,b\sigma_1(\Theta_2 + 3\,\theta_1)}}{b\sigma_1} > 0. \tag{59}$$

Product IIID:

Via using (17), one obtains

$$q_{10}(x,t) = \left\{ \frac{H_0 + \left(H_1/\left(\rho_2^2(x+(\alpha_1+2ak)t+\varpi_0)\right)/(2\rho_2(x+(\alpha_1+2ak)t+\varpi_0))+4)\right)\right) + \left(\left(\sqrt{-2\,b\sigma_1(\Theta_2+3\,\theta_1)}N_1/b\sigma_1\right)/\left(\rho_2^2(x+(\alpha_1+2ak)t+\varpi_0))/(2\rho_2(x+(\alpha_1+2ak)t+\varpi_0))+4)\right)^2\right)}{(H_1 N_1/2H_2) + \left(\left(N_1/\left(\rho_2^2(x+(\alpha_1+2ak)t+\varpi_0))/(2\rho_2(x+(\alpha_1+2ak)t+\varpi_0))+4)\right)\right)\right)} \right\} \cdot e^{i\left(-kx - \frac{\Lambda_1+\Lambda_2+\Lambda_3}{2N_1^2\sigma_1^2(\Theta_2+6\theta_1)^2 H_2^2} t\right)}, \tag{60}$$

in which

$$k = \frac{\lambda_1 \sigma_1 H_2 + \sqrt{-8\,a N_1{}^2 \Theta_1 \sigma_1{}^2 + \lambda_1{}^2 \sigma_1{}^2 H_2{}^2 + 8\,N_1{}^2 \Theta_1 \Theta_2 \sigma_1 + 24\,N_1{}^2 \Theta_1 \sigma_1 \theta_1}}{2\Theta_1 \sigma_1 H_2}, \tag{61}$$

where H_0, ϖ_0 are free amounts. This solution is valid for the following case:

$$24N_1{}^2\Theta_1\sigma_1\theta_1 - 8\,aN_1{}^2\Theta_1\sigma_1{}^2 + \lambda_1{}^2\sigma_1{}^2H_2{}^2 + 8\,N_1{}^2\Theta_1\Theta_2\sigma_1 > 0,\, b\sigma_1(9\theta_1 + \theta_2 + \theta_3) < 0. \tag{62}$$

Product IV:

$$\begin{gathered}
H_1 = \frac{\sqrt{-\sigma_1(\Theta_2 + 6\,\theta_1)\left(k^2H_2{}^2\Theta_1\sigma_1 - kH_2{}^2\lambda_1\sigma_1 + 2\,N_1{}^2\sigma_1 a - 4\,H_0H_2\Theta_2\sigma_1 - 24\,H_0H_2\sigma_1\theta_1 - 2\,N_1{}^2\Theta_2 - 6\,N_1{}^2\theta_1\right)}}{\sigma_1(\Theta_2 + 6\,\theta_1)},\\
H_0 = H_0, H_2 = \frac{\sqrt{-2\,b\sigma_1(\Theta_2 + 3\,\theta_1)}N_1}{b\sigma_1}, N_0 = \frac{H_1N_1}{2H_2}, N_1 = N_1, k = k,\\
\rho_2 = \frac{H_1}{H_2}, \rho_1 = -\frac{k^2H_2{}^2\Theta_1\sigma_1 - kH_2{}^2\lambda_1\sigma_1 + 2\,aN_1{}^2\sigma_1 - 4\,H_0H_2\Theta_2\sigma_1 - 24\,H_0H_2\sigma_1\theta_1 - 2\,N_1{}^2\Theta_2 - 6\,N_1{}^2\theta_1}{4H_2{}^2\sigma_1(\Theta_2 + 6\,\theta_1)},\\
\rho_2^2 - 4\rho_1 = \frac{H_1^2 - 4H_0H_2}{H_2^2}, \Lambda_1 = ak^2H_2{}^2\sigma_1 + kH_2{}^2\alpha_1\sigma_1 - 2\,aN_1{}^2,\\
\omega = -\frac{ak^2H_2{}^2\sigma_1\Theta_1 + \Theta_2\Lambda_1 + 6\,\theta_1\left(ak^2H_2{}^2\sigma_1 + kH_2{}^2\alpha_1\sigma_1 - aN_1{}^2\right) + a\sigma_1\left(-kH_2{}^2\lambda_1 + 2\,aN_1{}^2\right)}{\sigma_1H_2{}^2(\Theta_2 + 6\,\theta_1)},
\end{gathered} \tag{63}$$

where H_0, N_0, and N_1 are free amounts. Imposing the solution set (36) into (35), we conclude the following.

Product IVA:
Via utilizing (14), one gets

$$q_{11}(x,t) = \left\{\frac{H_0 - \left(H_1/\left(\left(\sqrt{\rho_2^2 - 4\rho_1}/2\rho_1\right)\tanh\left(\left(\sqrt{\rho_2^2 - 4\rho_1}/2\right)(\varpi + \varpi_0)\right) + (\rho_2/2\rho_1)\right)\right) + \left(\left(\sqrt{-2\,b\sigma_1(\Theta_2 + 3\,\theta_1)}N_1/b\sigma_1\right)/\left(\left(\sqrt{\rho_2^2 - 4\rho_1}/2\rho_1\right)\tanh\left(\left(\sqrt{\rho_2^2 - 4\rho_1}/2\right)(\varpi + \varpi_0)\right) + (\rho_2/2\rho_1)\right)^2\right)}{(H_1N_1/2H_2) - \left(N_1/\left(\left(\sqrt{\rho_2^2 - 4\rho_1}/2\rho_1\right)\tanh\left(\left(\sqrt{\rho_2^2 - 4\rho_1}/2\right)(\varpi + \varpi_0)\right) + (\rho_2/2\rho_1)\right)\right)}\right\} \times e^{i(-kx+\omega t)}, \tag{64}$$

where N_0, N_1, ϖ_0 are free arbitraries. This solution is valid for the following case:

$$H_1^2 - 4H_0H_2 > 0. \tag{65}$$

Product IVB:
Via utilizing (15), one concludes

$$q_{12}(x,t) = \left\{\frac{H_0 + \left(H_1/\left(\left(\sqrt{-\rho_2^2 + 4\rho_1}/2\rho_1\right)\tan\left(\left(\sqrt{-\rho_2^2 + 4\rho_1}/2\right)(\varpi + \varpi_0)\right) - (\rho_2/2\rho_1)\right)\right) + \left(\left(\sqrt{-2\,b\sigma_1(\Theta_2 + 3\,\theta_1)}N_1/b\sigma_1\right)/\left(\left(\sqrt{-\rho_2^2 + 4\rho_1}/2\rho_1\right)\tan\left(\left(\sqrt{-\rho_2^2 + 4\rho_1}/2\right)(\varpi + \varpi_0)\right) - (\rho_2/2\rho_1)\right)^2\right)}{(H_1N_1/2H_2) + \left(N_1/\left(\left(\sqrt{-\rho_2^2 + 4\rho_1}/2\rho_1\right)\tan\left(\left(\sqrt{-\rho_2^2 + 4\rho_1}/2\right)(\varpi + \varpi_0)\right) - (\rho_2/2\rho_1)\right)\right)}\right\} \times e^{i(-kx+\omega t)}, \tag{66}$$

where N_0, N_1, ϖ_0 are free amounts. This solution is valid for the following case:

$$H_1^2 - 4H_0H_2 < 0. \tag{67}$$

Product IVC:
Via utilizing (16), we obtain

$$q_{12}(x,t) = \left\{ \frac{H_0 + (H_1/(\rho_2/(\exp(\rho_2(x+(\alpha_1+2ak)t+\varpi_0))-1))) + \left(\left(\sqrt{-2b\sigma_1(\Theta_2+3\theta_1)}N_1/b\sigma_1\right)/((\rho_2/(\exp(\rho_2(x+(\alpha_1+2ak)t+\varpi_0))-1)))^2\right)}{(H_1N_1/2A_2) + (N_1/(\rho_2/(\exp(\rho_2(x+(\alpha_1+2ak)t+\varpi_0))-1)))} \right\} \times e^{i(-kx+\omega t)}, \tag{68}$$

in which

$$k = \frac{H_2{}^2\lambda_1\sigma_1 + \sqrt{-H_2{}^2\sigma_1\left(8aN_1{}^2\Theta_1\sigma_1 - 16H_0H_2\Theta_1\Theta_2\sigma_1 - 96H_0H_2\Theta_1\sigma_1\theta_1 - H_2{}^2\lambda_1{}^2\sigma_1 - 8N_1{}^2\Theta_1\Theta_2 - 24N_1{}^2\Theta_1\theta_1\right)}}{H_2{}^2\Theta_1\sigma_1}, \tag{69}$$

where H_0, ϖ_0 are free amounts. This solution is valid for the following case:

$$\frac{H_0H_1\sqrt{-2b\sigma_1(\Theta_2+3\theta_1)}}{b\sigma_1} > 0. \tag{70}$$

Product IVD:
Via employing (17), one obtains

$$q_{13}(x,t) = \left\{ \frac{H_0 + \left(H_1/\left(\rho_2^2(x+(\alpha_1+2ak)t+\varpi_0)\right)\right)/(2\rho_2(x+(\alpha_1+2ak)t+\varpi_0))+4))) + \left(\left(\sqrt{-2b\sigma_1(\Theta_2+3\theta_1)}N_1/b\sigma_1\right)/\left(\rho_2^2(x+(\alpha_1+2ak)t+\varpi_0))/(2\rho_2(x+(\alpha_1+2ak)t+\varpi_0))+4)\right)^2\right)}{(H_1N_1/2H_2) + \left(N_1/\left(\lambda^2(x+(\alpha_1+2ak)t+\varpi_0))/2\rho_2(x+(\alpha_1+2ak)t+\varpi_0))+4\right)\right)} \right\} \times e^{i(-kx+\omega t)}, \tag{71}$$

in which

$$k = \frac{\lambda_1\sigma_1H_2 + \sqrt{-8aN_1{}^2\Theta_1\sigma_1{}^2 + \lambda_1{}^2\sigma_1{}^2H_2{}^2 + 8N_1{}^2\Theta_1\Theta_2\sigma_1 + 24N_1{}^2\Theta_1\sigma_1\theta_1}}{2\Theta_1\sigma_1H_2}, \tag{72}$$

where H_0, E are free amounts. This solution is valid for the following case:

$$24N_1{}^2\Theta_1\sigma_1\theta_1 - 8aN_1{}^2\Theta_1\sigma_1{}^2 + \lambda_1{}^2\sigma_1{}^2H_2{}^2 + 8N_1{}^2\Theta_1\Theta_2\sigma_1 > 0,\ b\sigma_1(9\theta_1+\theta_2+\theta_3) < 0. \tag{73}$$

Set V:

$$
\begin{aligned}
H_1 &= \frac{\sqrt{-\left(6kH_2{}^2\sigma_1(2\Theta_2-3\theta_1)(k\Theta_1-\lambda_1)+12N_1{}^2\left(4a\Theta_2\sigma_1+9a\sigma_1\theta_1-2\Theta_2{}^2-3\Theta_2\theta_1+9\theta_1{}^2\right)\right)\sigma_1\left(\Theta_2{}^2-36\theta_1{}^2\right)}}{6\sigma_1\left(\Theta_2{}^2-36\theta_1{}^2\right)},\\
H_0 &= \frac{kH_2{}^2\sigma_1(4\Theta_2-33\theta_1)(k\Theta_1-\lambda_1)+2N_1{}^2\left(2a\Theta_2\sigma_1-45a\sigma_1\theta_1-4\Theta_2{}^2+21\Theta_2\theta_1+99\theta_1{}^2\right)}{24H_2\sigma_1\left(\Theta_2{}^2-36\theta_1{}^2\right)},\\
H_2 &= \frac{\sqrt{-2b\sigma_1(\Theta_2+3\theta_1)}N_1}{b\sigma_1}, N_0=\frac{H_1N_1}{2H_2}, N_1=N_1, k=k,\\
\rho_2 &= \frac{H_1}{H_2}, \rho_1=\frac{kH_2{}^2\sigma_1(4\Theta_2-33\theta_1)(k\Theta_1-\lambda_1)+2N_1{}^2\left(2a\Theta_2\sigma_1-45a\sigma_1\theta_1-4\Theta_2{}^2+21\Theta_2\theta_1+99\theta_1{}^2\right)}{24(\Theta_2+6\theta_1)\sigma_1H_2{}^2(\Theta_2-6\theta_1)},\\
\rho_2^2-4\rho_1 &= -\frac{k^2H_2{}^2\Theta_1\sigma_1-kH_2{}^2\lambda_1\sigma_1+2N_1{}^2\sigma_1a-2N_1{}^2\Theta_2-6N_1{}^2\theta_1}{(\Theta_2+6\theta_1)\sigma_1H_2{}^2},\\
\omega &= -\frac{kH_2{}^2\sigma_1(ak\Theta_1+ak\Theta_2+6ak\theta_1-a\lambda_1+\Theta_2\alpha_1+6\alpha_1\theta_1)+2aN_1{}^2(a\sigma_1-\Theta_2-3\theta_1)}{\sigma_1H_2{}^2(\Theta_2+6\theta_1)},
\end{aligned}
\tag{74}
$$

where H_0, N_0, and N_1 are free values. Imposing the solution set (36) into (35), we conclude the following.

Product VA:
Via using (14), one gains

$$
q_{14}(x,t)=\left\{\frac{H_0-\left(H_1/\left(\left(\sqrt{\rho_2^2-4\rho_1}/2\rho_1\right)\tanh\left(\left(\sqrt{\rho_2^2-4\rho_1}/2\right)(\varpi+\varpi_0)\right)+(\rho_2/2\rho_1)\right)\right)+\left(\left(\sqrt{-2b\sigma_1(\Theta_2+3\theta_1)}N_1/b\sigma_1\right)/\left(\left(\sqrt{\rho_2^2-4\rho_1}/2\rho_1\right)\tanh\left(\left(\sqrt{\rho_2^2-4\rho_1}/2\right)(\varpi+\varpi_0)\right)+(\rho_2/2\rho_1)\right)^2\right)}{(H_1N_1/2H_2)-\left(N_1/\left(\left(\sqrt{\rho_2^2-4\rho_1}/2\rho_1\right)\tanh\left(\left(\sqrt{\rho_2^2-4\rho_1}/2\right)(\varpi+\varpi_0)\right)+(\rho_2/2\rho_1)\right)\right)}\right\}\times e^{i(-kx+\omega t)},
\tag{75}
$$

where N_0, N_1, ϖ_0 are free values. This solution is valid for the following case:

$$
\left(k^2H_2{}^2\Theta_1\sigma_1-kH_2{}^2\lambda_1\sigma_1+2B_1{}^2\sigma_1a-2N_1{}^2\Theta_2-6N_1{}^2\theta_1\right)(\Theta_2+6\theta_1)\sigma_1<0.
\tag{76}
$$

Product VB:
Via utilizing (15), we get

$$
q_{15}(x,t)=\left\{\frac{H_0+\left(H_1/\left(\left(\sqrt{-\rho_2^2+4\rho_1}/2\rho_1\right)\tan\left(\left(\sqrt{-\rho_2^2+4\rho_1}/2\right)(\varpi+\varpi_0)\right)-(\rho_2/2\rho_1)\right)\right)+\left(\left(\sqrt{-2b\sigma_1(\Theta_2+3\theta_1)}N_1/b\sigma_1\right)/\left(\left(\sqrt{-\rho_2^2+4\rho_1}/2\rho_1\right)\tan\left(\left(\sqrt{-\rho_2^2+4\rho_1}/2\right)(\varpi+\varpi_0)\right)-(\rho_2/2\rho_1)\right)^2\right)}{(H_1N_1/2H_2)+\left(N_1/\left(\left(\sqrt{-\rho_2^2+4\rho_1}/2\rho_1\right)\tan\left(\left(\sqrt{-\rho_2^2+4\rho_1}/2\right)(\varpi+\varpi_0)\right)-(\rho_2/2\rho_1)\right)\right)}\right\}
\times e^{i(-kx+\omega t)},
\tag{77}
$$

where N_0, N_1, ϖ_0 are free arbitraries. This solution is valid for the following case:

$$\left(k^2 H_2{}^2 \Theta_1 \sigma_1 - k H_2{}^2 \lambda_1 \sigma_1 + 2 B_1{}^2 \sigma_1 a - 2 N_1{}^2 \Theta_2 - 6 N_1{}^2 \theta_1\right)(\Theta_2 + 6\theta_1)\sigma_1 > 0. \quad (78)$$

Product VC:

Via employing (16), one obtains

$$q_{16}(x,t) = \left\{ \frac{H_0 + (H_1/(\rho_2/(\exp(\rho_2(x + (\alpha_1 + 2ak)t + \varpi_0)) - 1))) + \left(\left(\sqrt{-2 b\sigma_1(\Theta_2 + 3\theta_1)} N_1/b\sigma_1\right)/(\rho_2/(\exp(\rho_2(x + (\alpha_1 + 2ak)t + \varpi_0)) - 1))^2\right)}{(H_1 N_1/2H_2) + (N_1/(\rho_2/(\exp(\rho_2(x + (\alpha_1 + 2ak)t + \varpi_0)) - 1)))} \right\} \times e^{i(-kx + \omega t)}, \quad (79)$$

in which

$$k = \frac{4 H_2{}^2 \Theta_2 \lambda_1 \sigma_1 - 33 H_2{}^2 \lambda_1 \sigma_1 \theta_1 - \sqrt{-H_2{}^2 \sigma_1 (4\Theta_2 - 33\theta_1)\left(8 N_1{}^2 \Theta_1 \Lambda_1 - H_2{}^2 \lambda_1{}^2 \sigma_1 (4\Theta_2 - 33\theta_1)\right)}}{2 H_2{}^2 \Theta_1 \sigma_1 (4\Theta_2 - 33\theta_1)},$$
$$\Lambda_1 = 2 a \Theta_2 \sigma_1 - 45 a \sigma_1 \theta_1 - 4\Theta_2{}^2 + 21\Theta_2 \theta_1 + 99\theta_1{}^2, \quad (80)$$

where H_0, ϖ_0 are free amounts. This solution is valid for the following case:

$$a(4\Theta_2 - 33\theta_1) < 0. \quad (81)$$

Product VD:
Via using (17), we conclude

$$q_{17}(x,t) = \left\{ \frac{H_0 + \left(H_1/\left(\rho_2^2(x + (\alpha_1 + 2ak)t + \varpi_0)\right)/(2\rho_2(x + (\alpha_1 + 2ak)t + \varpi_0)) + 4)\right)) + \left(\left(\sqrt{-2 b\sigma_1(\Theta_2 + 3\theta_1)} N_1/b\sigma_1\right)/\left(\rho_2^2(x + (\alpha_1 + 2ak)t + \varpi_0))/(2\rho_2(x + (\alpha_1 + 2ak)t + \varpi_0)) + 4)\right)^2\right)}{(H_1 N_1/2H_2) + \left(N_1/\left(\rho_2^2(x + (\alpha_1 + 2ak)t + \varpi_0))/(2\rho_2(x + (\alpha_1 + 2ak)t + \varpi_0)) + 4\right)\right)} \right\} \times e^{i(-kx + \omega t)}, \quad (82)$$

in which

$$k = \frac{\lambda_1 \sigma_1 H_2 + \sqrt{-8 a N_1{}^2 \Theta_1 \sigma_1{}^2 + \lambda_1{}^2 \sigma_1{}^2 H_2{}^2 + 8 N_1{}^2 \Theta_1 \Theta_2 \sigma_1 + 24 B_1{}^2 \Theta_1 \sigma_1 \theta_1}}{2\Theta_1 \sigma_1 H_2}, \quad (83)$$

where H_0, ϖ_0 are free values. This solution is valid for the following case:

$$-8 a N_1{}^2 \Theta_1 \sigma_1{}^2 + H_2{}^2 \lambda_1{}^2 \sigma_1{}^2 + 8 N_1{}^2 \Theta_1 \Theta_2 \sigma_1 + 24 N_1{}^2 \Theta_1 \sigma_1 \theta_1 > 0, \sigma_1 H_2(3\theta_1 + \theta_2 + \theta_3) \neq 0. \quad (84)$$

Set VI:

$$H_1 = \frac{1}{2}\,\frac{\sqrt{-2\,\sigma_1\left(\Theta_2{}^2 - 36\,\theta_1{}^2\right)\left(\sigma_1 k H_2{}^2(2\,\Theta_2 - 3\,\theta_1)(k\Theta_1 - \lambda_1) + 2\,N_1{}^2\left(4\,a\Theta_2\sigma_1 + 9\,a\sigma_1\theta_1 - 2\,\Theta_2{}^2 - 3\,\Theta_2\theta_1 + 9\,\theta_1{}^2\right)\right)}}{\sigma_1\left(\Theta_2{}^2 - 36\,\theta_1{}^2\right)},$$
$$H_0 = -\frac{1}{8}\,\frac{9\,\sigma_1 k H_2{}^2\theta_1(k\Theta_1 - \lambda_1) + 2\,N_1{}^2\left(2\,a\Theta_2\sigma_1 + 21\,a\sigma_1\theta_1 - 9\,\Theta_2\theta_1 - 27\,\theta_1{}^2\right)}{\sigma_1 H_2\left(\Theta_2{}^2 - 36\,\theta_1{}^2\right)},\ H_2 = \frac{\sqrt{-2\,b\sigma_1(\Theta_2 + 3\,\theta_1)}N_1}{b\sigma_1},$$
$$N_0 = \frac{H_1 N_1}{2H_2}, N_1 = N_1, k = k, \rho_2 = \frac{H_1}{H_2}, \rho_1 = \frac{H_0}{H_2}, \rho_2^2 - 4\rho_1 = -\frac{4\,H_0 H_2 - H_1{}^2}{H_2{}^2},$$
$$\omega = -\frac{\sigma_1 k H_2{}^2(ak\Theta_1 + ak\Theta_2 + 6\,ak\theta_1 - a\lambda_1 + \Theta_2\alpha_1 + 6\,\alpha_1\theta_1) + 2\,aN_1{}^2(a\sigma_1 - \Theta_2 - 3\,\theta_1)}{\sigma_1 H_2{}^2(\Theta_2 + 6\,\theta_1)}, \tag{85}$$

where N_1 is a free value. Imposing the solution set (36) into (35), we conclude the following.

Product VIA:
Via using (14), one gains

$$q_{18}(x,t) = \left\{\frac{H_0 - \left(H_1/\left(\left(\sqrt{\rho_2^2 - 4\rho_1}/2\rho_1\right)\tanh\left(\left(\sqrt{\rho_2^2 - 4\rho_1}/2\right)(\varpi + \varpi_0)\right) + (\rho_2/2\rho_1)\right)\right) + \left(\left(\sqrt{-2\,b\sigma_1(\Theta_2 + 3\,\theta_1)}N_1/b\sigma_1\right)/\left(\left(\sqrt{\rho_2^2 - 4\rho_1}/2\rho_1\right)\tanh\left(\left(\sqrt{\rho_2^2 - 4\rho_1}/2\right)(\varpi + \varpi_0)\right) + (\rho_2/2\rho_1)\right)^2\right)}{(H_1 N_1/2H_2) - \left(N_1/\left(\left(\sqrt{\rho_2^2 - 4\rho_1}/2\rho_1\right)\tanh\left(\left(\sqrt{\rho_2^2 - 4\rho_1}/2\right)(\varpi + \varpi_0)\right) + (\rho_2/2\rho_1)\right)\right)}\right\} \times e^{i(-kx + \omega t)}, \tag{86}$$

where N_0, N_1, ϖ_0 are free values. This solution is valid for the following case:

$$\begin{gathered}\sigma_1\left(\Theta_2{}^2 - 36\,\theta_1{}^2\right)\left(\sigma_1 k H_2{}^2(2\,\Theta_2 - 3\,\theta_1)(k\Theta_1 - \lambda_1) + 2\,N_1{}^2\left(4\,a\Theta_2\sigma_1 + 9\,a\sigma_1\theta_1 - 2\,\Theta_2{}^2 - 3\,\Theta_2\theta_1 + 9\,\theta_1{}^2\right)\right) < 0,\\ 4H_0 H_2 - H_1^2 < 0.\end{gathered} \tag{87}$$

Product VIB:
Via utilizing (15), we get

$$q_{19}(x,t) = \left\{\frac{H_0 + \left(H_1/\left(\left(\sqrt{-\rho_2^2 + 4\rho_1}/2\rho_1\right)\tan\left(\left(\sqrt{-\rho_2^2 + 4\rho_1}/2\right)(\varpi + \varpi_0)\right) - (\rho_2/2\rho_1)\right)\right) + \left(\left(\sqrt{-2\,b\sigma_1(\Theta_2 + 3\,\theta_1)}N_1/b\sigma_1\right)/\left(\left(\sqrt{-\rho_2^2 + 4\rho_1}/2\rho_1\right)\tan\left(\left(\sqrt{-\rho_2^2 + 4\rho_1}/2\right)(\varpi + \varpi_0)\right) - (\rho_2/2\rho_1)\right)^2\right)}{(H_1 N_1/2H_2) + \left(N_1/\left(\left(\sqrt{-\rho_2^2 + 4\rho_1}/2\rho_1\right)\tan\left(\left(\sqrt{-\rho_2^2 + 4\rho_1}/2\right)(\varpi + \varpi_0)\right) - (\rho_2/2\rho_1)\right)\right)}\right\} \times e^{i(-kx + \omega t)}, \tag{88}$$

where N_0, N_1, ϖ_0 are free arbitraries. This solution is valid for the following case:

$$\begin{gathered}\sigma_1\left(\Theta_2{}^2 - 36\,\theta_1{}^2\right)\left(\sigma_1 k H_2{}^2(2\,\Theta_2 - 3\,\theta_1)(k\Theta_1 - \lambda_1) + 2\,N_1{}^2\left(4\,a\Theta_2\sigma_1 + 9\,a\sigma_1\theta_1 - 2\,\Theta_2{}^2 - 3\,\Theta_2\theta_1 + 9\,\theta_1{}^2\right)\right) < 0,\\ 4H_0 H_2 - H_1^2 > 0.\end{gathered} \tag{89}$$

Product VIC: Via employing (16), one obtains

$$q_{20}(x,t)=\left\{\frac{H_0+(H_1/(\rho_2/(\exp(\rho_2(x+(\alpha_1+2ak)t+\varpi_0))-1)))+\left(\left(\sqrt{-2b\sigma_1(\Theta_2+3\theta_1)}N_1/b\sigma_1\right)/(\rho_2/\exp(\rho_2(x+(\alpha_1+2ak)t+\varpi_0))-1)^2\right)}{(H_1N_1/2H_2)+(N_1/\rho_2/(\exp(\rho_2(x+(\alpha_1+2ak)t+\varpi_0))-1))}\right\}\times e^{i(-kx+\omega t)}, \tag{90}$$

in which

$$k=\frac{1}{6}\,\frac{3\Theta_2\lambda_1\theta_1+9\lambda_1\theta_1^{\,2}+\sqrt{4b\Theta_1\Theta_2^{\,2}\theta_1\Delta_1+1108\,b\Theta_1\Theta_2\theta_1^{\,2}\Delta_2+36\,b\Theta_1\theta_1^{\,3}\Delta_3+9\lambda_1^{\,2}\theta_1^{\,2}(\Theta_2+3\theta_1)^2}}{\Theta_1\theta_1(\Theta_2+3\theta_1)},$$
$$\Delta_1=2a\sigma_1-9\theta_1,\ \Delta_2=a\sigma_1-2\theta_1,\ \Delta_3=7a\sigma_1-9\theta_1, \tag{91}$$

where N_0,ϖ_0 are free amounts. This solution is valid for the following case:

$$4b\Theta_1\Theta_2^{\,2}\theta_1\Delta_1+1108\,b\Theta_1\Theta_2\theta_1^{\,2}\Delta_2+36\,b\Theta_1\theta_1^{\,3}\Delta_3+9\lambda_1^{\,2}\theta_1^{\,2}(\Theta_2+3\theta_1)^2>0. \tag{92}$$

Product VID: Via using (17), we conclude

$$q_{21}(x,t)=\left\{\frac{H_0+\left(H_1/\left(\rho_2^2(x+(\alpha_1+2ak)t+\varpi_0)\right)/(2\rho_2(x+(\alpha_1+2ak)t+\varpi_0)+4)\right)+\left(\left(\sqrt{-2b\sigma_1(\Theta_2+3\theta_1)}N_1/b\sigma_1\right)/\left(\rho_2^2(x+(\alpha_1+2ak)t+\varpi_0)\right)/(2\rho_2(x+(\alpha_1+2ak)t+\varpi_0))+4)^2\right)}{(H_1N_1/2H_2)+\left(N_1/\left(\rho_2^2(x+(\alpha_1+2ak)t+\varpi_0)\right)/(2\rho_2(x+(\alpha_1+2ak)t+\varpi_0))+4)\right)}\right\}\times e^{i(-kx+\omega t)}, \tag{93}$$

in which

$$k=\frac{1}{2}\,\frac{\lambda_1\sigma_1H_2+\sqrt{\lambda_1^{\,2}\sigma_1^{\,2}H_2^{\,2}-8N_1^{\,2}\Theta_1\sigma_1(a\sigma_1-\Theta_2-3\theta_1)}}{\Theta_1\sigma_1H_2}, \tag{94}$$

where N_0,ϖ_0 are free values. This solution is valid for the following case:

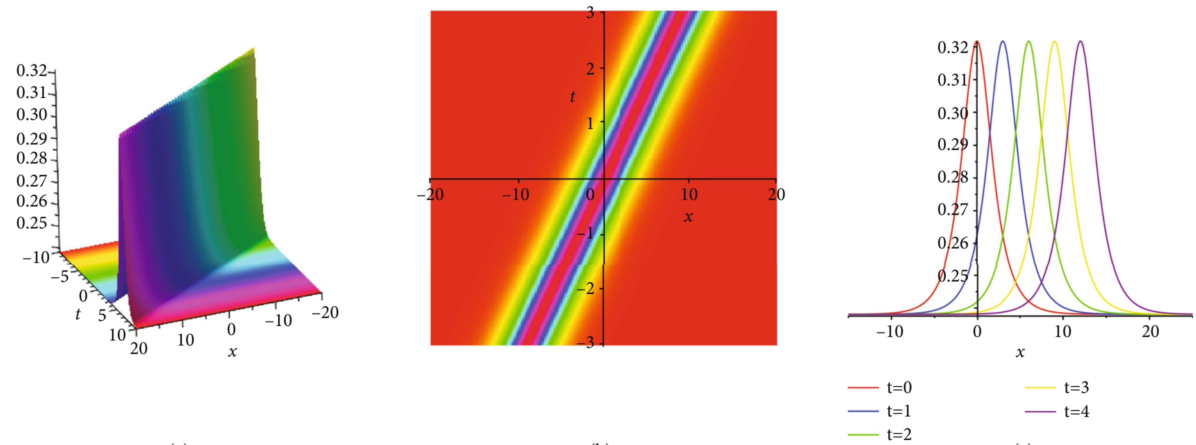
$$\lambda_1^{\,2}\sigma_1^{\,2}H_2^{\,2}-8N_1^{\,2}\Theta_1\sigma_1(a\sigma_1-\Theta_2-3\theta_1)>0. \tag{95}$$

Inserting $\theta_1=-1,\lambda_1=k=b=\theta_2=\theta_3=1,N_1=2,b=5,\alpha_1=1,a=-2,\sigma_1=-3$, the solution $u(x,t)$ given by (86) shows the soliton wave solution. Moreover, the solution finds to the soliton solution as suggested in Figures 5(a)–5(c). Likewise, by taking $\theta_1=1,\lambda_1=b=\theta_2=\theta_3=1,N_1=2,b=5,\alpha_1=1,a=-2,k=3,\sigma_1=-3$, the solution $u(x,t)$ given by (87) denotes the periodic wave solution. Moreover, the solution transforms to the periodic solution as mentioned in Figures 6(a)–6(c). Also, by plugging $\theta_1=1,\lambda_1=b=\theta_2=\theta_3=1,N_1=2,b=5,\alpha_1=1,a=-2,k=3,\sigma_1=-3$, the solution $u(x,t)$ given by (88) shows the kink-singular wave solution. Moreover, the solution finds to the soliton solution as found in Figures 7(a)–7(c). Finally, by putting $\theta_1=1,\lambda_1=b=\theta_2=\theta_3=1,N_1=2,b=5,\alpha_1=1,a=-2,k=3,\sigma_1=-3$, the solution $u(x,t)$ given by (89) finds the cupson-singular wave solution. Moreover, the solution finds to the rational-singular solution as offered in Figures 8(a)–8(c).

4. Description of the Second Proposed Method

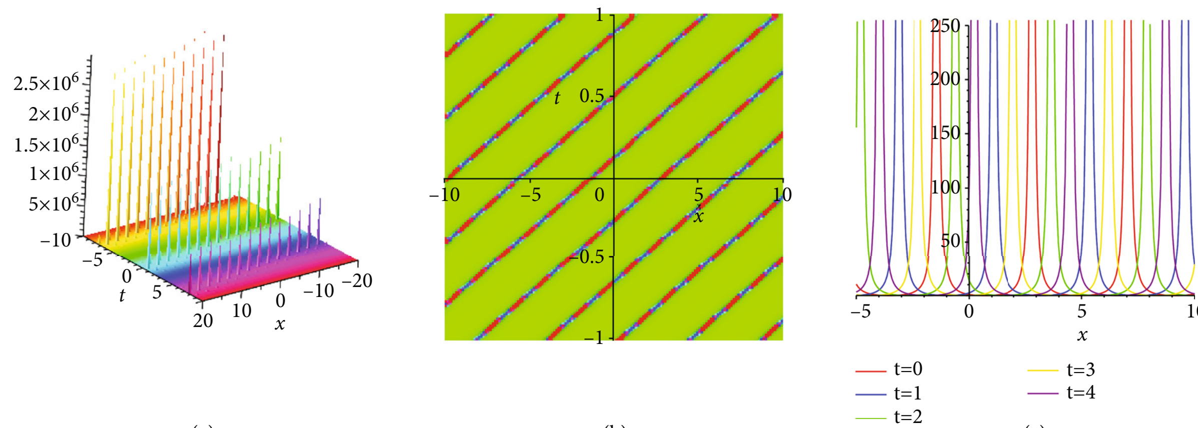
Take the improved simple equation method (ISEM). Let the solution of equation (11) be

$$U(\varpi)=\frac{\sum_{i=0}^{P_1}H_i(\Gamma(\varpi))^i}{\sum_{i=0}^{P_1}N_i(\Gamma(\varpi))^i}. \tag{96}$$



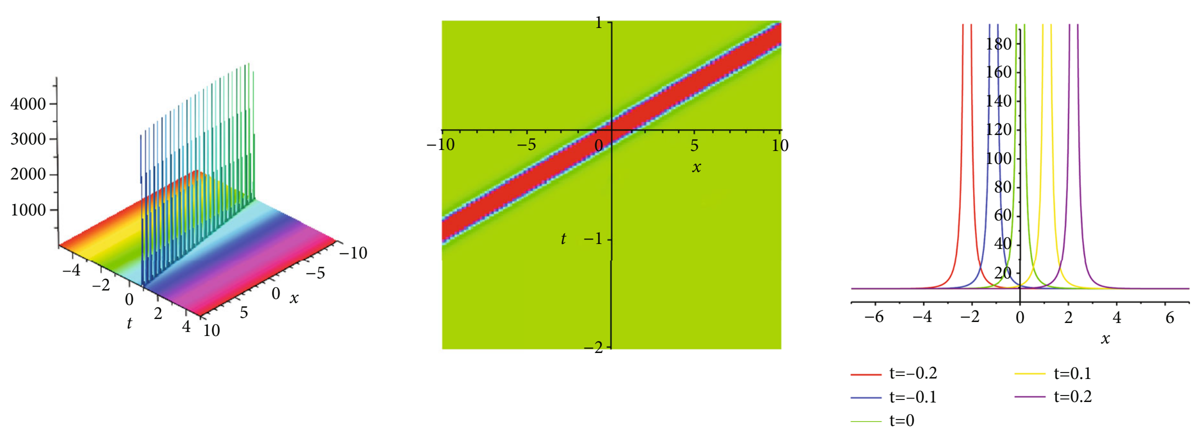
(a) (b) (c)

Figure 5: Graph of (86) by taking $\theta_1 = -1, \lambda_1 = k = b = \theta_2 = \theta_3 = 1, N_1 = 2, b = 5, \alpha_1 = 1, a = -2, \sigma_1 = -3$: (a) 3D graph, (b) density plot, and (c) 2D plot.



(a) (b) (c)

Figure 6: Graph of (87) by taking $\theta_1 = 1, \lambda_1 = b = \theta_2 = \theta_3 = 1, N_1 = 2, b = 5, \alpha_1 = 1, a = -2, k = 3, \sigma_1 = -3$: (a) 3D graph, (b) density plot, and (c) 2D plot.



(a) (b) (c)

Figure 7: Graph of (88) by taking $\theta_1 = 1, \lambda_1 = b = \theta_2 = \theta_3 = 1, N_1 = 2, b = 5, \alpha_1 = 1, a = -2, k = 3, \sigma_1 = -3$: (a) 3D graph, (b) density plot, and (c) 2D plot.

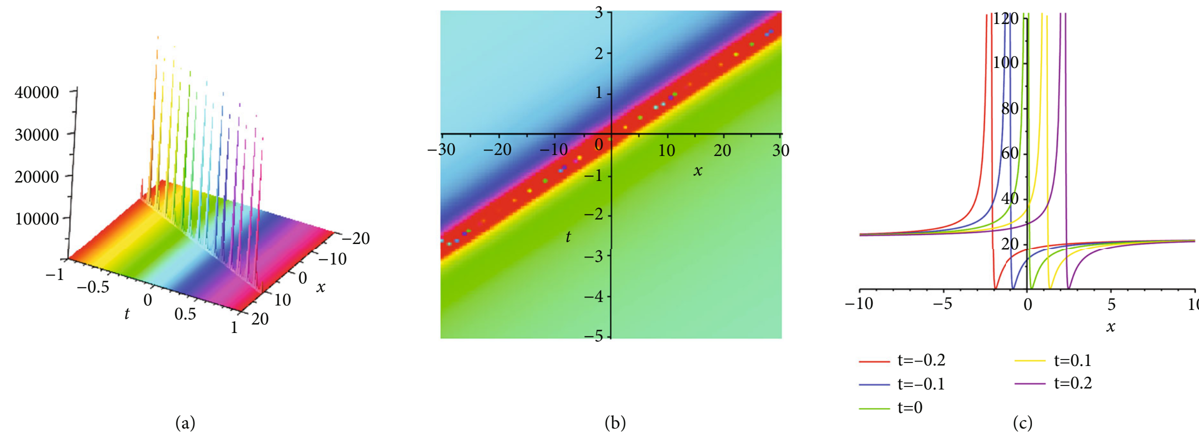


Figure 8: Graph of (89) by taking $\theta_1 = 1, \lambda_1 = b = \theta_2 = \theta_3 = 1, N_1 = 2, b = 5, \alpha_1 = 1, a = -2, k = 3, \sigma_1 = -3$: (a) 3D graph, (b) density plot, and (c) 2D plot.

Let $\Gamma = \Gamma(\varpi)$ satisfy

$$\Gamma' = \rho_0 + \rho_1 \Gamma(\varpi) + \rho_2 (\Gamma(\varpi))^2 + \rho_3 (\Gamma(\varpi))^3. \tag{97}$$

The solutions of equation (97) will be as follows:

Product 1: with $\rho_3 = 0$, then we have

$$\Gamma(\varpi) = \frac{1}{2} \frac{\tan\left((1/2)(\varpi + \varpi_0)\sqrt{4\rho_0\rho_2 - \rho_1^2}\right)\sqrt{4\rho_0\rho_2 - \rho_1^2} - \rho_1}{\rho_2}, \tag{98}$$

where ϖ_0 is the integral constant.

Product 2: with $\rho_0 = \rho_3 = 0$, then we have

$$\Gamma(\varpi) = \frac{\rho_1 e^{\rho_1(\varpi+\varpi_0)}}{C_0\rho_1 - \rho_2 e^{\rho_1(\varpi+\varpi_0)}}, \tag{99}$$

where C_0 is the integral constant.

Product 3: with $\rho_0 = \rho_2 = 0$, then we have

$$\Gamma(\varpi) = \frac{\sqrt{\left(C_0 e^{-2\rho_1(\varpi+\varpi_0)}\rho_1 - \rho_3\right)\rho_1}}{C_0 e^{-2\rho_1(\varpi+\varpi_0)}\rho_1 - \rho_3}. \tag{100}$$

Product 4: with $\rho_0 = \rho_1 = \rho_2 = 0$, then we have

$$\Gamma(\varpi) = \frac{1}{\sqrt{-2(\varpi + \varpi_0)\rho_3 + C_0}}. \tag{101}$$

where $H_i, N_i (0 \le i \le P_1)$ are also the values to be explored later.

4.1. Application of ISEM on Equation (1). The IEFM allows us to recruit the substitutions:

$$v(\varpi) = \frac{H_0 + H_1\Gamma(\varpi) + H_2\Gamma(\varpi)^2}{N_0 + N_1\Gamma(\varpi) + N_2\Gamma(\varpi)^2}. \tag{102}$$

Plug (35) along with (97) into equation (6).

With case (98), we have the following:

Product IA:

$$\begin{aligned}
b &= -\frac{2a\left(N_1^2\rho_2 - N_2^2\rho_0\right)^2 + H_2^2N_1^2\left(k^2\Theta_1 + 2\Theta_2\rho_0\rho_2 + 12\rho_0\rho_2\theta_1 - k\lambda_1\right) - 2H_2^2N_2^2\rho_0^2(\Theta_2 + 6\theta_1)}{H_2^2N_1^2},\\
k &= k, \omega = -\frac{N_1^2\left(ak^2 - 2a\rho_0\rho_2 + k\alpha_1\right) + 2\rho_0^2\left(aN_2^2 - 3H_2^2\theta_1\right)}{N_1^2}, H_0 = H_1 = N_0 = 0, N_1 = N_1, \rho_1 = 2\frac{N_2\rho_0}{N_1},\\
N_2 &= N_2, \sigma_1 = \frac{2\left(N_1^2\rho_2 - N_2^2\rho_0\right)^2(\Theta_2 + 3\theta_1)}{2a\left(N_1^2\rho_2 - N_2^2\rho_0\right)^2 + H_2^2N_1^2\left(k^2\Theta_1 + 2\Theta_2\rho_0\rho_2 + 12\rho_0\rho_2\theta_1 - k\lambda_1\right) - 2H_2^2N_2^2\rho_0^2(\Theta_2 + 6\theta_1)}.
\end{aligned} \tag{103}$$

Through (102) and (103), then we can find the below solution:

By using (98), we have

$$q_1(x,t)=\left\{\frac{H_2\left((1/2)\left(\tan\left((1/2)(\varpi+\varpi_0)\sqrt{4\rho_0\rho_2-\rho_1^2}\right)\sqrt{4\rho_0\rho_2-\rho_1^2}-\rho_1/\rho_2\right)\right)^2}{N_1\left((1/2)\left(\tan\left((1/2)(\varpi+\varpi_0)\sqrt{4\rho_0\rho_2-\rho_1^2}\right)\sqrt{4\rho_0\rho_2-\rho_1^2}-\rho_1/\rho_2\right)\right)+N_2\left((1/2)\left(\tan\left((1/2)(\varpi+\varpi_0)\sqrt{4\rho_0\rho_2-\rho_1^2}\right)\sqrt{4\rho_0\rho_2-\rho_1^2}-\rho_1/\rho_2\right)\right)^2}\right\}\times e^{i(-kx+\omega t)}, \tag{104}$$

where N_1, N_2, ϖ_0 are free amounts.

Product IB:

$$\begin{aligned}
b&=-\frac{2aN_0N_1\rho_0^2(H_0N_1-H_1N_0)+kH_0^3N_0(k\Theta_1-\lambda_1)-H_0H_1\rho_0^2(\Theta_2+6\theta_1)(H_0N_1-H_1N_0)}{H_0^3N_0},\\
k&=k,\ \omega=-\frac{ak^2H_0^2N_0+aH_0H_1N_1\rho_0^2-aH_1^2N_0\rho_0^2+kH_0^2N_0\alpha_1}{N_0H_0^2},\ H_0=H_0,\ H_1=H_1,\\
N_0&=N_0,\ N_1=N_1,\ N_2=\frac{1}{4}\,\frac{H_1N_1}{H_0},\ \rho_1=\frac{H_1\rho_0}{H_0},\ \rho_2=\frac{1}{4}\,\frac{H_1N_1\rho_0}{N_0H_0},\\
\sigma_1&=2\,\frac{N_1\rho_0^2(\Theta_2+3\theta_1)(H_0N_1-H_1N_0)N_0}{2aH_0N_0N_1^2\rho_0^2-2aH_1N_0^2N_1\rho_0^2+kH_0^3N_0(k\Theta_1-\lambda_1)-H_0H_1\rho_0^2(\Theta_2+6\theta_1)(H_0N_1-H_1N_0)}.
\end{aligned} \tag{105}$$

Through (102) and (103), then we can find the below solution:

By using (98), we have

$$q_2(x,t)=e^{i(-kx+\omega t)}\times\left\{\frac{H_0\left(H_1N_1\left(\tan\left((1/2)(\xi)\sqrt{4\rho_0\rho_2-\rho_1^2}\right)\sqrt{4\rho_0\rho_2-\rho_1^2}-\rho_1/2\rho_2\right)^2+4H_1\left(\tan\left((1/2)(\xi)\sqrt{4\rho_0\rho_2-\rho_1^2}\right)\sqrt{4\rho_0\rho_2-\rho_1^2}-\rho_1/2\rho_2\right)N_0+4N_0H_0\right)}{\left(H_1N_1\left(\left(\tan\left((1/2)(\xi)\sqrt{4\rho_0\rho_2-\rho_1^2}\right)\sqrt{4\rho_0\rho_2-\rho_1^2}-\rho_1/2\rho_2\right)\right)^2+4N_1\left(\tan\left((1/2)(\xi)\sqrt{4\rho_0\rho_2-\rho_1^2}\right)\sqrt{4\rho_0\rho_2-\rho_1^2}-\rho_1/2\rho_2\right)H_0+4N_0H_0\right)N_0}\right\}, \tag{106}$$

where $N_0, H_0, N_1, H_1, \varpi_0$ are free amounts.

With case (99), we have the following:

Product IIA:

$$\begin{aligned}
b&=8\,\frac{N_0\rho_1^2(\Theta_2+3\theta_1)\left(H_1^2N_2-H_2^2N_0\right)}{H_1^4\sigma_1},\ \omega=-ak^2+a\rho_1^2-k\alpha_1,\ H_0=0,\ N_1=2\,\frac{H_2N_0}{H_1},\\
\lambda_1&=-\frac{8aN_0\rho_1^2\sigma_1\left(H_1^2N_2-H_2^2N_0\right)-H_1^4\sigma_1\left(k^2\Theta_1+\Theta_2\rho_1^2+6\rho_1^2\theta_1\right)-8N_0\rho_1^2(\Theta_2+3\theta_1)\left(H_1^2N_2-H_2^2N_0\right)}{kH_1^4\sigma_1},\\
k&=k,\ H_1=H_1,\ H_2=H_2,\ N_2=N_2,\ \rho_2=\frac{H_2\rho_1}{H_1}.
\end{aligned} \tag{107}$$

Through (102) and (107), then we can find the below solution:

By using (99), we have

$$q_1(x,t)=\left\{\frac{H_1\left(\rho_1 e^{\rho_1(\varpi+\varpi_0)}/C_0\rho_1-\rho_2 e^{\rho_1(\varpi+\varpi_0)}\right)\left(H_2\left(\rho_1 e^{\rho_1(\varpi+\varpi_0)}/C_0\rho_1-\rho_2 e^{\rho_1(\varpi+\varpi_0)}\right)+H_1\right)}{H_1N_2\left(\rho_1 e^{\rho_1(\varpi+\varpi_0)}/C_0\rho_1-\rho_2 e^{\rho_1(\varpi+\varpi_0)}\right)^2+2H_2N_0\left(\rho_1 e^{\rho_1(\varpi+\varpi_0)}/C_0\rho_1-\rho_2 e^{\rho_1(\varpi+\varpi_0)}\right)+H_1N_0}\right\}e^{i\left(-kx+\left(-ak^2+a\rho_1{}^2-k\alpha_1\right)t\right)}, \tag{108}$$

where N_0, H_1, N_2, ϖ_0 are free amounts.

With case (100), we have the following:

Product IIIA:

$$\begin{aligned}
&a=-\frac{1}{8}\frac{H_2{}^2\left(k^2\Theta_1-k\lambda_1+b\right)-8H_0{}^2\rho_3{}^2(\Theta_2+6\theta_1)}{N_0{}^2\rho_3{}^2}, H_1=N_1=N_2=0, \rho_1=\frac{2\rho_3H_0}{H_2}, \sigma_1=-\frac{8N_0{}^2\rho_3{}^2(\Theta_2+3\theta_1)}{bH_2{}^2},\\
&\omega=\frac{-64H_0{}^4\rho_3{}^4(\Theta_2+3\theta_1)+8k^2H_0{}^2H_2{}^2\rho_3{}^2(\Theta_1-\Theta_2-6\theta_1)+k^2H_2{}^4S_1+8H_0{}^2H_2{}^2\rho_3{}^2S_2-8kH_2{}^2N_0{}^2\alpha_1\rho_3{}^2}{8N_0{}^2\rho_3{}^2H_2{}^2},\\
&k=k, H_0=H_0, N_0=N_0, S_1=k^2\Theta_1-k\lambda_1+b, S_2=-k\lambda_1+b.
\end{aligned} \tag{109}$$

Through (102) and (109), then we can find the below solution:

By using (100), we have

$$q_1(x,t)=\left\{\frac{H_2\left(\sqrt{\left(C_0e^{-2\rho_1(\varpi+\varpi_0)}\rho_1-\rho_3\right)\rho_1}/C_0e^{-2\rho_1(\varpi+\varpi_0)}\rho_1-\rho_3\right)^2+H_0}{N_0}\right\}e^{i(-kx+\omega t)}, \tag{110}$$

where C_0, N_0, H_0, ϖ_0 are free amounts.

In reference to case (101) and the solution function $\Gamma(\varpi) = 1/\sqrt{-2(\varpi+\varpi_0)\rho_3+C_0}$, we can get to the below results:

Product IVA:

$$\begin{aligned}
a&=-\frac{1}{8}\frac{kH_2{}^2\sigma_1(k\Theta_1-\lambda_1)-8N_0{}^2\rho_3{}^2(\Theta_2+3\theta_1)}{N_0{}^2\rho_3{}^2\sigma_1},\\
b&=-8\frac{N_0{}^2\rho_3{}^2(\Theta_2+3\theta_1)}{H_2{}^2\sigma_1}, H_0=H_1=N_1=N_2=0,\\
k=k, \omega&=\frac{1}{8}\frac{k\left(k^2H_2{}^2\sigma_1(k\Theta_1-\lambda_1)-8N_0{}^2\rho_3{}^2(k\Theta_2+3k\theta_1+\alpha_1\sigma_1)\right)}{N_0{}^2\rho_3{}^2\sigma_1}.
\end{aligned} \tag{111}$$

Through (102) and (111), then we can find the below solution:

By using (101), we have

$$q_1(x,t)=\frac{H_2\left(1/\sqrt{-2(\varpi+\varpi_0)\rho_3+C_0}\right)^2}{N_0}\cdot e^{i\left(-kx+(1/8)\left(k\left(k^2H_2{}^2\sigma_1(k\Theta_1-\lambda_1)-8N_0{}^2\rho_3{}^2(k\Theta_2+3k\theta_1+\alpha_1\sigma_1)\right)/N_0{}^2\rho_3{}^2\sigma_1\right)t\right)}, \tag{112}$$

where C_0, N_0, ϖ_0 are free amounts.

Product IVB:

$$\begin{aligned}
a&=-\frac{1}{8}\frac{H_2{}^2\left(k^2\Theta_1-k\lambda_1+b\right)}{N_0{}^2\rho_3{}^2}, H_0=H_1=N_1=0, \sigma_1\\
&=-8\frac{N_0{}^2\rho_3{}^2(\Theta_2+3\theta_1)}{bH_2{}^2},\\
k=k, \omega&=\frac{1}{8}\frac{k\left(k^3H_2{}^2\Theta_1-k^2H_2{}^2\lambda_1-8N_0{}^2\alpha_1\rho_3{}^2+bkH_2{}^2\right)}{N_0{}^2\rho_3{}^2}.
\end{aligned} \tag{113}$$

Through (102) and (113), then we can find the below solution:

By using (101), we have

$$q_2(x,t) = \left\{ \frac{H_2\left(1/\sqrt{-2\,(\varpi+\varpi_0)\rho_3 + C_0}\right)^2}{N_0 + N_2\left(1/\sqrt{-2\,(\varpi+\varpi_0)\rho_3 + C_0}\right)^2} \right\} \cdot e^{i\left(-kx+(1/8)\left(k\left(k^3 {H_2}^2 \Theta_1 - k^2 {H_2}^2 \lambda_1 - 8\,{N_0}^2 \alpha_1 {\rho_3}^2 + bk{H_2}^2\right)/{N_0}^2{\rho_3}^2\right)t\right)}, \tag{114}$$

where C_0, N_0, N_2, ϖ_0 are free amounts.

5. Discussion and Remark

The physical explanation of the established traveling wave solutions to the generalized nonlinear Schrödinger equation with parabolic law of nonlinearity will be discussed in this section. The collected traveling wave solutions of this equation are discussed in the three-dimensional plotline, the plot of density, three-dimensional list point plotline, and plot of vector which are designed via Maple. Using those eight sorts of pictorial descriptions, we may more explicitly characterize the physical sketch. As can be seen from the diagrams above, various structures, including those of the well-known form of solitons, such as the solutions of $q_1(x,t)$ (37) in Figure 1 within the duration $-30 < x < 30, -2 < t < 2$ which is the solution of the generalized nonlinear Schrödinger equation, represented shape traveling wave solution. Moreover, the solution transforms to the periodic solution as mentioned in $q_2(x,t)$ (38) in Figure 2 within the duration $-5 < x < 5, 0 < t < 5$. Also, by plugging the suitable values, the solution $q_3(x,t)$ given by (39) in Figure 3 shows the kink-singular wave solution within the duration $-3 < x < 3, -5 < t < 5$. Finally, by putting the specified amounts, the solution $q_4(x,t)$ given by (60) in Figure 4 finds the cupson-singular wave solution within the duration $-15 < x < 15, -1 < t < 1$. In addition, the solutions of $q_{18}(x,t)$ (86) in Figure 5 within the duration $-20 < x < 20, -10 < t < 10$ which is the solution of the generalized nonlinear Schrödinger equation represented shape traveling wave solution. Moreover, the solution transforms to the periodic solution as mentioned in $q_{19}(x,t)$ (87) in Figure 6 within the duration $-20 < x < 20, -10 < t < 10$. Also, by plugging the suitable values, the solution $q_{20}(x,t)$ given by (88) in Figure 7 shows the kink-singular wave solution within the duration $-10 < x < 10, -5 < t < 5$. Finally, by putting the specified amounts, the solution $q_{21}(x,t)$ given by (89) in Figure 8 finds the cupson-singular wave solution within the duration $-20 < x < 20, -1 < t < 1$.

The graphics for the generalized nonlinear Schrödinger equation solutions are developed here. We represent the different types of the picture which are gotten by those equations and also avoid the same shape which overlaps with the presented picture. We will discover all of the most recent exact solutions in our paper. However, for the sake of convenience, we will ignore it in this post. We have seen that the solutions to the aforementioned equation are relatively novel in literature as compared to previous solutions. Figures 1–8 are generated utilizing a computation package program, namely, Maple.

6. Conclusion

We proposed the novel schemes, namely, the improved $\exp(-\Gamma(\varpi))$ function method and the improved simple equation method, which are utilized to build the exact solutions of the generalized nonlinear Schrödinger equation. In addition, multifarious transformation was operated to renovate the nonlinear ordinary transformation equation. We found a slide of the closed form of solutions to the suggested equations, including kink-shaped, anti-kink-shaped, bell-shaped, singular bell-shaped, singular periodic-shaped, multiple soliton-shaped, single soliton-shaped, and other types of solutions with a variety of free parameters. These free parameters have significant consequences, for example, the ability to find specific solutions by changing the free parameter values of an individual solution. The activities of solitary waves were graphically illustrated concerning space and time, revealing the higher efficiency and validity of the claimed schemes. The retrieved solutions could be used to study the transmission of metamaterial waves, nonlinear optic, and so on. The constructed results have extensive potential to comprehend the interior configurations of the usual manifestation that arise in physics, mathematics, and other different fields. Hence, it is worth declaring that the execution of our schemes is extremely steady and well organized and for nonlinear models. The adopted method is direct, trustworthy, conformable, and effective, and it provides many novel physical model solutions to NLPEEs that arise in mathematical physics, applied mathematics, and engineering.

Data Availability

The datasets supporting the conclusions of this article are included in the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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