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Research Article

Certain Analysis of Solution for the Nonlinear Two-Point Boundary Value Problem with Caputo Fractional Derivative

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In this paper, the existence and uniqueness of solutions for a nonlinear fractional differential equation with a two-point boundary condition in a Banach space are investigated by using the contraction mapping principle and the Brouwer fixed-point theorem with Bielecki norm. The iterative scheme of the numerical solution for the nonlinear two-point boundary value problem will be discussed and illustrated by solving some problems. The well-known Ulam-Hyers and Ulam-Hyers-Rassias stability theorems are employed to establish the stability of solutions to the boundary value problem. In the end, we provided a couple of examples to support our results.

1. Introduction

Fractional calculus is an important mathematical topic due to its theoretical foundation and multiple applications in physical, chemical processes, and engineering; for instance, see [1–6]. The two-point boundary value problem occurs in applied mathematics, theoretical physics, engineering, control, and optimization theory (see [7]). The existence of solutions of initial and boundary value problems of fractional differential equations by the help of different fixed-point theorems has been discussed by many mathematicians, and the readers are referred to see the monographs [8–21]. In [22], the authors give the existence results for two-point boundary value problem of fractional differential equations at resonance by means of the coincidence degree theory. Mongkolkeha and Gopal [23] proposed new common fixed-point theorems for the Ciric type generalized F-contraction in metric spaces with the w-distance. The fixed points for the monotone γ-nonexpansive and generalized $\wp$-nonexpansive mappings in hyperbolic space have been approximated by [24].

In fact, the subject of numerical methods for solving fractional differential equations has gained prominence and has been discussed by several authors, including a series of papers [25–31] and references cited therein, which include some recent studies on the approximation method for differential equations of fractional order. El-Ajou et al. [32] extended the application of the homotopy analysis method (HAM) to provide symbolic approximate solution for two-point boundary value problems of fractional order. Lyons et al. [33] prove an extension of Picard's iterative existence and uniqueness theorem to Caputo fractional ordinary differential equations, when the nonhomogeneous term satisfies the usual Lipschitz's condition. In [34], the successive approximation method was applied to solve the temperature field based on the given Mittag-Leffler-type.

Nie et al. [35] investigated the existence and numerical method of two-point boundary value problems for fractional differential equations with Caputo's derivative or Riemann-Liouville derivative. The solutions can be deduced by the contraction mapping principle and fractional Green function. For the Caputo's derivative case, it has the form

$$^{C}Ɖ^{\gamma}\varpi(t) + \mho(t, \varpi(t), Ɖ^{\wp}\varpi(t)) = 0, \tag{1}$$

$$\varpi(a) = A, \quad \varpi(b) = B. \tag{2}$$

For Riemann-Liouville derivative case, it has the form

$$\begin{aligned} {}^{R}Đ^{\gamma}\varpi(t) + \mho(t, \varpi(t), Đ^{\wp}\varpi(t)) = 0, \\ \varpi(a) = 0, \quad \varpi(b) = B. \end{aligned} \tag{3}$$

More recently, Hyers-Ulam type stability theorems for nonlinear fractional differential equations have attracted a lot of attention as an interesting field and have been investigated in many papers (see [36–40]). Murad et al. [41] studied the existence, Ulam-Hyers, and Ulam-Hyers-Rassias theorems of solutions to a differential equation of mixed Caputo-Riemann fractional derivatives.

Dai et al. [42] discussed the existence and Hyers-Ulam and Hyers-Ulam-Rassias stability of solutions for the fractional differential equation with boundary condition. Prasad et al. [43] investigated the existence and Ulam stability of the fractional-order iterative two-point boundary value problem which has the form

$$\begin{aligned} {}^{C}Đ^{\gamma}\varpi(t) = \mho\left(t, \varpi(t), \varpi^{[2]}(t)\right), t \in [0, 1], 1 < \gamma \le 2, \\ \varpi(0) = A, \quad \varpi(1) = B, \end{aligned} \tag{4}$$

where ${}^{C}Đ^{\gamma}$ is the Caputo fractional derivative and $1 < \gamma \le 2$ and $0 \le A \le B \le 1$.

In this paper, we consider the nonlinear fractional differential equation which has the form

$$Đ^{\gamma}\varpi(t) = \mho(t, \varpi(t), Đ^{\wp}\varpi(t)), I = [a, b], \tag{5}$$

with boundary conditions

$$\begin{aligned} \varpi(a) = A, \\ \varpi(b) = B, \end{aligned} \tag{6}$$

where $\mho : I \times R \times R \longrightarrow R$, $1 < \gamma \le 2$, $0 < \wp \le 1$, $Đ^{\gamma}$ and $Đ^{\wp}$ are the Caputo fractional derivatives, and a, b, A, B are constants. The main objective is to study the existence of a solution to the boundary value problem (5) and (6). The results are based on Brouwer's fixed-point theorem and Banach contraction mapping principle. The analytical approximate technique to obtain the solution is a part of this work, and some examples are illustrated to explain the algorithm. Furthermore, we discuss the Ulam-Hyers stability and Ulam-Hyers-Rassias stability of the boundary value problem (5) and (6). Some examples are also constructed to illustrate and validate the main results.

2. Preliminaries

Let us give some definitions and lemmas that are basic and needed at various places in this work.

Definition 1 (see [44]). Let $\mho$ be a function which is defined almost everywhere ($a.e.$) on $[a, b]$. If $\gamma > 0$, then

$${}_{a}^{b}I^{\gamma}\mho = \int_{a}^{b} \mho(s) \frac{(b-s)^{\gamma-1}}{\Gamma(\gamma)} ds, \tag{7}$$

provided that this integral (Lebesgue) exists.

Definition 2 (see [5]). For a continuous function $\mho : [0,\infty) \longrightarrow \mathbb{R}$, the Caputo derivative of fractional order γ is defined as

$${}^{C}Đ^{\gamma}\mho(t) = \frac{1}{\Gamma(n-\gamma)} \int_{0}^{t} (t-s)^{n-\gamma-1} \mho^{(n)}(s) ds, n-1 < \gamma \le n, \tag{8}$$

provided that $\mho^{(n)}$ exists, where $n = [\gamma] + 1$, $[\gamma]$ denotes the integer part of the real number γ.

Lemma 3 (see [1]). *Let $\gamma > 0$. If we assume $\varpi \in C(0, 1) \cap L_1(0, 1)$, then the Caputo fractional differential equation*

$$Đ^{\gamma}\varpi(t) = 0 \tag{9}$$

has the solution

$$\varpi(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}, \tag{10}$$

where $c_i \in \mathbb{R}$, $i = 0, 1, 2, \cdots, n-1$, and $n = [\gamma] + 1$.

Lemma 4 (see [1]). *Let $\varpi \in C(0, 1) \cap L_1(0, 1)$ with fractional derivative of order $\gamma > 0$ that belongs to $C(0, 1) \cap L_1(0, 1)$. Then,*

$$I^{\gamma C}Đ^{\gamma}\varpi(t) = \varpi(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}, \tag{11}$$

for $c_i \in \mathbb{R}$, $i = 0, 1, 2, \cdots, n-1$, where n is the smallest integer greater than or equal to γ.

Theorem 5 (see [45]) (Brouwer's fixed-point theorem). *Let M be a nonempty compact (closed and bounded) convex set in R^n and $T : M \longrightarrow M$ be a continuous self-mapping. Then, T has (at least) one fixed point in M.*

Theorem 6 (see [45]) (Banach contraction mapping principle). *Let M be a Banach space. If $T : M \longrightarrow M$ is a contraction, then T has a unique fixed point in M.*

Lemma 7. *Let $\varpi(t) \in C(I, \mathbb{R})$ and $1 < \gamma \le 2$, $0 < \wp \le 1$; then, the solution of the boundary value problem (5) and (6) is given by*

$$\varpi(t) = A + \frac{(B-A)}{(b-a)}(t-a) - \frac{(t-a)}{(b-a)\Gamma(\gamma)}\int_a^b (b-s)^{\gamma-1} \cdot \mho(s, \varpi(s), Ꟈ^{\wp}\varpi(s))ds + \frac{1}{\Gamma(\gamma)}\int_a^t (t-s)^{\gamma-1} \cdot \mho(s, \varpi(s), Ꟈ^{\wp}\varpi(s))ds. \quad (12)$$

Proof. Byapplying the Lemma 4, we may reduce equation (5) to an equivalent equation

$$\varpi(t) = {}_a^t I^{\gamma}\mho(t, \varpi(t), Ꟈ^{\wp}\varpi(t)) + c_0 + c_1(t-a). \quad (13)$$

Using the boundary condition (6), we find that

$$c_0 = A \text{ and } c_1 = \frac{(B-a)}{(b-a)} - \frac{1}{(b-a)} {}_a^b I^{\gamma}\mho(t, \varpi(t), Ꟈ^{\wp}\varpi(t)). \quad (14)$$

Substituting the values of c_0 and c_1 in equation (13), the result is

$$\varpi(t) = A + \frac{(B-A)}{(b-a)}(t-a) - \frac{(t-a)}{(b-a)\Gamma(\gamma)}\int_a^b (b-s)^{\gamma-1} \cdot \mho(s, \varpi(s), Ꟈ^{\wp}\varpi(s))ds + \frac{1}{\Gamma(\gamma)}\int_a^t (t-s)^{\gamma-1} \cdot \mho(s, \varpi(s), Ꟈ^{\wp}\varpi(s))ds. \quad (15)$$

The derivative $Ꟈ^{\wp}\varpi(s)$ can be written as

$$Ꟈ^{\wp}\varpi(t) = \frac{(B-A)(t-a)^{1-\wp}}{(b-a)\Gamma(2-\wp)} - \frac{(t-a)^{1-\wp}}{(b-a)\Gamma(2-\wp)\Gamma(\gamma)} \cdot \int_a^b (b-s)^{\gamma-1}\mho(s, \varpi(s), Ꟈ^{\wp}\varpi(s))ds + \frac{1}{\Gamma(\gamma-\wp)}\int_a^t (t-s)^{\gamma-\wp-1}\mho(s, \varpi(s), Ꟈ^{\wp}\varpi(s))ds. \quad (16)$$

□

3. The Existence of Solution

This section deals with the existence and uniqueness of solution for the fractional differential equation (5) with boundary condition (6). Let $C([a, b], R)$ be the Banach space endowed with a Bielecki norm

$$\|\varpi\|_B = \int_a^b e^{-Nt}|\varpi(t)|dt. \quad (17)$$

Let the space $\Theta = \{\varpi(t) \in C[a, b]: Ꟈ^{\wp}\varpi(t) \in C[a, b]\}$, equipped with the norm

$$\|\varpi\|_{\Theta} = \max_{t\in[a,b]}\int_a^b e^{-Nt}|\varpi(t)|dt + \max_{t\in[a,b]}\int_a^b e^{-Nt}|Ꟈ^{\wp}\varpi(t)|dt, \quad (18)$$

which is a Banach space, where $N > 0$ is a fixed constant. Consider the following assumptions:

(H1) There exists a function $a_1(t) \in L_1(I)$ such that $|\mho(t, x, \varpi)| \le a_1(t) + \gamma_1 |x| + \gamma_2 |\varpi|$, where $\gamma_1, \gamma_2 \ge 0$.

(H2) There exists constants $\Omega_1, \Omega_2 > 0$ such that

$$|\mho(t, \varpi_1(t), Ꟈ^{\wp}\varpi_1(t)) - \mho(t, \varpi_2(t), Ꟈ^{\wp}\varpi_2(t))| \le \Omega_1|\varpi_1 - \varpi_2| + \Omega_2|Ꟈ^{\wp}\varpi_1 - Ꟈ^{\wp}\varpi_2|, \quad (19)$$

for each $t \in I$ and all $\varpi_1, \varpi_2 \in \mathbb{R}$. Let us set the following notation for convenience:

$$\beth_1 = \frac{1}{\Gamma(\gamma)}\int_a^b e^{-Nt}G(t, s)dt + \left(e^{-Na} - e^{-Nb}\right)\frac{(B\Gamma(\gamma) + D(b-a))}{N\Gamma(\gamma)},$$
$$\beth_2 = \frac{(b-a)\left(e^{-Na} - e^{-Nb}\right)}{N\Gamma(\gamma)}\left((b-a) + \frac{1}{\gamma}\right),$$
$$\zeta_1 = \frac{1}{\Gamma(\gamma-\wp)}\int_a^b e^{-Nt}G(t, s)dt + \left(e^{-Na} - e^{-Nb}\right)\frac{((B-A)\Gamma(\gamma) + (b-a)D)}{N\Gamma(2-\wp)\Gamma(\gamma)},$$
$$\zeta_2 = \frac{\left(e^{-Na} - e^{-Nb}\right)(b-a)}{N}\left(\frac{(b-a)}{\Gamma(2-\wp)\Gamma(\gamma)} + \frac{1}{\Gamma(\gamma-\wp+1)}\right). \quad (20)$$

Our results are based on the Brouwer's fixed-point theorem and Banach contraction principle.

Theorem 8. *Suppose that (H1) holds. Then, the boundary value problem (5) and (6) has at least one solution on $C(I, \mathbb{R})$.*

Proof. Define the operator $J: C(I, \mathbb{R}) \longrightarrow C(I, \mathbb{R})$ as

$$J\varpi(t) = A + \frac{(B-A)}{(b-a)}(t-a) - \frac{(t-a)}{(b-a)\Gamma(\gamma)}\int_a^b (b-s)^{\gamma-1} \cdot \mho(s, \varpi(s), Ꟈ^{\wp}\varpi(s))ds + \frac{1}{\Gamma(\gamma)}\int_a^t (t-s)^{\gamma-1} \cdot \mho(s, \varpi(s), Ꟈ^{\wp}\varpi(s))ds, \quad (21)$$

and the operator $Ꟈ^{\wp}J\varpi(t)$ can be written as

$$Ꟈ^{\wp}J\varpi(t) = \frac{(B-A)(t-a)^{1-\wp}}{(b-a)\Gamma(2-\wp)} - \frac{(t-a)^{1-\wp}}{(b-a)\Gamma(2-\wp)\Gamma(\gamma)} \cdot \int_a^b (b-s)^{\gamma-1}\mho(s, \varpi(s), Ꟈ^{\wp}\varpi(s))ds + \frac{1}{\Gamma(\gamma-\wp)} \cdot \int_a^t (t-s)^{\gamma-\wp-1}\mho(s, \varpi(s), Ꟈ^{\wp}\varpi(s))ds. \quad (22)$$

Firstly, we will prove that $JH_d \subset H_d$, where $H_d = \{\varpi \in C : \|\varpi\|_{\theta} \le d\}$, and choose $d \ge \eta_1/(1 - \eta_2(\gamma_1 + \gamma_2))$, for $\varpi \in H_d$, and the following is obtained:

$$\int_a^b e^{-Nt}|J(\varpi)(t)|dt \le \int_a^b e^{-Nt}\left(A+\frac{(B-A)}{(b-a)}(t-a)\right)dt + \frac{1}{(b-a)\Gamma(\gamma)}\int_a^b (t-a)e^{-Nt}\int_a^b (b-s)^{\gamma-1} \cdot |\mho(s,\varpi(s),Ð^{\wp}\varpi(s))|dsdt + \frac{1}{\Gamma(\gamma)}\int_a^b e^{-Nt}\int_a^t (t-s)^{\gamma-1}|\mho(s,\varpi(s),Ð^{\wp}\varpi(s))|dsdt, \tag{23}$$

and by using (H1), then

$$\int_a^b e^{-Nt}|J(\varpi)(t)|dt \le B\int_a^b e^{-Nt}dt + \frac{(b-a)}{\Gamma(\gamma)}\int_a^b e^{-Nt}\int_a^b \cdot (a_1(s)+\gamma_1|\varpi(s)|+\gamma_2|Ð^{\wp}\varpi(s)|)dsdt + \frac{1}{\Gamma(\gamma)}\int_a^b e^{-Nt}\int_a^t (t-s)^{\gamma-1} \cdot (a_1(s)+\gamma_1|\varpi(s)|+\gamma_2|Ð^{\wp}\varpi(s)|)dsdt,$$

$$\int_a^b e^{-Nt}|J(\varpi)(t)|dt \le \frac{1}{\Gamma(\gamma)}\int_a^b e^{-Nt}G(t,s)dt + \frac{\left(e^{-Na}-e^{-Nb}\right)(B\Gamma(\gamma)+D(b-a))}{N\Gamma(\gamma)} + \frac{(b-a)\left(e^{-Na}-e^{-Nb}\right)}{N\Gamma(\gamma)}\left((b-a)+\frac{1}{\gamma}\right)(\gamma_1 d+\gamma_2 d), \tag{24}$$

where

$$D=\int_a^b a_1(s)ds, \quad G(t,s)=\int_a^t (t-s)^{\gamma-1}a_1(s)ds. \tag{25}$$

Thus, we have

$$\Omega_1\int_a^b e^{-Nt}|\varpi(t)|dt \le \Omega_1 ℶ_1 + \Omega_1 ℶ_2(\gamma_1 d+\gamma_2 d). \tag{26}$$

Now to establish the bounded of equation (22) by (H1), get

$$\int_a^b e^{-Nt}|Ð^{\wp}J(\varpi)(t)|dt \le \frac{(B-A)}{\Gamma(2-\wp)}\int_a^b e^{-Nt}dt + \frac{(b-a)}{\Gamma(2-\wp)\Gamma(\gamma)} \cdot \int_a^b e^{-Nt}\int_a^b (a_1(s)+\gamma_1|\varpi(s)|+\gamma_2|Ð^{\wp}\varpi(s)|)dsdt + \frac{1}{\Gamma(\gamma-\wp)} \cdot \int_a^b e^{-Nt}\int_a^t (t-s)^{\gamma-\wp-1}(a_1(s)+\gamma_1|\varpi(s)|+\gamma_2|Ð^{\wp}\varpi(s)|)dsdt,$$

$$\int_a^b e^{-Nt}|Ð^{\wp}J(\varpi)(t)|dt \le \frac{1}{\Gamma(\gamma-\wp)}\int_a^b e^{-Nt}G(t,s)dt + \left(e^{-Na}-e^{-Nb}\right)\frac{((B-A)\Gamma(\gamma)+(b-a)D)}{N\Gamma(2-\wp)\Gamma(\gamma)} + \frac{\left(e^{-Na}-e^{-Nb}\right)(b-a)}{N}\left(\frac{(b-a)}{\Gamma(2-\wp)\Gamma(\gamma)}+\frac{1}{\Gamma(\gamma-\wp+1)}\right) \cdot (\gamma_1 d+\gamma_2 d),$$

$$\Omega_2\int_a^b e^{-Nt}|Ð^{\wp}J(\varpi)(t)|dt \le \Omega_2\zeta_1 + \Omega_2\zeta_2(\gamma_1 d+\gamma_2 d),$$

$$\|J\varpi\|_\theta \le \eta_1 + d\eta_2(\gamma_1+\gamma_2) \le d, \tag{27}$$

where $\eta_1=\Omega_2\zeta_1+\Omega_1\omega_1$ and $\eta_2=\Omega_1\zeta_2+\Omega_1\omega_2$, and we have $\|Jy\|_\theta \le d$. Hence, $J: H_d \longrightarrow H_d$.

According to the Brouwer fixed-point theorem, the boundary value problem (5) and (6) has at least one solution. □

Theorem 9. *Assume that (H2) holds. If*

$$(\lambda_2\Omega_1+\lambda_3\Omega_2)<1, \tag{28}$$

where

$$\lambda_1=\frac{(b-a)\left(2-e^{-N(a-b)}-e^{-N(b-a)}\right)}{N^2\Gamma(\gamma)}, \quad \lambda_2=\left(\lambda_1+\frac{1}{N^\gamma}\right), \quad \lambda_3=\left(\frac{\lambda_1}{\Gamma(2-\wp)}+\frac{1}{N^{\gamma-\wp}}\right), \tag{29}$$

then the boundary value problem (5) and (6) has a unique solution.

Proof. We prove that J is a contraction. Let $\varpi_1,\varpi_2\in C(I,\mathbb{R})$. Then,

$$|J(\varpi_1)(t)-J(\varpi_2)(t)| \le \frac{(t-a)}{(b-a)\Gamma(\gamma)}\int_a^b (b-s)^{\gamma-1} \cdot |\mho(s,\varpi_1(s),Ð^{\wp}\varpi_1(s))-\mho(s,\varpi_2(s),Ð^{\wp}\varpi_2(s))|ds + \frac{1}{\Gamma(\gamma)}\int_a^t (t-s)^{\gamma-1}|\mho(s,\varpi_1(s),Ð^{\wp}\varpi_1(s)) - \mho(s,\varpi_2(s),Ð^{\wp}\varpi_2(s))|ds, \tag{30}$$

and by the condition (H2), the result is

$$|J(\varpi_1)(t)-J(\varpi_2)(t)|\le \frac{(t-a)}{(b-a)\Gamma(\gamma)}\int_a^b (b-s)^{\gamma-1}[\Omega_1|\varpi_1(s)-\varpi_2(s)|+\Omega_2|Đ^{\wp}\varpi_1(s)-Đ^{\wp}\varpi_2(s)|]ds+\frac{1}{\Gamma(\gamma)}\int_a^t (t-s)^{\gamma-1}\cdot[\Omega_1|\varpi_1(s)-\varpi_2(s)|+\Omega_2|Đ^{\wp}\varpi_1(s)-Đ^{\wp}\varpi_2(s)|]ds, \tag{31}$$

$$\int_a^b e^{-Nt}|J(\varpi_1)(t)-J(\varpi_2)(t)|dt\le \frac{(b-a)}{\Gamma(\gamma)}\int_a^b e^{-N(t-s)}\cdot\int_a^b e^{-Ns}[\Omega_1|\varpi_1(s)-\varpi_2(s)|+\Omega_2|Đ^{\wp}\varpi_1(s)-Đ^{\wp}\varpi_2(s)|]dsdt+\frac{1}{\Gamma(\gamma)}\int_a^b e^{-N(t-s)}\int_a^t (t-s)^{\gamma-1}e^{-Ns}[\Omega_1|\varpi_1(s)-\varpi_2(s)|+\Omega_2|Đ^{\wp}\varpi_1(s)-Đ^{\wp}\varpi_2(s)|]dsdt, \tag{32}$$

and using Holder inequality, the first term of equation (32) becomes

$$\int_a^b e^{-Nt}|J(\varpi_1)(t)-J(\varpi_2)(t)|dt\le \lambda_1\int_a^b e^{-Ns}[\Omega_1|\varpi_1(s)-\varpi_2(s)|+\Omega_2|Đ^{\wp}\varpi_1(s)-Đ^{\wp}\varpi_2(s)|]ds+\frac{1}{\Gamma(\gamma)}\int_a^b\int_s^b e^{-N(t-s)}\cdot(t-s)^{\gamma-1}dt e^{-Ns}[\Omega_1|\varpi_1(s)-\varpi_2(s)|+\Omega_2|Đ^{\wp}\varpi_1(s)-Đ^{\wp}\varpi_2(s)|]ds, \tag{33}$$

and for the second term, let $z=N(t-s)$, $dz=Ndt$, and it follows

$$\int_a^b e^{-Nt}|J(\varpi_1)(t)-J(\varpi_2)(t)|dt\le \lambda_1\int_a^b e^{-Ns}[\Omega_1|\varpi_1(s)-\varpi_2(s)|+\Omega_2|Đ^{\wp}\varpi_1(s)-Đ^{\wp}\varpi_2(s)|]ds+\frac{1}{N^{\gamma}\Gamma(\gamma)}\cdot\int_a^b\int_0^{N(b-s)} e^{-Nz}z^{\gamma-1}dz\, e^{-Ns}[\Omega_1|\varpi_1(s)-\varpi_2(s)|+\Omega_2|Đ^{\wp}\varpi_1(s)-Đ^{\wp}\varpi_2(s)|]ds,\le \lambda_1\int_a^b e^{-Ns}[\Omega_1|\varpi_1(s)-\varpi_2(s)|+\Omega_2|Đ^{\wp}\varpi_1(s)-Đ^{\wp}\varpi_2(s)|]ds+\frac{1}{N^{\gamma}}\int_a^b e^{-Ns}[\Omega_1|\varpi_1(s)-\varpi_2(s)|+\Omega_2|Đ^{\wp}\varpi_1(s)-Đ^{\wp}\varpi_2(s)|]ds, \tag{34}$$

$$\Omega_1\int_a^b e^{-Nt}|J(\varpi_1)(t)-J(\varpi_2)(t)|dt\le \Omega_1\lambda_2\cdot\int_a^b e^{-Ns}[\Omega_1|\varpi_1(s)-\varpi_2(s)|+\Omega_2|Đ^{\wp}\varpi_1(s)-Đ^{\wp}\varpi_2(s)|]ds. \tag{35}$$

By the same way,

$$\int_a^b e^{-Nt}|Đ^{\wp}J(\varpi_1)(t)-Đ^{\wp}J(\varpi_2)(t)|dt\le \frac{(b-a)}{\Gamma(2-\wp)\Gamma(\gamma)}\cdot\int_a^b\int_a^b e^{-N(t-s)}e^{-Ns}[\Omega_1|\varpi_1(s)-\varpi_2(s)|+\Omega_2|Đ^{\wp}\varpi_1(s)-Đ^{\wp}\varpi_2(s)|]dsdt+\frac{1}{\Gamma(\gamma-\wp)}\int_a^b\int_a^t e^{-N(t-s)}(t-s)^{\gamma-\wp-1}e^{-Ns}\cdot[\Omega_1|\varpi_1(s)-\varpi_2(s)|+\Omega_2|Đ^{\wp}\varpi_1(s)-Đ^{\wp}\varpi_2(s)|]dsdt,$$

$$\Omega_2\int_a^b e^{-Nt}|Đ^{\wp}J(\varpi_1)(t)-Đ^{\wp}J(\varpi_2)(t)|dt\le \Omega_2\lambda_3\int_a^b e^{-Ns}\cdot[\Omega_1|\varpi_1(s)-\varpi_2(s)|+\Omega_2|Đ^{\wp}\varpi_1(s)-Đ^{\wp}\varpi_2(s)|]ds. \tag{36}$$

Then, we have

$$\|J(\varpi_1)(t)-J(\varpi_2)(t)\|_{\theta}\le(\lambda_2\Omega_1+\lambda_3\Omega_2)\|\varpi_1-\varpi_2\|. \tag{37}$$

Hence, we conclude that the problem (5) and (6) has a unique solution by the contraction mapping principle. □

Example 1. Consider the following fractional boundary value problem:

$$\begin{cases} Đ^{1.2}\varpi=\dfrac{e^t}{10(1+e^t)}\varpi+\dfrac{5}{(6+t)}Đ^{0.3}\varpi,\ t\in[0,1],\\ \varpi(0)=\varpi(1)=1.\end{cases} \tag{38}$$

Here, $\gamma=1.2$, $\wp=0.3$, and $N=0.47$; then (H2) is satisfied with $\Omega_1=0.073105857$ and $\Omega_2=0.833333$, and one can arrive at the following results:

$$\lambda_1=\frac{(b-a)\left(2-e^{-N(a-b)}-e^{-N(b-a)}\right)}{N^2\Gamma(\gamma)}=-1.109321597426112,$$
$$\lambda_2\Omega_1=\left(\lambda_1+\frac{1}{N^{\gamma}}\right)\Omega_1=0.099800501185966,$$
$$\lambda_3\Omega_2=\left(\frac{\lambda_1}{\Gamma(2-\wp)}+\frac{1}{N^{\gamma-\wp}}\right)\Omega_2=0.626725087609827,$$
$$\lambda_2\Omega_1+\lambda_3\Omega_2=0.726525588795793<1. \tag{39}$$

Hence, by Theorem 9, the boundary value problem (38) has a unique solution on $[0,1]$.

Example 2. Consider the following fractional boundary value problem:

$$\begin{cases} Đ^{3/2}\varpi=\dfrac{(t+1)e^{-t}}{(t+3)^2}\dfrac{\varpi}{(2+\varpi)}+\dfrac{e^t}{(te^t+5)}\dfrac{Đ^{1/2}\varpi}{\left(5+Đ^{1/2}\varpi\right)},\ t\in[0,1],\\ \varpi(0)=1\quad \varpi(1)=1.\end{cases} \tag{40}$$

Here, $\gamma = 1.5$, $\wp = 0.5$, and $N = 1/2$, and according to the Lipschitz condition, we have

$$\begin{gathered} |\mho(t, \varpi_1(t), \text{Đ}^{\wp}\varpi_1(t)) - \mho(t, \varpi_2(t), \text{Đ}^{\wp}\varpi_2(t))| \\ \leq \frac{(t+1)e^{-t}}{(t+3)^2}|\varpi_1 - \varpi_2| + \frac{e^t}{(te^t+5)}|\text{Đ}^{\wp}\varpi_1 - \text{Đ}^{\wp}\varpi_2|, \\ \Omega_1 = 0.1111111, \\ \Omega_2 = 0.352187428, \\ \lambda_1 = -1.152083842554695. \end{gathered} \tag{41}$$

Finally, the following obtained $\lambda_2\Omega_1 + \lambda_3\Omega_2 = 0.432795980868753 < 1$. Therefore, from Theorem 9, we conclude that boundary value problem (40) has a unique solution.

4. Iterative Numerical Schema

For the solution of the boundary value problem (5) and (6), an iterative schema is provided. Starting with $\varpi_0(t)$ and $\text{Đ}^{\wp}\varpi_0(t)$, the width $h = (b-a)/N$, and $t_i = a + ih, i = 0, 1, \cdots N$, the integral equations (21) and (22) are numerically evaluated to obtain the sequence $\{\varpi_n(t)\}$, $a \leq t \leq b$. The trapezoidal rule is most convenient for estimating $\{\varpi_n(t_k)\}$, $k = 0, 1..\cdots N$, and the approximation for (21) and (22) becomes

$$\varpi_{n+1}(t_k) = A + \frac{(B-A)k}{N} - \frac{k}{N}\aleph_{a,b,n} + \aleph_{a,t_k,n}, \tag{42}$$

$$\text{Đ}^{\wp}\varpi_{n+1}(t_k) = \frac{(B-A)k^{1-\wp}h^{-\wp}}{N\Gamma(2-\wp)} - \frac{k^{1-\wp}h^{-\wp}}{N\Gamma(2-\wp)}\aleph_{a,b,n} + J_{a,t_k,n}, \tag{43}$$

where

$$\begin{aligned} \aleph_{a,b,n} &= {}_a^b I^{\gamma}\mho(s, \varpi, \text{Đ}^{\wp}\varpi), \\ \aleph_{a,t_k,n} &= {}_a^{t_k} I^{\gamma}\mho(s, \varpi, \text{Đ}^{\wp}\varpi), \\ J_{a,t_k,n} &= {}_a^{t_k} I^{\gamma-\wp}\mho(s, \varpi, \text{Đ}^{\wp}\varpi), \end{aligned} \tag{44}$$

and the process of iteration may be terminated by setting a criterion

$$\max|\varpi_{n+1}(t_k) - \varpi_n(t_k)| \leq \varepsilon, a \leq t \leq b, \tag{45}$$

where ε is a constant which is to be taken smaller than the discretization error $O(h^2)$ in $\varpi(t_k)$ for the solution of the boundary value problem (5) and (6).

5. Convergence of $\{\varpi_n(t_k)\}$ and $\{\text{Đ}^{\gamma}\varpi_n(t_k)\}$

To prove the convergence of the iteration schema (42) and (43), let us suppose that $\mho : I \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ be continuous function that satisfies the conditions.

(H3) There exists a constant $\chi, \chi_1, \chi_2 > 0$ such that $|\mho(t, \varpi(t), \text{Đ}^{\wp}\varpi(t))| \leq \chi$,

$$\begin{aligned} \left|\frac{d^2}{ds^2}(b-s)^{\gamma-1}\mho(s, \varpi(s), \text{Đ}^{\wp}\varpi(s))\right| &\leq \chi_1, \\ \left|\frac{d^2}{ds^2}(b-s)^{\gamma-\wp-1}\mho(s, \varpi(s), \text{Đ}^{\wp}\varpi(s))\right| &\leq \chi_2, \end{aligned} \tag{46}$$

for each $t, s \in I$ and all $\varpi, \varpi_1, \varpi_2, \cdots \in R$. After simplifying equations (42) and (43), we have

$$\begin{aligned} \varpi_{n+1}(t_k) = {} & A + \frac{(B-A)k}{N} + \frac{h^{\gamma}}{2\Gamma(\gamma)}\left[k^{\gamma-1} - \frac{k}{N}N^{\gamma-1}\right]\mho_n(a) \\ & + \frac{kh^{\gamma}}{\Gamma(\gamma)}\sum_{j=1}^{k-1}\left[(k-j)^{\gamma-1} - \frac{k}{N}(N-j)^{\gamma-1}\right]\mho_n(t_j) \\ & - \frac{kh^{\gamma}}{N\Gamma(\gamma)}\sum_{j=k}^{N-1}(N-j)^{\gamma-1}\mho_n(t_j) \\ & - \frac{h^3k}{12\Gamma(\gamma)}\left((t_k-s)^{\gamma-1}\mho_n(t)\right)'', \end{aligned} \tag{47}$$

$$\begin{aligned} \text{Đ}^{\wp}\varpi_{n+1}(t_k) = {} & \frac{(B-A)k^{1-\wp}h^{-\wp}}{N\Gamma(2-\wp)} \\ & - h^{\gamma-\wp}\sum_{j=1}^{k}\left[\frac{(k-j)^{1-\wp}}{\Gamma(\gamma-\wp)} - \frac{k^{1-\wp}(N-j)^{\gamma-1}}{N\Gamma(2-\wp)\Gamma(\gamma)}\right]\mho_n(t_j) \\ & + \frac{h^{\gamma-\wp}}{2\Gamma(2-\wp)\Gamma(\gamma-\wp)\Gamma(\gamma)} \\ & \cdot\left[Nk^{\gamma-\wp-1}\Gamma(2-\wp)\Gamma(\gamma) - N^{\gamma-1}k^{1-\wp}\Gamma(\gamma-\wp)\right]\mho_n(a) \\ & + \frac{k^{1-\wp}h^{\gamma-\wp}}{N\Gamma(2-\wp)\Gamma(\gamma)}\sum_{j=k+1}^{N-1}(N-j)^{\gamma-1}\mho_n(t_j) \\ & - \frac{h^3k}{12\Gamma(\gamma-\wp)}\left((t_k-s)^{\gamma-\wp-1}\mho_n(t)\right)'', \end{aligned} \tag{48}$$

$$\varpi_{0,k} = A + \frac{(B-A)}{(b-a)}(t_k - a), \quad \text{Đ}^{\wp}\varpi_{0,k} = A + \frac{(B-A)(t_k-a)^{1-\wp}}{(b-a)\Gamma(2-\wp)}. \tag{49}$$

Now, Computing $|\varpi_{n+1,k} - \varpi_{n,k}|$ and $|\text{Đ}^{\wp}\varpi_{n+1,k} - \text{Đ}^{\wp}\varpi_{n,k}|$ from equations (47) and (48) and by using (H2) and (H3), the result is

$$\begin{gathered} |\varpi_{1,k}(t_k) - \varpi_{0,k}(t_k)| \leq \frac{kh^{\gamma}|N^{\gamma-1} - k^{\gamma-1}|}{\Gamma(\gamma+1)}\chi + \Delta_1, \\ |\text{Đ}^{\wp}\varpi_{1,k}(t_k) - \text{Đ}^{\wp}\varpi_{0,k}(t_k)| \\ \leq \frac{h^{\gamma-\wp}k^{1-\wp}|\Gamma(\gamma+1)\Gamma(2-\wp)k^{\gamma-1} - \Gamma(\gamma-\wp+1)N^{\gamma-1}|}{\Gamma(\gamma+1)\Gamma(2-\wp)\Gamma(\gamma-\wp+1)}\chi + \Delta_2. \end{gathered} \tag{50}$$

Let

$$P = \frac{kh^{\gamma}\left|N^{\gamma-1} - k^{\gamma-1}\right|}{\Gamma(\gamma+1)},$$
$$Q = \frac{h^{\gamma-\wp}k^{1-\wp}\left|\Gamma(\gamma+1)\Gamma(2-\wp)k^{\gamma-1} - \Gamma(\gamma-\wp+1)N^{\gamma-1}\right|}{\Gamma(\gamma+1)\Gamma(2-\wp)\Gamma(\gamma-\wp+1)},$$
$$\Delta_1 = \frac{kh^3}{6\Gamma(\gamma)}\chi_1,$$
$$\Delta_2 = \frac{kh^3}{6\Gamma(\gamma-\wp)}\chi_2,$$
$$\left|\varpi_{2,k}(t_k) - \varpi_{1,k}(t_k)\right| \le P(\Omega_1|\varpi_1 - \varpi_0| + \Omega_2|Đ^{\wp}\varpi_1 - Đ^{\wp}\varpi_0|) + \Delta_1,$$
$$\left|Đ^{\wp}\varpi_{2,k}(t_k) - Đ^{\wp}\varpi_{1,k}(t_k)\right| \le Q(\Omega_1|\varpi_1 - \varpi_0| + \Omega_2|Đ^{\wp}\varpi_1 - Đ^{\wp}\varpi_0|) + \Delta_2, \tag{51}$$

and so on, in general

$$\begin{aligned}\left|\varpi_{n+1,k}(t_k) - \varpi_{n,k}(t_k)\right| &\le P(P\Omega_1 + Q\Omega_2)^{n-1} \\ &\cdot [\Omega_1|\varpi_1 - \varpi_0| + \Omega_2|Đ^{\wp}\varpi_1 - Đ^{\wp}\varpi_0|] \\ &+ P\sum_{i=0}^{n-2}(P\Omega_1 + Q\Omega_2)^i(\Omega_1\Delta_1 + \Omega_2\Delta_2) + \Delta_1,\end{aligned}$$

$$\begin{aligned}\left|Đ^{\wp}\varpi_{n+1,k}(t_k) - Đ^{\wp}\varpi_{n,k}(t_k)\right| &\le Q(P\Omega_1 + Q\Omega_2)^{n-1} \\ &\cdot [\Omega_1|\varpi_1 - \varpi_0| + \Omega_2|Đ^{\wp}\varpi_1 - Đ^{\wp}\varpi_0|] \\ &+ P\sum_{i=0}^{n-2}(P\Omega_1 + Q\Omega_2)^i(\Omega_1\Delta_1 + \Omega_2\Delta_2) + \Delta_2.\end{aligned} \tag{52}$$

If $(P\Omega_1 + Q\Omega_2) < 1$, then the process of iteration is convergent and the bound of truncation error is given by

$$\begin{aligned}|E_n(t_k)| &= |\varpi(t_k) - \varpi_n(t_k)| \\ &= \sum_{i=n}^{\infty}\left|\varpi_{i+1,k} - \varpi_{i,k}\right| \le \Delta_1 \le \psi_1 h^2, \\ |Đ^{\wp}E_n(t_k)| &= |Đ^{\wp}\varpi(t_k) - Đ^{\wp}\varpi_n(t_k)| \\ &= \sum_{i=n}^{\infty}\left|Đ^{\wp}\varpi_{i+1,k} - Đ^{\wp}\varpi_{i,k}\right| \le \Delta_2 \le \psi_2 h^2.\end{aligned} \tag{53}$$

6. Numerical Illustrations

To show the efficiency of this method, we will approximate the solutions for some fractional differential equations of order $1 < \gamma \le 2$ and $0 < \wp \le 1$, using a prepared program in Matlab. To solve these problems, we used equations (42) and (43) to obtain the sequences $\{y_n(t_k)\}, \{Đ^{\wp}y_n(t_k)\}$. By using the exact solutions, we computed the error at each pivotal point. The partial output of these error terms is presented in Tables 1 and 2.

Table 1: Absolute error for the numerical solutions of Examples 3–5.

t	Example 3 iteration 8		Example 4 iteration 6		Example 5 iteration 7	
	$h = 1/20$	$h = 1/40$	$h = 1/20$	$h = 1/40$	$h = 0.01$	$h = 0.005$
0.1	.243e-2	.876e-3	.109e-2	.393e-3	.362e-3	.185e-3
0.2	.278e-2	.996e-3	.228e-2	.822e-3	.677e-3	.347e-3
0.3	.276e-2	.986e-3	.352e-2	.126e-2	.925e-3	.473e-3
0.4	.255e-2	.910e-3	.469e-2	.168e-2	.108e-2	.556e-3
0.5	.224e-2	.797e-3	.564e-2	.202e-2	.115e-2	.592e-3
0.6	.185e-2	.659e-3	.618e-2	.221e-2	.113e-2	.577e-3
0.7	.142e-2	.505e-3	.610e-2	.218e-2	.999e-3	.509e-3
0.8	.964e-3	.342e-3	.520e-2	.185e-2	.765e-3	.390e-3
0.9	.486e-3	.172e-3	.324e-2	.115e-2	.430e-3	.219e-3

Table 2: Error for the numerical solution of Example 6.

t	Example6 iteration 8	
	$h = 0.01$	$h = 0.005$
1.1	.194e-6	.208e-7
1.2	.262e-6	.566e-7
1.3	.299e-6	.560e-7
1.4	.338e-6	.103e-7
1.5	.376e-6	.603e-7
1.6	.392e-6	.125e-6
1.7	.367e-6	.160e-6
1.8	.290e-6	.150e-6
1.9	.164e-6	.935e-7

Example 3. Consider the fractional boundary value problem

$$Đ^{3/2}\varpi + \frac{1}{3}\varpi + \frac{1}{4}Đ^{1/2}\varpi = -\frac{7t^{0.5}}{2\sqrt{\pi}} + \frac{t}{3} - \frac{2t^{1.5}}{3\sqrt{\pi}} - \frac{t^2}{3}, t \in [0, 1], \tag{54}$$

$\varpi(0) = 0, \varpi(1) = 0$. The exact solution is $\varpi(t) = t(1 - t)$.

A comparison of the absolute errors, computed by the proposed method with $h = 1/20$, $h = 1/40$, and $N = 8$ at $t \in [0, 1]$, is calculated in Table 1. The approximate solutions obtained with the exact solution of corresponding fractional-order equation when $\gamma = 3/2$, $\wp = 1/2$, and $h = 1/20$ are given in Figure 1(a).

Example 4. Consider the fractional boundary value problem

$$Đ^{3/2}\varpi + \varpi = t^5 - t^4 + \frac{128t^{3.5}}{7\sqrt{\pi}} - \frac{64t^{2.5}}{5\sqrt{\pi}}, t \in [0, 1], \tag{55}$$

$\varpi(0) = 0, \varpi(1) = 0$. The exact solution is $\varpi(t) = t^5 - t^4$.

In Figure 1(b), the approximate solutions obtained with $h = 1/40$ together with the exact solution of this problem are plotted. Furthermore, by considering $h = 1/20$, $h = 1/40$, and $N = 6$, the absolute errors at some selected points are reported in Table 1.

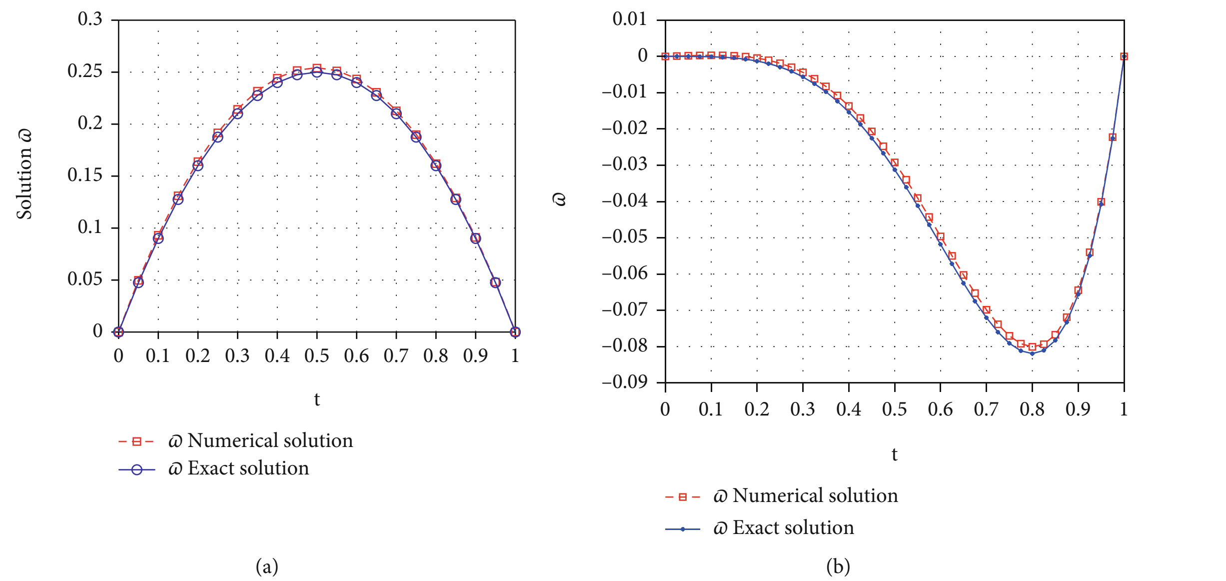


Figure 1: A comparison between exact and approximate solutions: (a) Example 3 when $\gamma = 3/2$, $\wp = 1/2$, and $h = 1/20$. (b) Example 4 when $\gamma = 3/2$ and $h = 1/40$.

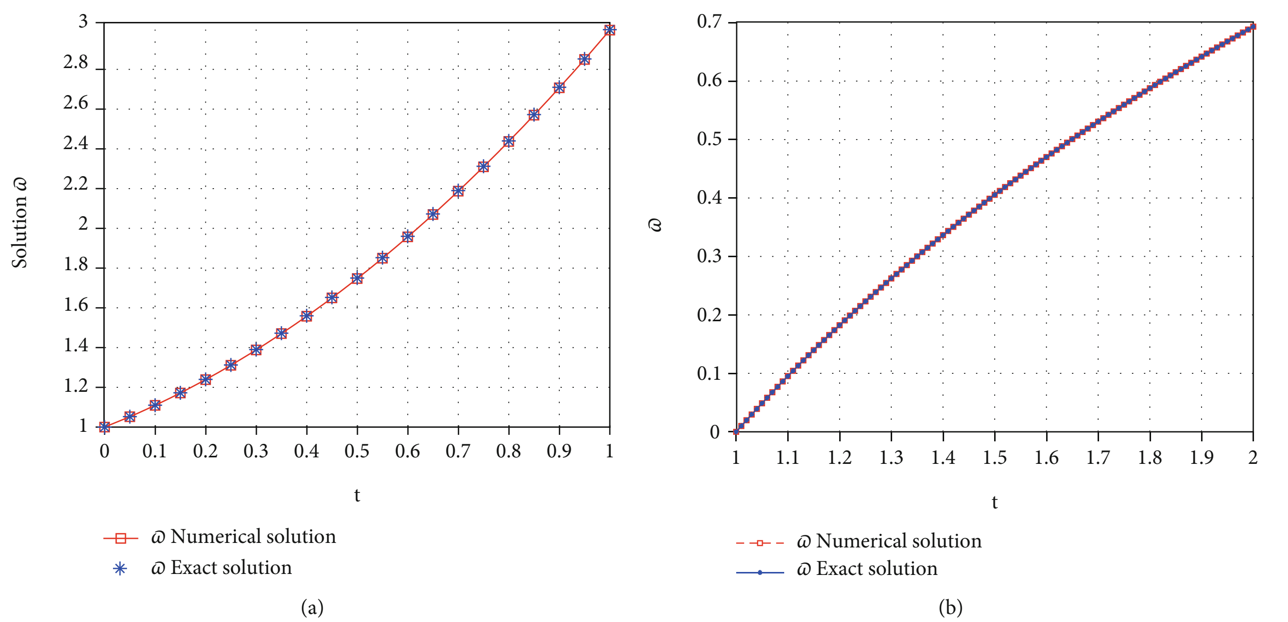


Figure 2: Exact and approximate solutions with $h = 0.01$. (a) Example 5 when $\gamma = 2$ and $\wp = 1/2$. (b) Example 6 when $\gamma = 2$ and $\wp = 1$.

Example 5. Consider the fractional boundary value problem

$$\varpi'' + \varpi + \mathcal{D}^{1/2}\varpi = t^2 + t + \frac{2t^{1.5}}{\Gamma(5/2)} + \frac{t^{0.5}}{\Gamma(3/2)} + 3, t \in [0, 1], \quad (56)$$

$\varpi(0) = 1, \varpi(1) = 3$. The exact solution is $\varpi(t) = t^2 + t + 1$.

Table 1 shows the absolute error between exact and numerical solutions when $h = 0.01$, $h = 0.005$, and $N = 7$. Figure 2(a) compares both the exact and numerical solutions for the fractional differential equation (56) with $\gamma = 2$, $\wp = 1/2$, and $h = 0.01$ in some points $t \in [0, 1]$.

Example 6. Consider the boundary value problem

$$\varpi'' = -\left(t\varpi'\right)^2 e^{-2\varpi}, t \in [1, 2], \quad (57)$$

$\varpi(1) = 0, \varpi(2) = \ln(2)$. The exact solution is $\varpi(t) = \ln(t)$.

The numerical results of Example 6 for the values $\gamma = 2$, $\wp = 1$ and $h = 0.01$ are shown in Figure 2(b). In addition,

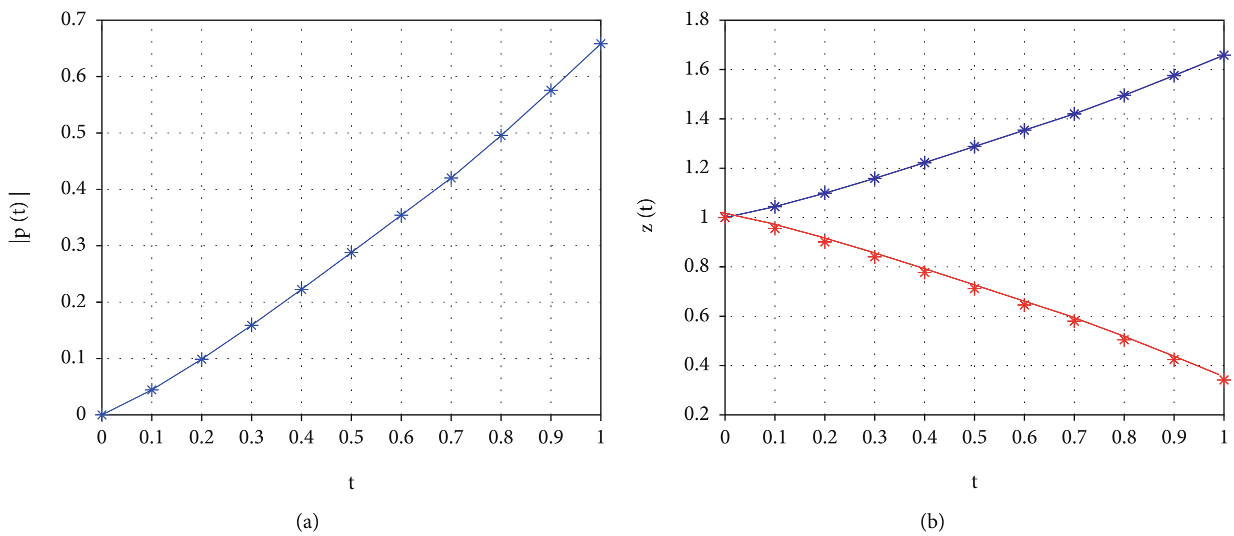


FIGURE 3: (a) The function $|p(t)|$ for $t \in [0, 1]$. (b) The function $z(t)$ for $t \in [0, 1]$.

the absolute error is presented in Table 2 when $h = 0.01$, $h = 0.005$, and $N = 8$.

7. Stability Theorems

In this section, we investigate the Ulam-Hyers and Ulam-Hyers-Rassias stability of the boundary value problem (5) and (6). For the definitions of Ulam-Hyers stability and Ulam-Hyers-Rassias stability, see [46]. For $\mho : I \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$, the norm is defined as

$$\|\varpi\| = \max_{t\in[a,b]} \Omega_1 |\varpi(t)| + \max_{t\in[a,b]} \Omega_2 |Đ^{\wp} \varpi(t)|. \tag{58}$$

Definition 10. Equation (5) is Ulam-Hyers stability if there exists a real number $c_f > 0$ such that for each $\varepsilon > 0$ and for each solution $z \in C^1(I, \mathbb{R})$ of the inequality

$$|{}^{c}Đ^{\gamma} z(t) - \mho(t, z(t), Đ^{\wp} z(t))| \leq \varepsilon, \quad t \in I, \tag{59}$$

there exists a solution $\varpi \in C^1(I, \mathbb{R})$ of equation (5) with

$$|z(t) - \varpi(t)| \leq c_f \varepsilon, \quad t \in I. \tag{60}$$

Definition 11. Equation (5) is Ulam-Hyers-Rassias stability with respect to $\varphi \in C^1(I, \mathbb{R}_+)$ if there exists a real number $c_f > 0$ such that for each $\varepsilon > 0$ and for each solution $z \in C(I, \mathbb{R})$ of the inequality

$$|{}^{c}Đ^{\gamma} z(t) - \mho(t, z(t), Đ^{\wp} z(t))| \leq \varepsilon \varphi(t), \quad t \in I, \tag{61}$$

there exists a solution $\varpi \in C^1(I, \mathbb{R})$ of equation (5) with

$$|z(t) - \varpi(t)| \leq c_f \varepsilon \varphi(t), \quad t \in I. \tag{62}$$

For proving, we need the following hypothesis.

(H4) There exists an increasing function $\varphi \in C(I, \mathbb{R}_+)$, and there exists $\Lambda_\varphi, \lambda_\varphi > 0$ such that for any $t \in I$, we have

$$\begin{gathered} \int_a^t \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)} \varphi(s) ds \leq \Lambda_\varphi \varphi(t), \\ \int_a^t \frac{(t-s)^{\gamma-\wp-1}}{\Gamma(\gamma-\wp)} \varphi(s) ds \leq \lambda_\varphi \varphi(t). \end{gathered} \tag{63}$$

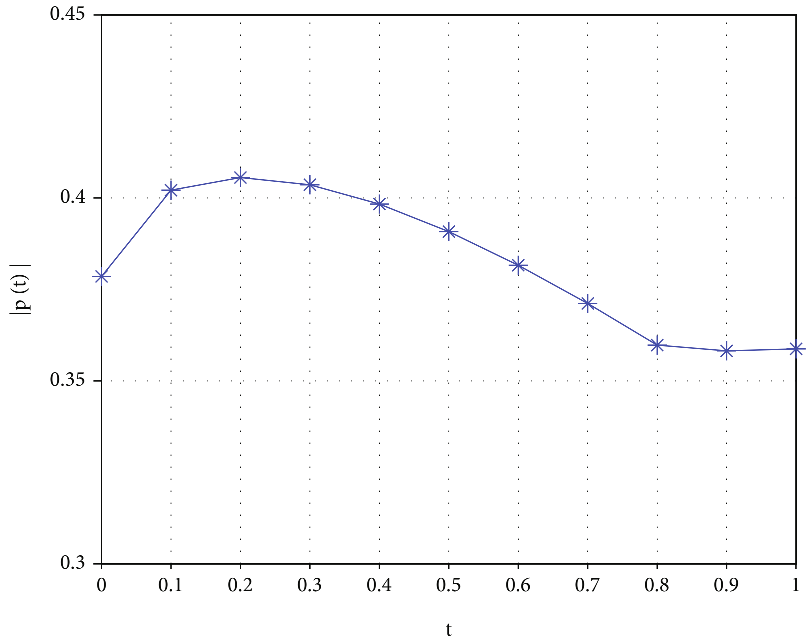
Theorem 12. *Assume that $\mho : I \times R \times R \longrightarrow R$ is a continuous function and (H2) holds with $(P\Omega_1 + Q\Omega_2) < 1$. Then, the problem (5) and (6) is Ulam-Hyers stability.*

Proof. Let $z(t_k) \in C(I, \mathbb{R})$ be a solution of the inequality (59), and there exists a solution $\varpi \in C(I, \mathbb{R})$ of equation (5). Then, we have

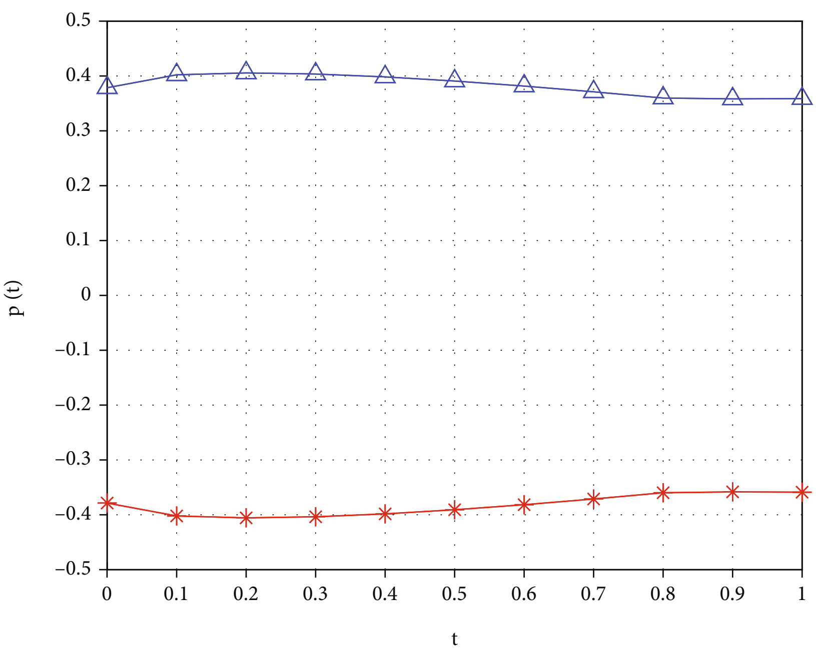
$$\begin{aligned} \varpi(t_k) = {} & A + \frac{(B-A)}{(b-a)}(t-a) - \frac{(t-a)}{(b-a)\Gamma(\gamma)} \int_a^b (b-s)^{\gamma-1} \\ & \cdot \mho(s, \varpi(s), Đ^{\wp} \varpi(s)) ds + \frac{1}{\Gamma(\gamma)} \int_a^{t_k} (t_k - s)^{\gamma-1} \\ & \cdot \mho(s, \varpi(s), Đ^{\wp} \varpi(s)) ds. \end{aligned} \tag{64}$$

From inequality (21), for each $t \in I$, we get

$$\begin{aligned} & \left| z(t_k) - A - \frac{(B-A)k}{N} + \frac{k}{N} {}^{b}_{a} I^{\gamma} \mho(t_k, z(t_k), Đ^{\wp} z(t_k)) \right. \\ & \left. - {}^{t_k}_{a} I^{\gamma} \mho(t_k, z(t_k), Đ^{\wp} z(t_k)) - \Delta_1 \right| \leq \frac{(kh)^{\gamma} \varepsilon}{\Gamma(\gamma+1)}, \end{aligned}$$

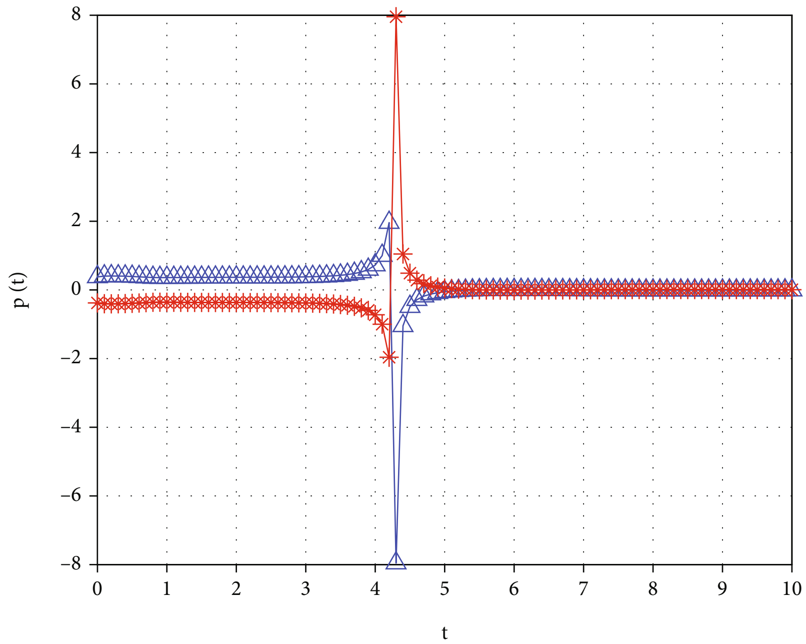


(a)



(b)

Figure 4: Continued.



(c)

Figure 4: (a) The value of $|p(t)|$ for $t \in [0,1]$. (b) The value $p(t)$ for $t \in [0,1]$. (c) The value of $p(t)$ for $t \in [0,10]$.

$$\left| Đ^{\wp} z(t_k) - \frac{(B-A)k^{1-\wp}h^{-\wp}}{N\Gamma(2-\wp)} + \frac{k^{1-\wp}h^{-\wp}}{N\Gamma(2-\wp)} {}_a^b \cdot I^{\gamma}\mho(t_k, z(t_k), Đ^{\wp}z(t_k)) - {}_a^{t_k}I^{\gamma-\wp}\mho(t_k, z(t_k), Đ^{\wp}z(t_k)) - \Delta_2 \right| \le \frac{(kh)^{\gamma-\wp}\varepsilon}{\Gamma(\gamma-\wp+1)}, \tag{65}$$

and by (H2), for each $t \in I$, one can arrive at the following results:

$$|z(t_k) - \varpi(t_k)| \le \frac{(kh)^{\gamma}\varepsilon}{\Gamma(\gamma+1)} + P(\Omega_1|z-\varpi| + \Omega_2|Đ^{\wp}z - Đ^{\wp}\varpi|), \tag{66}$$

$$|Đ^{\wp}z(t_k) - Đ^{\wp}\varpi(t_k)| \le \frac{(kh)^{\gamma-\wp}\varepsilon}{\Gamma(\gamma-\wp+1)} + Q(\Omega_1|z-\varpi| + \Omega_2|Đ^{\wp}z - Đ^{\wp}\varpi|), \tag{67}$$

$$\|z(t_k) - \varpi(t_k)\| = \max_{t\in I}\Omega_1|z(t_k) - \varpi(t_k)| + \max_{t\in I}\Omega_2|Đ^{\wp}z(t_k) - Đ^{\wp}\varpi(t_k)|. \tag{68}$$

Then, from equation (68), we conclude that

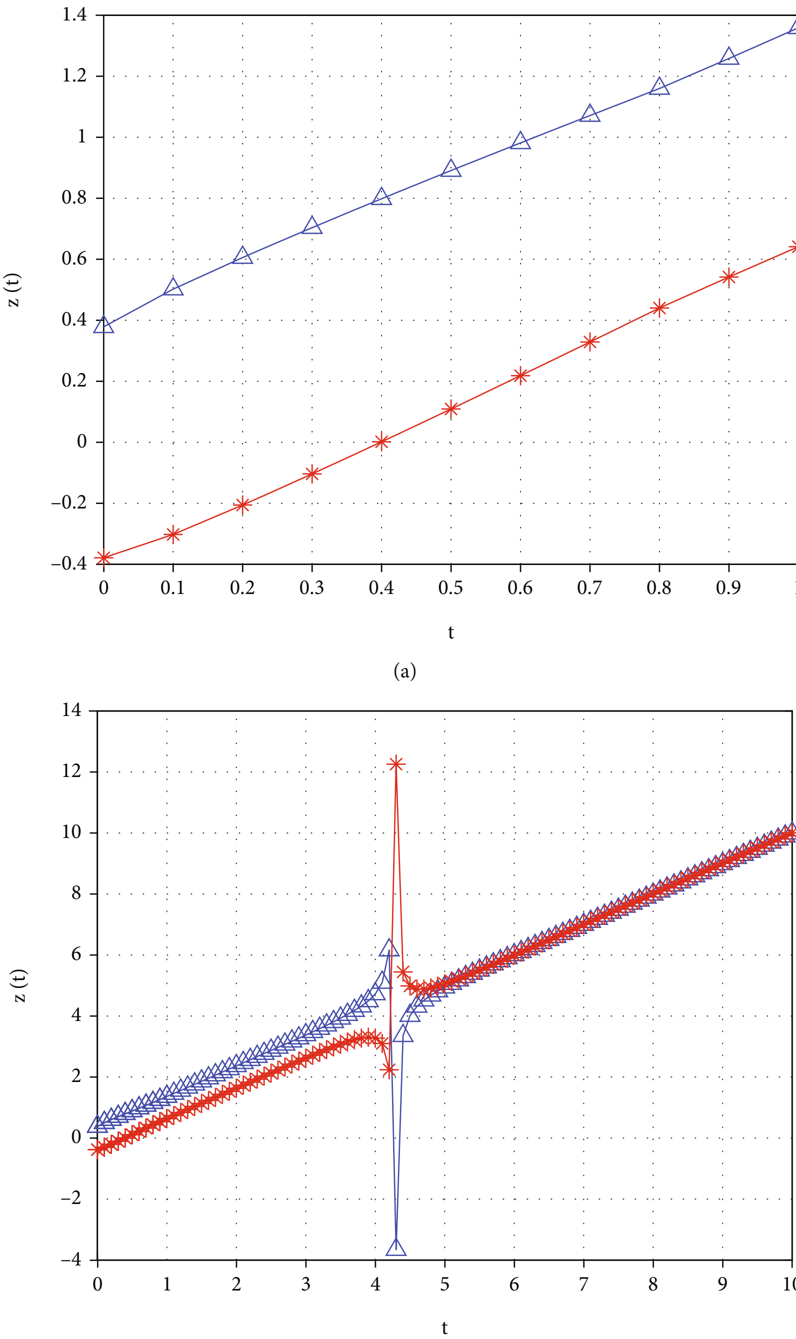
$$\|z(t_k) - \varpi(t_k)\| \le c_f\varepsilon, \quad t \in I, \quad \text{where} \quad c_f = \frac{(\Gamma(\gamma-\wp+1)(kh)^{\gamma}\Omega_1 + \Gamma(\gamma+1)(kh)^{\gamma-\wp}\Omega_2)}{\Gamma(\gamma+1)\Gamma(\gamma-\wp+1)(1-(P\Omega_1 + Q\Omega_2))}. \tag{69}$$

Thus, the problem (5) and (6) is Ulam-Hyers stability. □

Theorem 13. *Suppose that conditions (H2) and (H4) are satisfied. Then, the problem (5) and (6) is Ulam-Hyers-Rassias stability.*

Proof. Let $z(t_k) \in C(I, \mathbb{R})$ be a solution of the inequality (61), and there exists a solution $y \in C(I, \mathbb{R})$ of equation (5). From inequality (61), for each $t \in I$, we have

$$\left| z(t_k) - A - \frac{(B-A)k}{N} + \frac{k}{N} {}_a^b I^{\gamma}\mho(t_k, z(t_k), Đ^{\wp}z(t_k)) - {}_a^{t_k}I^{\gamma}\mho(t_k, z(t_k), Đ^{\wp}z(t_k)) - \Delta_1 \right| \le \varepsilon\Lambda_{\varphi}\varphi(t_k),$$
$$\left| Đ^{\wp}z(t_k) - \frac{(B-A)k^{1-\wp}h^{-\wp}}{N\Gamma(2-\wp)} + \frac{k^{1-\wp}h^{-\wp}}{N\Gamma(2-\wp)} {}_a^b \cdot I^{\gamma}\mho(t_k, z(t_k), Đ^{\wp}z(t_k)) - {}_a^{t_k}I^{\gamma-\wp}\mho(t_k, z(t_k), Đ^{\wp}z(t_k)) - \Delta_2 \right| \le \varepsilon\lambda_{\varphi}\varphi(t_k), \tag{70}$$



(a)

(b)

Figure 5: The graphic of the function $z(t)$: (a) for $t \in [0, 1]$ and (b) for $t \in [0, 10]$.

and by using the hypothesis (H2), for each $t \in I$, the result is

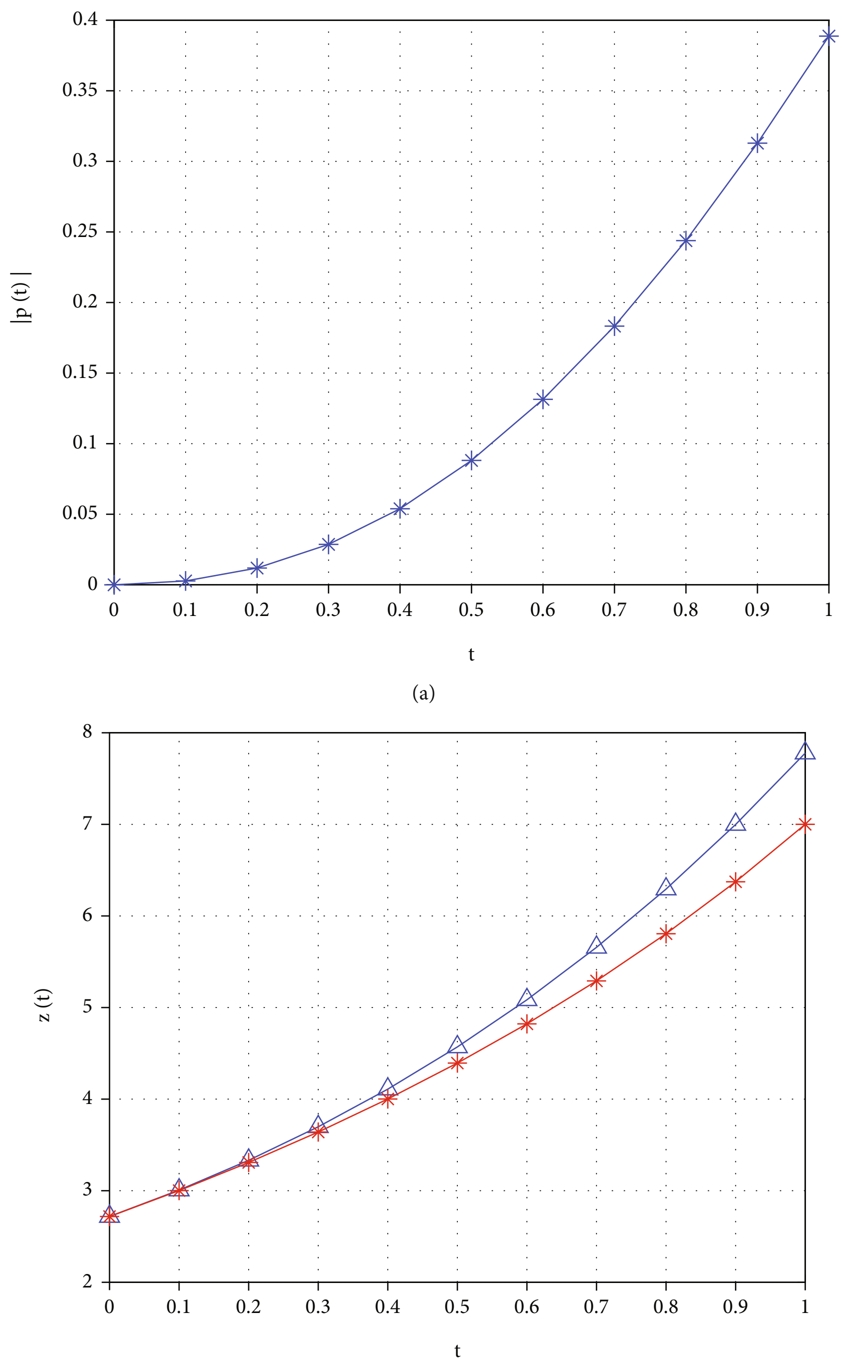
$$
\begin{aligned}
|z(t_k) - \varpi(t_k)| &\le \varepsilon \Lambda_\varphi \varphi(t_k) + P(\Omega_1 |z - \varpi| + \Omega_2 |\mathcal{D}^{\wp} z - \mathcal{D}^{\wp} \varpi|), \\
|\mathcal{D}^{\wp} z(t_k) - \mathcal{D}^{\wp} \varpi(t_k)| &\le \varepsilon \lambda_\varphi \varphi(t_k) + Q(\Omega_1 |z - \varpi| + \Omega_2 |\mathcal{D}^{\wp} z - \mathcal{D}^{\wp} \varpi|).
\end{aligned} \tag{71}
$$

Then, the use of equation (68) implies that

$$
\begin{aligned}
&\| z(t_k) - \varpi(t_k) \| \le c_f \varepsilon \varphi(t_k), \quad t \in I, \\
&\text{where} \quad c_f = \frac{(\Omega_1 \Lambda_\varphi + \Omega_2 \lambda_\varphi)}{(1 - (P\Omega_1 + Q\Omega_2))}.
\end{aligned} \tag{72}
$$

Then, the problem (5) and (6) is Ulam-Hyers-Rassias stability.

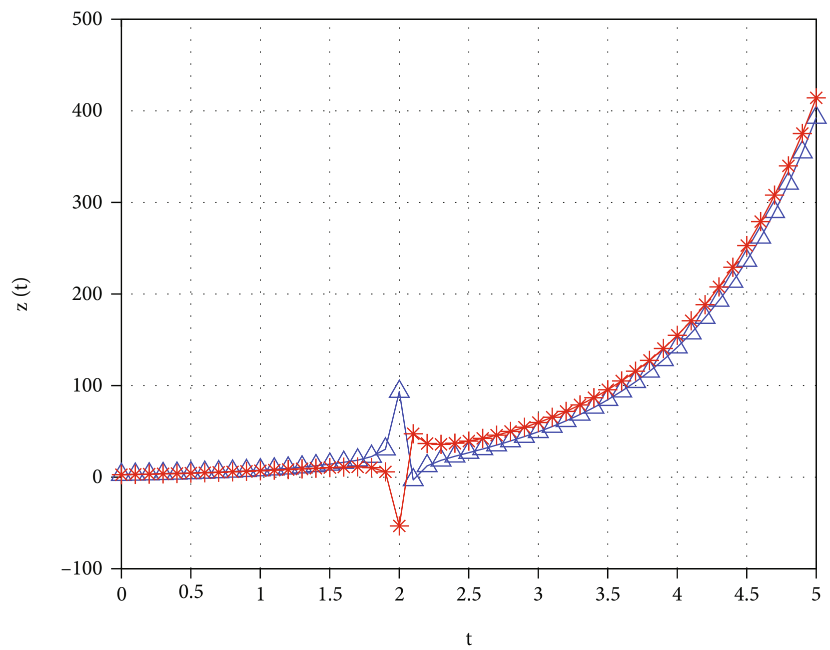
□



(a)

(b)

Figure 6: Continued.



(c)

FIGURE 6: (a) Function $|p(t)|$ for $t \in [0, 1]$. (b) Function $z(t)$ for $t \in [0, 1]$. (c) Function $z(t)$ for $t \in [0, 5]$.

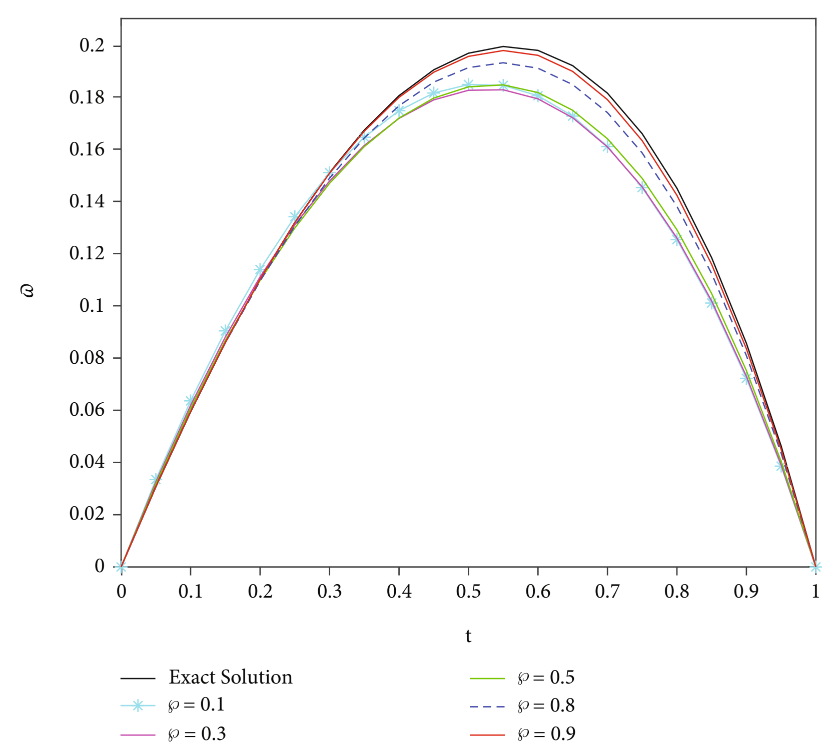


FIGURE 7: Exact and approximate solution for Example 10, when $\theta = -1$.

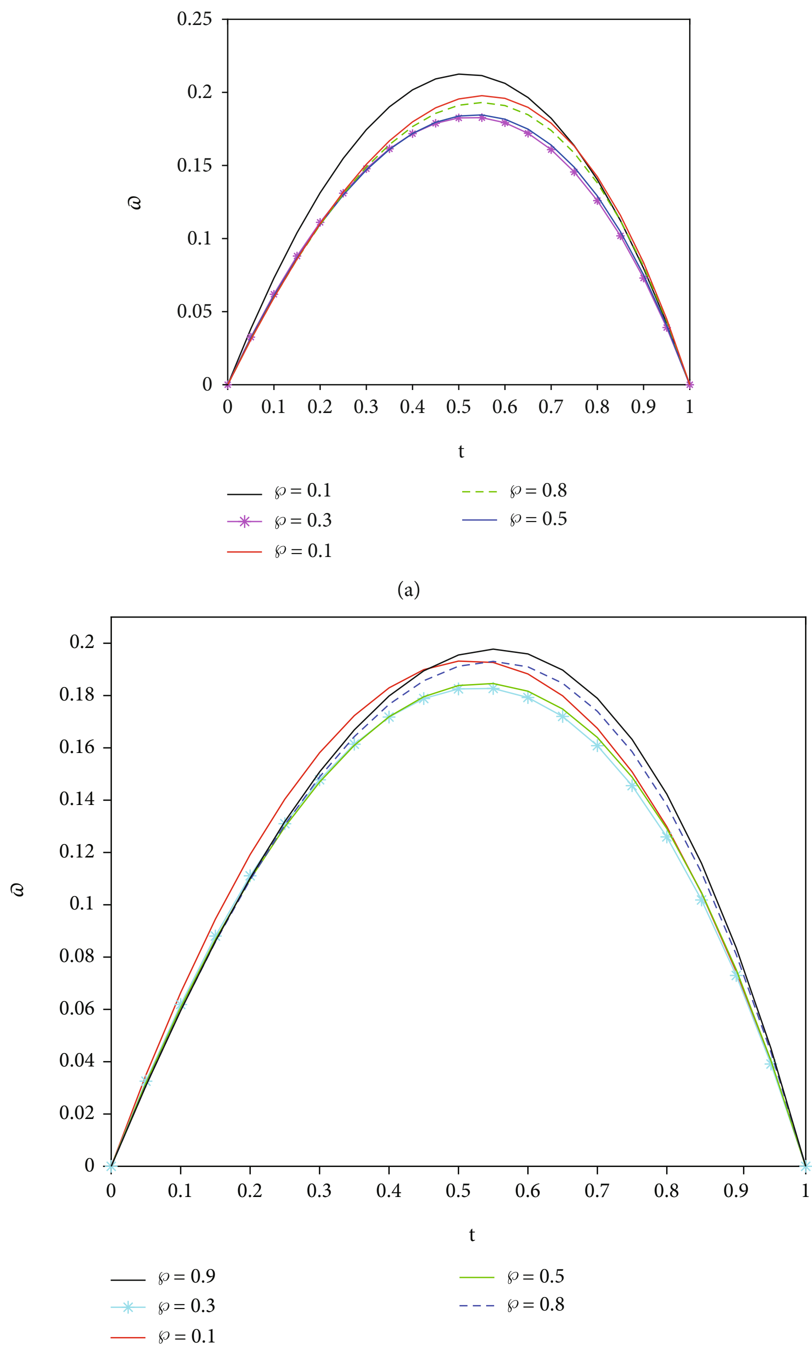


FIGURE 8: Approximate solution for Example 10 when (a) $\theta = -0.5$ and (b) $\theta = 0.5$.

8. Examples

In this section, we give some examples to illustrate the usefulness of our main results.

Example 7. Consider the following fractional boundary value problem:

$$\begin{cases} Đ^{3/2}\varpi + \dfrac{1}{3}\varpi + \dfrac{1}{4}Đ^{1/2}\varpi = -\dfrac{7t^{0.5}}{2\sqrt{\pi}} + \dfrac{t}{3} - \dfrac{2t^{1.5}}{3\sqrt{\pi}} - \dfrac{t^2}{3}, t \in [0, 1], \\ \varpi(0) = 0, \varpi(1) = 0. \end{cases} \tag{73}$$

Here, $\gamma = 3/2$, and $\wp = 1/2$. By Lipschitz condition, we obtain $\Omega_1 = 1/3$ and $\Omega_2 = 1/4$. To estimate the Ulam stability, let $\varpi = 1$ and $h = 0.1$, and by Theorem 12, we have

$$\begin{gathered} \left| Đ^{3/2}\varpi + \frac{1}{3}\varpi + \frac{1}{4}Đ^{1/2}\varpi + \frac{7t^{0.5}}{2\sqrt{\pi}} - \frac{t}{3} + \frac{2t^{1.5}}{3\sqrt{\pi}} + \frac{t^2}{3} \right| \leq 1.2648822308 \leq \varepsilon, \\ |z(t_k) - 1| \leq \zeta\varepsilon, \zeta = \frac{\Omega_1 (kh)^{3/2} + (kh)\Gamma(5/2)\Omega_2}{\Gamma(5/2)(1 - (P\Omega_1 + Q\Omega_2))}, \end{gathered} \tag{74}$$

which shows the problem (73) is Ulam-Hyers stability. Using Matlab, the function $|p(t)| = |z - \varpi|$ is computed and depicted in Figure 3(a), while the function $z(t)$ for $t \in [0, 1]$ is given in Figure 3(b).

Example 8. Consider the following fractional boundary value problem:

$$\begin{cases} Đ^{1.2}\varpi = \dfrac{e^t}{10(1+e^t)}\varpi + \dfrac{5}{(6+t)}Đ^{0.3}\varpi, t \in [0,1], \\ \varpi(0) = 1,\ \varpi(1) = 1. \end{cases} \quad (75)$$

Here, $\gamma = 1.2$, and $\wp = 0.3$. By Lipschitz condition, we obtain $\Omega_1 = 0.073105857$ and $\Omega_2 = 0.833333$. Now, we investigate the Ulam-Hyers-Rassias stability for equation (75), let $\varpi = t, \varphi = \sin t$ and $h = 0.1$, and by Theorem 13, the following is obtained:

$$\begin{aligned} &\left|Đ^{1.2}\varpi - \frac{e^t}{10(1+e^t)}\varpi - \frac{5}{(6+t)}Đ^{0.3}\varpi\right| \le 0.778612319543994 \le \varepsilon, \\ &|z(t_k) - \varpi(t_k)| \le \frac{\left(\Omega_1 t_k^{2.2} E_{2,3.2}\left(-t_k^2\right) + \Omega_2 t_k^{1.9} E_{2,2.9}\left(-t_k^2\right)\right)\varepsilon}{(1 - (P\Omega_1 + Q\Omega_2))}. \end{aligned} \quad (76)$$

The function $|p(t)| = |z - \varpi|$ is depicted in Figure 4(a) for $t \in [0, 1]$, while the function $p(t)$ for $t \in [0, 1]$ is given in Figure 4(b), and the function $p(t)$ for $t \in [0, 10]$ is plotted in Figure 4(c).

Moreover, Figures 5(a) and 5(b) show the function $z(t)$ for $t \in [0, 1]$ and $t \in [0, 10]$, respectively.

Example 9. Consider the following fractional boundary value problem:

$$\begin{cases} Đ^{2}\varpi = \dfrac{e^{-t}}{20}\varpi + \dfrac{9}{10}Đ^{0.5}\varpi, t \in [0,1], \\ \varpi(0) = \varpi(1) = 1. \end{cases} \quad (77)$$

Here, $\gamma = 2$, and $\wp = 0.5$. By Lipschitz condition, we obtain $\Omega_1 = 0.05$ and $\Omega_2 = 0.9$. Now, we investigate the Ulam-Hyers-Rassias stability for equation (77), let $\varpi = e^{t+1}$, and $\varphi = e^t$, and by Theorem 13, the results are

$$\begin{aligned} &\left|Đ^{2}\varpi - \frac{e^{-t}}{20}\varpi - \frac{5}{7}Đ^{0.5}\varpi\right| \le \varepsilon, \\ &|z(t) - \varpi(t)| \le \frac{\varepsilon\left(\Omega_1 t^2 E_{1,3}(t) + \Omega_2 t^{1.5} E_{1,2.5}(t)\right)}{(1 - (P\Omega_1 + Q\Omega_2))}. \end{aligned} \quad (78)$$

The behavior of the function $|p(t)| = |z - \varpi|$ is depicted in Figure 6(a) for $t \in [0, 1]$. Furthermore, the behavior of $z(t)$ for $t \in [0, 1]$ and for $t \in [0, 5]$ is plotted in Figures 6(b) and 6(c), respectively.

Example 10 (see [47, 48]). Consider the following Bagley-Torvik equation:

$$\begin{cases} D^2\varpi + \theta D^{\wp}\varpi = -1 - e^{t-1}, t \in [0,1], \\ \varpi(0) = \varpi(1) = 0. \end{cases} \quad (79)$$

The exact solution of Example 10 is not known for general values of $\wp$. Only, the exact solution is

$$\varpi(t) = t\left(1 - e^{(t-1)}\right), \quad (80)$$

for $\theta = -1$ and $\wp = 1$. This example was solved for different values of $0 < \wp \le 1$, and the numerical solutions are shown in (Figures 7 and 8).

9. Conclusions

In this article, the author analyzed the existence, uniqueness, and stability of solutions for nonlinear two-point boundary value problems in the sense of Caputo fractional derivative. Banach's fixed-point theorem was used to prove the uniqueness of the solution, while Brouwer's fixed-point theorem was employed to study the existence results. In addition, the numerical solution for the fractional differential equation with boundary condition was studied by using successive approximation method. Discussion on the convergence and error analysis of the proposed method is presented. Some numerical examples are given to show the accuracy of the suggested method. Finally, the Hyers-Ulam and Hyers-Ulam-Rassias stability of the problem (5) and (6) are studied. Examples are presented to illustrate our theoretical results.

Data Availability

The results in Tables 1 and 2 have been obtained by conducting the numerical procedure.

Conflicts of Interest

The author declares no conflicts of interest.

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