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Research Article

On ψ-Caputo Partial Hyperbolic Differential Equations with a Finite Delay

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In this work, we are concerned with some qualitative analyses of fractional-order partial hyperbolic functional differential equations under the ψ-Caputo type. To be precise, we investigate the existence and uniqueness results based on the nonlinear alternative of the Leray-Schauder type and Banach contraction mapping. Moreover, we present two similar results to nonlocal problems. Then, the guarantee of the existence of solutions is shown by Ulam-Hyer's stability. Two examples will be given to illustrate the abstract results. Eventually, some known results in the literature are extended.

1. Introduction

Since fractional calculus (FC) has a decent global correlation execution to reflect the historical reliance process of the improvement of system functions and can likewise describe the characteristics of the dynamic system itself, it turned into a strong mathematical gadget to describe a few complex developments, unpredictable phenomena, memory highlights, and other aspects. FC theory was vastly utilized by mathematicians as well as scientific experts, engineers, financial analysts, scholars, and physicists (see [1–4]). Riemann, in 1876, suggested the definition of the Riemann-Liouville (RL) fractional derivative (FD). Caputo originally proposed one more definition of FD through a changed RL fractional integral (FI) toward the start of the twentieth century, to be specific, a Caputo FD. One issue in this field is the major and extraordinary number of possible various definitions of FD and FI; settling on the choice of the best operator for every specific framework is a significant issue. One method for conquering this issue is to consider overall definitions, of which the classical ones can be viewed as specific cases [5, 6].

In this regard, Almeida [7] and Sousa and de Oliveira [8] recently introduced ψ-Caputo FD and ψ-Hilfer FD of one variable, respectively, from which it is feasible to obtain a wide class of FDs already well established. Sousa and de Oliveira [9] have very recently expanded ψ-Hilfer FD with two variables. Therefore, one of the aims of this work is to introduce some qualitative analyses of solutions based on ψ-Caputo FD with two variables.

Then again, functional differential equations (FDEs) and fractional FDEs with finite delay show up frequently in applications as models of equations, and consequently, the investigation of these kinds of equations has gotten incredible consideration somewhat recently; see, for instance, [10–14] and the references in those. The literature connected with the existence of solutions of fractional partial FDEs with a finite delay was processed very slowly; see, for instance, [15–19].

The background and survey in the literature relative to classical fractional partial hyperbolic FDEs can be found in the monograph of Abbas et al. [19]. Sousa and de Oliveira [9] discussed the stability of fractional partial hyperbolic DEs without delay under the ψ-Hilfer operator. Baitiche et al. [20] established the existence result of coupled systems of fractional partial hyperbolic DEs without delay.

This work is concerned with the existence, uniqueness, and Ulam-Hyer (HU) stability of the solution to the ψ-Caputo-type fractional partial hyperbolic FDE with finite delay:

$$\begin{gathered}
{}^{C}\mathscr{D}_{0+}^{r;\psi}\varkappa(\varkappa,\tau)=\mathfrak{F}\left(\varkappa,\tau,\varkappa_{(\varkappa,\tau)}\right),(\varkappa,\tau)\in J_1:=[0,c]\times[0,d],\\
\varkappa(\varkappa,\tau)=\varphi(\varkappa,\tau),(\varkappa,\tau)\in J_2:=[-\kappa_1,c]\times[-\kappa_2,d]\setminus(0,c]\times(0,d],\\
\varkappa(\varkappa,0)=\phi_1(\varkappa),\varkappa(0,\tau)=\phi_2(\tau),\varkappa\in[0,c],\tau\in[0,d],
\end{gathered} \tag{1}$$

and the ψ-Caputo-type fractional nonlocal partial hyperbolic FDE with finite delay:

$$\begin{gathered}
{}^{C}\mathscr{D}_{\ell+}^{r;\psi}\varkappa(\varkappa,\tau)=\mathfrak{F}\left(\varkappa,\tau,\varkappa_{(\varkappa,\tau)}\right),(\varkappa,\tau)\in J_1:=[0,c]\times[0,d],\\
\varkappa(\varkappa,\tau)=\varphi(\varkappa,\tau),(\varkappa,\tau)\in J_2:=[-\kappa_1,c]\times[-\kappa_2,d]\setminus(0,c]\times(0,d],\\
\varkappa(\varkappa,0)+h_1(\varkappa)=\phi_1(\tau),\varkappa(0,\tau)+h_1(\varkappa)=\phi_2(\tau),\varkappa\in[0,c],\tau\in[0,d],
\end{gathered} \tag{2}$$

where $c, d, \kappa_1, \kappa_2 > 0, r = (\mu, \nu) \in (0,1]\times(0,1]$, ${}^{C}\mathscr{D}_{\ell^+}^{r;\psi}$ is the ψ-Caputo FD of order r with respect to another function ψ, which is increasing, and $\partial\psi/\partial u, \partial\psi/\partial\tau \neq 0$, for $(\varkappa,\tau)\in J_1$, $\ell=(0,0)$, $\varphi(\cdot,\cdot)\in\mathscr{C}:=\mathscr{C}([-\kappa_1,0]\times[-\kappa_2,0],\mathbb{R})$, $\mathfrak{F}: J_1\times\mathscr{C}\longrightarrow\mathbb{R}$, $\phi_1:[0,c]\longrightarrow\mathbb{R}$, $\phi_2:[0,d]\longrightarrow\mathbb{R}$ are absolutely continuous with $\phi_1(\varkappa)=\varphi(\varkappa,0)$, $\phi_2(\tau)=\varphi(0,\tau)$, $\forall\varkappa\in[0,c]$, $\forall\tau\in[0,d]$, and $h_1,h_2: C(J_1,\mathbb{R})\longrightarrow\mathbb{R}$ are continuous.

This paper is concerned with the qualitative analyses of fractional partial hyperbolic FDEs, which are very new, and the implementation of the ψ-fractional operator makes it more general and novel, unlike the classical fractional operators. To be precise, we are interested in investigating the existence, uniqueness, and Ulam-Hyer's stability results for our problems (1)–(2). These results initiate the investigation of ψ-Caputo fractional partial hyperbolic FDEs with a finite delay, which mainly includes a more general fractional operator based on another function ψ. To be certain, in the analysis of our results, we essentially use fixed point theorems (FPTs) of the Leray-Schauder type and Banach type. Our outcomes can be interpreted as extensions of preceding results that Abbas et al. [19] and Sousa and de Oliveira [9] obtained for classical FHDEs, which can be considered a contribution to the literature.

The rest of the work has been organized as follows. Section 2 is devoted to some essential connotations of ψ-fractional calculus with auxiliary lemmas to problems at hand. The existence, uniqueness, and UH stability results based on fixed point techniques are provided in Section 3. Suitable examples are given in Section 4. In Section 5, we present the conclusions.

2. Preliminary Results

In this section, we give some notations and essential definitions of fractional partial integrals and derivatives (FPIs and FPDs) and some function spaces to simplify the forthcoming analysis. Let $J_1=[0,c]\times 0,d]$, $J_2:=[-\kappa_1,c]\times-\kappa_2,d]\setminus(0,c]\times(0,d]$, where $c,d,\kappa_1,\kappa_2>0,\ell=(0,0)$, and $r=(\mu,\nu)\in(0,1]\times(0,1]$. Denote $\mathscr{C}:=\mathscr{C}([-\kappa_1,0]\times-[\kappa_2,0],\mathbb{R})$ the space of continuous functions on $[-\kappa_1,0]\times-\kappa_2,0]$. Note that $\mathscr{C}$ is the Banach space with the norm

$$\|z\|_{\mathscr{C}}=\sup_{(\varkappa,\tau)\in[-\kappa_1,0]\times[-\kappa_2,0]}|z(\varkappa,\tau)|, \tag{3}$$

and let $C(J_1,\mathbb{R})$ be the Banach space with the norm

$$\|z\|_{\infty}=\sup_{(\varkappa,\tau)\in[0,c]\times[0,d]}|z(\varkappa,\tau)|. \tag{4}$$

The space $\mathscr{L}^1(J_1,\mathbb{R})$ is endowed with the norm

$$\|z\|_{\mathscr{L}^1}=\int_0^c\int_0^d|z(\varkappa,\tau)|d\varkappa d\tau. \tag{5}$$

For any $z_{(\varkappa,\tau)}:[-\kappa_1,c]\times-\kappa_2,d]\longrightarrow\mathbb{R}$, where $(\varkappa,\tau)\in J_1$, we have

$$z_{(\varkappa,\tau)}(\theta,\theta)=z(\varkappa+\theta,\tau+\theta),\text{for}(\theta,\theta)\in[-\kappa_1,0]\times[-\kappa_2,0]. \tag{6}$$

Define the space $\mathscr{C}([-\kappa_1,c]\times-\kappa_2,d],\mathbb{R})$ as

$$\mathscr{C}_{(c,d)}=\left\{z:[-\kappa_1,c]\times-\kappa_2,d]\longrightarrow\mathbb{R}:z|_{J_2}=\varphi\in\mathscr{C},z|_{J_1}\in C(J_1,\mathbb{R})\right\}, \tag{7}$$

where $z|_{J_1}$ is the restriction of z to J_1, which is a Banach space with the norm

$$\|z\|_{\mathscr{C}_{(c,d)}}=\sup_{(\varkappa,\tau)\in[-\kappa_1,c]\times[-\kappa_2,d]}|z(\varkappa,\tau)|. \tag{8}$$

In the forthcoming analysis, let us consider $\psi(\cdot)$ to be an increasing and positive monotone function on J_1 with $\psi_\varkappa(\cdot)$, $\psi_\tau(\cdot)\neq 0$ on J_1, where $\psi_\varkappa=\partial\psi/\partial\varkappa$ and $\psi_\tau=\partial\psi/\partial\tau$. On the whole paper, keep in mind $\psi^{\theta-1}(y,m):=(\psi(y)-\psi(m))^{\theta-1}$.

Definition 1 (see [9]). Let $\ell=(0,0)$, $r=(\mu,\nu)$, where $\mu,\nu>0$. Then, the ψ-RL FPI of a function of two variables $z(\varkappa,\tau)\in$

$\mathscr{L}^1(J_1, \mathbb{R})$ of order r is given by

$$\mathscr{I}_{\ell^+}^{r;\psi} z(\varkappa,\tau) = \frac{1}{\Gamma(\mu)\Gamma(\nu)} \int_0^\varkappa \int_0^\tau \psi_\varkappa(\theta)\psi_\tau(\zeta)\psi^{\mu-1}(\varkappa,\theta)\psi^{\nu-1}(\tau,\zeta) z(\theta,\zeta) d\theta d\zeta. \tag{9}$$

Also, we have

$$\begin{aligned} \mathscr{I}_{0^+}^{\mu} z(\varkappa,\tau) &= \frac{1}{\Gamma(\mu)} \int_0^\varkappa \psi_\varkappa(\theta)\psi^{\mu-1}(\varkappa,\theta) z(\theta,\tau) d\theta, \\ \mathscr{I}_{0^+}^{\nu} z(\varkappa,\tau) &= \frac{1}{\Gamma(\nu)} \int_0^\tau \psi_\tau(\theta)\psi^{\nu-1}(\tau,\theta) z(\varkappa,\theta) d\theta. \end{aligned} \tag{10}$$

Definition 2 (see [9]). Let $\ell = (0,0)$, and $r = (\mu, \nu)$, where $0 < \mu, \nu \le 1$. Then, the ψ-RL FPD of a function $z(\varkappa,\tau) \in \mathscr{L}^1(J_1, \mathbb{R})$ of order r is defined by

$$\mathscr{D}_{\ell^+}^{r;\psi} z(\varkappa,\tau) = \left(\frac{1}{\psi_\varkappa \psi_\tau} \frac{\partial^2}{\partial\varkappa\partial\tau} \right) \mathscr{I}_{\ell^+}^{1-r;\psi} z(\varkappa,\tau). \tag{11}$$

Definition 3 (see [9]). Let $\ell = (0,0)$,$r = (\mu, \nu)$, where $0 < \mu, \nu \le 1$, and $\psi \in C^1(J_1, \mathbb{R})$. Then, the ψ-Caputo FPD of a function $z(\varkappa,\tau) \in C^1(J_1, \mathbb{R})$ of order r is defined by

$${}^C\mathscr{D}_{\ell^+}^{r;\psi} z(\varkappa,\tau) = \mathscr{I}_{\ell^+}^{1-r;\psi} \left(\frac{1}{\psi_\varkappa \psi_\tau} \frac{\partial^2}{\partial\varkappa\partial\tau} \right) z(\varkappa,\tau). \tag{12}$$

Lemma 4 (see [9]). *Let $r = (\mu, \nu) \in (0,\infty) \times (0,\infty)$, and ξ_1, $\xi_2 > -1$. Then,*

$$\mathscr{I}_{\ell^+}^{r;\psi} \psi^{\xi_1-1}(\varkappa,0)\psi^{\xi_2-1}(\tau,0) = \frac{\Gamma(\xi_1)}{\Gamma(\mu+\xi_1)} \frac{\Gamma(\xi_2)}{\Gamma(\nu+\xi_2)} \psi^{\mu+\xi_1-1}(\varkappa,0)\psi^{\nu+\xi_2-1}(\tau,0). \tag{13}$$

Lemma 5 (see [9]). *Let $r = (\mu, \nu) \in (0,1] \times (0,1]$, and $\xi_1, \xi_2 > -1$. Then,*

$$\mathscr{D}_{\ell^+}^{r;\psi} \psi^{\xi_1-1}(\varkappa,0)\psi^{\xi_2-1}(\tau,0) = \frac{\Gamma(\xi_1)}{\Gamma(\mu-\xi_1)} \frac{\Gamma(\xi_2)}{\Gamma(\nu-\xi_2)} \psi^{\mu-\xi_1-1}(\varkappa,0)\psi^{\nu-\xi_2-1}(\tau,0). \tag{14}$$

Lemma 6 (see [7]). *Let $0 < r < 1$, and $h : [c, d] \longrightarrow \mathbb{R}$ is continuous. Then,*

$${}^C\mathscr{D}_{c^+}^{r,\psi} \mathscr{I}_{c^+}^{r,\psi} h(\varkappa) = h(\varkappa), \mathscr{I}_{c^+}^{r,\psi} {}^C\mathscr{D}_{c^+}^{r,\psi} h(\varkappa) = h(\varkappa) - h(c). \tag{15}$$

Lemma 7. *The following problem*

$$\begin{gathered} {}^C\mathscr{D}_{0+}^{r;\psi} z(\varkappa,\tau) = f(\varkappa,\tau), (\varkappa,\tau) \in [0,c] \times [0,d], \\ z(\varkappa,0) = \phi_1(\varkappa), z(0,\tau) = \phi_2(\tau), \varkappa \in [0,c], \tau \in [0,d], \end{gathered} \tag{16}$$

with $\phi_1(0) = \phi_2(0)$ which has a solution $z(\varkappa,\tau) \in \mathscr{C}([0,c] \times 0, d], \mathbb{R})$ if and only if $z(\varkappa,\tau)$ satisfies

$$z(\varkappa,\tau) = \eta(\varkappa,\tau) + \mathscr{I}_{0^+}^{r;\psi} f(\varkappa,\tau), (\varkappa,\tau) \in 0, c] \times 0, d], \tag{17}$$

where $\eta(\varkappa,\tau) = \phi_1(\varkappa) + \phi_2(\tau) - \phi_1(0)$.

Proof. The proof is primitive and similar to the proof of Lemma 3.2 given in [21], so it can be omitted. □

Here, we only refer to source [22] of the results of Leray-Schauder and Banach FPT.

3. Main Results

Let us begin by describing what we mean by a solution to problem (1).

Definition 8. A function z is a solution of (1), if $z \in \mathscr{C}_{(c,d)}$ and $(D^2 z_{\varkappa\tau})(\varkappa,\tau)$ exists and is integrable.

Theorem 9. *Let the following assumptions hold:*
(A1) $\mathfrak{F} : J_1 \times \mathscr{C} \longrightarrow \mathbb{R}$ is continuous.
(A2) There exists $L_\mathfrak{F} > 0$ such that

$$|\mathfrak{F}(\varkappa,\tau,z) - \mathfrak{F}(\varkappa,\tau,\upsilon)| \le L_\mathfrak{F} \|z-\upsilon\|_\mathscr{C}, (\varkappa,\tau) \in J_1, z, \upsilon \in \mathscr{C}. \tag{18}$$

If

$$\sigma \coloneqq \frac{\psi^\mu(c,0)\psi^\nu(d,0)}{\Gamma(\mu+1)\Gamma(\nu+1)} L_\mathfrak{F} < 1, \tag{19}$$

then there exists a unique solution for the ψ-Caputo problem (1) on $[-\kappa_1, c] \times -\kappa_2, d)$.

Proof. Consider the operator $\mathscr{K} : \mathscr{C}_{(c,d)} \longrightarrow \mathscr{C}_{(c,d)}$ defined by $(\mathscr{K}z)(\varkappa,\tau) = z(\varkappa,\tau)$, i.e.,

$$(\mathscr{K}z)(\varkappa,\tau) = \begin{cases} \varphi(\varkappa,\tau), & (\varkappa,\tau) \in J_2, \\ \eta(\varkappa,\tau) + \dfrac{1}{\Gamma(\mu)\Gamma(\nu)} \displaystyle\int_0^\varkappa \int_0^\tau \psi_\mathrm{u}(\theta)\psi^{\mu-1}(\varkappa,\theta)\psi_\tau(\zeta)\psi^{\nu-1}(\tau,\zeta) \times \mathfrak{F}\left(\theta,\zeta,z_{(\theta,\zeta)}\right) d\zeta d\theta, & (\varkappa,\tau) \in J_1, \end{cases} \tag{20}$$

where $\eta(\varkappa, \tau) := \phi_1(\varkappa) + \phi_2(\tau) - \phi_1(0)$.

Let $z, \omega \in \mathscr{C}_{(c,d)}$, and $(\varkappa, \tau) \in [-\kappa_1, c] \times [-\kappa_2, d]$. Then,

$$\begin{aligned}
&|(\mathscr{K}z)(\varkappa,\tau) - (\mathscr{K}\omega)(\varkappa,\tau)| \\
&\le \frac{1}{\Gamma(\mu)\Gamma(\nu)} \int_0^\varkappa \int_0^\tau \psi_\varkappa(\theta)\psi^{\mu-1}(\varkappa,\theta)\psi_\tau(\zeta)\psi^{\nu-1}(\tau,\zeta) \\
&\quad \times \left|\mathfrak{F}\left(\theta,\zeta,z_{(\theta,\zeta)}\right) - \mathfrak{F}\left(\theta,\zeta,\omega_{(\theta,\zeta)}\right)\right| d\zeta d\theta \\
&\le \frac{L_{\mathfrak{F}}}{\Gamma(\mu)\Gamma(\nu)} \int_0^\varkappa \int_0^\tau \psi_\varkappa(\theta)\psi^{\mu-1}(\varkappa,\theta)\psi_\tau(\zeta)\psi^{\nu-1}(\tau,\zeta) \\
&\quad \times \left\| z_{(\theta,\zeta)} - \omega_{(\theta,\zeta)} \right\|_{\mathscr{C}} d\zeta d\theta \\
&\le \frac{L_{\mathfrak{F}}}{\Gamma(\mu)\Gamma(\nu)} \|z-\omega\|_{\mathscr{C}_{(c,d)}} \int_0^\varkappa \psi_{\mathrm{u}}(\theta)\psi^{\mu-1}(\varkappa,\theta) \int_0^\tau \psi_\tau(\zeta)\psi^{\nu-1}(\tau,\zeta) d\zeta d\theta \\
&\le \frac{L_{\mathfrak{F}}}{\Gamma(\mu)\Gamma(\nu)} \|z-\omega\|_{\mathscr{C}_{(c,d)}} \frac{\psi^\nu(d,0)}{\nu} \int_0^\varkappa \psi_\varkappa(\theta)\psi^{\mu-1}(\varkappa,\theta) d\theta \\
&\le \frac{L_{\mathfrak{F}}\psi^\mu(c,0)\,\psi^\nu(d,0)}{\Gamma(\mu+1)\Gamma(\nu+1)} \|z-\omega\|_{\mathscr{C}_{(c,d)}}.
\end{aligned} \tag{21}$$

which implies

$$\|(\mathscr{K}z) - (\mathscr{K}\omega)\|_{\mathscr{C}_{(c,d)}} \le \sigma \|z-\omega\|_{\mathscr{C}_{(c,d)}}. \tag{22}$$

Since $\sigma < 1$, the operator $\mathscr{K}$ is a contraction. This means that $\mathscr{K}$ has a unique fixed point by Banach's FPT. □

Theorem 10. *Let (A1) and the following assumption hold:*

(A3) There exist $p, q \in C(J_1, \mathbb{R})$ such that

$$|\mathfrak{F}(\varkappa,\tau,z)| \le p(\varkappa,\tau) + q(\varkappa,\tau)\|z\|_{\mathscr{C}}, (\varkappa,\tau) \in J_1, z \in \mathscr{C}. \tag{23}$$

If $\rho := \|q\|_\infty \psi^\mu(c,0)\psi^\nu(d,0)/\Gamma(\mu+1)\Gamma(\nu+1) < 1$, then there exists at least one solution for the ψ-Caputo problem (1) on $[-\kappa_1, c] \times -\kappa_2, d]$.

Proof. Consider the operator $\mathscr{K} : \mathscr{C}_{(c,d)} \longrightarrow \mathscr{C}_{(c,d)}$ defined by (20); then, we show that$\mathscr{K}$is completely continuous.

Step 1: $\mathscr{K}$ is continuous. Let $\{z_n\}$ be a sequence such that $z_n \longrightarrow z$ in $\mathscr{C}_{(c,d)}$. Then,

$$\begin{aligned}
&|(\mathscr{K}z_n)(\varkappa,\tau) - (\mathscr{K}z)(\varkappa,\tau)| \\
&\le \frac{1}{\Gamma(\mu)\Gamma(\nu)} \int_0^\varkappa \int_0^\tau \psi_\varkappa(\theta)\psi^{\mu-1}(\varkappa,\theta)\psi_\tau(\zeta)\psi^{\nu-1}(\tau,\zeta) \\
&\quad \times \left|\mathfrak{F}\left(\theta,\zeta,z_{n_{(\theta,\zeta)}}\right) - \mathfrak{F}\left(\theta,\zeta,z_{(\theta,\zeta)}\right)\right| d\zeta d\theta \\
&\le \frac{1}{\Gamma(\mu)\Gamma(\nu)} \int_0^c \int_0^d \psi_\varkappa(\theta)\psi^{\mu-1}(\varkappa,\theta)\psi_\tau(\zeta)\psi^{\nu-1}(\tau,\zeta) \\
&\quad \times \sup_{(\theta,\zeta)\in J_1} \left|\mathfrak{F}\left(\theta,\zeta,z_{n_{(\theta,\zeta)}}\right) - \mathfrak{F}\left(\theta,\zeta,z_{(\theta,\zeta)}\right)\right| d\zeta d\theta \\
&\le \frac{\psi^\mu(c,0)\,\psi^\nu(d,0)}{\Gamma(\mu+1)\Gamma(\nu+1)} \left\|\mathfrak{F}\left(.,.,z_{n_{(.,.)}}\right) - \mathfrak{F}\left(.,.,z_{(.,.)}\right)\right\|_\infty.
\end{aligned} \tag{24}$$

Since $\mathfrak{F}$ is continuous, $\|(\mathscr{K}z_n) - (\mathscr{K}z)\|_{\mathscr{C}_{(c,d)}} \longrightarrow 0$, as $n \longrightarrow \infty$.

Step 2: $\mathscr{K}(B_\xi)$ is bounded in $\mathscr{C}_{(c,d)}$, where $B_\xi = \{z \in \mathscr{C}_{(c,d)} : \|z\|_{\mathscr{C}_{(c,d)}} \le \xi\}$, for any $\xi > 0$.

Set

$$\xi > \max\left\{\|\varphi\|_{\mathscr{C}}, \frac{\lambda}{1-\rho}\right\}, \tag{25}$$

where

$$\lambda := \|\eta\|_\infty + \left(\frac{\|p\|_\infty \psi^\mu(c,0)\,\psi^\nu(d,0)}{\Gamma(\mu+1)\Gamma(\nu+1)}\right). \tag{26}$$

For $(\varkappa, \tau) \in J_2$, we get

$$|(\mathscr{K}z)(\varkappa,\tau)| \le \sup_{(\varkappa,\tau)\in J_2} |\varphi(\varkappa,\tau)| = \|\varphi\|_{\mathscr{C}}. \tag{27}$$

Let $(\varkappa, \tau) \in J_1$, and $z \in B_\xi$. Then,

$$\begin{aligned}
|(\mathscr{K}z)(\varkappa,\tau)| &\le |\eta(\varkappa,\tau)| + \frac{1}{\Gamma(\mu)\Gamma(\nu)} \int_0^\varkappa \int_0^\tau \psi_\varkappa(\theta)\psi^{\mu-1}(\varkappa,\theta)\psi_\tau(\zeta)\psi^{\nu-1}(\tau,\zeta) \\
&\quad \times \left|\mathfrak{F}\left(\theta,\zeta,z_{(\theta,\zeta)}\right)\right| d\zeta d\theta \\
&\le |\eta(\varkappa,\tau)| + \frac{1}{\Gamma(\mu)\Gamma(\nu)} \int_0^\varkappa \int_0^\tau \psi_\varkappa(\theta)\psi^{\mu-1}(\varkappa,\theta)\psi_\tau(\zeta)\psi^{\nu-1}(\tau,\zeta)p(\theta,\zeta) d\zeta d\theta \\
&\quad + \frac{1}{\Gamma(\mu)\Gamma(\nu)} \int_0^\varkappa \int_0^\tau \psi_\varkappa(\theta)\psi^{\mu-1}(\varkappa,\theta)\psi_\tau(\zeta)\psi^{\nu-1}(\tau,\zeta)q(\theta,\zeta)\left\|z_{(\theta,\zeta)}\right\|_{\mathscr{C}} d\zeta d\theta \\
&\le \|\eta\|_\infty + \frac{\|p\|_\infty}{\Gamma(\mu)\Gamma(\nu)} \int_0^\varkappa \int_0^\tau \psi_\varkappa(\theta)\psi^{\mu-1}(\varkappa,\theta)\psi_\tau(\zeta)\psi^{\nu-1}(\tau,\zeta) d\zeta d\theta \\
&\quad + \frac{\|q\|_\infty \|z\|_{\mathscr{C}(c,d)}}{\Gamma(\mu)\Gamma(\nu)} \int_0^\varkappa \int_0^\tau \psi_\varkappa(\theta)\psi^{\mu-1}(\varkappa,\theta)\psi_\tau(\zeta)\psi^{\nu-1}(\tau,\zeta) d\zeta d\theta \\
&\le \|\eta\|_\infty + \left(\|p\|_\infty + \|q\|_\infty \xi\right) \frac{\psi^\mu(c,0)\,\psi^\nu(d,0)}{\Gamma(\mu+1)\Gamma(\nu+1)} = \lambda + \rho\xi.
\end{aligned} \tag{28}$$

Due to (25), (27), and (28), $\|(\mathscr{K}z)\|_{\mathscr{C}_{(c,d)}} \le \xi$, or $(\mathscr{K}z) \in B_\xi$, which implies that $\mathscr{K}(B_\xi)$ is bounded in $\mathscr{C}_{(c,d)}$.

Step 3: $\mathscr{K}(B_\xi)$ is equicontinuous in $\mathscr{C}_{(c,d)}$. Let $z \in B_\xi$, and $(\varkappa_1, \tau_1), (\varkappa_2, \tau_2) \in [-\kappa_1, c] \times [-\kappa_2, d]$ with $\varkappa_1 < \varkappa_2, \tau_1 < \tau_2$. If $(\varkappa_1, \tau_1), (\varkappa_2, \tau_2) \in J_1$ and $z \in B_\xi$. Then,

$$\begin{aligned}
&|(\mathscr{K}z)(\varkappa_2,\tau_2) - (\mathscr{K}z)(\varkappa_1,\tau_1)| \\
&\le |\eta(\varkappa_2,\tau_2) - \eta(\varkappa_1,\tau_1)| + \frac{1}{\Gamma(\mu)\Gamma(\nu)} \int_0^{\varkappa_1} \int_0^{\tau_1} \psi_\varkappa(\theta)\psi_\tau(\zeta)\left[\psi^{\mu-1}(\varkappa_2,\theta)\psi^{\nu-1}(\tau_2,\zeta)\right. \\
&\quad \left. - \psi^{\mu-1}(\varkappa_1,\theta)\psi^{\nu-1}(\tau_1,\zeta)\right] \times \left|\mathfrak{F}\left(\theta,\zeta,z_{(\theta,\zeta)}\right)\right| d\zeta d\theta \\
&\quad + \frac{1}{\Gamma(\mu)\Gamma(\nu)} \int_{\varkappa_1}^{\varkappa_2} \int_{\tau_1}^{\tau_2} \psi_\varkappa(\theta)\psi_\tau(\zeta)\left[\psi^{\mu-1}(\varkappa_2,\theta)\psi^{\nu-1}(\tau_2,\zeta)\right]\left|\mathfrak{F}\left(\theta,\zeta,z_{(\theta,\zeta)}\right)\right| d\zeta d\theta \\
&\quad + \frac{1}{\Gamma(\mu)\Gamma(\nu)} \int_0^{\varkappa_1} \int_{\tau_1}^{\tau_2} \psi_\varkappa(\theta)\psi_\tau(\zeta)\left[\psi^{\mu-1}(\varkappa_2,\theta)\psi^{\nu-1}(\tau_2,\zeta)\right]\left|\mathfrak{F}\left(\theta,\zeta,z_{(\theta,\zeta)}\right)\right| d\zeta d\theta \\
&\quad + \frac{1}{\Gamma(\mu)\Gamma(\nu)} \int_{\varkappa_1}^{\varkappa_2} \int_0^{\tau_1} \psi_\varkappa(\theta)\psi_\tau(\zeta)\left[\psi^{\mu-1}(\varkappa_2,\theta)\psi^{\nu-1}(\tau_2,\zeta)\right]\left|\mathfrak{F}\left(\theta,\zeta,z_{(\theta,\zeta)}\right)\right| d\zeta d\theta \\
&\le \|\eta(\varkappa_2,\tau_2) - \eta(\varkappa_1,\tau_1)\|_\infty + \frac{\|p\|_\infty + \|q\|_\infty\xi}{\Gamma(\mu)\Gamma(\nu)} \int_0^{\varkappa_1} \int_0^{\tau_1} \psi_{\mathrm{u}}(\theta)\psi_\tau(\zeta)\left[\psi^{\mu-1}(\varkappa_2,\theta)\psi^{\nu-1}(\tau_2,\zeta)\right. \\
&\quad \left. - \psi^{\mu-1}(\varkappa_1,\theta)\psi^{\nu-1}(\tau_1,\zeta)\right] d\zeta d\theta + \frac{\|p\|_\infty + \|q\|_\infty\xi}{\Gamma(\mu)\Gamma(\nu)} \int_{\varkappa_1}^{\varkappa_2} \int_{\tau_1}^{\tau_2} \psi_\varkappa(\theta)\psi_\tau(\zeta) \\
&\quad \times \left[\psi^{\mu-1}(\varkappa_2,\theta)\psi^{\nu-1}(\tau_2,\zeta)\right] d\zeta d\theta + \frac{\|p\|_\infty + \|q\|_\infty\xi}{\Gamma(\mu)\Gamma(\nu)} \int_0^{\varkappa_1} \int_{\tau_1}^{\tau_2} \psi_\varkappa(\theta)\psi_\tau(\zeta) \\
&\quad \times \left[\psi^{\mu-1}(\varkappa_2,\theta)\psi^{\nu-1}(\tau_2,\zeta)\right] d\zeta d\theta + \frac{\|p\|_\infty + \|q\|_\infty\xi}{\Gamma(\mu)\Gamma(\nu)} \int_{\varkappa_1}^{\varkappa_2} \int_0^{\tau_1} \psi_\varkappa(\theta)\psi_\tau(\zeta) \\
&\quad \times \left[\psi^{\mu-1}(\varkappa_2,\theta)\psi^{\nu-1}(\tau_2,\zeta)\right] d\zeta d\theta \le \|\eta(\varkappa_2,\tau_2) - \eta(\varkappa_1,\tau_1)\|_\infty \\
&\quad + \frac{\|p\|_\infty + \|q\|_\infty\xi}{\Gamma(\mu+1)\Gamma(\nu+1)}\left[2\psi^\nu(\tau_2,0)\psi^\mu(\varkappa_2,\varkappa_1) + 2\psi^\mu(\varkappa_2,0)\psi^\nu(\tau_2,\tau_1)\right. \\
&\quad \left. + \psi^\nu(\tau_1,0)\psi^\mu(\varkappa_1,0) - \psi^\nu(\tau_2,0)\psi^\mu(\varkappa_2,0) - 2\psi^\nu(\tau_2,\tau_1)\psi^\mu(\varkappa_2,\varkappa_1)\right].
\end{aligned} \tag{29}$$

If $-\kappa_1 \le \varkappa_1 \le \varkappa_2 \le 0$, and $-\kappa_2 \le \tau_1 \le \tau_2 \le 0$, then

$$|(\mathscr{K}z)(\varkappa_2,\tau_2)-(\mathscr{K}z)(\varkappa_1,\tau_1)| \le |\varphi(\varkappa_2,\tau_2)-\varphi(\varkappa_1,\tau_1)|. \tag{30}$$

If $-\kappa_1 \le \varkappa_1 < 0 < \varkappa_2 \le c$, and $-\kappa_2 \le \tau_1 < 0 < \tau_2 \le d$, then

$$\begin{aligned}
&|(\mathscr{K}z)(\varkappa_2,\tau_2)-(\mathscr{K}z)(\varkappa_1,\tau_1)| \\
&\quad \le |(\mathscr{K}z)(\varkappa_2,\tau_2)-(\mathscr{K}z)(0,0)| + |(\mathscr{K}z)(0,0)-(\mathscr{K}z)(\varkappa_1,\tau_1)| \\
&\quad \le \|\eta(\varkappa_2,\tau_2)-\eta(0,0)\|_\infty + \frac{\|p\|_\infty+\|q\|_\infty\xi}{\Gamma(\mu+1)\Gamma(\nu+1)}[2\psi^\nu(\tau_2,0)\psi^\mu(\varkappa_2,0) \\
&\qquad + 2\psi^\mu(\varkappa_2,0)\psi^\nu(\tau_2,0)+\psi^\nu(0,0)\psi^\mu(0,0)-\psi^\nu(\tau_2,0)\psi^\mu(\varkappa_2,0) \\
&\qquad -2\psi^\nu(\tau_2,0)\psi^\mu(\varkappa_2,0)] + |\varphi(0,0)-\varphi(\varkappa_1,\tau_1)|.
\end{aligned} \tag{31}$$

In all previous cases, as $\varkappa_1 \longrightarrow \varkappa_2$, $\tau_1 \longrightarrow \tau_2$, and the uniform continuity of η on J_1 and φ on J_2 implies that for any $\varepsilon>0$, there exists $\delta>0$, independent of $\varkappa_1,\varkappa_2$, τ_1,τ_2 and z, such that $|(\mathscr{K}z)(\varkappa_2,\tau_2)-(\mathscr{K}z)(\varkappa_1,\tau_1)| \le \varepsilon$ whenever $|\psi(\varkappa_2)-\psi(\varkappa_1)| \le \delta/2$ and $|\psi(\tau_2)-\psi(\tau_1)| \le \delta/2$. Therefore, $\mathscr{K}(B_\xi)$ is equicontinuous. It follows from the Arzela–Ascoli theorem that $\mathscr{K}$ is compact.

Step 4: $\mathscr{K}(B_\xi)$ a priori bounds. $\exists$ an open set $\Omega \subseteq \mathscr{C}_{(c,d)}$ with $z \ne \aleph\mathscr{K}z$, for $\aleph \in (0,1)$, and $z \in \partial\Omega$. Let $(\varkappa,\tau) \in [-\kappa_1, c] \times [-\kappa_2, d]$ and $z \in \mathscr{C}_{(c,d)}$ with $z \ne \aleph\mathscr{K}z$, for some $\aleph \in (0,1)$. Then,

$$\begin{aligned}
|z(\varkappa,\tau)| &\le \aleph|\eta(\varkappa,\tau)| + \frac{\aleph}{\Gamma(\mu)\Gamma(\nu)}\int_0^\varkappa\int_0^\tau \psi_\varkappa(\theta)\psi^{\mu-1}(\varkappa,\theta)\psi_\tau(\zeta)\psi^{\nu-1}(\tau,\zeta)\left|\mathfrak{F}\left(\theta,\zeta,z_{(\theta,\zeta)}\right)\right|d\zeta d\theta \\
&\le |\eta(\varkappa,\tau)| + \frac{1}{\Gamma(\mu)\Gamma(\nu)}\int_0^\varkappa\int_0^\tau \psi_\varkappa(\theta)\psi^{\mu-1}(\varkappa,\theta)\psi_\tau(\zeta)\psi^{\nu-1}(\tau,\zeta) \\
&\quad \cdot \left[p(\theta,\zeta)+q(\theta,\zeta)\left\|z_{(\theta,\zeta)}\right\|_{\mathscr{C}}\right]d\zeta d\theta \\
&\le \|\eta\|_\infty + \frac{\|p\|_\infty\psi^\mu(c,0)\,\psi^\nu(d,0)}{\Gamma(\mu+1)\Gamma(\nu+1)} \\
&\quad + \frac{\|q\|_\infty}{\Gamma(\mu)\Gamma(\nu)}\int_0^\varkappa\int_0^\tau \psi_{\mathrm{u}}(\theta)\psi^{\mu-1}(\varkappa,\theta)\psi_\tau(\zeta)\psi^{\nu-1}(\tau,\zeta)\|z\|_{\mathscr{C}_{(c,d)}}d\zeta d\theta.
\end{aligned} \tag{32}$$

If $(\varkappa,\tau) \in J_1$, then (32) becomes

$$\begin{aligned}
\|z(\varkappa,\tau)\|_\infty &\le \|\eta\|_\infty + \frac{\|p\|_\infty\psi^\mu(c,0)\,\psi^\nu(d,0)}{\Gamma(\mu+1)\Gamma(\nu+1)} \\
&\quad + \frac{\|q\|_\infty}{\Gamma(\mu)\Gamma(\nu)}\int_0^\varkappa\int_0^\tau \psi_\varkappa(\theta)\psi^{\mu-1}(\varkappa,\theta)\psi_\tau(\zeta)\psi^{\nu-1}(\tau,\zeta)\|z(\theta,\zeta)\|_\infty d\zeta d\theta \\
&\le \|\eta\|_\infty + \frac{\|p\|_\infty+\|q\|_\infty\|z(\varkappa,\tau)\|_\infty}{\Gamma(\mu+1)\Gamma(\nu+1)}\psi^\mu(c,0)\,\psi^\nu(d,0),
\end{aligned} \tag{33}$$

which implies

$$\|z(\varkappa,\tau)\|_\infty \le \frac{\lambda}{1-\rho} := M. \tag{34}$$

For $(\varkappa,\tau) \in J_2$,$\|z(\varkappa,\tau)\|_\infty = \|\varphi\|_{\mathscr{C}}$. Consequently,

$$\|z\|_\infty = \max\left\{M, \|\varphi\|_{\mathscr{C}}\right\} := \xi^*. \tag{35}$$

Set

$$\Omega = \left\{z \in \mathscr{C}_{(c,d)} : \|z\|_\infty < \xi^*+1\right\}. \tag{36}$$

□

Through our choice Ω, nothing $z \in \partial\Omega$ such that $z=\aleph\mathscr{K}z$,$0<\aleph<1$.

As conclusion, the Leray-Schauder FPT shows that $\mathscr{K}$ has a fixed point $z \in \Omega \subset \mathscr{C}_{(c,d)}$ such that $z=\mathscr{K}z$ which is a solution to problem.

We now provide two results on the nonlocal problem (2), and their proofs are quite similar to the preceding results. In addition, the results in Theorems 9 and 10 can be presented by

$$(\mathscr{K}z)(\varkappa,\tau) = \begin{cases} \varphi(\varkappa,\tau), & (\varkappa,\tau) \in J_2, \\ \eta(\varkappa,\tau)+h_1(z)+h_2(z)+\dfrac{1}{\Gamma(\mu)\Gamma(\nu)} \times \displaystyle\int_0^\varkappa\int_0^\tau \psi_\varkappa(\theta)\psi^{\mu-1}(\varkappa,\theta)\psi_\tau(\zeta)\psi^{\nu-1}(\tau,\zeta)\mathfrak{F}\left(\theta,\zeta,z_{(\theta,\zeta)}\right)d\zeta d\theta & (\varkappa,\tau) \in J_1. \end{cases} \tag{37}$$

Theorem 11. *Let (A1) and (A2) be satisfied. If there exist $L_{h_1}, L_{h_2} > 0$ such that*

$$\begin{aligned}
|h_1(z)-h_1(v)| &\le L_{h_1}\|z-v\|_\infty, \quad \text{for } z,v \in \mathscr{C}(J_1,\mathbb{R}), \\
|h_2(z)-h_2(v)| &\le L_{h_2}\|z-v\|_\infty, \quad \text{for } z,v \in \mathscr{C}(J_1,\mathbb{R}),
\end{aligned} \tag{38}$$

with $\Lambda := L_{h_1}+L_{h_2}+\sigma<1$, where σ is defined by (19); then, there exists a unique solution for the ψ-Caputo problem (2) on $[-\kappa_1, c] \times -\kappa_2, d]$.

Theorem 12. *Let (A1) and (A3) be satisfied. If there exist $d_{h_1}, d_{h_2} > 0$ such that*

$$\begin{aligned}
\|h_1(z)\| &\le d_{h_1}\left(1+\|z\|_\infty\right), \quad \text{for } z \in \mathscr{C}(J_1,\mathbb{R}^n), \\
\|h_2(z)\| &\le d_{h_2}\left(1+\|z\|_\infty\right), \quad \text{for } z \in \mathscr{C}(J_1,\mathbb{R}^n),
\end{aligned} \tag{39}$$

with $\rho < 1$, where ρ is defined by (A3); then, there exists at least one solution for the ψ-Caputo problem (2) on $[-\kappa_1, c] \times [-\kappa_2, d]$.

Now, we provide the UH and GUH stability of the ψ-problem (2).

Definition 13. (see (2)). Problem ((2)) is UH stable if there exists a $\chi_\varphi > 0$ such that $\forall \varepsilon > 0$ and each solution $\omega(\varkappa, \tau) \in \mathscr{C}_{(c,d)}$ of the inequality

$$\begin{aligned} &\left| {}^C\mathscr{D}^{r;\psi}_{\ell^+} \omega(\varkappa,\tau) - \mathfrak{F}\left(\varkappa, \tau, \omega_{(\varkappa,\tau)}\right) \right| \le \varepsilon, \quad (\varkappa,\tau) \in J_1, \\ &|\omega(\varkappa,\tau) - \varphi(\varkappa,\tau)| \le \varepsilon, \quad (\varkappa,\tau) \in J_2, \end{aligned} \tag{40}$$

there exists a solution $z(\varkappa, \tau) \in \mathscr{C}_{(c,d)}$ of (2) satisfies

$$\|\omega(\varkappa,\tau) - z(\varkappa,\tau)\|_{\mathscr{C}_{(c,d)}} \le \chi_\varphi \varepsilon. \tag{41}$$

Remark 14. $\omega(\varkappa, \tau) \in \mathscr{C}_{(c,d)}$ satisfies (40) iff there exists $\varsigma(\varkappa, \tau) \in \mathscr{C}_{(c,d)}$ with

(i) $|\varsigma(\varkappa,\tau)| \le \varepsilon, \varkappa \in J_1$

(ii) for all $\varkappa \in J_1$

$$ {}^C\mathscr{D}^{r;\psi}_{\ell^+} \omega(\varkappa,\tau) = \mathfrak{F}\left(\varkappa, \tau, \omega_{(\varkappa,\tau)}\right) + \varsigma(\varkappa,\tau). \tag{42}$$

Lemma 15. *Let $r = (\mu, \nu) \in (0, 1] \times (0, 1]$, and $\omega(\varkappa, \tau) \in \mathscr{C}_{(c,d)}$ is a solution of (40). Then, $\omega(\varkappa, \tau)$ satisfies*

$$\begin{aligned} &\left|\omega(\varkappa,\tau) - \omega_0(\varkappa,\tau) - \frac{1}{\Gamma(\mu)\Gamma(\nu)} \int_0^\varkappa \psi_\varkappa(\theta)\psi^{\mu-1}(\varkappa,\theta) \int_0^\tau \psi_\tau(\zeta)\psi^{\nu-1}(\tau,\zeta)\mathfrak{F} \right. \\ &\quad \left. \times \left(\theta, \zeta, \omega_{(\theta,\zeta)}\right) d\zeta d\theta \right| \le \varepsilon \frac{\psi^\nu(d,0)}{\Gamma(\nu+1)} \frac{\psi^\mu(c,0)}{\Gamma(\mu+1)}, \end{aligned} \tag{43}$$

for $(\varkappa, \tau) \in J_1$, where $\omega_0(\varkappa, \tau) = \eta(\varkappa, \tau) + h_1(\omega) + h_2(\omega)$. Moreover, $|\omega(\varkappa, \tau) - \varphi(\varkappa, \tau)| = 0$, for $(\varkappa, \tau) \in J_2$.

Proof. Let $\omega(\varkappa, \tau)$ is a solution of (40). It follows from (ii) of Remark 14that

$$\begin{aligned} &{}^C\mathscr{D}^{r;\psi}_{\ell+} \omega_{(\varkappa,\tau)} = \mathfrak{F}\left(\varkappa, \tau, \omega_{(\varkappa,\tau)}\right) + \varsigma(\varkappa,\tau), (\varkappa,\tau) \in J_1, \\ &\omega(\varkappa,\tau) = \varphi(\varkappa,\tau), \ (\varkappa,\tau) \in J_2, \\ &\omega(\varkappa, 0) + h_1(\omega) = \phi_1(\varkappa), \omega(0,\tau) + h_2(\omega) = \phi_2(\tau), (\varkappa,\tau) \in J_1. \end{aligned} \tag{44}$$

Then, the solution of problem (44) is

$$\omega(\varkappa,\tau) = \begin{cases} |\varphi(\varkappa,\tau)|, & (\varkappa,\tau) \in J_2, \\ \eta(\varkappa,\tau) + h_1(\omega) + h_2(\omega) + \dfrac{1}{\Gamma(\mu)\Gamma(\nu)} \times \displaystyle\int_0^\varkappa \int_0^\tau \psi_\varkappa(\theta)\psi^{\mu-1}(\varkappa,\theta)\psi_\tau(\zeta)\psi^{\nu-1}(\tau,\zeta) \left[\mathfrak{F}\left(\theta, \zeta, \omega_{(\theta,\zeta)}\right) + \varsigma(\theta,\zeta)\right] d\zeta d\theta, & (\varkappa,\tau) \in J_1. \end{cases} \tag{45}$$

Once more by (i) of Remark 14, we get

$$\begin{aligned} &\left|\omega(\varkappa,\tau) - \omega_0(\varkappa,\tau) - \frac{1}{\Gamma(\mu)\Gamma(\nu)} \int_0^\varkappa \psi_\varkappa(\theta)\psi^{\mu-1}(\varkappa,\theta) \int_0^\tau \psi_\tau(\zeta)\psi^{\nu-1}(\tau,\zeta)\mathfrak{F}\left(\theta, \zeta, \omega_{(\theta,\zeta)}\right) d\zeta d\theta \right| \\ &\le \frac{1}{\Gamma(\mu)\Gamma(\nu)} \int_0^\varkappa \int_0^\tau \psi_\varkappa(\theta)\psi^{\mu-1}(\varkappa,\theta)\psi_\tau(\zeta)\psi^{\nu-1}(\tau,\zeta)|\varsigma(\theta,\zeta)| d\zeta d\theta \\ &\le \frac{\varepsilon}{\Gamma(\mu)\Gamma(\nu)} \int_0^\varkappa \psi_\varkappa(\theta)\psi^{\mu-1}(\varkappa,\theta) \int_0^\tau \psi_\tau(\zeta)\psi^{\nu-1}(\tau,\zeta) d\zeta d\theta \\ &= \varepsilon \frac{\psi^\nu(\tau,0)}{\Gamma(\nu+1)} \frac{1}{\Gamma(\mu)} \int_0^\varkappa \psi_\varkappa(\theta)\psi^{\mu-1}(\varkappa,\theta) d\theta \\ &= \varepsilon \frac{\psi^\nu(\tau,0)}{\Gamma(\nu+1)} \frac{\psi^\mu(\varkappa,0)}{\Gamma(\mu+1)} \le \varepsilon \frac{\psi^\nu(d,0)}{\Gamma(\nu+1)} \frac{\psi^\mu(c,0)}{\Gamma(\mu+1)}, \end{aligned} \tag{46}$$

for $(\varkappa, \tau) \in J_2$. For $(\varkappa, \tau) \in J_2$, we obtain $|\omega(\varkappa, \tau) - \varphi(\varkappa, \tau)| = |\varphi(\varkappa, \tau) - \varphi(\varkappa, \tau)| = 0$. □

Theorem 16. *Under assumptions of Theorem 9, the solution of the problem (2) is HU and GHU stable on $[-\kappa_1, c] \times -\kappa_2, d]$.*

Proof. Let $\omega(\varkappa, \tau) \in \mathscr{C}$ be a solution of (40), and $z(\varkappa, \tau) \in \mathscr{C}_{(c,d)}$ is a unique solution of the following problem:

$$\begin{aligned} &{}^C\mathscr{D}^{r;\psi}_{\ell+} z(\varkappa,\tau) = \mathfrak{F}\left(\varkappa, \tau, z_{(\varkappa,\tau)}\right), (\varkappa,\tau) \in J_1, \\ &z(\varkappa,\tau) = \varphi(\varkappa,\tau), \ (\varkappa,\tau) \in J_2, \\ &z(\varkappa, 0) + h_1(z) = \omega(\varkappa, 0) + h_1(\omega), z(0,\tau) + h_2(z) \\ &\quad = \omega(0,\tau) + h_2(\omega), (\varkappa,\tau) \in J_1. \end{aligned} \tag{47}$$

The previous problem has a solution

$$z(\varkappa,\tau) = \begin{cases} \varphi(\varkappa,\tau), & (\varkappa,\tau) \in J_2, \\ z_0(\varkappa,\tau) + \dfrac{1}{\Gamma(\mu)\Gamma(\nu)} \displaystyle\int_0^\varkappa \psi_\varkappa(\theta)\psi^{\mu-1}(\varkappa,\theta) \int_0^\tau \psi_\tau(\zeta)\psi^{\nu-1}(\tau,\zeta)\mathfrak{F}\left(\theta, \zeta, z_{(\theta,\zeta)}\right) d\zeta d\theta & (\varkappa,\tau) \in J_1, \end{cases} \tag{48}$$

where $z_0(\varkappa, \tau) \coloneqq \eta(\varkappa, \tau) + h_1(z) + h_2(z)$.

Since $z(\varkappa, 0) + h_1(z) = \omega(\varkappa, 0) + h_1(\omega)$ and $z(0, \tau) + h_2(z) = \omega(0, \tau) + h_2(\omega)$, we have $z_0(\varkappa, \tau) = \omega_0(\varkappa, \tau)$. Indeed,

$$\begin{aligned} z_0(\varkappa, \tau) &= \eta(\varkappa, \tau) + h_1(z) + h_2(z) \\ &= \eta(\varkappa, \tau) - z(\varkappa, 0) + \omega(\varkappa, 0) + h_1(\omega) - z(0, \tau) + \omega(0, \tau) + h_2(\omega) \\ &= \eta(\varkappa, \tau) - \phi_1(\varkappa) + \phi_1(\varkappa) + h_1(\omega) - \phi_2(\tau) + \phi_2(\tau) + h_2(\omega) \\ &= \eta(\varkappa, \tau) + h_1(\omega) + h_2(\omega) = \omega_0(\varkappa, \tau). \end{aligned} \tag{49}$$

Hence, (48) becomes

$$z(\varkappa, \tau) = \begin{cases} \varphi(\varkappa, \tau), & (\varkappa, \tau) \in J_2, \\ \omega_0(\varkappa, \tau) + \dfrac{1}{\Gamma(\mu)\Gamma(\nu)} \displaystyle\int_0^\varkappa \psi_\varkappa(\theta)\psi^{\mu-1}(\varkappa, \theta) \int_0^\tau \psi_\tau(\zeta)\psi^{\nu-1}(\tau, \zeta)\mathfrak{F}\left(\theta, \zeta, z_{(\theta\zeta)}\right) d\zeta d\theta & (\varkappa, \tau) \in J_1. \end{cases} \tag{50}$$

Note that, $|\omega(\varkappa, \tau) - z(\varkappa, \tau)| = 0$, for all $(\varkappa, \tau) \in J_2$.

Using Lemma 15 and (A2), for $(\varkappa, \tau) \in J_1$, we have

$$\begin{aligned} |\omega(\varkappa, \tau) - z(\varkappa, \tau)| &= \left| \omega(\varkappa, \tau) - \omega_0(\varkappa, \tau) - \frac{1}{\Gamma(\mu)\Gamma(\nu)} \int_0^\varkappa \psi_\varkappa(\theta)\psi^{\mu-1}(\varkappa, \theta) \int_0^\tau \psi_\tau \right. \\ &\quad \left. \times (\zeta)\psi^{\nu-1}(\tau, \zeta)\mathfrak{F}\left(\theta, \zeta, z_{(\theta\zeta)}\right) d\zeta d\theta \right| \\ &\le \left| \omega(\varkappa, \tau) - \omega_0(\varkappa, \tau) - \frac{1}{\Gamma(\mu)\Gamma(\nu)} \int_0^\varkappa \psi_\varkappa(\theta)\psi^{\mu-1}(\varkappa, \theta) \int_0^\tau \psi_\tau(\zeta)\psi^{\nu-1}(\tau, \zeta)\mathfrak{F} \right. \\ &\quad \left. \times \left(\theta, \zeta, \omega_{(\theta\zeta)}\right) d\zeta d\theta \right| + \frac{1}{\Gamma(\mu)\Gamma(\nu)} \int_0^\varkappa \psi_\varkappa(\theta)\psi^{\mu-1}(\varkappa, \theta) \int_0^\tau \psi_\tau(\zeta)\psi^{\nu-1}(\tau, \zeta) |\mathfrak{F} \\ &\quad \times \left(\theta, \zeta, \omega_{(\theta\zeta)}\right) - \mathfrak{F}\left(\theta, \zeta, z_{(\theta\zeta)}\right) \Big| d\zeta d\theta \\ &\le \varepsilon \frac{\psi^\nu(d, 0)}{\Gamma(\nu+1)} \frac{\psi^\mu(c, 0)}{\Gamma(\mu+1)} + L_{\mathfrak{F}} \left\| \omega_{(\theta\zeta)} - z_{(\theta\zeta)} \right\|_{\mathscr{C}} \frac{\psi^\nu(d, 0)}{\Gamma(\nu+1)} \frac{\psi^\mu(c, 0)}{\Gamma(\mu+1)} \le \varepsilon \frac{\sigma}{L_{\mathfrak{F}}} + \sigma \|\omega - z\|_{\mathscr{C}(c,d)}, \end{aligned} \tag{51}$$

which implies

$$\|\omega - z\|_{\mathscr{C}(c,d)} \le \frac{\sigma}{L_{\mathfrak{F}}(1-\sigma)} \varepsilon. \tag{52}$$

Taking $\chi_\varphi \coloneqq \sigma / L_{\mathfrak{F}}(1-\sigma)$ such that $\sigma < 1$, then (51) becomes

$$\|\omega - z\|_{\mathscr{C}(c,d)} \le \chi_\varphi \varepsilon. \tag{53}$$

Hence, problem (2) is UH stable. Moreover, if there exists a nondecreasing function $\Psi : [0,\infty) \longrightarrow 0, \infty)$ such that $\Psi(\varepsilon) = \varepsilon$, then we have with $\Psi(0) = 0$,

$$\|\omega - z\|_{\mathscr{C}(c,d)} \le \Psi(\varepsilon), \tag{54}$$

which proves that problem (2) is also GUH stable. □

4. Examples

In this portion, we provide two examples of partial hyperbolic FDEs having fractional order and satisfying the obtained results. All computational work will be performed through MATLAB.

Example 1. Consider a ψ-Caputo fractional partial hyperbolic FDE

$$\begin{aligned} {}^C\mathscr{D}_{0+}^{r;\psi} z(\varkappa, \tau) &= \frac{1}{2e^{\varkappa+\tau+2}(1 + |z(\varkappa-1, \tau-2)|)}, (\varkappa, \tau) \in [0, 1] \times [0, 1], \\ z(\varkappa, \tau) &= \varkappa + \tau^2, \ (\varkappa, \tau) \in [-1, 1] \times [-2, 1] \setminus (0, 1] \times (0, 1], \\ z(\varkappa, 0) &= \varkappa, z(0, \tau) = \tau^2, \varkappa, \tau \in [0, 1], \end{aligned} \tag{55}$$

where $r = (\mu, \nu)$, $\mu = 1/2, \nu = 1/3, \varphi(\varkappa, \tau) = \varkappa + \tau^2$, $c = d = 1, \kappa_1 = 1, \kappa_2 = 2, \phi_1(\varkappa) = \varkappa, \phi_2(\tau) = \tau^2$. Consider $\mathfrak{F}(\varkappa, \tau, \bar{z}) = 1/(3e^{\varkappa+\tau+2}(1 + \bar{z}(\varkappa-1, \tau-2)))$, for $(\varkappa, \tau, \bar{z}) \in 0, 1] \times 0, 1] \times \mathscr{C}([-1, 0] \times -2, 0], \mathbb{R})$. Let $z, v \in \mathscr{C}([-1, 0] \times -2, 0], \mathbb{R})$, and $(\varkappa, \tau) \in 0, 1] \times 0, 1]$. Then,

$$\begin{aligned} &\left| \mathfrak{F}\left(\varkappa, \tau, z_{(\varkappa,\tau)}\right) - \mathfrak{F}\left(\varkappa, \tau, v_{(\varkappa,\tau)}\right) \right| \\ &\quad \le \frac{1}{3e^2} |z(\varkappa-1, \tau-2) - v(\varkappa-1, \tau-2)| \le \frac{1}{3e^2} \|z - v\|_{\mathscr{C}}. \end{aligned} \tag{56}$$

So, assumptions (A1) and (A2) are satisfied with $L_{\mathfrak{F}} = 1/3e^2$. Moreover, the condition $\sigma = 2/(3^{5/6} e^2 \sqrt{\pi} \Gamma(1/3)) < 1$ with $\psi(\varkappa) = \varkappa/3, \psi(\tau) = \tau e^{\tau-1}/3$ and $c = d = 1$. Hence, Theorem 9 shows that problem (55) has a unique solution defined on $[-1, 1] \times [-2, 1]$.

Example 2. Consider a ψ-Caputo fractional partial hyperbolic FDE

$$\begin{aligned} {}^C\mathscr{D}_{0+}^{r;\psi} z(\varkappa, \tau) &= \frac{e^{-\varkappa-\tau}}{4 + e^{\varkappa+\tau}} \left(1 + \frac{|z(\varkappa-1, \tau-2)|}{(1 + |z(\varkappa-1, \tau-2)|)}\right), \\ &\quad \cdot (\varkappa, \tau) \in \left[0, \frac{1}{3}\right] \times \left[0, \frac{1}{3}\right], \\ z(\varkappa, \tau) &= \varkappa^2 + \tau, \ (\varkappa, \tau) \in \left[-1, \frac{1}{3}\right] \times \left[-2, \frac{1}{3}\right] \setminus \left(0, \frac{1}{3}\right] \times \left(0, \frac{1}{3}\right], \\ z(\varkappa, 0) &= \varkappa^2, z(0, \tau) = \tau, \varkappa, \tau \in \left[0, \frac{1}{3}\right], \end{aligned} \tag{57}$$

where $r = (\mu, \nu)$, $\mu = 1/2, \nu = 1/3, \varphi(\varkappa, \tau) = \varkappa^2 + \tau$, $c = d = 1/3, \kappa_1 = 1, \kappa_2 = 2, \phi_1(\varkappa) = \varkappa^2, \phi_2(\tau) = \tau$. Consider $\mathfrak{F}(\varkappa, \tau, \bar{z}) = (e^{-\varkappa-\tau}/(4 + e^{\varkappa+\tau}))(1 + \bar{z}/(1 + \bar{z}))$, for $(\varkappa, \tau, \bar{z}) \in [0, 1/3] \times [0, 1/3] \times \mathscr{C}([-1, 0] \times -2, 0], \mathbb{R})$. Let $z \in \mathscr{C}([-1, 0] \times -2, 0], \mathbb{R})$, and $(\varkappa, \tau) \in [0, 1/3] \times [0, 1/3]$. Then,

$$\begin{aligned} \left| \mathfrak{F}\left(\varkappa, \tau, z_{(\varkappa,\tau)}\right) \right| &\le \frac{e^{-\varkappa-\tau}}{4 + e^{\varkappa+\tau}} \left| 1 + \frac{z_{(\varkappa,\tau)}}{1 + z_{(\varkappa,\tau)}} \right| \le \frac{e^{-\varkappa-\tau}}{4 + e^{\varkappa+\tau}} + \frac{e^{-\varkappa-\tau}}{4 + e^{\varkappa+\tau}} \left| z_{(\varkappa,\tau)} \right| \\ &\le \frac{e^{-\varkappa-\tau}}{4 + e^{\varkappa+\tau}} + \frac{e^{-\varkappa-\tau}}{4 + e^{\varkappa+\tau}} \|z\|_{\mathscr{C}}. \end{aligned} \tag{58}$$

Thus, (A3) holds with $p(\varkappa, \tau) = q(\varkappa, \tau) = e^{-\varkappa-\tau}/(4 + e^{\varkappa+\tau})$, where $\|q\|_\infty = 1/5$. To verify that $\rho < 1$, we select $\psi(\varkappa) = e^{\varkappa/3}$

and $\psi(\tau)=\sqrt{\tau+1}$, then find that

$$\psi^{\mu}(c,0)=(\psi(c)-\psi(0))^{\mu}=\left(e^{c/3}\right)^{1/2}=\sqrt{e^{1/9}},$$
$$\psi^{\nu}(d,0)=(\psi(d)-\psi(0))^{\nu}=\left(\sqrt{d+1}\right)^{1/3}=\left(2\sqrt{\frac{2}{3}}\right)^{1/3}, \tag{59}$$

and $\rho\approx 0.314<1$. So, all assumptions of Theorem 10 are satisfied. Hence, Theorem 10 shows that problem (57) has a solution defined on $[-1,1/3]\times[-2,1/3]$.

Remark 3. Our current outcomes on problems (1) and (2) can be interpreted as extensions of preceding results of Abbas et al. [19], for $\psi(\varkappa)=\varkappa$.

Remark 4. As special cases, it is possible to obtain other results for similar problems involving various FDs such as Caputo-Katugampola FD (for $(\varkappa)=(\varkappa^{\rho}),\rho>0$), Caputo-Hadamard FD (for $\psi(\varkappa)=\ln(\varkappa)$), and other FDs, for different choices of $\psi(\cdot)$.

5. Conclusion

Somewhat recently, several fractional definitions have been proposed to describe the behaviors of some complex world problems arising in many scientific fields. In this regard, Sousa and de Oliveira [9] introduced the concept of the multivariate partial fractional derivative with respect to another function. As an additional contribution to this topic, existence and uniqueness results have been obtained for two types of Cauchy and nonlocal fractional partial hyperbolic FDEs (1) and (2) involving ψ-Caputo FD with two variables. We have presented several results based on Banach's and Leray-Schauder's fixed point theorem. In light of our present results, special cases of similar problems containing several partial fractional operators have been presented according to different choices of the ψ function. Moreover, we have provided the stability results in UH and GUH sense. Lastly, two suitable examples that validate the obtained results were given.

It is interesting to approach current problems with infinite delay, and this is what we are thinking of in future research. One can also study the same present problem in terms of the generalized fractional derivative that was recently proposed in [23, 24].

Data Availability

Data are available upon request.

Conflicts of Interest

No conflicts of interest are related to this work.

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