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Research Article

Applications of Lehmer's Infinite Series Involving Reciprocals of the Central Binomial Coefficients

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The main objective of this paper is to establish several new closed-form evaluations of the generalized hypergeometric function ${}_{q+1}F_q(z)$ for $q = 2, 3, 4, 5$. This is achieved by means of separating the generalized hypergeometric function ${}_{q+1}F_q(z)$ $(q = 2, 3, 4, 5)$ into even and odd components together with the use of several known infinite series involving reciprocals of the central binomial coefficients obtained earlier by Lehmer.

1. Lehmer's Series Involving Central Binomial Coefficients

The binomial coefficients are defined by

$$\binom{n}{k} = \begin{cases} \dfrac{n!}{k!(n-k)!}, & n \geq k, \\ 0, & n < k, \end{cases} \tag{1}$$

for nonnegative integers n and k. The central binomial coefficients are defined by

$$\binom{2n}{n} = \frac{(2n)!}{(n!)^2} \quad (n = 0, 1, 2, \cdots). \tag{2}$$

It is well-known that the binomial and reciprocal of binomial coefficients play an important role in many areas of mathematics (including number theory, probability, and statistics). Actually, the sums containing the central binomial coefficients and reciprocals of the central binomial coefficients have been studied for a long time. A large number of very interesting results can be seen in the research papers by Lehmer [1], Mansour [2], Pla [3], Rockett [4], Sprugnoli [5, 6], Sury [7], Sury et al. [8], Trif [9], Wheelon [10], and Zhao and Wang [11]. Many facts about the central binomial coefficients and the reciprocals of the central binomial coefficients can be found in the book of Koshy [12]. Gould [13] has collected numerous identities involving central binomial coefficients. Riordan [14] is also a good reference. However, in our present investigation, we are interested in a very interesting paper due to Lehmer [1] in which he studied the following two types of the series, viz,

$$\sum_{n=0}^{\infty} a_n \binom{2n}{n}, \sum_{n=0}^{\infty} \frac{a_n}{\binom{2n}{n}}, \tag{3}$$

where a_n are of very simple functions of n and deduced several interesting series involving the central binomial coefficients and reciprocals of the central binomial coefficients, with Golden ratio $\varphi = (\sqrt{5} + 1)/2$ as follows:

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{n4^n} = \ln 4, \tag{4}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\binom{2n}{n}}{n4^n} = 2\ \ln\left(\frac{\sqrt{2}+1}{2}\right), \tag{5}$$

$$\sum_{n=1}^{\infty} \frac{1}{n\binom{2n}{n}} = \frac{\pi\sqrt{3}}{9}, \tag{6}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n\binom{2n}{n}} = \frac{2}{\sqrt{5}} \ln\varphi, \tag{7}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2\binom{2n}{n}} = \frac{\pi^2}{18}, \tag{8}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2\binom{2n}{n}} = 2(\ln\varphi)^2, \tag{9}$$

$$\sum_{n=1}^{\infty} \frac{1}{\binom{2n}{n}} = \frac{1}{3}\left(1+\frac{2\pi\sqrt{3}}{9}\right), \tag{10}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\binom{2n}{n}} = \frac{1}{5}\left(1+\frac{4\sqrt{5}}{5}\ln\varphi\right), \tag{11}$$

$$\sum_{n=1}^{\infty} \frac{n}{\binom{2n}{n}} = \frac{2}{3}\left(1+\frac{\sqrt{3}\pi}{9}\right), \tag{12}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}n}{\binom{2n}{n}} = \frac{2}{125}\left(15+2\sqrt{5}\ln\varphi\right), \tag{13}$$

$$\sum_{n=1}^{\infty} \frac{n^2}{\binom{2n}{n}} = \frac{2}{81}\left(5\sqrt{3}\pi+54\right), \tag{14}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}n^2}{\binom{2n}{n}} = \frac{4}{125}\left(5-\sqrt{5}\ln\varphi\right), \tag{15}$$

$$\sum_{n=1}^{\infty} \frac{n^3}{\binom{2n}{n}} = \frac{2}{243}\left(37\sqrt{3}\pi+405\right), \tag{16}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n}n^3}{\binom{2n}{n}} = \frac{2}{625}\left(5+14\sqrt{5}\ln\varphi\right), \tag{17}$$

$$\sum_{n=1}^{\infty} \frac{2^n}{n\binom{2n}{n}} = \frac{\pi}{2}, \tag{18}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}2^n}{n\binom{2n}{n}} = \frac{1}{\sqrt{3}}\ln\left(2+\sqrt{3}\right), \tag{19}$$

$$\sum_{n=1}^{\infty} \frac{2^n}{\binom{2n}{n}} = \frac{\pi}{2}+1, \tag{20}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}2^n}{\binom{2n}{n}} = \frac{1}{3}+\frac{\sqrt{3}}{9}\ln\left(2+\sqrt{3}\right), \tag{21}$$

$$\sum_{n=1}^{\infty} \frac{n2^n}{\binom{2n}{n}} = \pi+3, \tag{22}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}n2^n}{\binom{2n}{n}} = \frac{1}{3}, \tag{23}$$

$$\sum_{n=1}^{\infty} \frac{n^2 2^n}{\binom{2n}{n}} = \frac{7\pi}{2}+11, \tag{24}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}n^2 2^n}{\binom{2n}{n}} = \frac{1}{9}\left(1-\frac{\sqrt{3}}{3}\ln\left(2+\sqrt{3}\right)\right), \tag{25}$$

$$\sum_{n=1}^{\infty} \frac{n^3 2^n}{\binom{2n}{n}} = \frac{35\pi}{2}+55, \tag{26}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}n^3 2^n}{\binom{2n}{n}} = -\frac{1}{81}\left(15+\sqrt{3}\ln\left(2+\sqrt{3}\right)\right), \tag{27}$$

We conclude this section by remarking that in the next section, the results (4), (5), (6), (7), (8), (9), (10), (11), (12), (13), (14), (15), (16), (17), (18), (19), (20), (21), (22), (23), (24), (25), (26), AND (27) will be written in terms of a generalized hypergeometric function.

2. Generalized Hypergeometric Function

The generalized hypergeometric function ${}_pF_q(z)$ with p numerator and q denominator parameters is defined by [15]

$$ {}_pF_q\left[\begin{matrix} a_1, a_2, \cdots, a_p \\ b_1, b_2, \cdots, b_q \end{matrix} ; z\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_p)_n}{(b_1)_n (b_2)_n \cdots (b_q)_n} \frac{z^n}{n!}, \tag{28} $$

where $(a)_n$ is the well-known Pochhammer's symbol defined by

$$ (a)_n = \begin{cases} a(a+1)\cdots(a+n-1), & n \in \mathbb{N}, \\ 1, & n = 0. \end{cases} \tag{29} $$

In terms of gamma function, we have

$$ (a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}. \tag{30} $$

Here, as usual, p and q are nonnegative integers, and the parameters $a_j (1 \le j \le p)$ and $b_j (1 \le j \le q)$ can have arbitrary complex values with zero or negative integer values of b_j excluded. The generalized hypergeometric function ${}_pF_q(z)$ converges for $|z| < \infty, (p \le q)$, $|z| < 1 (p = q + 1)$, and $|z| = 1 (p = q + 1 \text{ and } \mathrm{Re}(s) > 0)$, where s is the parametric excess defined by

$$ s = \sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j. \tag{31} $$

It is interesting to mention here that the generalized hypergeometric function occurs in many theoretical and practical applications such as mathematics, theoretical physics, engineering, and statistics.

For more details about this function, we refer the standard texts [16–21]. Furthermore, it is not difficult to see that the results (4), (5), (6), (7), (8), (9), (10), (11), (12), (13), (14), (15), (16), (17), (18), (19), (20), (21), (22), (23), (24), (25), (26), and (27) given in the previous section can be written in terms of the following generalized hypergeometric function that will be required in our present investigations. These are

$$ {}_3F_2\left[\begin{matrix} 1, 1, \frac{3}{2} \\ 2, 2 \end{matrix} ; 1\right] = 4 \ln 2, \tag{32} $$

$$ {}_3F_2\left[\begin{matrix} 1, 1, \frac{3}{2} \\ 2, 2 \end{matrix} ; -1\right] = 4 \ln \left(\frac{\sqrt{2}+1}{2}\right), \tag{33} $$

$$ {}_2F_1\left[\begin{matrix} 1, 1, \\ \frac{3}{2} \end{matrix} ; \frac{1}{4}\right] = \frac{2\sqrt{3}\pi}{9}, \tag{34} $$

$$ {}_2F_1\left[\begin{matrix} 1, 1, \\ \frac{3}{2} \end{matrix} ; -\frac{1}{4}\right] = \frac{4}{\sqrt{5}} \ln \varphi, \tag{35} $$

$$ {}_3F_2\left[\begin{matrix} 1, 1, 1 \\ \frac{3}{2}, 2 \end{matrix} ; \frac{1}{4}\right] = \frac{\pi^2}{9}, \tag{36} $$

$$ {}_3F_2\left[\begin{matrix} 1, 1, 1 \\ \frac{3}{2}, 2 \end{matrix} ; -\frac{1}{4}\right] = 4(\ln \varphi)^2, \tag{37} $$

$$ {}_2F_1\left[\begin{matrix} 1, 2, \\ \frac{3}{2} \end{matrix} ; \frac{1}{4}\right] = \frac{2}{3}\left(1 + \frac{2\sqrt{3}\pi}{9}\right), \tag{38} $$

$$ {}_2F_1\left[\begin{matrix} 1, 2, \\ \frac{3}{2} \end{matrix} ; -\frac{1}{4}\right] = \frac{2}{5}\left(1 + \frac{4\sqrt{5}}{5} \ln \varphi\right), \tag{39} $$

$$ {}_2F_1\left[\begin{matrix} 2, 2, \\ \frac{3}{2} \end{matrix} ; \frac{1}{4}\right] = \frac{4}{27}\left(\sqrt{3}\pi + 9\right), \tag{40} $$

$$ {}_2F_1\left[\begin{matrix} 2, 2, \\ \frac{3}{2} \end{matrix} ; -\frac{1}{4}\right] = \frac{4}{125}\left(15 + 2\sqrt{5} \ln \varphi\right), \tag{41} $$

$$ {}_3F_2\left[\begin{matrix} 2, 2, 2 \\ 1, \frac{3}{2} \end{matrix} ; \frac{1}{4}\right] = \frac{4}{81}\left(5\sqrt{3}\pi + 54\right), \tag{42} $$

$$ {}_3F_2\left[\begin{matrix} 2, 2, 2 \\ 1, \frac{3}{2} \end{matrix} ; -\frac{1}{4}\right] = \frac{8}{125}\left(5 - \sqrt{5} \ln \varphi\right), \tag{43} $$

$$ {}_4F_3\left[\begin{matrix} 2, 2, 2, 2 \\ 1, 1, \frac{3}{2} \end{matrix} ; \frac{1}{4}\right] = \frac{4}{243}\left(37\sqrt{3}\pi + 405\right), \tag{44} $$

$$ {}_4F_3\left[\begin{matrix} 2, 2, 2, 2 \\ 1, 1, \frac{3}{2} \end{matrix} ; -\frac{1}{4}\right] = -\frac{4}{625}\left(5 + 14\sqrt{5} \ln \varphi\right), \tag{45} $$

$$ {}_2F_1\left[\begin{matrix} 1, 1 \\ \frac{3}{2} \end{matrix} ; \frac{1}{2}\right] = \frac{\pi}{2}, \tag{46} $$

$$ {}_2F_1\left[\begin{matrix} 1, 1 \\ \frac{3}{2} \end{matrix} ; -\frac{1}{2}\right] = \frac{1}{\sqrt{3}} \ln \left(2 + \sqrt{3}\right), \tag{47} $$

$$ {}_2F_1\left[\begin{matrix} 1, 2 \\ \frac{3}{2} \end{matrix} ; \frac{1}{2}\right] = \frac{\pi}{2} + 1, \tag{48} $$

$$ {}_2F_1\left[\begin{matrix} 1, 2 \\ \frac{3}{2} \end{matrix} ; -\frac{1}{2}\right] = \frac{1}{3} + \frac{\sqrt{3}}{9} \ln \left(2 + \sqrt{3}\right), \tag{49} $$

$$ {}_2F_1\left[\begin{matrix} 2,2 \\ \frac{3}{2} \end{matrix};\frac{1}{2}\right]=\pi+3, \tag{50}$$

$$ {}_2F_1\left[\begin{matrix} 2,2 \\ \frac{3}{2} \end{matrix};-\frac{1}{2}\right]=\frac{1}{3}, \tag{51}$$

$$ {}_3F_2\left[\begin{matrix} 2,2,2 \\ 1,\frac{3}{2} \end{matrix};\frac{1}{2}\right]=\frac{7\pi}{2}+11, \tag{52}$$

$$ {}_3F_2\left[\begin{matrix} 2,2,2 \\ 1,\frac{3}{2} \end{matrix};-\frac{1}{2}\right]=\frac{1}{9}-\frac{\sqrt{3}}{27}\ln\left(2+\sqrt{3}\right), \tag{53}$$

$$ {}_4F_3\left[\begin{matrix} 2,2,2,2 \\ 1,1,\frac{3}{2} \end{matrix};\frac{1}{2}\right]=\frac{35\pi}{2}+55, \tag{54}$$

$$ {}_4F_3\left[\begin{matrix} 2,2,2,2 \\ 1,1,\frac{3}{2} \end{matrix};-\frac{1}{2}\right]=-\frac{1}{81}\left(15+\sqrt{3}\ln\left(2+\sqrt{3}\right)\right). \tag{55}$$

Also, it is well-known that the process of resolving a generalized hypergeometric function ${}_pF_q(z)$ into even and odd components can lead to two new results. We shall employ this procedure combined with the results (32), (33), (34), (35), (36), (37), (38), (39), (40), (41), (42), (43), (44), (45), (46), (47), (48), (49), (50), (51), (52), (53), (54), and (55) to obtain several new and interesting closed form evaluations of the series ${}_{p+1}F_q(z)$ for $q = 1, 2, 3, 4$, and 5 with the arguments 1/16, 1/4, 1/2, and 1.

3. Several Closed-Form Evaluations

In this section, we shall establish the following twenty-four new closed-form evaluations for the generalized hypergeometric function.

$$ {}_3F_2\left[\begin{matrix} \frac{1}{2},\frac{3}{4},\frac{5}{4} \\ \frac{3}{2},\frac{3}{2} \end{matrix};1\right]=2\ln\left(\sqrt{2}+1\right), \tag{56}$$

$$ {}_4F_3\left[\begin{matrix} 1,1,\frac{5}{4},\frac{7}{4} \\ \frac{3}{2},2,2 \end{matrix};1\right]=\frac{16}{3}\left(2\ln 2-\ln\left(\sqrt{2}+1\right)\right), \tag{57}$$

$$ {}_3F_2\left[\begin{matrix} \frac{1}{2},1,1 \\ \frac{3}{4},\frac{5}{4} \end{matrix};\frac{1}{16}\right]=\frac{\sqrt{3}\pi}{9}+\frac{2}{\sqrt{5}}\ln\varphi, \tag{58}$$

$$ {}_3F_2\left[\begin{matrix} 1,1,\frac{3}{2} \\ \frac{5}{4},\frac{7}{4} \end{matrix};\frac{1}{16}\right]=2\left(\frac{\pi}{\sqrt{3}}-\frac{6}{\sqrt{5}}\ln\varphi\right), \tag{59}$$

$$ {}_4F_3\left[\begin{matrix} \frac{1}{2},\frac{1}{2},1,1 \\ \frac{3}{4},\frac{5}{4},\frac{3}{2} \end{matrix};\frac{1}{16}\right]=\frac{1}{2}\left(\frac{\pi^2}{9}+4(\ln\varphi)^2\right), \tag{60}$$

$$ {}_4F_3\left[\begin{matrix} 1,1,1,\frac{3}{2} \\ \frac{5}{4},\frac{7}{4},2 \end{matrix};\frac{1}{16}\right]=6\left(\frac{\pi^2}{9}-4(\ln\varphi)^2\right), \tag{61}$$

$$ {}_3F_2\left[\begin{matrix} 1,1,\frac{3}{2} \\ \frac{3}{4},\frac{5}{4} \end{matrix};\frac{1}{16}\right]=\frac{8}{15}+\frac{2\sqrt{3}\pi}{27}+\frac{4\sqrt{5}}{25}\ln\varphi, \tag{62}$$

$$ {}_3F_2\left[\begin{matrix} 1,\frac{3}{2},2 \\ \frac{5}{4},\frac{7}{4} \end{matrix};\frac{1}{16}\right]=6\left(\frac{1}{15}+\frac{\sqrt{3}\pi}{27}-\frac{2\sqrt{5}}{25}\ln\varphi\right), \tag{63}$$

$$ {}_4F_3\left[\begin{matrix} 1,1,\frac{3}{2},\frac{3}{2} \\ \frac{1}{2},\frac{3}{4},\frac{5}{4} \end{matrix};\frac{1}{16}\right]=\frac{2\sqrt{3}\pi}{27}+\frac{68}{75}+\frac{4\sqrt{5}}{125}\ln\varphi, \tag{64}$$

$$ {}_3F_2\left[\begin{matrix} \frac{3}{2},2,2 \\ \frac{5}{4},\frac{7}{4} \end{matrix};\frac{1}{16}\right]=\frac{\sqrt{3}\pi}{9}+\frac{16}{25}-\frac{6\sqrt{5}}{125}\ln\varphi, \tag{65}$$

$$ {}_5F_4\left[\begin{matrix} 1,1,\frac{3}{2},\frac{3}{2},\frac{3}{2} \\ \frac{1}{2},\frac{1}{2},\frac{3}{4},\frac{5}{4} \end{matrix};\frac{1}{16}\right]=\frac{10\sqrt{3}\pi}{81}+\frac{112}{75}-\frac{4\sqrt{5}}{125}\ln\varphi, \tag{66}$$

$$ {}_4F_3\left[\begin{matrix} \frac{3}{2},2,2,2 \\ 1,\frac{5}{4},\frac{7}{4} \end{matrix};\frac{1}{16}\right]=\frac{5\sqrt{3}\pi}{54}+\frac{22}{25}+\frac{3\sqrt{5}}{125}\ln\varphi, \tag{67}$$

$$ {}_6F_5\left[\begin{matrix} 1,1,\frac{3}{2},\frac{3}{2},\frac{3}{2},\frac{3}{2} \\ \frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{3}{4},\frac{5}{4} \end{matrix};\frac{1}{16}\right]=\frac{74\sqrt{3}\pi}{243}+\frac{1244}{375}-\frac{28\sqrt{5}}{625}\ln\varphi, \tag{68}$$

$$ {}_5F_4\left[\begin{matrix} \frac{3}{2},2,2,2,2 \\ 1,1,\frac{5}{4},\frac{7}{4} \end{matrix};\frac{1}{16}\right]=\frac{37\sqrt{3}\pi}{324}+\frac{157}{125}+\frac{21\sqrt{5}}{1250}\ln\varphi, \tag{69}$$

$$ {}_3F_2\left[\begin{matrix}1,1,\frac{3}{2}\\ \frac{3}{4},\frac{5}{4}\end{matrix};\frac{1}{4}\right]=\frac{1}{2}\left(\frac{\pi}{2}+\frac{\sqrt{3}}{9}\ln\left(2+\sqrt{3}\right)+\frac{4}{3}\right), \quad (70) $$

$$ {}_3F_2\left[\begin{matrix}1,\frac{3}{2},2\\ \frac{5}{4},\frac{7}{4}\end{matrix};\frac{1}{4}\right]=\frac{3}{4}\left(\frac{\pi}{2}-\frac{\sqrt{3}}{9}\ln\left(2+\sqrt{3}\right)+\frac{2}{3}\right), \quad (71) $$

$$ {}_3F_2\left[\begin{matrix}\frac{1}{2},1,1\\ \frac{3}{4},\frac{5}{4}\end{matrix};\frac{1}{4}\right]=\frac{1}{2}\left(\frac{\pi}{2}+\frac{1}{\sqrt{3}}\ln\left(2+\sqrt{3}\right)\right), \quad (72) $$

$$ {}_3F_2\left[\begin{matrix}1,1,\frac{3}{2}\\ \frac{5}{4},\frac{7}{4}\end{matrix};\frac{1}{4}\right]=\frac{3}{2}\left(\frac{\pi}{2}-\frac{1}{\sqrt{3}}\ln\left(2+\sqrt{3}\right)\right), \quad (73) $$

$$ {}_4F_3\left[\begin{matrix}1,1,\frac{3}{2},\frac{3}{2}\\ \frac{1}{2},\frac{3}{4},\frac{5}{4}\end{matrix};\frac{1}{4}\right]_4=\frac{\pi}{2}+\frac{5}{3}, \quad (74) $$

$$ {}_3F_2\left[\begin{matrix}\frac{3}{2},2,2\\ \frac{5}{4},\frac{7}{4}\end{matrix};\frac{1}{4}\right]=\frac{3\pi}{8}+1, \quad (75) $$

$$ {}_5F_4\left[\begin{matrix}1,1,\frac{3}{2},\frac{3}{2},\frac{3}{2}\\ \frac{1}{2},\frac{1}{2},\frac{3}{4},\frac{5}{4}\end{matrix};\frac{1}{4}\right]=\frac{1}{2}\left(\frac{7\pi}{2}+\frac{100}{9}-\frac{\sqrt{3}}{27}\ln\left(2+\sqrt{3}\right)\right), \quad (76) $$

$$ {}_4F_3\left[\begin{matrix}\frac{3}{2},2,2,2\\ 1,\frac{5}{4},\frac{7}{4}\end{matrix};\frac{1}{4}\right]=\frac{3}{16}\left(\frac{7\pi}{2}+\frac{98}{9}+\frac{\sqrt{3}}{27}\ln\left(2+\sqrt{3}\right)\right), \quad (77) $$

$$ {}_6F_5\left[\begin{matrix}1,1,\frac{3}{2},\frac{3}{2},\frac{3}{2},\frac{3}{2}\\ \frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{3}{4},\frac{5}{4}\end{matrix};\frac{1}{4}\right]=\frac{35\pi}{4}+\frac{740}{27}-\frac{\sqrt{3}}{162}\ln\left(2+\sqrt{3}\right), \quad (78) $$

$$ {}_5F_4\left[\begin{matrix}\frac{3}{2},2,2,2,2\\ 1,1,\frac{5}{4},\frac{7}{4}\end{matrix};\frac{1}{4}\right]=\frac{1}{16}\left(\frac{105\pi}{4}+\frac{745}{9}+\frac{\sqrt{3}}{54}\ln\left(2+\sqrt{3}\right)\right), \quad (79) $$

Proof. In order to establish the results (56), (57), (58), (59), (60), (61), (62), (63), (64), (65), (66), (67), (68), (69), (70), (71), (72), (73), (74), (75), (76), (77), (78), and (79), we shall employ the procedure of resolving a generalized hypergeometric function ${}_pF_q(z)$ into even and odd components. This decomposition is facilitated by use of the identities:

$$ \begin{aligned} (a)_{2n}&=2^{2n}\left(\frac{a}{2}\right)_n\left(\frac{a}{2}+\frac{1}{2}\right)_n,\\ (a)_{2n+1}&=a2^{2n}\left(\frac{a}{2}+\frac{1}{2}\right)_n\left(\frac{a}{2}+1\right)_n, \end{aligned} \quad (80) $$

then for the generalized hypergeometric function

$$ {}_{q+1}F_q\left[\begin{matrix}a_1,a_2,\cdots,a_{q+1}\\ b_1,b_2,\cdots,b_q\end{matrix};z\right]. \quad (81) $$

It is not difficult to see the following relations:

$$ \begin{aligned} &{}_{q+1}F_q\left[\begin{matrix}a_1,a_2,\cdots,a_{q+1}\\ b_1,b_2,\cdots,b_q\end{matrix};z\right]+{}_{q+1}F_q\left[\begin{matrix}a_1,a_2,\cdots,a_{q+1}\\ b_1,b_2,\cdots,b_q\end{matrix};-z\right]\\ &=2{}_{2q+2}F_{2q+1}\left[\begin{matrix}\frac{a_1}{2},\frac{a_1}{2}+\frac{1}{2},\cdots,\frac{a_{q+1}}{2},\frac{a_{q+1}}{2}+\frac{1}{2}\\ \frac{1}{2},\frac{b_1}{2},\frac{b_1}{2}+\frac{1}{2},\cdots,\frac{b_q}{2},\frac{b_q}{2}+\frac{1}{2}\end{matrix};z^2\right], \end{aligned} \quad (82) $$

$$ \begin{aligned} &{}_{q+1}F_q\left[\begin{matrix}a_1,a_2,\cdots,a_{q+1}\\ b_1,b_2,\cdots,b_q\end{matrix};z\right]-{}_{q+1}F_q\left[\begin{matrix}a_1,a_2,\cdots,a_{q+1}\\ b_1,b_2,\cdots,b_q\end{matrix};-z\right]\\ &=\frac{2za_1a_2\cdots a_{q+1}}{b_1b_2\cdots b_q}{}_{2q+2}F_{2q+1}\\ &\cdot\left[\begin{matrix}\frac{a_1}{2}+\frac{1}{2},\frac{a_1}{2}+1,\cdots,\frac{a_{q+1}}{2}+\frac{1}{2},\frac{a_{q+1}}{2}+1\\ \frac{3}{2},\frac{b_1}{2}+\frac{1}{2},\frac{b_1}{2}+1,\cdots,\frac{b_{q+1}}{2}+\frac{1}{2},\frac{b_q}{2}+1\end{matrix};z^2\right], \end{aligned} \quad (83) $$

provided for $z=1$, the convergence condition for the series ${}_{q+1}F_q\,(1)$ should be

$$ \sum_{j=1}^{q}b_j-\sum_{j=1}^{q+1}\alpha_j>0. \quad (84) $$

Therefore, for the derivation of the results (56) and (57), we substitute the results (32) and (33) into the results (82) and (83) with $q=2$ and setting $a_1=a_2=1, a_3=3/2, b_1=b_2=2, z=1$, and we immediately obtain the results ((56)) and ((57)), respectively. The remaining results (58), (59), (60), (61), (62), (63), (64), (65), (66), (67), (68), (69), (70), (71), (72), (73), (74), (75), (76), (77), (78), and (79) can be proven on similar lines, so we left this as an exercise to the interested reader.

We conclude this section by mentioning that the results (56), (57), (58), (59), (60), (61), (62), (63), (64), (65), (66), (67), (68), (69), (70), (71), (72), (73), (74), (75), (76), (77), (78), and (79) established in this paper have been verified by using MAPLE. □

4. Concluding Remark

In this paper, several new closed-form evaluations of the generalized hypergeometric functions ${}_{q+1}F_q(z)$ for $q = 1, 2, 3, 4, 5$ with arguments 1/16, 1/4, 1/2, and 1 have been established. This is achieved by means of separating the generalized hypergeometric function ${}_{q+1}F_q(z)$ into even and odd components together with the use of the several known results of interesting series involving central binomial coefficients obtained earlier by Lehmer. We believe that the results established in this paper have not appeared in the literature and represent a definite contribution to the theory of generalized hypergeometric function. It is hoped that the results could be of potential use in the area of mathematics, statistics, and mathematical physics.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there is no conflict of interest.

Authors' Contributions

All authors contributed equally to this work. All authors have read and agreed to the published version of the manuscript.

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