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Research Article

On Solutions of Hybrid–Sturm-Liouville–Langevin Equations with Generalized Versions of Caputo Fractional Derivatives

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The main intention of this research article is to introduce a new class of generalized fractional differential equations that fall into the categories of Sturm-Liouville's, Langevin's, and hybrid's problems involving Y-Caputo fractional derivatives. The existence of the solutions of the proposed equations is discussed by using the technique of the measure of noncompactness related to the fixed point theorem, which is a generalization of Darbo's fixed point theorem. Additionally, pertinent examples are provided along with the different values of the function Y to confirm the validity of the reported results.

1. Introduction

Fractional differential equations (FDEs) with their various branches such as Hybrid Equation (HE), Langevin Equation (LE), and Sturm-Liouville Equation (SLE) are currently well established, due to the number of papers and books edited worldwide. These types of equations have been applied in many applications in different fields, such as engineering and science. Since in recent years, it has achieved a great deal of development and interest by many researchers, for some of these developments in the theory of fractional differential equations, one can look at the monographs of Kilbas et al. [1] and Podlubny [2], where they presented some properties and applications appropriate for various types of fractional operators. Dhage and Lakshmikantham [3] and Dhage et al. [4] made excellent results on hybrid problems, as did Zhao et al. [5] and Ahmad and Ntouyas [6]. The LE [7] is formulated to be a powerful tool for describing the evolution of physical phenomena in volatile environments. Some of recent Langevin's problem is studied through [8–10]. However, SLE has many applications in distinct areas of technical knowledge and engineering [11, 12]. The mix of both fractional SLE and fractional LE might give an adequate description of the dynamic processes described in a fractal medium where fractal and memory properties are inserted with a scattered memory kernel. Recently, the authors in [13] suggested an approach to the fractional model of the SLE and LE. Indeed, they discussed the existence of solutions to the considered systems through fixed point techniques and mathematical inequalities. Muensawat et al. [14] studied antiperiodic BVPs for fractional systems of generalized SL and LE. Boutiara et al. [15] considered fractional LE under Caputo function-dependent kernel fractional derivatives. Existence theorem for psi-fractional HEs has been proven by Suwan et al. [16]. Some qualitative analyses for multiterm LEs with generalized Caputo FDs and diffusion FDE with ABC operators can be found in [17, 18]. The authors in [19] considered a hybrid LE involving Caputo FD and Riemann-Liouville (RL) fractional integral (FI) as follows:

$$\begin{cases} c\mathscr{D}^{\zeta}\left[{}^{c}\mathscr{D}^{\xi}\left[\dfrac{\vartheta(\sigma)}{\mathscr{M}(\sigma,\vartheta(\nu(\sigma)))}\right]-\lambda\vartheta(\sigma)\right]=\mathscr{N}(\sigma,\vartheta(\mu(\sigma)),\mathscr{I}^{\gamma}\vartheta(\mu(\sigma)),\ \sigma\in 0,1], \\ \vartheta(0)=0,\ c\mathscr{D}\xi\left[\dfrac{\vartheta(\sigma)}{\mathscr{M}(\sigma,\vartheta(\nu(\sigma)))}\right]_{\sigma=0}=0,\ \vartheta(1)=\zeta\vartheta(\kappa), \quad 0<\kappa<1,\zeta,\lambda\in\mathbb{R}. \end{cases} \tag{1}$$

Motivated by the above works aforesaid and inspired by [19, 21], in this paper, we deal with the existence of solutions for the following BVP to the nonlinear fractional hybrid–Sturm-Liouville–Langevin differential equation:

$$\begin{cases} c\mathscr{D}^{\zeta,Y}\left[p(\sigma)^{c}\mathscr{D}^{\xi,Y}\left[\dfrac{\vartheta(\sigma)}{\mathscr{M}(\sigma,\vartheta(\nu(\sigma)))}\right]-q(\sigma)\vartheta(\sigma)\right]=\mathscr{N}(\sigma,\vartheta(\mu(\sigma))), \quad \sigma\in\Pi=[a,b], \\ \vartheta(a)=0,\ p(b)^{c}\mathscr{D}^{\xi,Y}\left[\dfrac{\vartheta(\sigma)}{\mathscr{M}(\sigma,\vartheta(\nu(\sigma)))}\right]_{\sigma=b}+q(b)\vartheta(b)=0, \end{cases} \tag{2}$$

where ${}^{c}\mathscr{D}^{r,Y}$ denotes the Y-Caputo FD of order $r\in\{\zeta,\xi\}$, $0<\zeta,\xi\le 1$. Here, $\mathscr{M}\in C(\Pi\times\mathbb{R},\mathbb{R}\setminus\{0\})$, $\mathscr{N}\in C(\Pi\times\mathbb{R},\mathbb{R})$, $\nu,\mu:\Pi\longrightarrow\Pi$ are given functions, $p\in C(\Pi,\mathbb{R}\setminus\{0\})$, and $q\in C(\Pi,\mathbb{R})$. As in Banach spaces, a closed and bounded set is not generally a compact set; just continuity of the function $\mathscr{M}$ does not ensure the existence of a solution to differential equations. Our arguments are principally founded on Darbo's fixed point technique mixed with the technique of measures of noncompactness to set up the existence of solutions for (2). In particular, problem (2) is formed as an overarching structure comprising both fractional SLE, LE, and HE, subjected to boundary conditions involving Y-Caputo FDs. In fact, choosing $q(\sigma)\equiv 0$ on the one hand and $p(\sigma)\equiv 1$, $q(\sigma)=\lambda$, and $\lambda\in\mathbb{R}$, on the other hand, reduces the problem (2) into the fractional Sturm-Liouville problem and the fractional Langevin problem, respectively. Besides, if we set $p(\sigma)\equiv 1$ and $q(\sigma)\equiv 0$, the problem (2) reduces to the fractional sequential hybrid problem.

Observe also that the current results are consistent with some of the literature results when $Y(\sigma)=\sigma$, and they are new even for the special case: $Y(\sigma)=\log\sigma$ and $Y(\sigma)=\sigma^{\rho}$.

Here is a brief outline of the paper. In Section 2, we provide some preliminary facts. Sections 3 and 4 handle the formulation of solutions and the existence of solutions for (2) by using the generalized Darbo's fixed point theorem (D'sFPT) along with the approach of measures of noncompactness in the Banach algebras. Lastly, we give pertinent examples.

2. Preliminaries

Let us start this section with some auxiliary results used in the forthcoming analysis.

Definition 1 (see [1]). The Y-RL FI of order $\zeta>0$ for an integrable function $\vartheta:\Pi\longrightarrow\mathbb{R}$ is given by

$$\mathscr{I}_{a^{+}}^{\zeta;Y}\vartheta(\sigma)=\frac{1}{\Gamma(\zeta)}\int_{a}^{\sigma}Y'(\varsigma)(Y(\sigma)-Y(\varsigma))^{\zeta-1}\vartheta(\varsigma)d\varsigma, \tag{3}$$

where Γ is the gamma function. One can deduce that

$$D_{\sigma}\left(\mathscr{I}_{a^{+}}^{\zeta;Y}\vartheta(\sigma)\right)=Y'(\sigma)\mathscr{I}_{a^{+}}^{\zeta-1;Y}\vartheta(\sigma),\zeta>1, \tag{4}$$

where $D_{\sigma}=d/dt$.

Definition 2 (see [20]). For $n-1<\zeta<n(n\in\mathbb{N})$ and $\vartheta,Y\in C^{n}(\Pi,\mathbb{R})$, the Y-Caputo FD of a function ϑ of order ζ is given by

$${}^{c}\mathscr{D}_{a^{+}}^{\zeta;Y}\vartheta(\sigma)=\mathscr{I}_{a^{+}}^{n-\zeta;Y}\left(\frac{D_{\sigma}}{Y'(\sigma)}\right)^{n}\vartheta(\sigma), \tag{5}$$

where $n=[\zeta]+1$ for $\zeta\notin\mathbb{N}$ and $n=\zeta$ for $\zeta\in\mathbb{N}$.

Also, we can express Y-Caputo FD by

$$ {}^{c}\mathscr{D}_{a^+}^{\zeta;Y}\vartheta(\sigma)=\begin{cases}\displaystyle\int_a^\sigma \frac{Y'(\varsigma)(Y(\sigma)-Y(\varsigma))^{n-\zeta-1}}{\Gamma(n-\zeta)}\left(\frac{D_\sigma}{Y'(\varsigma)}\right)^n\vartheta(\varsigma)d\varsigma, & \text{if } \zeta\notin\mathbb{N},\\ \left(\dfrac{D_\sigma}{Y'(\sigma)}\right)^n\vartheta(\sigma), & \text{if } \zeta\in\mathbb{N}.\end{cases} \tag{6}$$

Lemma 3 (see [1, 20]). *Let $\zeta,\xi>0$ and $\vartheta\in L^1(\Pi,\mathbb{R})$. Then,*

$$\mathscr{I}_{a^+}^{\zeta;Y}\mathscr{I}_{a^+}^{\xi;Y}\vartheta(\sigma)=\mathscr{I}_{a^+}^{\zeta+\xi;Y}\vartheta(\sigma),\ a.e.\sigma\in\Pi. \tag{7}$$

In particular, if $\vartheta\in C(\Pi,\mathbb{R})$, then $\mathscr{I}_{a^+}^{\zeta;Y}\mathscr{I}_{a^+}^{\xi;Y}\vartheta(\sigma)=\mathscr{I}_{a^+}^{\zeta+\xi;Y}\vartheta(\sigma)$, $\sigma\in\Pi$.

Lemma 4 (see [20]). *Let $\zeta>0$. Then, the following holds:*
If $\vartheta\in C(\Pi,\mathbb{R})$, then

$${}^{c}\mathscr{D}_{a^+}^{\zeta;Y}\mathscr{I}_{a^+}^{\zeta;Y}\vartheta(\sigma)=\vartheta(\sigma),\quad \sigma\in\Pi. \tag{8}$$

If $\in C^n(\Pi,\mathbb{R})$, $n-1<\zeta<n$. Then,

$$\mathscr{I}_{a^+}^{\zeta;Y}{}^{c}\mathscr{D}_{a^+}^{\zeta;Y}\vartheta(\sigma)=\vartheta(\sigma)-\sum_{k=0}^{n-1}\frac{\vartheta_Y^{[k]}(a)}{k!}[Y(\sigma)-Y(a)]^k,\quad \sigma\in\Pi. \tag{9}$$

Lemma 5 (see [1, 20]). *Let $\sigma>a$, $\zeta\geq0$, and $\xi>0$. Then,*

(i) $\mathscr{I}_{a^+}^{\zeta;Y}(Y(\sigma)-Y(a))^{\xi-1}=(\Gamma(\xi)/\Gamma(\xi+\zeta))(Y(\sigma)-Y(a))^{\xi+\zeta-1}$

(ii) ${}^{c}\mathscr{D}_{a^+}^{\zeta;Y}(Y(\sigma)-Y(a))^{\xi-1}=(\Gamma(\xi)/\Gamma(\xi-\zeta))(Y(\sigma)-Y(a))^{\xi-\zeta-1}$

(iii) ${}^{c}\mathscr{D}_{a^+}^{\zeta;Y}(Y(\sigma)-Y(a))^{k}=0$, *for $k<n$, $n\in\mathbb{N}$*

Let $B(\upsilon,\tilde r)$ be the closed ball in the Banach space $\mathbb{E}$; if $\upsilon=0$, then $B_r\equiv B(0,\tilde r)$. Let $\tilde{\mathbb{X}}\subset\mathbb{E}$, such that $\tilde{\mathbb{X}}$ and Conv $\tilde{\mathbb{X}}$ are a closure and a convex closure of $\tilde{\mathbb{X}}$, respectively. And let $M_{\mathbb{E}}$ be the family of the nonempty and bounded subsets of $\mathbb{E}$, while $P_{\mathbb{E}}$ denotes the subfamily of all relatively compact subsets of $M_{\mathbb{E}}$.

Definition 6 (see [22]). We say that $\tilde\chi : M_{\mathbb{E}}\longrightarrow 0,\infty)$ is a noncompactness measure in $\mathbb{E}$ if all the assumptions below hold:

(i) $\ker\tilde\chi=\{\tilde{\mathbb{X}}\in\mathfrak{M}_{\mathbb{E}} : \tilde\chi(\tilde{\mathbb{X}})=0\}$ is nonempty and $\ker\tilde\chi\subset P_{\mathbb{E}}$

(ii) $\tilde{\mathbb{Y}}\subset\tilde{\mathbb{X}}$, then $\tilde\chi(\tilde{\mathbb{Y}})\leq\tilde\chi(\tilde{\mathbb{X}})$

(iii) $\tilde\chi(\tilde{\mathbb{Y}})=\tilde\chi(\tilde{\mathbb{Y}})=\tilde\chi(\text{Conv}\tilde{\mathbb{Y}})$

(iv) $\tilde\chi(\lambda_1\tilde{\mathbb{Y}}+\lambda_2\tilde{\mathbb{X}})\leq\lambda_1\tilde\chi(\tilde{\mathbb{Y}})+\lambda_2\tilde\chi(\tilde{\mathbb{X}})$, $\lambda_1+\lambda_2=1$

(v) In the case of $(\tilde{\mathbb{Y}}_n)$ being a sequence of closed subsets of $M_{\mathbb{E}}$ with $\tilde{\mathbb{Y}}_{n+1}\subset\tilde{\mathbb{Y}}_n(n\geq1)$ and $\lim_{n\longrightarrow\infty}\tilde\chi(\tilde{\mathbb{Y}}_n)=0$, then $\cap_{n=1}^{\infty}\tilde{\mathbb{Y}}_n\neq\varnothing$

Definition 7 (see [22]). Let $\tilde{\mathbb{Y}}$ be a nonempty bounded set and $\subset\tilde{\mathscr{C}}$ be a Banach space. We say that $\mathscr{M}\in\tilde{\mathbb{Y}}$ is a modulus of continuous function, denoted by $\omega(\mathscr{M},\varepsilon)$; if $\forall\mathscr{M}\in\tilde{\mathbb{Y}}$ and $\forall e>0$, we have

$$\omega(\mathscr{M},\varepsilon)=\sup\{|\mathscr{M}(\sigma)-\mathscr{M}(\varsigma)|: \sigma,\varsigma\in\Pi, |\sigma-\varsigma|\leq\varepsilon\}. \tag{10}$$

Moreover,

$$\begin{aligned}\omega\left(\tilde{\mathbb{Y}},\varepsilon\right)&=\sup\left\{\omega(\mathscr{M},\varepsilon): \mathscr{M}\in\tilde{\mathbb{Y}}\right\},\\ \omega_0\left(\tilde{\mathbb{Y}}\right)&=\lim_{\varepsilon\longrightarrow0}\omega\left(\tilde{\mathbb{Y}},e\right).\end{aligned} \tag{11}$$

Definition 8 (see [23]). A noncompactness measure $\tilde\chi$ in $\tilde{\mathscr{C}}$ satisfies the condition (m) if

$$\tilde\chi(\mathscr{M}\mathscr{N})\leq\|\mathscr{M}\|\tilde\chi(\mathscr{N})+\|\mathscr{N}\|\tilde\chi(\mathscr{M}), \tag{12}$$

for all $\mathscr{M},\mathscr{N}\in M_{C(\Pi)}$, where $\tilde{\mathscr{C}}\coloneqq C(\Pi)$ is the Banach algebra.

Lemma 9 (see [24]). *The condition (m) may be grasped by the noncompactness measure ϑ_0 on $\tilde{\mathscr{C}}$.*
Set

$$S=\left\{Y : (0,\infty)\longrightarrow(b,\infty): \forall(\upsilon_n)\subset(0,\infty), \lim_{n\longrightarrow\infty}Y(\upsilon_n) = b\Longleftrightarrow \lim_{n\longrightarrow\infty}\upsilon_n=0\right\}. \tag{13}$$

Now, we present D'sFPT and generalized D'sFPT to prove that there exists at least one fixed point.

Theorem 10 (see [25, 26]). *Let $\tilde{\mathscr{C}}$ be a Banach space and $\Xi\subset\tilde{\mathscr{C}}$ be a nonempty, bounded, convex, and closed set. Let $\mathscr{K}:\Xi\longrightarrow\Xi$ be continuous. Assume that there is $0\leq$*

$\theta < 1$ with υ as a noncompactness measure in $\tilde{\mathscr{C}}$ meeting the following requirements:

$$\upsilon\left(\mathscr{K}\tilde{\mathbb{Y}}\right) \leq \theta\tilde{\chi}\left(\tilde{\mathbb{Y}}\right), \Theta \neq \tilde{\mathbb{Y}} \subseteq \Xi. \tag{14}$$

Then, $\mathscr{K}$ has a fixed point in Ξ.

Theorem 11 (see [26]). *Let $\tilde{\mathscr{C}}$ be a Banach space and $V \subset \tilde{\mathscr{C}}$ be a nonempty, bounded, convex, and closed set, and let $\mathscr{K} : V \longrightarrow V$ be continuous. Assume there exist $\Theta \in S$ and $0 \leq \theta < 1$ such that for each nonempty subset D of V with $\tilde{\chi}(KD) > 0$,*

$$\Theta(\tilde{\chi}(\mathscr{K}\mathscr{D})) \leq (\Theta(\tilde{\chi}(\mathscr{D})))^{\theta}, \tag{15}$$

where $\tilde{\chi}$ is a noncompactness measure in $\tilde{\mathscr{C}}$. Then, $\mathscr{K}$ has a fixed point in V.

3. Solution Formulation

This section presents a formulation of the solution to problem (2) along with the assumptions required in the forthcoming analysis. Foremost, we denote by $(\tilde{\mathscr{C}}, \|.\|)$ the space of real valued continuous functions defined on a unit interval Π. It is clearly the Banach space with the norm:

$$\|\vartheta\| = \sup_{\sigma\in\Pi}|\vartheta(\sigma)|, \text{for } \vartheta \in \tilde{\mathscr{C}}. \tag{16}$$

Multiplication is defined as the usual product of real functions.

To prove the existence of solutions to (2), we need the following lemma:

Lemma 12. *The problem (2) is equivalent to the following fractional integral equation:*

$$\vartheta(\sigma) = \mathscr{M}(\sigma, \vartheta(\nu(\sigma)))\left\{\mathscr{I}^{\xi,Y}\left(\frac{1}{p}\mathscr{I}^{\zeta,Y}\mathscr{N}\right)(\sigma, \vartheta(\mu(\sigma))) - \mathscr{I}^{\xi,Y}\left(\frac{q}{p}\vartheta\right)(\sigma) - \frac{(Y(\sigma) - Y(a))^{\xi}}{p(\sigma)\Gamma(\xi + 1)}\mathscr{I}^{\zeta,Y}\mathscr{N}(b, \vartheta(\mu(b)))\right\}. \tag{17}$$

Proof. Applying the ζ^{th}-Y-RL integral on (2), we obtain

$${}^{c}\mathscr{D}^{\xi,Y}\left[\frac{\vartheta(\sigma)}{\mathscr{M}(\sigma, \vartheta(\nu(\sigma)))}\right] = \frac{\mathscr{I}^{\zeta,Y}\mathscr{N}(\sigma, \vartheta(\mu(\sigma))) - q(\sigma)\vartheta(\sigma) + k_1}{p(\sigma)}, \tag{18}$$

where $k_1 \in \mathbb{R}$. From the BCs of (2), we get

$$k_1 = -\mathscr{I}^{\zeta,Y}\mathscr{N}(b, \vartheta(\mu(b))). \tag{19}$$

Taking the ξ^{th}-Y-RL integral of (18), one has

$$\left[\frac{\vartheta(\sigma)}{\mathscr{M}(\sigma, \vartheta(\nu(\sigma)))}\right] = \left\{\mathscr{I}^{\xi,Y}\left(\frac{1}{p}\mathscr{I}^{\zeta,Y}\mathscr{N}\right)(\sigma, \vartheta(\mu(\sigma))) - \mathscr{I}^{\xi,Y}\left(\frac{q}{p}\vartheta\right)(\sigma) - \frac{(Y(\sigma) - Y(a))^{\xi}}{p(\sigma)\Gamma(\xi + 1)}\mathscr{I}^{\zeta,Y}\mathscr{N}(b, \vartheta(\mu(b)))\right\} + k_2, \tag{20}$$

where $k_2 \in \mathbb{R}$. The BCs of (2) give $k_2 = 0$. In this regard, if we apply the ξ^{th}-Y-Caputo FD and ζ^{th}-Y-Caputo FD to both sides of (17) and use Lemma 5, then the problem (2) immediately is established. □

Before giving the essential result, we shall investigate formula (17) under the following assumptions:

(i) (AS1) Both functions $\nu, \mu : \Pi \longrightarrow \Pi$ are continuous

(ii) (AS2) $\mathscr{M} \in C(\Pi \times \mathbb{R}, \mathbb{R} \setminus \{0\})$, and $\mathscr{N} \in C(\Pi \times \mathbb{R}^2, \mathbb{R})$

(iii) (AS3) There exists a real number $\rho \in (a, b)$ with

$$|\mathscr{M}(\sigma, \nu) - \mathscr{M}(\sigma, \vartheta)| \leq (|\nu - \vartheta| + d)^{\rho} - d^{\rho}, \quad \forall\sigma \in \Pi, \vartheta, \nu \in \mathbb{R}, d \in \mathbb{R}^{+}. \tag{21}$$

(iv) (AS4) There exists a continuous nondecreasing function $\varphi : \mathbb{R}^{+} \longrightarrow \mathbb{R}^{+}$ with $\varphi(0) = 0$ such that

$$|\mathscr{N}(\sigma, \vartheta)| \leq \varphi(\|\vartheta\|), \quad \sigma \in \Pi, \vartheta \in \mathbb{R}. \tag{22}$$

(v) (AS5) There exists $r_0 > 0$ such that

$$[(r_0 + d)^{\rho} - d^{\rho} + N]\left\{\frac{\varphi(r_0)}{\tilde{p}}\frac{(Y(b) - Y(a))^{\zeta+\xi}}{\Gamma(\zeta + \xi + 1)} + \frac{\tilde{q}}{\tilde{p}}\frac{r_0(Y(b) - Y(a))^{\zeta}}{\Gamma(\zeta + 1)} + \frac{1}{\tilde{p}}\frac{\varphi(r_0)(Y(b) - Y(a))^{\zeta+\xi}}{\Gamma(\zeta + 1)\Gamma(\xi + 1)}\right\} \leq r_0,$$

where

$$\Lambda \coloneqq \left\{\frac{\varphi(r_0)}{\tilde{p}}\frac{(Y(b) - Y(a))^{\zeta+\xi}}{\Gamma(\zeta + \xi + 1)} + \frac{\tilde{q}}{\tilde{p}}\frac{r_0(Y(b) - Y(a))^{\zeta}}{\Gamma(\zeta + 1)} + \frac{1}{\tilde{p}}\frac{\varphi(r_0)(Y(b) - Y(a))^{\zeta+\xi}}{\Gamma(\zeta + 1)\Gamma(\xi + 1)}\right\} \leq 1,$$

$$N = \sup\{|\mathscr{M}(\sigma, 0)|: \sigma \in \Pi\},$$

$$\tilde{p} = \sup_{\sigma\in\Pi}p(\sigma), \tilde{q} = \sup_{\sigma\in\Pi}q(\sigma). \tag{24}$$

4. Existence Result

The aim of this section is to discuss the existence of solutions to the problem (2). For this end, we apply Theorems 10 and 11.

Theorem 13 Under hypotheses (AS1)–(AS5). *Then, the problem (2) has a least one solution in the Banach algebra* $\tilde{\mathscr{C}}$.

Proof. Consider the operator $\mathscr{K}:\tilde{\mathscr{C}}\longrightarrow\tilde{\mathscr{C}}$ on the Banach algebra $\tilde{\mathscr{C}}$ as

$$(\mathscr{K}\vartheta)(\sigma)=(\mathscr{F}\vartheta)(\sigma)(\mathscr{G}\vartheta)(\sigma), \tag{25}$$

where

$$\begin{aligned}(\mathscr{F}\vartheta)(\sigma)&=\mathscr{M}(\sigma,\vartheta(\nu(\sigma))),\\ (\mathscr{G}\vartheta)(\sigma)&=\mathscr{I}^{\xi,Y}\left(\frac{1}{p}\mathscr{I}^{\zeta,Y}\mathscr{N}\right)(\sigma,\vartheta(\mu(\sigma)))\\ &\quad-\mathscr{I}^{\xi,Y}\left(\frac{q}{p}\vartheta\right)(\sigma)-\frac{(Y(\sigma)-Y(a))^{\xi}}{p(\sigma)\Gamma(\xi+1)}G(\mathscr{N}),\\ G(\mathscr{N})&=\mathscr{I}^{\zeta,Y}\mathscr{N}(b,\vartheta(\mu(b))).\end{aligned} \tag{26}$$

From (AS4), we have

$$\|G(\mathscr{N})\|\leq\frac{(Y(b)-Y(a))^{\zeta}}{\Gamma(\zeta+1)}Y(\|\vartheta\|). \tag{27}$$

For the sake of simplicity, we put

$$\mathscr{Q}_Y^{\chi}(\sigma,\varsigma)=\frac{Y'(\varsigma)(Y(\sigma)-Y(\varsigma))^{\chi-1}}{\Gamma(\chi)},\quad \chi>0. \tag{28}$$

□

Now, we divide the proof into several steps.

Step 1. $\mathscr{K}$ transforms $\tilde{\mathscr{C}}$ into itself.

At first, we show that $\forall\vartheta\in\tilde{\mathscr{C}}$ implies that $(\mathscr{K}\vartheta)\in\tilde{\mathscr{C}}$, i.e., $(\mathscr{F}\vartheta)(\mathscr{G}\vartheta)\in\tilde{\mathscr{C}}$ for all $\vartheta\in\tilde{\mathscr{C}}$. Certainly, (AS1) and (AS2) guarantee that if $\vartheta\in\tilde{\mathscr{C}}$, then $(\mathscr{F}\vartheta)\in\tilde{\mathscr{C}}$. It remains to prove if $\vartheta\in\tilde{\mathscr{C}}$, then $(\mathscr{G}\vartheta)\in\tilde{\mathscr{C}}$. Let $\vartheta\in\tilde{\mathscr{C}}$ and $\sigma_2,\sigma_1\in\Pi$ with $\sigma_2>\sigma_1$. By hypothesis (AS4), we get

$$\begin{aligned}&|\mathscr{G}\vartheta(\sigma_1)-\mathscr{G}\vartheta(\sigma_2)|\\ &=\frac{1}{\tilde{\tilde{p}}}\left|\int_a^{\sigma_1}\mathscr{Q}_Y^{\zeta+\xi}(\sigma_1,\varsigma)\mathscr{M}(\varsigma,\vartheta(\varsigma))d\varsigma\right.\\ &\quad\left.-\int_a^{\sigma_2}\mathscr{Q}_Y^{\zeta+\xi}(\sigma_2,\varsigma)\mathscr{M}(\varsigma,\vartheta(\varsigma))d\varsigma\right|\\ &\quad+\frac{\tilde{q}}{\tilde{\tilde{p}}}\left|\int_a^{\sigma_1}\mathscr{Q}_Y^{\zeta}(\sigma_1,\varsigma)\vartheta(\varsigma)d\varsigma-\int_a^{\sigma_2}\mathscr{Q}_Y^{\zeta+\xi}(\sigma_2,\varsigma)\vartheta(\varsigma)d\varsigma\right|\\ &\quad+\frac{1}{\tilde{\tilde{p}}}\frac{|G(\mathscr{N})|}{\Gamma(\xi+1)}\left((Y(\sigma_2)-Y(a))^{\xi}-(Y(\sigma_1)-Y(a))^{\xi}\right)\\ &\leq\frac{1}{\tilde{\tilde{p}}}\left|\int_a^{\sigma_2}\left[\mathscr{Q}_Y^{\zeta+\xi}(\sigma_1,\varsigma)-\mathscr{Q}_Y^{\zeta+\xi}(\sigma_2,\varsigma)\right]\mathscr{M}(\varsigma,\vartheta(\varsigma))d\varsigma\right|\\ &\quad+\frac{1}{\tilde{\tilde{p}}}\left|\int_{\sigma_2}^{\sigma_1}\mathscr{Q}_Y^{\zeta+\xi}(\sigma_1,\varsigma)\mathscr{M}(\varsigma,\vartheta(\varsigma))d\varsigma\right|\\ &\quad+\frac{\tilde{q}}{\tilde{\tilde{p}}}\left|\int_a^{\sigma_2}\left[\mathscr{Q}_Y^{\zeta}(\sigma_1,\varsigma)-\mathscr{Q}_Y^{\zeta}(\sigma_2,\varsigma)\right]\mathscr{M}(\varsigma,\vartheta(\varsigma))d\varsigma\right|\\ &\quad+\frac{\tilde{q}}{\tilde{\tilde{p}}}\left|\int_{\sigma_2}^{\sigma_1}\mathscr{Q}_Y^{\zeta}(\sigma_1,\varsigma)\mathscr{M}(\varsigma,\vartheta(\varsigma))d\varsigma\right|\\ &\quad+\frac{1}{\tilde{\tilde{p}}}\frac{|G(\mathscr{N})|}{\Gamma(\xi+1)}\left((Y(\sigma_2)-Y(a))^{\xi}-(Y(\sigma_1)-Y(a))^{\xi}\right)\\ &\leq\frac{1}{\tilde{\tilde{p}}}\frac{\varphi(\|\vartheta\|)}{\Gamma(\zeta+\xi+1)}\left[\left|(Y(\sigma_1)-Y(a))^{\zeta+\xi}\right.\right.\\ &\quad\left.-(Y(\sigma_2)-Y(a))^{\zeta+\xi}-(Y(\sigma_1)-Y(\sigma_2))^{\zeta+\xi}\right|\\ &\quad\left.+(Y(\sigma_1)-Y(\sigma_2))^{\zeta+\xi}\right]\\ &\quad+\frac{\tilde{q}}{\tilde{\tilde{p}}}\frac{\|\vartheta\|}{\Gamma(\zeta+1)}\left[\left|(Y(\sigma_1)-Y(a))^{\zeta}-(Y(\sigma_2)-Y(a))^{\zeta}\right.\right.\\ &\quad\left.\left.-(Y(\sigma_1)-Y(\sigma_2))^{\zeta}\right|+(Y(\sigma_1)-Y(\sigma_2))^{\zeta}\right]\\ &\quad+\frac{1}{\tilde{\tilde{p}}}\frac{\varphi(\|\vartheta\|)(Y(b)-Y(a))^{\zeta}}{\Gamma(\zeta+1)\Gamma(\xi+1)}\\ &\quad\cdot\left((Y(\sigma_2)-Y(a))^{\xi}-(Y(\sigma_1)-Y(a))^{\xi}\right),\end{aligned} \tag{29}$$

which tends to be zero uniformly once $\sigma_2\longrightarrow\sigma_1$. It is clear that $\mathscr{G}\vartheta\in\tilde{\mathscr{C}}$ for all $\vartheta\in\tilde{\mathscr{C}}$.

Step 2. An estimate of $\|\mathscr{K}\vartheta\|$ for $\vartheta\in\tilde{\mathscr{C}}$.

Let $\vartheta\in\tilde{\mathscr{C}}$ and $\sigma\in\Pi$. Then, by using our hypothesis, we have

$$\begin{aligned}|(\mathscr{K}\vartheta)(\sigma)|&=|(\mathscr{F}\vartheta)(\sigma)(\mathscr{G}\vartheta)(\sigma)|\\ &\leq(|\mathscr{M}(\sigma,\vartheta(\nu(\sigma)))-\mathscr{M}(\sigma,0)|+|\mathscr{M}(\sigma,0)|)\\ &\quad\times\left\{\frac{1}{\tilde{\tilde{p}}}\int_a^{\sigma}\mathscr{Q}_Y^{\zeta+\xi}(\sigma,\varsigma)|\mathscr{N}(\varsigma,\vartheta(\mu(\varsigma)))|d\varsigma\right.\\ &\quad\left.+\frac{\tilde{q}}{\tilde{\tilde{p}}}\int_a^{\sigma}\mathscr{Q}_Y^{\zeta}(\sigma,\varsigma)|\vartheta(\varsigma)|d\varsigma+\frac{1}{\tilde{\tilde{p}}}\frac{|G(\mathscr{N})|}{\Gamma(\xi+1)}(Y(\sigma)-Y(a))^{\xi}\right\}\\ &\leq[(\|\vartheta\|+d)^{\rho}-d^{\rho}+N]\\ &\quad\cdot\left\{\frac{1}{\tilde{\tilde{p}}}\int_a^{\sigma}\mathscr{Q}_Y^{\zeta+\xi}(\sigma,\varsigma)Y(\|\vartheta\|)d\varsigma+\frac{\tilde{q}}{\tilde{\tilde{p}}}\int_a^{\sigma}\mathscr{Q}_Y^{\zeta}(\sigma,\varsigma)\|\vartheta\|d\varsigma\right.\\ &\quad\left.+\frac{1}{\tilde{\tilde{p}}}\frac{\varphi(\|\vartheta\|)(Y(b)-Y(a))^{\zeta+\xi}}{\Gamma(\zeta+1)\Gamma(\xi+1)}\right\}.\end{aligned} \tag{30}$$

Therefore,

$$\|\mathscr{K}\vartheta\| \leq [(\|\vartheta\| + d)^{\rho} - d^{\rho} + N]\left\{\frac{\varphi(\|\vartheta\|)}{\tilde{p}}\frac{(Y(b) - Y(\varsigma))^{\zeta+\xi}}{\Gamma(\zeta+\xi+1)} + \frac{\tilde{q}}{\tilde{p}}\frac{\|\vartheta\|(Y(b) - Y(\varsigma))^{\zeta}}{\Gamma(\zeta+1)} + \frac{1}{\tilde{p}}\frac{\varphi(\|\vartheta\|)(Y(b) - Y(a))^{\zeta+\xi}}{\Gamma(\zeta+1)\Gamma(\xi+1)}\right\}. \tag{31}$$

Step 3. The operator $\mathscr{K}$ is continuous on $\mathscr{B}_{r_0}$. Here, $\mathscr{B}_{r_0}$ is a subset of $\tilde{\mathscr{C}}$ defined by

$$\mathscr{B}_{r_0} = \left\{\vartheta(\sigma) \in \tilde{\mathscr{C}} : \|\vartheta\| \leq r_0 : \sigma \in \Pi\right\}, \tag{32}$$

with a fixed radius r_0, which satisfies the inequality (AS5).

We shall need to show the continuity of $\mathscr{F}$ and $\mathscr{G}$ on $\mathscr{B}_{r_0}$, separately. For any $\varepsilon > 0$ and $\vartheta, \nu \in \mathscr{B}_{r_0}$, there exists $0 < \delta < (\varepsilon + d^{\rho})^{1/\rho} - d, \ni \|\vartheta - \nu\| \leq \delta$; it follows for $\sigma \in \Pi$ that

$$\begin{aligned}|\mathscr{F}\vartheta(\sigma) - \mathscr{F}\nu(\sigma)| &= |\mathscr{M}(\sigma, \vartheta(\nu(\sigma))) - \mathscr{M}(\sigma, \nu(\nu(\sigma)))| \\ &\leq (|\vartheta(\nu(\sigma)) - (\nu(\sigma))| + d)^{\rho} - d^{\rho} \\ &\leq (\|\vartheta - \nu\| + d)^{\rho} - d^{\rho} \leq (\delta + d)^{\rho} - d^{\rho} < \varepsilon.\end{aligned} \tag{33}$$

Therefore, $\mathscr{F}$ is continuous on $\mathscr{B}_{r_0}$. The continuity of the operator $\mathscr{G}$ is obtained by Lebesgue dominated convergence (LDC) theorem. Indeed, let (ϑ_n) be a sequence such that $\vartheta_n \longrightarrow \vartheta$ in $\mathscr{B}_{r_0}$ with $\|\vartheta_n - \vartheta\| \longrightarrow 0$ as $n \longrightarrow 0$. As $\mu : \Pi \longrightarrow \Pi$ is continuous, we obtain

$$|\vartheta_n(\mu(\sigma))| \leq r_0, \quad \forall n \in \mathbb{N}, \forall \sigma \in \Pi. \tag{34}$$

Since $\mathscr{N}$ is continuous on $\Pi \times [-r_0, r_0]$, it is uniformly continuous on $\Pi \times [-r_0, r_0]$. Now, we set

$$G_0 = \max_{(\sigma,\vartheta)\in\Pi\times[-r_0,r_0]} |(\mathscr{G}\vartheta)(\sigma)|, \tag{35}$$

$$\kappa_0 = \frac{G_0(Y(b) - Y(a))^{\zeta}}{\Gamma(\zeta+1)}. \tag{36}$$

Applying the LDC theorem, we get

$$\begin{aligned}\lim_{n\longrightarrow\infty}(\mathscr{G}\vartheta_n)(\sigma) &= \lim_{n\longrightarrow\infty}\left\{\frac{1}{\tilde{p}}\int_a^{\sigma}\mathscr{Q}_Y^{\zeta+\xi}(\sigma,\varsigma)|\mathscr{N}(\varsigma,\vartheta_n(\mu(\varsigma)))|d\varsigma + \frac{\tilde{q}}{\tilde{p}}\int_a^{\sigma}\mathscr{Q}_Y^{\zeta}(\sigma,\varsigma)|\vartheta_n(\varsigma)|d\varsigma + \frac{|G(\vartheta_n)|}{\Gamma(\xi+1)}(Y(\sigma) - Y(a))^{\xi}\right\} \\ &= \left\{\frac{1}{\tilde{p}}\int_a^{\sigma}\mathscr{Q}_Y^{\zeta+\xi}(\sigma,\varsigma)|\mathscr{N}(\varsigma,\vartheta(\mu(\varsigma)))|d\varsigma + \frac{\tilde{q}}{\tilde{p}}\int_a^{\sigma}\mathscr{Q}_Y^{\zeta}(\sigma,\varsigma)|\vartheta(\varsigma)|d\varsigma + \frac{1}{\tilde{p}}\frac{|G(\vartheta)|}{\Gamma(\xi+1)}(Y(\sigma) - Y(a))^{\xi}\right\} = (\mathscr{G}\vartheta)(\sigma).\end{aligned} \tag{37}$$

Thus, $\mathscr{G}$ is continuous in $\mathscr{B}_{r_0}$.

Due to the continuity of $\mathscr{F}$ and $\mathscr{G}$, the operator $\mathscr{K}$ is continuous in $\mathscr{B}_{r_0}$.

Step 4. We estimate $\vartheta_0(\mathscr{F}\Xi)$ and $\vartheta_0(\mathscr{G}\Xi)$ for $\varnothing \neq \Xi \subset \mathscr{B}_{r_0}$.

At first, we estimate $\vartheta_0(\mathscr{F}\Xi)$. Since $\nu : \Pi \longrightarrow \Pi$ is uniformly continuous, we obtain for any $\varepsilon > 0, \exists \delta > 0$ with $(\delta < \varepsilon), \forall \sigma_1, \sigma_2 \in \Pi$ with $|\sigma_2 - \sigma_1| < \delta$, which implies $|\nu(\sigma_2) - \nu(\sigma_1)| < \varepsilon$. Taking $\vartheta \in \Xi$ and $\sigma_1, \sigma_2 \in \Pi$ with $|\sigma_2 - \sigma_1| < \delta$, under hypothesis (AS5), we get

$$\begin{aligned}|(\mathscr{F}\vartheta)(\sigma_2) - (\mathscr{F}\vartheta)(\sigma_1)| &= |\mathscr{M}(\sigma_2, \vartheta(\nu(\sigma_2))) - \mathscr{M}(\sigma_1, \vartheta(\nu(\sigma_1)))| \\ &\leq |\mathscr{M}(\sigma_2, \vartheta(\nu(\sigma_2))) - \mathscr{M}(\sigma_2, \vartheta(\nu(\sigma_1)))| + |\mathscr{M}(\sigma_2, \vartheta(\nu(\sigma_1))) - \mathscr{M}(\sigma_1, \vartheta(\nu(\sigma_1)))| \\ &\leq [(|\vartheta(\nu(\sigma_2)) - \vartheta(\nu(\sigma_1))| + d)^{\rho} - d^{\rho}] + \omega(\mathscr{M}, \varepsilon) \\ &\leq [(\omega(\Xi, \varepsilon) + d)^{\rho} - d^{\rho}] + \omega(\mathscr{M}, \varepsilon).\end{aligned} \tag{38}$$

Considering

$$\omega(\mathscr{M}, \varepsilon) = \sup\{|\mathscr{M}(\sigma_2, \vartheta) - \mathscr{M}(\sigma_1, \vartheta)| : \sigma_1, \sigma_2 \in \Pi, |\sigma_2 - \sigma_1| < \varepsilon, \vartheta \in [-r_0, r_0]\}, \tag{39}$$

then we can write (38) as

$$\omega(\mathscr{F}\Xi, \varepsilon) \leq \left[(\omega(\Xi, \varepsilon) + b)^{\rho} - b^{\rho}\right] + \omega(\mathscr{M}, \varepsilon). \tag{40}$$

Obviously, $\mathscr{M}(\sigma, \vartheta)$ is uniformly continuous on $\Pi \times [-r_0, r_0]$, and $\omega(M, \varepsilon) \longrightarrow 0$ once $\varepsilon \longrightarrow 0$. Hence, (40) becomes as follows:

$$\omega_0(\mathscr{F}\Xi) \leq (\omega_0(\Xi) + b)^{\rho} - b^{\rho}. \tag{41}$$

Next, since $\mu : \Pi \longrightarrow \Pi$ is uniformly continuous, we have $\forall \varepsilon > 0, \exists \delta > 0$ with $(\delta = \delta(\varepsilon)), \forall \sigma_1, \sigma_2 \in \Pi$ with $|\sigma_2 - \sigma_1| < \delta$, which implies $|\mu(\sigma_2) - \mu(\sigma_1)| < \varepsilon$. Take into account equations (32), (35), and (36) for each $\varepsilon > 0$. Set

$$\delta = \min\left\{\frac{1}{2}, \frac{\Gamma(\xi+1)\varepsilon}{\kappa_0}, \frac{p^*\Gamma(\zeta+1)\varepsilon}{q^* r_0}, \frac{p^*\Gamma(\zeta+\xi+1)\varepsilon}{4G_0}\right\}. \tag{42}$$

Choosing $\vartheta \in \Xi$ and $\sigma_1, \sigma_2 \in \Pi$ with $|\sigma_2 - \sigma_1| \leq \delta$ yields

$$
\begin{aligned}
|\mathcal{G}\vartheta(\sigma_1)-\mathcal{G}\vartheta(\sigma_2)| &= \frac{1}{\tilde{p}}\left|\int_a^{\sigma_1}\mathcal{Q}_Y^{\zeta+\xi}(\sigma_1,\varsigma)\mathcal{M}(\sigma,\vartheta(\varsigma))d\varsigma-\int_a^{\sigma_2}\mathcal{Q}_Y^{\zeta+\xi}(\sigma_2,\varsigma)\mathcal{M}(\varsigma,\vartheta(\varsigma))d\varsigma\right| + \frac{\tilde{q}}{\tilde{p}}\left|\int_a^{\sigma_1}\mathcal{Q}_Y^{\zeta}(\sigma_1,\varsigma)\vartheta(\varsigma)d\varsigma-\int_a^{\sigma_2}\mathcal{Q}_Y^{\zeta}(\sigma_2,\varsigma)\vartheta(\varsigma)d\varsigma\right| \\
&\quad + \frac{1}{\tilde{p}}\frac{|G(\mathcal{N})|}{\Gamma(\xi+1)}\left((Y(\sigma_2)-Y(a))^{\xi}-(Y(\sigma_1)-Y(a))^{\xi}\right) \\
&\le \frac{1}{\tilde{p}}\frac{G_0}{\Gamma(\zeta+\xi+1)}\left[\left|(Y(\sigma_1)-Y(a))^{\zeta+\xi}-(Y(\sigma_2)-Y(a))^{\zeta+\xi}-(Y(\sigma_1)-Y(\sigma_2))^{\zeta+\xi}\right|+(Y(\sigma_1)-Y(\sigma_2))^{\zeta+\xi}\right] \\
&\quad + \frac{\tilde{q}}{\tilde{p}}\frac{r_0}{\Gamma(\zeta+1)}\left[\left|(Y(\sigma_1)-Y(a))^{\zeta}-(Y(\sigma_2)-Y(a))^{\zeta}-(Y(\sigma_1)-Y(\sigma_2))^{\zeta}\right|\right. \\
&\quad \left.+(Y(\sigma_1)-Y(\sigma_2))^{\zeta}\right] + \frac{1}{\tilde{p}}\frac{\kappa_0}{\Gamma(\xi+1)}\left((Y(\sigma_2)-Y(a))^{\xi}-(Y(\sigma_1)-Y(a))^{\xi}\right).
\end{aligned} \tag{43}
$$

For simplicity's sake, we set

$$
\mathcal{H}_Y^{\chi}(\sigma)=(Y(\sigma)-Y(a))^{\chi},\quad \chi>0. \tag{44}
$$

The factors $\mathcal{H}_Y^{\zeta}(\sigma_2)-\mathcal{H}_Y^{\zeta}(\sigma_1)$, $\mathcal{H}_Y^{\xi}(\sigma_2)-\mathcal{H}_Y^{\xi}(\sigma_1)$, and $\mathcal{H}_Y^{\zeta+\xi}(\sigma_2)-\mathcal{H}_Y^{\zeta+\xi}(\sigma_1)$ can be estimated as in the following cases:

Case 1. If $0\le\mathcal{H}_Y(\sigma_1)<\delta,\mathcal{H}_Y(\sigma_2)\le 2\delta$, then

$$
\begin{aligned}
\mathcal{H}_Y^{\zeta}(\sigma_2)-\mathcal{H}_Y^{\zeta}(\sigma_1)&\le\mathcal{H}_Y^{\zeta}(\sigma_2)<(2\delta)^{\zeta}\le 2^{\zeta}\delta\le 2\delta,\\
\mathcal{H}_Y^{\xi}(\sigma_2)-\mathcal{H}_Y^{\xi}(\sigma_1)&\le\mathcal{H}_Y^{\xi}(\sigma_2)<(2\delta)^{\xi}\le 2^{\xi}\delta\le 2\delta,\\
\mathcal{H}_Y^{\zeta+\xi}(\sigma_2)-\mathcal{H}_Y^{\zeta+\xi}(\sigma_1)&\le\mathcal{H}_Y^{\zeta+\xi}(\sigma_2)<(2\delta)^{\zeta+\xi}\le 2^{\zeta+\xi}\delta\le 4\delta.
\end{aligned} \tag{45}
$$

Case 2. If $0<\mathcal{H}_Y(\sigma_1)<\mathcal{H}_Y(\sigma_2)\le\delta$, then

$$
\begin{aligned}
\mathcal{H}_Y^{\zeta}(\sigma_2)-\mathcal{H}_Y^{\zeta}(\sigma_1)&\le\mathcal{H}_Y^{\zeta}(\sigma_2)<\delta^{\zeta}\le\zeta\delta<2\delta,\\
\mathcal{H}_Y^{\xi}(\sigma_2)-\mathcal{H}_Y^{\xi}(\sigma_1)&\le\mathcal{H}_Y^{\xi}(\sigma_2)<\delta^{\xi}\le\xi\delta<2\delta,\\
\mathcal{H}_Y^{\zeta+\xi}(\sigma_2)-\mathcal{H}_Y^{\zeta+\xi}(\sigma_1)&\le\mathcal{H}_Y^{\zeta+\xi}(\sigma_2)<\delta^{\zeta+\xi}\le(\zeta+\xi)\delta<4\delta.
\end{aligned} \tag{46}
$$

Case 3. If $\delta\le\mathcal{H}_Y(\sigma_1)<\mathcal{H}_Y(\sigma_2)\le 1$, then

$$
\begin{aligned}
&\mathcal{H}_Y^{\zeta}(\sigma_2)-\mathcal{H}_Y^{\zeta}(\sigma_1)<\zeta\delta<2\delta,\\
&\mathcal{H}_Y^{\xi}(\sigma_2)-\mathcal{H}_Y^{\xi}(\sigma_1)<\xi\delta<2\delta \text{ and } \sigma_2^{a+\xi}-\sigma_1^{a+\xi}<(\zeta+\xi)\delta<4\delta.
\end{aligned} \tag{47}
$$

Accordingly, we obtain $|\mathcal{G}\vartheta(\sigma_2)-\mathcal{G}\vartheta(\sigma_1)|\le\varepsilon$, which implies that $\omega(\mathcal{G}\vartheta,\varepsilon)\le\varepsilon$.

Let $\varepsilon\longrightarrow 0$. Then,

$$
\omega_0(\mathcal{G}\Xi)=0. \tag{48}
$$

Step 5. We estimate $\omega_0(\mathcal{K}\Xi)$ for $\varnothing=\Xi\in\mathcal{B}_{r_0}$.

By Lemma 9 and equations (32), (41), and (48), we obtain

$$
\begin{aligned}
\omega_0(\mathcal{K}\Xi)&=\omega_0(\mathcal{F}\Xi.\mathcal{G}\Xi)\|\mathcal{F}\Xi\|\omega_0(\mathcal{G}\Xi)+\|\mathcal{G}\Xi\|\omega_0(\mathcal{F}\Xi)\\
&\le\left\|\mathcal{F}\left(\mathcal{B}_{r_0}\right)\right\|\omega_0(\mathcal{G}\Xi)+\left\|\mathcal{G}\left(\mathcal{B}_{r_0}\right)\right\|\omega_0(\mathcal{F}\Xi)\\
&\le[(\omega_0(\Xi)+d)^{\rho}-d^{\rho}]\left\{\frac{\varphi(r_0)}{\tilde{p}}\frac{(Y(\sigma)-Y(\varsigma))^{\zeta+\xi}}{\Gamma(\zeta+\xi+1)}\right.\\
&\quad\left.+\frac{\tilde{q}}{\tilde{p}}\frac{r_0(Y(\sigma)-Y(\varsigma))^{\zeta}}{\Gamma(\zeta+1)}+\frac{1}{\tilde{p}}\frac{\varphi(r_0)(Y(b)-Y(a))^{\zeta+\xi}}{\Gamma(\zeta+1)\Gamma(\xi+1)}\right\}\\
&=[(\omega_0(\Xi)+d)^{\rho}-d^{\rho}]\Lambda.
\end{aligned} \tag{49}
$$

Since $\Lambda\le 1$, the assumption (AS5) gives

$$
\omega_0(\mathcal{K}\Xi)+d^{\rho}\le(\omega_0(\Xi)+d)^{\rho}. \tag{50}
$$

Thanks to Theorem 10, the contractive condition is fulfilled with $\varphi(\vartheta)=\vartheta+b$, where $\varphi\in S$. By applying Theorem 11, $\mathcal{K}$ has at least fixed point in $\mathcal{B}_{r_0}$. Hence, the problem (2) has at least one solution in $\mathcal{B}_{r_0}$.

5. Examples

Here, we provide two examples to illustrate previous results.

Example 14. Consider the problem (2) with following specific data:

$$
p(\sigma)=1, q(\sigma)=\lambda=\frac{1}{100}, Y(\sigma)=\sigma. \tag{51}
$$

Then, the problem (2) reduces to

Table 1: Examples with some special cases of the Y function.

$Y(\sigma)$	$[a, b]$	ζ	ξ	$\tilde{p}$	$\tilde{q}$	r_0	Λ
σ	$[0, 1]$	$\frac{1}{4}$	$\frac{1}{2}$	1	$\frac{1}{100}$	$0 < r_0 \leq 1.7924$	$0.5445 < 1$
e^{σ}	$[0, 1]$	$\frac{1}{3}$	$\frac{3}{4}$	2	$\frac{2}{25}$	$0 < r_0 \leq 0.385$	$0.1613 < 1$
$\ln(\sigma)$	$[1, e]$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{3}{5}$	$\frac{1}{25}$	$0 < r_0 \leq 1.2603$	$0.3034 < 1$
2^{σ}	$[1, 2]$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{1}{35}$	$0 < r_0 \leq 0.1807$	$0.0449 < 1$

$$\begin{cases} {}^{c}\mathscr{D}^{1/4}\left[{}^{c}\mathscr{D}^{1/2}\dfrac{3}{\sqrt{\vartheta\left(e^{(\sigma-1)}/2\right)+4^2}}\right]-\dfrac{1}{100}\vartheta(\sigma)\right]=\dfrac{\sigma}{10}\left[\sin\vartheta\left(\sqrt{\sigma}\right)\right], & \sigma\in\Pi=[0,1], \\ \vartheta(0)=0,\ \dfrac{2}{2}\left[\dfrac{\vartheta(\sigma)}{\sqrt{\vartheta\left(e^{(\sigma-1)}/2\right)+4^2}}\right]_{\sigma=1}+\dfrac{1}{100}\vartheta(1)=0, & \end{cases} \tag{52}$$

where

$$a=0, b=1, \zeta=\frac{1}{4}, \xi=\frac{1}{2}, \lambda=\frac{1}{100}, \nu(\sigma)=\frac{e^{(\sigma-1)}}{2}, \mu(\sigma)=\sqrt{\sigma}. \tag{53}$$

$\mathscr{M}(\sigma,\vartheta)=\sqrt{|\vartheta|+4^2}$,$\mathscr{N}(\sigma,\vartheta)=(1/10)\sin(\vartheta)$, $d=16$, and $N=\sup_{\sigma\in0,1]}|\mathscr{M}(\sigma,0)|=4$. Thus, (AS1) and (AS2) hold. For (AS3), we obtain $\rho=1/2$. Furthermore, let $z(\vartheta)=\sqrt{|\vartheta|+4^2}-2^2$. Then, $z(0)=0$, and it is a concave function. Since $z(\sigma)$ is concave. As a result, the subadditive property of the concave function allows us to conclude

$$\begin{aligned} |\mathscr{M}(\sigma,\vartheta_2)-\mathscr{M}(\sigma,\vartheta_1)| &= |z(\vartheta_2)-z(\vartheta_1)| \\ &\leq z(\vartheta_2-\vartheta_1)=\sqrt{|\vartheta_2-\vartheta_1|+4^2}-2^2. \end{aligned} \tag{54}$$

Thus, (AS3) holds, with $\rho=1/2$. Moreover, for every $\sigma\in\Pi$ and $\vartheta\in\mathbb{R}$, we obtain

$$|\mathscr{N}(\sigma,\vartheta)|=\left|\frac{\sigma}{10}[\sin\vartheta(\sigma)]\right|\leq\frac{1}{10}|\vartheta(\sigma)|, \quad \forall\sigma\in\Pi. \tag{55}$$

Hence, (AS4) holds with $\varphi(\|\vartheta\|)=(1/10)\vartheta$. Finally, (AS5) permitted to provide us the range of r_0 which is obviously

$$0<r_0\leq1.7924. \tag{56}$$

Accordingly, (AS5) confirms that the illustrated example (52) has a solution in $\tilde{\mathscr{C}}$ due to

$$\Lambda=0.544529299<1. \tag{57}$$

Example 15. Depending on the previous example, we present some special cases of Y with different values for some parameters as in Table 1.

6. Conclusions

In this work, we have successfully studied some qualitative properties of the solution to a fractional problem that integrates three different types of BVP; more precisely, we have investigated the existence of the solutions of the Sturm-Liouville–Langevin–hybrid-type FDEs. Our analysis has been based on the technique of the measure of noncompactness along with the generalized Darbo's fixed point theorem. The results were consistent with some of the literature results when $Y(\sigma)=\sigma$, and they are new even for the special case: $Y(\sigma)=\log\sigma$ and $Y(\sigma)=\sigma^{\rho}$.

The problem studied can be extended to a more general problem containing Y-Hilfer FD, and this is what we are considering in future research.

Data Availability

No real data were used to support this study.

Conflicts of Interest

No conflicts of interest are related to this work.

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