Research Article

A β -Convolution Theorem Associated with the General Quantum Difference Operator

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In this paper, we prove some properties of the β -partial derivative. We define the β -convolution of two functions associated with the general quantum difference operator, $D_{\beta}f(t) = (f(\beta(t)) - f(t))/(\beta(t) - t)$; β is a strictly increasing continuous function. Moreover, we study the shift, the associative law, and the β -differentiability of the β -convolution. Furthermore, we prove the β -convolution theorem and give some applications.

1. Introduction and Preliminaries

The classical convolution of two real functions u and v is defined by

$$(u * v)(t) = \int_0^t u(t-s)v(s)ds, \quad (1)$$

where $u(t-s) = f(t, s)$ is the usual shift of the function u defined on $\mathbb{R}$ which is considered as the unique solution of the first-order partial differential equation

$$\frac{\partial}{\partial t}f(t, s) = -\frac{\partial}{\partial s}f(t, s), \quad t, s \in \mathbb{R}, f(t, 0) = u(t), \quad t \in \mathbb{R}. \quad (2)$$

The convolution theorem highly extends the potential of Laplace transform in solving ordinary differential equations. Furthermore, the operation of the convolution is needed in understanding many topics such as partial differential equations, Green's functions, and forming the general solution of some kinds of boundary value problems. See [1–4] for some applications of the convolution in electrical engineering, physics, and theory of distribution. The classical convolution theorem says that the Laplace transform of the convolution

$u * v$ of the two functions u and v is equal to the Laplace transform of u multiplied by the Laplace transform of v . Recently, there are many versions of the convolution theorem such as h -convolution theorem, q -convolution theorem, convolution theorem on time scale, and (q, h) -convolution theorem on discrete time scale, see, e.g., [5–8]. Bohner and Guseinov [9] studied the convolution theorem on a time scale $\mathbb{T}$, where the convolution of two functions $u, v : \mathbb{T} \rightarrow \mathbb{R}$ is defined by

$$(u * v)(t) = \int_{t_0}^t \hat{u}(t, \sigma(s))v(s)\Delta s, \quad t, t_0 \in \mathbb{T}. \quad (3)$$

Here, $\hat{u}(t, s)$ is the shift of the given function u defined on $\mathbb{T}$ which is considered as the unique solution of the shifting problem

$$u\Delta^{\Delta_t}(t, \sigma(s)) = -u\Delta^{\Delta_s}(t, s), \quad t, s \in \mathbb{T}, \hat{u}(t, t_0) = u(t), \quad t, t_0 \in \mathbb{T}, \quad (4)$$

where Δ is the delta differentiation and σ is the forward jump operator in $\mathbb{T}$. In [7], the convolution theorem of two

positive real functions $u_1(t)$ and $u_2(t)$ is defined by

$$\mathcal{L}_q\{u_1(t) * u_2(t)\} = \mathcal{L}_q\{u_1(t)\} \mathcal{L}_q\{u_2(t)\}, \quad (5)$$

where $\mathcal{L}_q\{u(t)\} = \int_0^\infty e_q^{-st} u(t) dt$, $s \in \mathbb{C}$, $R(s) > 0$, and

$$u_1(t) * u_2(t) = \int_0^t u_1(\tau) u_2(t - \tau) d_q \tau. \quad (6)$$

It should be noted that, here in [7], the Laplace transform denoted by $\mathcal{L}_q$ and the related convolution are defined by the usual integral; however, the q -exponential e_q^{-st} is used in the form $[1 + (q - 1)st]^{-1/(q-1)}$, $t \geq 0$, $q > 1$. In [10], the q -convolution is defined by

$$u_1(t) * u_2(t) = \int_0^t u_1(\tau) u_2(t - \tau) d_q \tau. \quad (7)$$

Hamza et al. [11] defined the general quantum β -difference operator D_β by:

$$D_\beta f(t) = \begin{cases} \frac{f(\beta(t)) - f(t)}{\beta(t) - t}, & t \neq s_0, \\ f'(s_0), & t = s_0, \end{cases} \quad (8)$$

where the function β is a strictly increasing and continuous defined on an interval $I \subseteq \mathbb{R}$, β has only one fixed point $s_0 \in I$, and the inequality $(t - s_0)(\beta(t) - t) \leq 0$ holds for all $t \in I$, accordingly $s_0 = \lim_{k \rightarrow \infty} \beta^k(t)$, $\beta^k(t) = \beta \circ \beta \circ \dots \circ \beta(t)$ (k -times). A function f is said to be β -differentiable on I , if the ordinary derivative $f'(s_0)$ exists. The β -difference operator D_β produces the Jackson q -difference operator, the Hahn difference operator, and the power quantum difference operator when $\beta(t) = qt$, $\beta(t) = qt + \omega$, and $\beta(t) = qt^n$, respectively; $t \in I$, $q \in (0, 1)$, $\omega > 0$, and n is odd number, see [12–14]. Quantum difference operators consider a good tool in dealing with sets of nondifferentiable functions in the usual concept; furthermore, it has an interesting role due to their applications in several sciences, see, e.g., [15–18]. Recent quantum calculus applications can be found in [19–21].

The β -integral of $f : I \rightarrow \mathbb{R}$ is defined in [11] by

$$\int_a^c f(t) d_\beta t = \int_{s_0}^c f(t) d_\beta t - \int_{s_0}^a f(t) d_\beta t, \quad a, c \in I, \quad (9)$$

where

$$F(x) = \int_{s_0}^x f(t) d_\beta t = \sum_{k=0}^{\infty} (\beta^k(x) - \beta^{k+1}(x)) f(\beta^k(x)), \quad x \in I, \quad (10)$$

provided that the series in the right hand side converges. Furthermore, if f is continuous at s_0 , then F exists and is continuous at s_0 , and $D_\beta F(x)$ exists for all $x \in I$, $D_\beta F(x) =$

$f(x)$. The β -exponential functions are defined, in [22], by

$$e_{\lambda, \beta}(t) = \frac{1}{\prod_{k=0}^{\infty} [1 - \lambda(\beta^k(t))(\beta^k(t) - \beta^{k+1}(t))]}, \quad (11)$$

$$E_{\lambda, \beta}(t) = \prod_{k=0}^{\infty} [1 + \lambda(\beta^k(t))(\beta^k(t) - \beta^{k+1}(t))].$$

If the function $f : I \rightarrow \mathbb{R}$ is β -differentiable and $D_\beta f(t) = 0$ for all $t \in I$, then f is constant function and

$$f(t) = f(s_0), \quad \text{for all } t \in I, \quad (12)$$

see [11]. In [23] (p. 126), the β -partial derivative of $f(t, y)$ with respect to t is defined by

$$\frac{\partial_\beta}{\partial_\beta t} f(t, y) = \frac{f(\beta(t), y) - f(t, y)}{\beta(t) - t}, \quad t \neq s_0. \quad (13)$$

Also, if $F(t) := \int_{s_0}^b f(t, y) d_\beta y$, $b, t \in I$. Then, $D_\beta F(t)$ at $t \neq s_0$ exists and is given by

$$D_\beta F(t) = \int_{s_0}^b \frac{\partial_\beta}{\partial_\beta t} f(t, y) d_\beta y. \quad (14)$$

Moreover, the Leibniz's rule is proved in [23] such that if $F(t) := \int_{s_0}^t f(t, y) d_\beta y$, then $D_\beta F(t)$ at $t \neq s_0$ exists and is given by

$$D_\beta F(t) = f(\beta(t), t) + \int_{s_0}^t \frac{\partial_\beta}{\partial_\beta t} f(t, y) d_\beta y. \quad (15)$$

The general quantum Laplace transform $\mathcal{L}_\beta$ and its inverse $\mathcal{L}_\beta^{-1}$ and some of their properties were deduced in [24]. The β -Laplace transform $\mathcal{L}_\beta$ is defined by:

$$\mathcal{L}_\beta\{f(t)\} := \int_{s_0}^{\infty} f(t) e_{\ominus_{\beta} p, \beta}(\beta(t)) d_\beta t, \quad (16)$$

such that the improper β -integral, see [24] (Definition 3.5), in (16) exists, where $s_0 \in I$, $f \in V([s_0, \infty), \mathbb{C})$, which is the set of β -integrable functions over each compact subinterval of $[s_0, \infty)$, and z satisfies $1 + z(\beta(t) - t) \neq 0$. The function f is said to be of exponential order $\lambda > 0$, if there exist a constant $M > 0$ such that $|f(t)| \leq M e_{\lambda, \beta}(t)$ for all $t \in [s_0, \infty)$. Let $p, q : I \rightarrow \mathbb{C}$ be continuous functions at s_0 and satisfy the condition $1 + (\beta(t) - t)p(t) \neq 0$ for all $t \in I$, then the following properties hold [24]:

- (i₁) $e_{\ominus_{\beta} p, \beta}(t) = 1/e_{p, \beta}(t)$
 - (i₂) $e_{\ominus_{\beta} p, \beta}(\beta(t)) = e_{\ominus_{\beta} p, \beta}(t)/(1 + (\beta(t) - t)p(t)) = -((\ominus_{\beta} p)(t)/p(t))e_{\ominus_{\beta} p, \beta}(t)$
 - (i₃) $D_\beta(e_{\ominus_{\beta} p, \beta}(t)) = (\ominus_{\beta} p)(t)e_{\ominus_{\beta} p, \beta}(t)$
- where $(\ominus_{\beta} p)(t) = -p(t)/(1 + (\beta(t) - t)p(t))$.

Recently, Cardoso [25] investigated the β -Lagrange's identity for the β -Sturm-Liouville eigenvalue problem and proved that it is self-adjoint in $L^2_\beta([a, b])$. For more results in β -calculus, we refer the readers to see [26–29].

The current paper is organized as follows. In Section 2, we introduce some properties of the β -partial derivative. In Section 3, we define the β -convolution of two functions and study their shift, associativity, and the β -differentiability. Moreover, we prove the β -convolution theorem and give some applications.

2. β -Partial Derivative

In this section, we introduce some properties of the β -partial derivative.

Theorem 1. Let $f, g : I \times I \rightarrow \mathbb{R}$. Assume that the β -partial derivatives of $f(t, \tau)$, $g(t, \tau)$, $t, \tau \in I$ with respect to t at $t \neq s_0$ exist. Then:

(i) The β -partial derivative of the product $(fg)(t, \tau)$ with respect to t at $t \neq s_0$ is given by

$$\begin{aligned} \frac{\partial_\beta}{\partial_\beta t} (fg)(t, \tau) &= \left[\frac{\partial_\beta}{\partial_\beta t} f(t, \tau) \right] g(\beta(t), \tau) + f(t, \tau) \left[\frac{\partial_\beta}{\partial_\beta t} g(t, \tau) \right] \\ &= \left[\frac{\partial_\beta}{\partial_\beta t} f(t, \tau) \right] g(t, \tau) + f(\beta(t), \tau) \left[\frac{\partial_\beta}{\partial_\beta t} g(t, \tau) \right], \end{aligned} \quad (17)$$

where $(fg)(t, \tau) = f(t, \tau)g(t, \tau)$.

(ii) The β -partial derivative of $(f/g)(t, \tau)$ with respect to t at $t \neq s_0$ is given by

$$\frac{\partial_\beta}{\partial_\beta t} \left(\frac{f}{g} \right)(t, \tau) = \frac{[(\partial_\beta / \partial_\beta t) f(t, \tau)] g(t, \tau) - f(t, \tau) [(\partial_\beta / \partial_\beta t) g(t, \tau)]}{g(t, \tau) g(\beta(t), \tau)}, \quad (18)$$

provided that $g(t, \tau)g(\beta(t), \tau) \neq 0$.

Proof. (i).

$$\begin{aligned} \frac{\partial_\beta}{\partial_\beta t} (fg)(t, \tau) &= \frac{(fg)(\beta(t), \tau) - (fg)(t, \tau)}{\beta(t) - t} \\ &= \frac{f(\beta(t), \tau) - f(t, \tau)}{\beta(t) - t} g(\beta(t), \tau) + f(t, \tau) \frac{g(\beta(t), \tau) - g(t, \tau)}{\beta(t) - t} \\ &= \left[\frac{\partial_\beta}{\partial_\beta t} f(t, \tau) \right] g(\beta(t), \tau) + f(t, \tau) \left[\frac{\partial_\beta}{\partial_\beta t} g(t, \tau) \right]. \end{aligned} \quad (19)$$

Similarly, we can prove that

$$\frac{\partial_\beta}{\partial_\beta t} (fg)(t, \tau) = \left[\frac{\partial_\beta}{\partial_\beta t} f(t, \tau) \right] g(t, \tau) + f(\beta(t), \tau) \left[\frac{\partial_\beta}{\partial_\beta t} g(t, \tau) \right]. \quad (20)$$

In the same way, we can prove (ii). $\square$

Lemma 2. Let $f : I \times I \rightarrow \mathbb{R}$ and assume that the β -partial derivative of $f(t, \tau)$, $t, \tau \in I$ with respect to τ exists and is continuous at (t, s_0) . Then

$$\int_\tau^{\beta(\tau)} f(t, y) d_\beta y = (\beta(\tau) - \tau) f(t, \tau). \quad (21)$$

Proof. Let $F(t, \tau) = \int_{s_0}^\tau f(t, y) d_\beta y$. Since $F(t, \tau)$ is continuous at (t, s_0) and $(\partial_\beta / \partial_\beta \tau) F(t, \tau) = f(t, \tau)$, then

$$\begin{aligned} \int_\tau^{\beta(\tau)} f(t, y) d_\beta y &= \int_{s_0}^{\beta(\tau)} f(t, y) d_\beta y - \int_{s_0}^\tau f(t, y) d_\beta y \\ &= F(t, \beta(\tau)) - F(t, \tau). \end{aligned} \quad (22)$$

Since $F(t, \beta(\tau)) - F(t, \tau) = (\beta(\tau) - \tau)(\partial_\beta / \partial_\beta \tau) F(t, \tau)$, then

$$\int_\tau^{\beta(\tau)} f(t, y) d_\beta y = (\beta(\tau) - \tau) f(t, \tau). \quad (23)$$

$\square$

Lemma 3. Let $f : I \times I \rightarrow \mathbb{R}$. If the β -partial derivative of $f(t, \tau)$, $t, \tau \in I$ with respect to t exists and is continuous at (s_0, τ) , then

$$\int_a^b \frac{\partial_\beta}{\partial_\beta t} f(t, \tau) d_\beta t = f(b, \tau) - f(a, \tau) \text{ for all } a, b \in I. \quad (24)$$

Proof. We have

$$\begin{aligned} \int_{s_0}^b \frac{\partial_\beta}{\partial_\beta t} f(t, \tau) d_\beta t &= \sum_{k=0}^{\infty} (\beta^k(b) - \beta^{k+1}(b)) \frac{\partial_\beta}{\partial_\beta t} f(\beta^k(b), \tau) \\ &= \sum_{k=0}^{\infty} [f(\beta^k(b), \tau) - f(\beta^{k+1}(b), \tau)] \\ &= \lim_{n \rightarrow \infty} \sum_{k=0}^n [f(\beta^k(b), \tau) - f(\beta^{k+1}(b), \tau)] \\ &= f(b, \tau) - f(s_0, \tau). \end{aligned} \quad (25)$$

Similarly,

$$\int_{s_0}^a \frac{\partial_\beta}{\partial_\beta t} f(t, \tau) d_\beta t = f(a, \tau) - f(s_0, \tau). \quad (26)$$

Hence,

$$\int_a^b \frac{\partial_\beta}{\partial_\beta t} f(t, \tau) d_\beta t = f(b, \tau) - f(a, \tau). \quad (27)$$

□

Lemma 4 (integration by parts). *Let $f, g : I \times I \rightarrow \mathbb{R}$. Assume that the β -partial derivatives of $f(t, \tau)$, $g(t, \tau)$, $t, \tau \in I$ with respect to t exist and are both continuous at (s_0, τ) . Then*

$$\begin{aligned} \int_a^b f(t, \tau) \left[\frac{\partial_\beta}{\partial_\beta t} g(t, \tau) \right] d_\beta t &= f(b, \tau) g(b, \tau) - f(a, \tau) g(a, \tau) \\ &\quad - \int_a^b \left[\frac{\partial_\beta}{\partial_\beta t} f(t, \tau) \right] g(\beta(t), \tau) d_\beta t, \quad a, b \in I. \end{aligned} \quad (28)$$

Proof. Since

$$\frac{\partial_\beta}{\partial_\beta t} (fg)(t, \tau) = f(t, \tau) \left[\frac{\partial_\beta}{\partial_\beta t} g(t, \tau) \right] + g(\beta(t), \tau) \left[\frac{\partial_\beta}{\partial_\beta t} f(t, \tau) \right]. \quad (29)$$

Then

$$\begin{aligned} \int_a^b f(t, \tau) \frac{\partial_\beta}{\partial_\beta t} g(t, \tau) d_\beta t &= \int_a^b \frac{\partial_\beta}{\partial_\beta t} (fg)(t, \tau) d_\beta t \\ &\quad - \int_a^b \left[\frac{\partial_\beta}{\partial_\beta t} f(t, \tau) \right] g(\beta(t), \tau) d_\beta t, \quad a, b \in I. \end{aligned} \quad (30)$$

By Lemma 3, we get

$$\int_a^b \frac{\partial_\beta}{\partial_\beta t} (fg)(t, \tau) d_\beta t = f(b, \tau) g(b, \tau) - f(a, \tau) g(a, \tau). \quad (31)$$

Therefore,

$$\begin{aligned} \int_a^b f(t, \tau) \left[\frac{\partial_\beta}{\partial_\beta t} g(t, \tau) \right] d_\beta t &= f(b, \tau) g(b, \tau) - f(a, \tau) g(a, \tau) \\ &\quad - \int_a^b \left[\frac{\partial_\beta}{\partial_\beta t} f(t, \tau) \right] g(\beta(t), \tau) d_\beta t, \quad a, b \in I. \end{aligned} \quad (32)$$

□

3. The β -Convolution Theorem

In the following, we define the β -convolution of two functions and prove some of its properties. Furthermore, we prove the β -convolution theorem and give some applications.

Definition 5. Consider the β -shifting problem

$$\frac{\partial_\beta}{\partial_\beta t} z(t, \beta(\tau)) = -\frac{\partial_\beta}{\partial_\beta \tau} z(t, \tau), \quad t, \tau \in I, t \geq \tau \geq s_0, \quad (33)$$

$$z(t, s_0) = f(t), \quad t \in I, t \geq s_0. \quad (34)$$

We denote the shift of a function $f \in V([s_0, \infty), \mathbb{C})$ by $\hat{f}(t, \tau)$, $t, \tau \in I$ which is the solution of (33).

We will assume in this paper that the problem (33) has a unique solution $z = \hat{f}$.

Definition 6 (β -convolution). The β -convolution $f * g$ of two functions $f, g \in V([s_0, \infty), \mathbb{C})$ is defined by

$$(f * g)(t) = \int_{s_0}^t \hat{f}(t, \beta(\tau)) g(\tau) d_\beta \tau, \quad t \in I, \quad (35)$$

where $\hat{f}$ is the shift of f given in Definition 5.

Note that in the case of $\beta(t) = qt$, $s_0 = 0$, we get the q -shifting problem

$$\frac{\partial_q}{\partial_q t} z(t, q\tau) = -\frac{\partial_q}{\partial_q \tau} z(t, \tau), \quad z(t, 0) = f(t), \quad (36)$$

and the q -convolution

$$(f * g)(t) = \int_0^t \hat{f}(t, q\tau) g(\tau) d_q \tau, \quad (37)$$

we refer the reader to see [6]. Furthermore, in the case of $\beta(t) = t$, we get the usual shifting problem

$$\frac{\partial}{\partial t} z(t, \tau) = -\frac{\partial}{\partial \tau} z(t, \tau), \quad z(t, s_0) = f(t), \quad (38)$$

with the unique solution $z(t, \tau) = f(t - \tau) = \hat{f}(t, \tau)$ which is the usual shift, and we get the classical convolution

$$(f * g)(t) = \int_0^t f(t - \tau) g(\tau) d\tau. \quad (39)$$

Lemma 7. Let $\hat{f}$ be the shift of f . Then, $\hat{f}(t, t) = f(s_0)$ for all $t \in I$.

Proof. Let $\hat{f}$ be the shift of f . Set $\hat{f}(t, t) = F(t)$. From the β -shifting problem (33), we have

$$\begin{aligned} 0 &= \frac{\partial_\beta}{\partial_\beta t} \hat{f}(t, \beta(\tau)) + \frac{\partial_\beta}{\partial_\beta \tau} \hat{f}(t, \tau) = \frac{\hat{f}(\beta(t), \beta(\tau)) - \hat{f}(t, \beta(\tau))}{\beta(t) - t} \\ &\quad + \frac{\hat{f}(t, \beta(\tau)) - \hat{f}(t, \tau)}{\beta(\tau) - \tau}. \end{aligned} \quad (40)$$

Take $t = \tau$, consequently

$$\begin{aligned} 0 &= \frac{\hat{f}(\beta(t), \beta(t)) - \hat{f}(t, \beta(t))}{\beta(t) - t} + \frac{\hat{f}(t, \beta(t)) - \hat{f}(t, t)}{\beta(t) - t} \\ &= \frac{\hat{f}(\beta(t), \beta(t)) - \hat{f}(t, t)}{\beta(t) - t} = \frac{F(\beta(t)) - F(t)}{\beta(t) - t} = D_\beta F(t), t \neq s_0. \end{aligned} \quad (41)$$

By equation (12), $\hat{f}(t, t) = F(t) = F(s_0)$, for all $t \in I$. Moreover, from the initial condition (33), $F(s_0) = f(s_0)$. Then, $\hat{f}(t, t) = f(s_0)$ for all $t \in I$. $\square$

Lemma 8. Define the function F by

$$F(t, \tau) := \int_\tau^t \hat{f}(t, \beta(y)) \hat{g}(y, \tau) d_\beta y. \quad (42)$$

Then, $(\partial_\beta / \partial_\beta t) F(t, \tau)$ and $(\partial_\beta / \partial_\beta \tau) F(t, \tau)$ exist and are given by:

$$\begin{aligned} \frac{\partial_\beta}{\partial_\beta t} F(t, \tau) &= \hat{f}(\beta(t), \beta(t)) \hat{g}(t, \tau) + \int_\tau^t \left[\frac{\partial_\beta}{\partial_\beta t} \hat{f}(t, \beta(y)) \right] \hat{g}(y, \tau) d_\beta y, \\ \frac{\partial_\beta}{\partial_\beta \tau} F(t, \tau) &= \int_\tau^t \hat{f}(t, \beta(y)) \left[\frac{\partial_\beta}{\partial_\beta \tau} \hat{g}(y, \tau) \right] d_\beta y - \hat{f}(t, \beta(\tau)) \hat{g}(\tau, \beta(\tau)). \end{aligned} \quad (43)$$

Proof. We have

$$F(t, \tau) = \int_{s_0}^t \hat{f}(t, \beta(y)) \hat{g}(y, \tau) d_\beta y - \int_{s_0}^\tau \hat{f}(t, \beta(y)) \hat{g}(y, \tau) d_\beta y. \quad (44)$$

Then,

$$\begin{aligned} \frac{\partial_\beta}{\partial_\beta t} F(t, \tau) &= \frac{\partial_\beta}{\partial_\beta t} \left[\int_{s_0}^t \hat{f}(t, \beta(y)) \hat{g}(y, \tau) d_\beta y \right] \\ &\quad - \frac{\partial_\beta}{\partial_\beta t} \left[\int_{s_0}^\tau \hat{f}(t, \beta(y)) \hat{g}(y, \tau) d_\beta y \right], \\ \frac{\partial_\beta}{\partial_\beta \tau} F(t, \tau) &= \frac{\partial_\beta}{\partial_\beta \tau} \left[\int_{s_0}^t \hat{f}(t, \beta(y)) \hat{g}(y, \tau) d_\beta y \right] \\ &\quad - \frac{\partial_\beta}{\partial_\beta \tau} \left[\int_{s_0}^\tau \hat{f}(t, \beta(y)) \hat{g}(y, \tau) d_\beta y \right]. \end{aligned} \quad (45)$$

Now,

$$\begin{aligned} &\frac{\partial_\beta}{\partial_\beta t} \left[\int_{s_0}^t \hat{f}(t, \beta(y)) \hat{g}(y, \tau) d_\beta y \right] \\ &= \frac{1}{\beta(t) - t} \left\{ \sum_{k=0}^{\infty} (\beta^{k+1}(t) - \beta^{k+2}(t)) \hat{f}(\beta(t), \beta^{k+2}(t)) \right. \\ &\quad \cdot \hat{g}(\beta^{k+1}(t), \tau) - \sum_{k=0}^{\infty} (\beta^k(t) - \beta^{k+1}(t)) \hat{f}(t, \beta^{k+1}(t)) \\ &\quad \cdot \hat{g}(\beta^k(t), \tau) \left. \right\} = \frac{1}{\beta(t) - t} \left\{ \sum_{k=1}^{\infty} (\beta^k(t) - \beta^{k+1}(t)) \right. \\ &\quad \cdot [\hat{f}(\beta(t), \beta^{k+1}(t)) \hat{g}(\beta^k(t), \tau) - \hat{f}(t, \beta^{k+1}(t)) \\ &\quad \cdot \hat{g}(\beta^k(t), \tau)] \left. \right\} - \frac{(t - \beta(t)) \hat{f}(t, \beta(t)) \hat{g}(t, \tau)}{\beta(t) - t} \\ &= \frac{1}{\beta(t) - t} \left\{ \sum_{k=0}^{\infty} (\beta^k(t) - \beta^{k+1}(t)) [\hat{f}(\beta(t), \beta^{k+1}(t)) \right. \\ &\quad \cdot \hat{g}(\beta^k(t), \tau) - \hat{f}(t, \beta^{k+1}(t)) \hat{g}(\beta^k(t), \tau)] \left. \right\} \\ &\quad - \frac{t - \beta(t)}{\beta(t) - t} [\hat{f}(\beta(t), \beta(t)) \hat{g}(t, \tau) - \hat{f}(t, \beta(t)) \hat{g}(t, \tau)] \\ &\quad + \hat{f}(t, \beta(t)) \hat{g}(t, \tau) = \hat{f}(\beta(t), \beta(t)) \hat{g}(t, \tau) \\ &\quad + \int_{s_0}^t \left[\frac{\partial_\beta}{\partial_\beta t} \hat{f}(t, \beta(y)) \right] \hat{g}(y, \tau) d_\beta y. \end{aligned} \quad (46)$$

Similarly,

$$\begin{aligned} &\frac{\partial_\beta}{\partial_\beta t} \left[\int_{s_0}^\tau \hat{f}(t, \beta(y)) \hat{g}(y, \tau) d_\beta y \right] \\ &= \int_{s_0}^\tau \left[\frac{\partial_\beta}{\partial_\beta t} \hat{f}(t, \beta(y)) \right] \hat{g}(y, \tau) d_\beta y. \end{aligned} \quad (47)$$

Therefore,

$$\begin{aligned} \frac{\partial_\beta}{\partial_\beta t} F(t, \tau) &= \hat{f}(\beta(t), \beta(t)) \hat{g}(t, \tau) + \int_{s_0}^t \left[\frac{\partial_\beta}{\partial_\beta t} \hat{f}(t, \beta(y)) \right] \\ &\quad \cdot \hat{g}(y, \tau) d_\beta y - \int_{s_0}^\tau \left[\frac{\partial_\beta}{\partial_\beta t} \hat{f}(t, \beta(y)) \right] \hat{g}(y, \tau) d_\beta y \\ &= \hat{f}(\beta(t), \beta(t)) \hat{g}(t, \tau) + \int_\tau^t \left[\frac{\partial_\beta}{\partial_\beta t} \hat{f}(t, \beta(y)) \right] \\ &\quad \cdot \hat{g}(y, \tau) d_\beta y. \end{aligned} \quad (48)$$

Using the same technique, we obtain

$$\begin{aligned} \frac{\partial_\beta}{\partial_\beta \tau} F(t, \tau) &= \int_{s_0}^\tau \widehat{f}(t, \beta(y)) \left[\frac{\partial_\beta}{\partial_\beta \tau} \widehat{g}(y, \tau) \right] d_\beta y - \widehat{f}(t, \beta(\tau)) \widehat{g}(\tau, \beta(\tau)) \\ &\quad - \int_{s_0}^t \widehat{f}(t, \beta(y)) \left[\frac{\partial_\beta}{\partial_\beta \tau} \widehat{g}(y, \tau) \right] d_\beta y \\ &= \int_\tau^t \widehat{f}(t, \beta(y)) \left[\frac{\partial_\beta}{\partial_\beta \tau} \widehat{g}(y, \tau) \right] d_\beta y - \widehat{f}(t, \beta(\tau)) \widehat{g}(\tau, \beta(\tau)). \end{aligned} \quad (49)$$

□

Theorem 9. The shift of the β -convolution of two functions f, g is given by

$$(\widehat{f * g})(t, \tau) = \int_\tau^t \widehat{f}(t, \beta(y)) \widehat{g}(y, \tau) d_\beta y. \quad (50)$$

Proof. We aim to prove that the integral in equation (50) is a solution of the β -shifting problem (33). Set

$$F(t, \tau) = \int_\tau^t \widehat{f}(t, \beta(y)) \widehat{g}(y, \tau) d_\beta y. \quad (51)$$

Take $\tau = s_0$, then

$$F(t, s_0) = \int_{s_0}^t \widehat{f}(t, \beta(y)) \widehat{g}(y, s_0) d_\beta y. \quad (52)$$

From the initial condition of the shift problem (33), we get $\widehat{g}(y, s_0) = g(y)$. Consequently,

$$F(t, s_0) = \int_{s_0}^t \widehat{f}(t, \beta(y)) g(y) d_\beta y = (f * g)(t). \quad (53)$$

Using Lemmas 7 and 8, we get

$$\begin{aligned} \frac{\partial_\beta}{\partial_\beta t} F(t, \beta(\tau)) + \frac{\partial_\beta}{\partial_\beta \tau} F(t, \tau) &= - \int_{\beta(\tau)}^t \left[\frac{\partial_\beta}{\partial_\beta y} \widehat{f}(t, y) \right] \widehat{g}(y, \beta(\tau)) d_\beta y + f(s_0) \widehat{g}(t, \beta(\tau)) \\ &\quad + \int_\tau^t \widehat{f}(t, \beta(y)) \left[\frac{\partial_\beta}{\partial_\beta \tau} \widehat{g}(y, \tau) \right] d_\beta y - \widehat{f}(t, \beta(\tau)) \widehat{g}(\tau, \beta(\tau)). \end{aligned} \quad (54)$$

We have, using the integration by parts,

$$\begin{aligned} \int_{\beta(\tau)}^t \left[\frac{\partial_\beta}{\partial_\beta y} \widehat{f}(t, y) \right] \widehat{g}(y, \beta(\tau)) d_\beta y &= \widehat{f}(t, t) \widehat{g}(t, \beta(\tau)) - \widehat{f}(t, \beta(\tau)) \widehat{g}(\beta(\tau), \beta(\tau)) \\ &\quad - \int_{\beta(\tau)}^t \widehat{f}(t, \beta(y)) \left[\frac{\partial_\beta}{\partial_\beta y} \widehat{g}(y, \beta(\tau)) \right] d_\beta y \\ &= f(s_0) \widehat{g}(t, \beta(\tau)) - \widehat{f}(t, \beta(\tau)) g(s_0) \\ &\quad + \int_{\beta(\tau)}^t \widehat{f}(t, \beta(y)) \left[\frac{\partial_\beta}{\partial_\beta \tau} \widehat{g}(y, \tau) \right] d_\beta y. \end{aligned} \quad (55)$$

Consequently,

$$\begin{aligned} \frac{\partial_\beta}{\partial_\beta t} F(t, \beta(\tau)) + \frac{\partial_\beta}{\partial_\beta \tau} F(t, \tau) &= -f(s_0) \widehat{g}(t, \beta(\tau)) + \widehat{f}(t, \beta(\tau)) g(s_0) - \int_{\beta(\tau)}^t \widehat{f}(t, \beta(y)) \\ &\quad \cdot \left[\frac{\partial_\beta}{\partial_\beta \tau} \widehat{g}(y, \tau) \right] d_\beta y + f(s_0) \widehat{g}(t, \beta(\tau)) - \widehat{f}(t, \beta(\tau)) \widehat{g}(\tau, \beta(\tau)) \\ &\quad + \int_\tau^t \widehat{f}(t, \beta(y)) \left[\frac{\partial_\beta}{\partial_\beta \tau} \widehat{g}(y, \tau) \right] d_\beta y = \widehat{f}(t, \beta(\tau)) g(s_0) \\ &\quad + \int_\tau^{\beta(\tau)} \widehat{f}(t, \beta(y)) \left[\frac{\partial_\beta}{\partial_\beta \tau} \widehat{g}(y, \tau) \right] d_\beta y - \widehat{f}(t, \beta(\tau)) \widehat{g}(\tau, \beta(\tau)). \end{aligned} \quad (56)$$

By using Lemma 2, we get

$$\begin{aligned} \frac{\partial_\beta}{\partial_\beta t} F(t, \beta(\tau)) + \frac{\partial_\beta}{\partial_\beta \tau} F(t, \tau) &= \widehat{f}(t, \beta(\tau)) g(s_0) + (\beta(\tau) - \tau) \\ &\quad \cdot \widehat{f}(t, \beta(\tau)) \left[\frac{\partial_\beta}{\partial_\beta \tau} \widehat{g}(\tau, \tau) \right] - \widehat{f}(t, \beta(\tau)) \widehat{g}(\tau, \beta(\tau)) \\ &= \widehat{f}(t, \beta(\tau)) g(s_0) + \widehat{f}(t, \beta(\tau)) [\widehat{g}(\tau, \beta(\tau)) - \widehat{g}(\tau, \tau)] \\ &\quad - \widehat{f}(t, \beta(\tau)) \widehat{g}(\tau, \beta(\tau)) = 0. \end{aligned} \quad (57)$$

Therefore,

$$\frac{\partial_\beta}{\partial_\beta t} F(t, \beta(\tau)) = - \frac{\partial_\beta}{\partial_\beta \tau} F(t, \tau). \quad (58)$$

Hence, $F(t, \tau) = (\widehat{f * g})(t, \tau)$ is the solution of equation (58). □

The proof of the following Lemma 10 will be omitted since it is a direct consequence of Definitions 5 and 6 and the linearity of the β -integral.

Lemma 10. The β -convolution $f * g$ defined in (35) satisfies the following properties:

- (i) $c(f * g) = cf * g = f * cg$, $c \in \mathbb{R}$
- (ii) $f * (g + h) = f * g + f * h$
- (iii) $f * (ag + bh) = a(f * g) + b(f * h)$, $a, b \in \mathbb{R}$

Lemma 11. Let $w : J \times J \longrightarrow \mathbb{R}$ be continuous function at (s_0, s_0) , $s_0 \in J$, and J is a compact subinterval of I , where $w(y, \tau) = \widehat{f}(t, \beta(y))\widehat{g}(y, \beta(\tau))h(\tau)$. Then,

$$\begin{aligned} \int_{s_0}^t \int_{\beta(\tau)}^y \widehat{f}(t, \beta(y))\widehat{g}(y, \beta(\tau))h(\tau)d_\beta y d_\beta \tau \\ = \int_{s_0}^t \int_{s_0}^y \widehat{f}(t, \beta(y))\widehat{g}(y, \beta(\tau))h(\tau)d_\beta \tau d_\beta y. \end{aligned} \quad (59)$$

Proof. First, we refer the reader to see Fubini's Theorem in [23] (p. 132). Now, let

$$\begin{aligned} W(t, \tau) &= \int_{\beta(\tau)}^t w(y, \tau)d_\beta y, \quad \phi_1(t) = \int_{s_0}^t W(t, \tau)d_\beta \tau, \\ G(y) &= \int_{s_0}^y w(y, \tau)d_\beta \tau, \quad \phi_2(t) = \int_{s_0}^t G(y)d_\beta y. \end{aligned} \quad (60)$$

Then,

$$D_\beta \phi_1(t) = \int_{s_0}^t \frac{\partial_\beta}{\partial_\beta t} W(t, \tau)d_\beta \tau,$$

$$\begin{aligned} \frac{\partial_\beta}{\partial_\beta t} W(t, \tau) &= \frac{1}{\beta(t) - t} \left\{ \sum_{k=0}^{\infty} (\beta^{k+1}(t) - \beta^{k+2}(t))w(\beta^{k+1}(t), \tau) \right. \\ &\quad \left. - \sum_{k=0}^{\infty} (\beta^k(t) - \beta^{k+1}(t))w(\beta^k(t), \tau) \right\} = w(t, \tau). \end{aligned} \quad (61)$$

Therefore, $D_\beta \phi_1(t) = \int_{s_0}^t w(t, \tau)d_\beta \tau$. Also,

$$D_\beta \phi_2(t) = D_\beta \left[\int_{s_0}^t G(y)d_\beta y \right] = G(t) = \int_{s_0}^t w(t, \tau)d_\beta \tau = D_\beta \phi_1(t). \quad (62)$$

Hence, $\phi_1(t) = \phi_2(t)$, $t \in I$. $\square$

Theorem 12 (Associativity of the β -convolution). The β -convolution is associative, that is,

$$(f * g) * h = f * (g * h). \quad (63)$$

Proof. Using Theorem 9 and Lemma 11, we have

$$\begin{aligned} ((f * g) * h)(t) &= \int_{s_0}^t (\widehat{f * g})(t, \beta(\tau))h(\tau)d_\beta \tau \\ &= \int_{s_0}^t \int_{\beta(\tau)}^t \widehat{f}(t, \beta(y))\widehat{g}(y, \beta(\tau))h(\tau)d_\beta y d_\beta \tau \\ &= \int_{s_0}^t \int_{s_0}^y \widehat{f}(t, \beta(y))\widehat{g}(y, \beta(\tau))h(\tau)d_\beta \tau d_\beta y \\ &= \int_{s_0}^t \widehat{f}(t, \beta(y))(g * h)(y)d_\beta y = (f * (g * h))(t). \end{aligned} \quad (64)$$

$\square$

Theorem 13. Let f and g be β -differentiable functions, then we have

$$D_\beta(f * g)(t) = f(t)g(s_0) + (f * D_\beta g)(t). \quad (65)$$

Proof. We have

$$(f * g)(t) = \int_{s_0}^t \widehat{f}(t, \beta(\tau))g(\tau)d_\beta \tau, \quad t \in I. \quad (66)$$

Using Lemma 8, then,

$$D_\beta(f * g)(t) = \int_{s_0}^t \left[\frac{\partial_\beta}{\partial_\beta t} \widehat{f}(t, \beta(\tau)) \right] g(\tau)d_\beta \tau + \widehat{f}(\beta(t), \beta(t))g(t). \quad (67)$$

By Definition 5 and Lemma 7

$$\begin{aligned} \frac{\partial_\beta}{\partial_\beta t} \widehat{f}(t, \beta(\tau)) &= -\frac{\partial_\beta}{\partial_\beta \tau} \widehat{f}(t, \tau), \\ \widehat{f}(\beta(t), \beta(t)) &= f(s_0), \end{aligned} \quad (68)$$

therefore,

$$D_\beta(f * g)(t) = -\int_{s_0}^t \left[\frac{\partial_\beta}{\partial_\beta \tau} \widehat{f}(t, \tau) \right] g(\tau)d_\beta \tau + f(s_0)g(t). \quad (69)$$

We get, by the integration by parts,

$$\begin{aligned} D_\beta(f * g)(t) &= -\widehat{f}(t, t)g(t) + \widehat{f}(t, s_0)g(s_0) + \int_{s_0}^t \widehat{f}(t, \beta(\tau)) \left[\frac{\partial_\beta}{\partial_\beta \tau} g(\tau) \right] \\ &\quad \cdot d_\beta \tau + f(s_0)g(t) = -f(s_0)g(t) + f(t)g(s_0) \\ &\quad + \int_{s_0}^t \widehat{f}(t, \beta(\tau)) \left[\frac{\partial_\beta}{\partial_\beta \tau} g(\tau) \right] d_\beta \tau + f(s_0)g(t) \\ &= f(t)g(s_0) + \int_{s_0}^t \widehat{f}(t, \beta(\tau)) \left[\frac{\partial_\beta}{\partial_\beta \tau} g(\tau) \right] d_\beta \tau. \end{aligned} \quad (70)$$

Since $(\partial_\beta/\partial_\beta\tau)g(\tau) = D_\beta g(\tau)$, then,

$$\begin{aligned} D_\beta(f * g)(t) &= f(t)g(s_0) + \int_{s_0}^t \hat{f}(t, \beta(\tau))(D_\beta g(\tau))d_\beta\tau \\ &= f(t)g(s_0) + (f * D_\beta g)(t). \end{aligned} \quad (71)$$

This completes the proof of equation of (65). $\square$

Corollary 14. For all $k \in \mathbb{N}_0$, we have

$$D_\beta^k(f * g) = f * (D_\beta^k g) + \sum_{r=0}^{k-1} (D_\beta^r f) (D_\beta^{k-1-r} g)(s_0). \quad (72)$$

Proof. Using the induction, for $k=1$ equation (72) is equation (65). Suppose that for $k=m \in \mathbb{N}_0$ equation (72) is true, i.e.

$$D_\beta^m(f * g)(t) = (f * D_\beta^m g)(t) + \sum_{r=0}^{m-1} (D_\beta^r f)(t) (D_\beta^{m-1-r} g)(s_0). \quad (73)$$

Then, by using equation (65) and from the linearity of D_β , we get

$$\begin{aligned} D_\beta^{m+1}(f * g)(t) &= D_\beta \left[(f * D_\beta^m g)(t) + \sum_{r=0}^{m-1} (D_\beta^r f)(t) (D_\beta^{m-1-r} g)(s_0) \right] \\ &= f(t) (D_\beta^m g)(s_0) + (f * D_\beta^{m+1} g)(t) + \sum_{r=0}^{m-1} (D_\beta^{r+1} f) \\ &\quad \cdot (t) (D_\beta^{m-1-r} g)(s_0) = (f * D_\beta^{m+1} g)(t) \\ &\quad + \sum_{r=0}^m (D_\beta^r f)(t) (D_\beta^{m-r} g)(s_0). \end{aligned} \quad (74)$$

Therefore, equation (72) is true for $k=m+1$, and thus, it holds for any $k \in \mathbb{N}$. $\square$

Lemma 15. Let Ψ be a function defined by

$$\Psi(\tau) = \int_\tau^\infty \frac{\hat{f}(t, \tau)}{e_{z,\beta}(\beta(t), \tau)} d_\beta t. \quad (75)$$

Then, Ψ is constant, where $e_{z,\beta}(\beta(t), \tau) = e_{z,\beta}(\beta(t))/e_{z,\beta}(\tau)$.

Proof. We will show that $D_\beta \Psi(\tau) \equiv 0$ for all τ , then Ψ is constant.

By using Theorem 1 and Lemma 8, we find

$$\begin{aligned} D_\beta \Psi(\tau) &= \int_\tau^\infty \left[\frac{(\partial_\beta/\partial_\beta t) \hat{f}(t, \tau)}{e_{z,\beta}(\beta(t), \tau) e_{z,\beta}(\beta(t), \beta(\tau))} - (\ominus_\beta z)(\tau) e_{z,\beta}(\beta(t), \tau) \hat{f}(t, \tau) \right. \\ &\quad \cdot d_\beta t - \frac{\hat{f}(\tau, \beta(\tau))}{e_{z,\beta}(\beta(\tau), \beta(\tau))} \cdot d_\beta t \left. \right]. \end{aligned} \quad (76)$$

Since $(\partial_\beta/\partial_\beta t) \hat{f}(t, \tau) = -(\partial_\beta/\partial_\beta t) \hat{f}(t, \beta(\tau))$, $e_{z,\beta}(\beta(\tau), \beta(\tau)) = 1$, we get

$$\begin{aligned} D_\beta \Psi(\tau) &= \int_\tau^\infty \left[\frac{-(\partial_\beta/\partial_\beta t) \hat{f}(t, \beta(\tau))}{e_{z,\beta}(\beta(t), \beta(\tau))} + \frac{[z/(1+z(\beta(\tau)-\tau))] \hat{f}(t, \tau)}{e_{z,\beta}(\beta(t), \beta(\tau))} \right] \\ &\quad \cdot d_\beta t - \hat{f}(\tau, \beta(\tau)) = - \int_\tau^\infty \frac{(\partial_\beta/\partial_\beta t) \hat{f}(t, \beta(\tau))}{e_{z,\beta}(\beta(t), \beta(\tau))} d_\beta t \\ &\quad + z \left[\int_\tau^\infty \frac{\hat{f}(t, \tau)}{e_{z,\beta}(\beta(t), \tau)} d_\beta t \right] - \hat{f}(\tau, \beta(\tau)) = z \Psi(\tau) \\ &\quad - \int_\tau^\infty \frac{(\partial_\beta/\partial_\beta t) \hat{f}(t, \beta(\tau))}{e_{z,\beta}(\beta(t), \beta(\tau))} d_\beta t - \hat{f}(\tau, \beta(\tau)) = z \Psi(\tau) \\ &\quad - \int_\tau^{\beta(\tau)} \frac{(\partial_\beta/\partial_\beta t) \hat{f}(t, \beta(\tau))}{e_{z,\beta}(\beta(t), \beta(\tau))} d_\beta t - \int_{\beta(\tau)}^\infty \frac{(\partial_\beta/\partial_\beta t) \hat{f}(t, \beta(\tau))}{e_{z,\beta}(\beta(t), \beta(\tau))} \\ &\quad \cdot d_\beta t - \hat{f}(\tau, \beta(\tau)). \end{aligned} \quad (77)$$

Also, since $\hat{f}(\beta(\tau), \beta(\tau)) = f(s_0)$ and

$$\begin{aligned} &\left[\frac{\partial_\beta}{\partial_\beta t} \hat{f}(t, \beta(\tau)) \right] e_{\ominus_\beta z, \beta}(\beta(t), \beta(\tau)) \\ &= \frac{\partial_\beta}{\partial_\beta t} \left[\hat{f}(t, \beta(\tau)) e_{\ominus_\beta z, \beta}(t, \beta(\tau)) \right] \\ &\quad - (\ominus_\beta z)(t) e_{\ominus_\beta z, \beta}(t, \beta(\tau)) \hat{f}(t, \beta(\tau)). \end{aligned} \quad (78)$$

Hence, by Lemma 3, we get

$$\int_\tau^{\beta(\tau)} \frac{(\partial_\beta/\partial_\beta t) \hat{f}(t, \beta(\tau))}{e_{z,\beta}(\beta(t), \beta(\tau))} d_\beta t - \hat{f}(\tau, \beta(\tau)) = f(s_0),$$

$$\begin{aligned} &\int_{\beta(\tau)}^\infty \left[\frac{\partial_\beta}{\partial_\beta t} \hat{f}(t, \beta(\tau)) \right] e_{\ominus_\beta z, \beta}(\beta(t), \beta(\tau)) d_\beta t \\ &= \int_{\beta(\tau)}^\infty \frac{\partial_\beta}{\partial_\beta t} \left[\hat{f}(t, \beta(\tau)) e_{\ominus_\beta z, \beta}(t, \beta(\tau)) \right] d_\beta t \\ &\quad - \int_{\beta(\tau)}^\infty (\ominus_\beta z)(t) e_{\ominus_\beta z, \beta}(t, \beta(\tau)) \hat{f}(t, \beta(\tau)) d_\beta t \\ &= -f(s_0) + z \int_{\beta(\tau)}^\infty \hat{f}(t, \beta(\tau)) e_{\ominus_\beta z, \beta}(\beta(t), \beta(\tau)) d_\beta t \\ &= -f(s_0) + z \Psi(\beta(\tau)). \end{aligned} \quad (79)$$

Therefore,

$$\begin{aligned} D_\beta \Psi(\tau) &= z\Psi(\tau) - f(s_0) + f(s_0) - z\Psi(\beta(\tau)) \\ &= -z[\Psi(\beta(\tau)) - \Psi(\tau)] = -z(\beta(\tau) - \tau)D_\beta \Psi(\tau), \end{aligned} \quad (80)$$

and then $[1 + (\beta(\tau) - \tau)z]D_\beta \Psi(\tau) = 0$, i.e., $D_\beta \Psi(\tau) = 0$. $\square$

Now, we prove the β -convolution theorem.

Theorem 16 (β -Convolution theorem). *Let $f, g \in V([s_0, \infty), \mathbb{C})$ be two functions of exponential order and their β -convolution $f * g$ is defined by (35). Then,*

$$\mathcal{L}_\beta\{f * g\} = \mathcal{L}_\beta\{f\} \cdot \mathcal{L}_\beta\{g\}. \quad (81)$$

Proof. We have

$$\begin{aligned} \mathcal{L}_\beta\{f * g\} &= \int_{s_0}^{\infty} \frac{(f * g)(t)}{e_{z,\beta}(\beta(t))} d_\beta t = \int_{s_0}^{\infty} \frac{1}{e_{z,\beta}(\beta(t))} \\ &\quad \cdot \int_{s_0}^t \hat{f}(t, \beta(\tau)) g(\tau) d_\beta \tau d_\beta t. \end{aligned} \quad (82)$$

Then,

$$\begin{aligned} \mathcal{L}_\beta\{f * g\} &= \int_{s_0}^{\infty} \frac{g(\tau)}{e_{z,\beta}(\beta(\tau))} \left[\int_{\beta(\tau)}^{\infty} \frac{\hat{f}(t, \beta(\tau)) e_{z,\beta}(\beta(\tau))}{e_{z,\beta}(\beta(t))} d_\beta t \right] d_\beta \tau \\ &= \int_{s_0}^{\infty} \frac{g(\tau)}{e_{z,\beta}(\beta(\tau))} \Psi(\beta(\tau)) d_\beta \tau \\ &= \int_{s_0}^{\infty} g(\tau) e_{\ominus_{\beta z}, \beta}(\beta(\tau)) \Psi(\beta(\tau)) d_\beta \tau. \end{aligned} \quad (83)$$

Using Lemma 15, equation (12), we get $\Psi(\tau) = \Psi(\beta(\tau)) = \Psi(s_0)$. Moreover,

$$\begin{aligned} \Psi(s_0) &= \int_{s_0}^{\infty} \frac{\hat{f}(t, s_0) e_{z,\beta}(s_0)}{e_{z,\beta}(\beta(t))} d_\beta t \\ &= \int_{s_0}^{\infty} f(t) e_{\ominus_{\beta z}, \beta}(\beta(t)) d_\beta t = \mathcal{L}_\beta\{f\}. \end{aligned} \quad (84)$$

Hence, $\mathcal{L}_\beta\{f * g\} = \mathcal{L}_\beta\{f\} \cdot \mathcal{L}_\beta\{g\}$. $\square$

Corollary 17. *Let $\hat{f}$ be the shift of f , $\tau \in I$ with $\tau \geq s_0$. Define the function χ_τ by*

$$\chi_\tau(t) = \begin{cases} 0, & t < \tau, \\ 1, & t \geq \tau. \end{cases} \quad (85)$$

Then

$$\mathcal{L}_\beta\{\chi_\tau \hat{f}(\cdot, \tau)\} = e_{\ominus_{\beta z}, \beta}(\tau) \mathcal{L}_\beta\{f\}. \quad (86)$$

Proof.

$$\begin{aligned} \mathcal{L}_\beta\{\chi_\tau \hat{f}(\cdot, \tau)\} &= \int_{s_0}^{\infty} \chi_\tau(t) \hat{f}(t, \tau) e_{\ominus_{\beta z}, \beta}(\beta(t)) d_\beta t \\ &= \int_{\tau}^{\infty} \hat{f}(t, \tau) e_{\ominus_{\beta z}, \beta}(\beta(t)) d_\beta t \\ &= e_{\ominus_{\beta z}, \beta}(\tau) \int_{\tau}^{\infty} \hat{f}(t, \tau) e_{\ominus_{\beta z}, \beta}(\beta(t), \tau) d_\beta t \\ &= e_{\ominus_{\beta z}, \beta}(\tau) \Psi(\tau), \end{aligned} \quad (87)$$

where Ψ is defined in equation (75). Then, by Lemma 15, $\Psi(\tau) \equiv \Psi(s_0) = \mathcal{L}_\beta\{f\}$. $\square$

3.1. Applications on the β -Convolution Theorem. Here, we apply the β -convolution theorem to find the following:

- (1) $e_{\lambda, \beta}(t) * e_{\mu, \beta}(t)$,
- (2) $t * e_{\lambda, \beta}(t)$,
- (3) $\sin_{\lambda, \beta}(t) * \sin_{\lambda, \beta}(t)$.

- (1) We have by the β -convolution Theorem 16 and [24] (Theorem 3.10) that

$$\begin{aligned} \mathcal{L}_\beta\{e_{\lambda, \beta}(t) * e_{\mu, \beta}(t)\} &= \frac{1}{(z - \lambda)(z - \mu)} = \frac{1}{(\lambda - \mu)(z - \lambda)} \\ &\quad - \frac{1}{(\lambda - \mu)(z - \mu)}. \end{aligned} \quad (88)$$

Then, by [24] (Corollary 3.22), we get

$$\begin{aligned} e_{\lambda, \beta}(t) * e_{\mu, \beta}(t) &= \frac{1}{(\lambda - \mu)} \mathcal{L}_\beta^{-1}\left\{\frac{1}{(z - \lambda)}\right\} - \frac{1}{(\lambda - \mu)} \\ &\quad \cdot \mathcal{L}_\beta^{-1}\left\{\frac{1}{(z - \mu)}\right\} = \frac{1}{(\lambda - \mu)} [e_{\lambda, \beta}(t) - e_{\mu, \beta}(t)]. \end{aligned} \quad (89)$$

- (2) We have $\mathcal{L}_\beta\{t\} = 1/z^2$ by [24] (Example 3.23), then

$$\mathcal{L}_\beta\{t * e_{\lambda, \beta}(t)\} = \mathcal{L}_\beta\{t\} \cdot \mathcal{L}_\beta\{e_{\lambda, \beta}(t)\} = \frac{1}{z^2(z - \lambda)}. \quad (90)$$

Therefore,

$$\begin{aligned} t * e_{\lambda, \beta}(t) &= \mathcal{L}_\beta^{-1} \left\{ \frac{1}{z^2(z-\lambda)} \right\} = -\frac{1}{\lambda^2} \mathcal{L}_\beta^{-1} \left\{ \frac{1}{z} \right\} - \frac{1}{\lambda} \mathcal{L}_\beta^{-1} \left\{ \frac{1}{z^2} \right\} \\ &+ \frac{1}{\lambda^2} \mathcal{L}_\beta^{-1} \left\{ \frac{1}{z-\lambda} \right\} = -\frac{1}{\lambda^2} - \frac{1}{\lambda} t + \frac{1}{\lambda^2} e_{\lambda, \beta}(t). \end{aligned} \quad (91)$$

(3) Since

$$\mathcal{L}_\beta \{ \sin_{\lambda, \beta}(t) * \sin_{\lambda, \beta}(t) \} = \mathcal{L}_\beta \{ \sin_{\lambda, \beta}(t) \} \cdot \mathcal{L}_\beta \{ \sin_{\lambda, \beta}(t) \}. \quad (92)$$

From [24] (Example 3.13), we have

$$\mathcal{L}_\beta \{ \sin_{\lambda, \beta}(t) * \sin_{\lambda, \beta}(t) \} = \frac{\lambda^2}{(z^2 + \lambda^2)^2}. \quad (93)$$

By [24] (Theorem 3.21),

$$\begin{aligned} \mathcal{L}_\beta^{-1} \left\{ \frac{1}{(z^2 + \lambda^2)^2} \right\} &= \frac{\sin_{\lambda, \beta}(t)}{2\lambda^3} - \frac{\cos_{\lambda, \beta}(t)}{2\lambda^2} \int_{s_0}^t \frac{1}{1 + \lambda^2(\beta(\tau) - \tau)^2} \\ &\cdot d_\beta \tau - \frac{\sin_{\lambda, \beta}(t)}{2\lambda} \int_{s_0}^t \frac{(\beta(\tau) - \tau)}{1 + \lambda^2(\beta(\tau) - \tau)^2} d_\beta \tau. \end{aligned} \quad (94)$$

Hence,

$$\begin{aligned} \sin_{\lambda, \beta}(t) * \sin_{\lambda, \beta}(t) &= \mathcal{L}_\beta^{-1} \left\{ \frac{\lambda^2}{(z^2 + \lambda^2)^2} \right\} = \lambda^2 \mathcal{L}_\beta^{-1} \left\{ \frac{1}{(z^2 + \lambda^2)^2} \right\} \\ &= \frac{\sin_{\lambda, \beta}(t)}{2\lambda} - \frac{\cos_{\lambda, \beta}(t)}{2} \int_{s_0}^t \frac{1}{1 + \lambda^2(\beta(\tau) - \tau)^2} d_\beta \tau \\ &- \frac{\lambda \sin_{\lambda, \beta}(t)}{2} \int_{s_0}^t \frac{(\beta(\tau) - \tau)}{1 + \lambda^2(\beta(\tau) - \tau)^2} d_\beta \tau. \end{aligned} \quad (95)$$

4. Conclusion

In this paper, some properties of the β -partial derivative corresponding to $D_\beta f(t) = (f(\beta(t)) - f(t))/(\beta(t) - t)$, $\beta(t) \neq t$, were introduced. The associated β -convolution of two functions was presented, and also the shift, associativity, and β -differentiability of the β -convolution were studied. Moreover, the β -convolution theorem and some applications were introduced.

Data Availability

All the data are included in the article.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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