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Research Article

Some Midpoint Inequalities for η-Convex Function via Weighted Fractional Integrals

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In this research, by using a weighted fractional integral, we establish a midpoint version of Hermite-Hadamrad Fejér type inequality for η-convex function on a specific interval. To confirm the validity, we considered some special cases of our results and relate them with already existing results. It can be observed that several existing results are special cases of our presented results.

1. Introduction

In the last few decades, the classical convexity has a rapid development in fractional calculus [1]. We can say that convexity plays a vital role in fractional integral inequalities because of its geometric features [2–4].

Take a function $f:I \longrightarrow \mathbb{R}$ be a continuous function. Then, this function is called convex if

$$f(tm + (1-t)n) \le tf(m) + (1-t)f(n), \tag{1}$$

$\forall m, n \in I$, and $t \in [0, 1]$.

There are many integral inequalities in the literature and one of the most common inequality is Hermite-Hadamarad or, shortly, the HH integral inequality, which is introduced by [5]:

$$f(m+n/2) \le 1/n-m\int_m^n f(x)dx \le f(m)+f(n)/2, m < n \in I. \tag{2}$$

In the literature, we can notice that Hermite-Hadamarad inequality (2) has been applied to distinct convexities like exponential convexity [6, 7], s-convexity [8], quasiconvexity [9, 10], GA-convexity [11], (α, m)-convexity [12], MT-convexity [13], and also, other types of convexity (see [14, 15]). Different forms of fractional integrals like Riemann-Liouville (RL), Caputo Fabrizio, Hadamrad, Riesz, Prabhakar, Ψ-RL, and weighted integrals [16–20] have been established. A lot of integer-order integral inequalities like Simpson [21], Ostrowski [22], Rozanova [23], Gagliardo-Nirenberg [24], Olsen [25], Hardy [26], Opial [27, 28], and Akdemir et al. [29, 30] have been developed and generalized from fractional point of view.

Definition 1. Let $I \subset \mathbb{R}$ be an interval and $f : I \longrightarrow \mathbb{R}$ be a continuous function. Then, the function f is called η-convex if

$$f(tm + (1-t)n) \le f(n) + t\eta(f(m), f(n)). \tag{3}$$

Definition 2. [18] Let f is positive convex function, continuous on closed interval $[m, n]$ and $x \in [m, n]$ when $f(x) \in L^1$

$[m, n]$ with $m < n$, where left- and right-side RL fractional integrals are defined by

$$\begin{aligned} {}^{RL}I^{\nu}_{m+}f(x) &= \frac{1}{\Gamma(\nu)}\int_m^x (x-t)^{\nu-1}f(t)dt, \\ {}^{RL}I^{\nu}_{n-}f(x) &= \frac{1}{\Gamma(\nu)}\int_x^n (t-x)^{\nu-1}f(t)dt, \end{aligned} \tag{4}$$

where Γ is famous Gamma function and for any positive integer $n, \Gamma(n) = (n-1)!$.

Definition 3 (see [19]). Let $[m, n] \subseteq \mathbb{R}$, $f : [m, n] \longrightarrow \mathbb{R}$ and $\phi : (m, n] \longrightarrow \mathbb{R}$ be monotonically increasing positive function with a continuous derivative $\phi'(x)$ on (m, n). Then, the left-sided and the right-sided weighted fractional integrals of f according to ϕ on $[m, n]$ are defined by:

$$\begin{aligned} \left({}_{m+}I^{\nu,\phi}_{g}f\right)(x) &= \frac{[g(x)]^{-1}}{\Gamma(\nu)}\int_m^x \phi'(t)(\phi(x)-\phi(t))^{\nu-1}f(t)g(t)dt, \\ \left({}_{n+}I^{\nu,\phi}_{g}f\right)(x) &= \frac{[g(x)]^{-1}}{\Gamma(\nu)}\int_x^n \phi'(t)(\phi(x)-\phi(t))^{\nu-1}f(t)g(t)dt, \nu > 0. \end{aligned} \tag{5}$$

In this research, we denote $[g(x)]^{-1} = 1/g(x)$ and the inverse of function $\phi(x)$ by $\phi^{-1}(x)$.

Remark 4. From Definition 3, we can see some special cases:

(i) If $\phi(x) = x$ and $g(x) = 1$, then weighted fractional integrals [14] deduce to the classical RL fractional integrals [9].

(ii) If $g(x)$=1, we get fractional integrals of function f with respect to function $\phi(x)$, which is defined by [16, 17]:

$$\begin{aligned} \left({}_{m+}I^{\nu:\phi}\right)f(x) &= \frac{1}{\Gamma(\nu)}\int_m^x \phi'(t)(\phi(x)-\phi(t))^{\nu-1}f(t)dt, \\ \left({}_{n+}I^{\nu:\phi}\right)f(x) &= \frac{1}{\Gamma(\nu)}\int_x^n \phi'(t)(\phi(x)-\phi(t))^{\nu-1}f(t)dt, \nu > 0. \end{aligned} \tag{6}$$

Lemma 5. *[31] Assume that $g : [m, n] \longrightarrow (0,\infty)$ is integrable function and symmetric with respect to $(m+n)/2, m < n$. Then,*

(i) For each $t \in [0, 1]$, we have

$$g\left(\frac{t}{2}m + \frac{2-t}{2}n\right) = g\left(\frac{2-t}{2}m + \frac{t}{2}n\right). \tag{7}$$

(ii) For $\nu > 0$, we have

$$\begin{aligned} &\left({}_{\phi^{-1}(m+n)/2+}I^{\nu:\phi}(go\phi)\right)\left(\phi^{-1}(n)\right) \\ &= \left(I^{\nu:\phi}_{\phi^{-1}((m+n)/2)-(go\phi)}\right)\left(\phi^{-1}(m)\right) \\ &= \frac{1}{2}\left[\left({}_{\phi^{-1}(m+n)/2+}I^{\nu:\phi}(go\phi)\right)\left(\phi^{-1}(n)\right)\right. \\ &= \left.\left(I^{\nu:\phi}_{\phi^{-1}((m+n)/2)-(go\phi)}\right)\left(\phi^{-1}(m)\right)\right]. \end{aligned} \tag{8}$$

2. Main Results

Theorem 6. *Let $0 \le m < n$ and $f : [m, n] \longrightarrow \mathbb{R}$ be an $L^1\eta$-convex function and $g : [m, n] \longrightarrow \mathbb{R}$ be an integrable, positive and weighted symmetric function with respect to $(m+n)/2$. If, in addition, ϕ is an increasing and positive function from $[m, n]$ onto itself such that its derivative $\phi'(x)$ is continues on (m, n), then for $\nu > 0$, the following inequalities are valid:*

$$\begin{aligned} &f\left(\frac{m+n}{2}\right) \times \left[\left({}_{\phi_{\phi-1((m+n)/2)+}}I^{\nu:\phi}(go\phi)\right)\left(\phi^{-1}(n)\right)\right. \\ &\quad \left.+ \left(I^{\nu:\phi}_{\phi_{\phi^{-1}((m+n)/2)-}}(go\phi)\right)\left(\phi^{-1}(m)\right)\right] \\ &\le \left[g(n)\left({}_{\phi^{-1}(m+n/2)+}I^{\nu:\phi}_{go\phi}(fo\phi)\right)\phi^{-1}(n)\right) \\ &\quad + \frac{\eta}{2}g(m)\left(f\left(\frac{t}{2}m + \frac{2-t}{2}n\right), f\left(\frac{2-t}{2}m + \frac{t}{2}n\right)\right) \\ &\quad \left.\cdot\left(\phi^{-1}(m)\right)\right]. \\ &\le \left[g(n)\left({}_{\phi^{-1}(m+n/2)+}I^{\nu:\phi}_{go\phi}(fo\phi)\right)\left(\phi^{-1}(n)\right) + \frac{\eta}{2}g(m)\right. \\ &\quad \left.\cdot\left(f\left(\frac{t}{2}m + \frac{2-t}{2}n\right), f\left(\frac{2-t}{2}m + \frac{t}{2}n\right)\right)\left(\phi^{-1}(m)\right)\right]. \end{aligned} \tag{9}$$

Proof. The η-convexity of f on $[m, n]$, for all $x, y \in [m, n]$ gives

$$f\left(\frac{x+y}{2}\right) \le \frac{f(y) + \eta(f(x), f(y))}{2}, \tag{10}$$

setting $x = (t/2)m + ((2-t)/2)n$ and $y = ((2-t)/2)m + (t/2)n$

$$\begin{aligned} 2f\left(\frac{m+n}{2}\right) &\le f\left(\frac{2-t}{2}m + \frac{t}{2}n\right) \\ &\quad + \eta\left(f\left(\frac{t}{2}m + \frac{2-t}{2}n\right), f\left(\frac{2-t}{2}m + \frac{t}{2}n\right)\right). \end{aligned} \tag{11}$$

Multiplying both sides of inequality (11) by $t^{\nu-1}g((t/2)m+((2-t)/2)n)$ and integrating over $[0,1]$, we get

$$\begin{aligned} 2f\left(\frac{m+n}{2}\right)\int_0^1 t^{\nu-1}g\left(\frac{t}{2}m+\frac{2-t}{2}n\right)dt &\le \int_0^1 t^{\nu-1}g\left(\frac{t}{2}m+\frac{2-t}{2}n\right)f\left(\frac{2-t}{2}m+\frac{t}{2}n\right)dt \\ &+\int_0^1 \eta t^{\nu-1}g\left(\frac{t}{2}m+\frac{2-t}{2}n\right) \\ &\cdot\left(f\left(\frac{t}{2}m+\frac{2-t}{2}n\right),f\left(\frac{2-t}{2}m+\frac{t}{2}n\right)\right). \end{aligned} \tag{12}$$

From the left side of inequality (12), we use

$$\begin{aligned} &\frac{2^{\nu-1}\Gamma(\nu)}{(n-m)^{\nu}}\left[\left({}_{\phi^{-1}(m+n/2)+}I^{\nu:\phi}(go\phi)\right)\left(\phi^{-1}(n)\right)+\left(I^{\nu:\phi}_{\phi_{\phi^{-1}((m+n)/2)-}}(go\phi)\right)\left(\phi^{-1}(m)\right)\right] \\ &=\frac{2^{\nu}\Gamma(\nu)}{(n-m)^{\nu}}\left({}_{\phi_{\phi-1((m+n)/2)+}}I^{\nu:\phi}(go\phi)\right)\left(\phi^{-1}(n)\right) \\ &=\frac{2^{\nu}}{(n-m)^{\nu}}\int_{\phi^{-1}((m+n)/2)}^{\phi^{-1}(n)}(n-\phi(x))^{\nu-1}(go\phi)(x)\phi'(x)dx \\ &=\int_{\phi^{-1}((m+n)/2)}^{\phi^{-1}(n)}\left(\frac{2(n-\phi(x))}{n-m}\right)^{\nu-1}(go\phi)(x)\phi'(x)\frac{2dx}{n-m} \\ &=\int_0^1 t^{\nu-1}g\left(\frac{t}{2}m+\frac{2-t}{2}n\right)dt, \end{aligned} \tag{13}$$

where $t=2(n-\phi(x))/(n-m)$. It follows that

$$\begin{aligned} 2f\left(\frac{m+n}{2}\right)\int_0^1 t^{\nu-1}g\left(\frac{t}{2}m+\frac{2-t}{2}n\right)dt &=\frac{2^{\nu}\Gamma(\nu)}{(n-m)^{\nu}}f\left(\frac{m+n}{2}\right)\times\left[\left({}_{\phi_{\phi-1((m+n)/2)+}}I^{\nu:\phi}(go\phi)\right)\left(\phi^{-1}(n)\right)\right. \\ &\left.+\left(I^{\nu:\phi}_{\phi_{\phi^{-1}((m+n)/2)-}}(go\phi)\right)\left(\phi^{-1}(m)\right)\right]. \end{aligned} \tag{14}$$

By evaluating the weighted fractional operators, we see that

$$\begin{aligned} &g(n)\left({}_{\phi_{\phi-1((m+n)/2)+}}I^{\nu:\phi}_{go\phi}(fo\phi)\right)\left(\phi^{-1}(n)\right)+g(m)\left({}_{go\phi}I^{\nu:\phi}_{\phi_{\phi^{-1}((m+n)/2)-}}(fo\phi)\right)\left(\phi^{-1}(m)\right) \\ &=g(n)\frac{(go\phi)^{-1}\left(\phi^{-1}(n)\right)}{\Gamma(\nu)}\int_{\phi^{-1}((m+n)/2)}^{\phi^{-1}(n)}(n-\phi(x))^{\nu-1}\cdot(fo\phi)(x)(go\phi)(x)\phi'(x)dx \\ &+g(m)\frac{(go\phi)^{-1}\left(\phi^{-1}(n)\right)}{\Gamma(\nu)}\int_{\phi^{-1}(m)}^{\phi^{-1}((m+n)/2)}\cdot(\phi(x)-m)^{\nu-1}(fo\phi)(x)(go\phi)(x)\phi'(x)dx \\ &=\frac{(n-m)^{\nu}}{2^{\nu}\Gamma(\nu)}\int_{\phi^{-1}((m+n)/2)}^{\phi^{-1}(n)}\left(\frac{2(n-\phi(x))}{n-m}\right)^{\nu-1}\cdot(fo\phi)(x)(go\phi)(x)\phi'(x)\frac{2dx}{n-m} \\ &+\frac{(n-m)^{\nu}}{2^{\nu}\Gamma(\nu)}\int_{\phi^{-1}(m)}^{\phi^{-1}((m+n)/2)}\left(\frac{2(\phi(x)-m)}{n-m}\right)^{\nu-1}\cdot(fo\phi)(x)(go\phi)(x)\phi'(x)\frac{2dx}{n-m}, \end{aligned} \tag{15}$$

where

$$\left[(go\phi)\left(\phi^{-1}(y)\right)\right]^{-1}=\frac{1}{(go\phi)\left(\phi^{-1}(y)\right)}=\frac{1}{g(y)}, \tag{16}$$

for $y=m,n$.

Setting $u_1=2(n-\phi(x))/(n-m)$ and $u_2=2(\phi(x)-m)/(n-m)$, one can deduce that

$$\begin{aligned} &g(n)\left({}_{\phi^{-1}(m+n/2)+}I^{\nu:\phi}_{go\phi}(fo\phi)\right)\left(\phi^{-1}(n)\right)+g(m)\left({}_{go\phi}I^{\nu:\phi}_{\phi^{-1}((m+n)/2)-}(fo\phi)\right)\left(\phi^{-1}(m)\right) \\ &=\frac{(n-m)^{\nu}}{2^{\nu}\Gamma(\nu)}\left[\int_0^1 u_1^{\nu-1}f\left(\frac{u_1}{2}m+\frac{2-u_1}{2}n\right)\cdot g\left(\frac{u_1}{2}m+\frac{2-u_1}{2}n\right)du_1\right. \\ &\left.+\int_0^1 u_2^{\nu-1}f\left(\frac{2-u_2}{2}m+\frac{u_2}{2}n\right)g\left(\frac{2-u_2}{2}m+\frac{u_2}{2}n\right)du_2\right] \\ &=\frac{(n-m)^{\nu}}{2^{\nu}\Gamma(\nu)}\left[\int_0^1 t^{\nu-1}f\left(\frac{t}{2}m+\frac{2-t}{2}n\right)g\left(\frac{t}{2}m+\frac{2-t}{2}n\right)dt\right. \\ &\left.+\int_0^1 t^{\nu-1}f\left(\frac{2-t}{2}m+\frac{t}{2}n\right)g\left(\frac{t}{2}m+\frac{2-t}{2}n\right)dt\right], \end{aligned} \tag{17}$$

$$\begin{aligned} &\int_0^1 t^{\nu-1}f\left(\frac{t}{2}m+\frac{2-t}{2}n\right)\Bigg)g\left(\frac{t}{2}m+\frac{2-t}{2}n\right)dt \\ &+\int_0^1 \eta t^{\nu-1}g\left(\frac{t}{2}m+\frac{2-t}{2}n\right)\cdot\left(f\left(\frac{t}{2}m+\frac{2-t}{2}n\right),f\left(\frac{2-t}{2}m+\frac{t}{2}n\right)\right) \\ &=\frac{2^{\nu}\Gamma(\nu)}{(n-m)^{\nu}}\left[g(n)\left({}_{\phi^{-1}(m+n/2)+}I^{\nu:\phi}_{go\phi}(fo\phi)\right)\left(\phi^{-1}(n)\right)\right) \\ &+\eta g(m)\left(f\left(\frac{t}{2}m+\frac{2-t}{2}n\right),f\left(\frac{2-t}{2}m+\frac{t}{2}n\right)\right) \\ &\left.\cdot\left(\phi^{-1}(m)\right)\right]. \end{aligned} \tag{18}$$

By using (14) and (18) in (12), we get

$$f\left(\frac{m+n}{2}\right)\times\left[\left({}_{\phi_{\phi-1((m+n)/2)+}}I^{\nu:\phi}(go\phi)\right)\left(\phi^{-1}(n)\right)+\left(I^{\nu:\phi}_{\phi_{\phi^{-1}((m+n)/2)-}}(go\phi)\right)\left(\phi^{-1}(m)\right)\right]\leq\left[g(n)\left({}_{\phi^{-1}(m+n/2)+}I^{\nu:\phi}_{go\phi}(fo\phi)\right)\left(\phi^{-1}(n)\right)\right)+\eta t g(m)\left(f\left(\frac{t}{2}m+\frac{2-t}{2}n\right),f\left(\frac{2-t}{2}m+\frac{t}{2}n\right)\right)\cdot\left(\phi^{-1}(m)\right)\Big]. \tag{19}$$

The left side of Theorem 6 is completed.

Now, we will prove right side of inequality (9) by using η-convexity.

$$f\left(\frac{2-t}{2}m+\frac{t}{2}n\right)+t\eta\left(f\left(\frac{t}{2}m+\frac{2-t}{2}n\right),f\left(\frac{2-t}{2}m+\frac{t}{2}n\right)\right)\leq f(n)+\frac{2-t}{2}\eta(f(m),f(n))+\eta\left(f\left(\frac{t}{2}m+\frac{2-t}{2}n\right),f\left(\frac{2-t}{2}m+\frac{t}{2}n\right)\right). \tag{20}$$

Multiply Equation (20) by $t^{\nu-1}g((t/2)m+((2-t)/2)n)$ and integrate over $[0,1]$ leads us to

$$\left[\int_0^1 t^{\nu-1}f\left(\frac{t}{2}m+\frac{2-t}{2}n\right)g\left(\frac{t}{2}m+\frac{2-t}{2}n\right)dt+\int_0^1 t^{\nu-1}f\left(\frac{2-t}{2}m+\frac{t}{2}n\right)g\left(\frac{t}{2}m+\frac{2-t}{2}n\right)dt\right].\leq\left[f(n)+\frac{2-t}{2}\eta(f(m),f(n))+\eta\left(f\left(\frac{t}{2}m+\frac{2-t}{2}n\right),f\left(\frac{2-t}{2}m+\frac{t}{2}n\right)\right).\right]\times\int_0^1 t^{\nu-1}g\left(\frac{t}{2}m+\frac{2-t}{2}n\right)dt. \tag{21}$$

By using (7) and (14) in (21), we get

$$g(n)\left({}_{\phi^{-1}(m+n/2)+}I^{\nu:\phi}_{go\phi}(fo\phi)\right)\left(\phi^{-1}(n)\right)\Big)+\eta t g(m)\left(f\left(\frac{t}{2}m+\frac{2-t}{2}n\right),f\left(\frac{2-t}{2}m+\frac{t}{2}n\right)\right)\cdot\left(\phi^{-1}(m)\right)\leq\left[g(n)\left({}_{\phi^{-1}(m+n/2)+}I^{\nu:\phi}_{go\phi}(fo\phi)\right)\left(\phi^{-1}(n)\right)\right)+\frac{\eta}{2}g(m)\left(f\left(\frac{t}{2}m+\frac{2-t}{2}n\right),f\left(\frac{2-t}{2}m+\frac{t}{2}n\right)\right)\cdot\left(\phi^{-1}(m)\right)\Big].$$

$$\leq\left[g(n)\left({}_{\phi^{-1}(m+n/2)+}I^{\nu:\phi}_{go\phi}(fo\phi)\right)\left(\phi^{-1}(n)\right)+\frac{\eta}{2}g(m)\left(f\left(\frac{t}{2}m+\frac{2-t}{2}n\right),f\left(\frac{2-t}{2}m+\frac{t}{2}n\right)\right)\cdot\left(\phi^{-1}(m)\right)\right]. \tag{22}$$

This completes our proof. □

Remark 7. From Theorem 6, we can get following special case:

If $\phi(x)=x$, then inequality (9) becomes

$$f\left(\frac{m+n}{2}\right)\left[{}^{RL}_{(m+n/2)+}I^{\nu}g(n)+{}^{RL}I^{\nu}_{((m+n)/2)-}g(m)\right]\leq g(n)\left({}^{RL}_{(m+n/2)+}I^{\nu}_{g}f\right)(n)+g(m)\frac{\eta}{2}\left({}^{RL}_{g}I^{\nu}_{((m+n)/2)-}f\right)(m)+\frac{\eta}{2}\left(f\left(\frac{t}{2}m+\frac{2-t}{2}n\right),f\left(\frac{2-t}{2}m+\frac{t}{2}n\right)\right)\leq g(n)\left({}^{RL}_{(m+n/2)+}I^{\nu}_{g}f\right)(n)+g(m)\frac{\eta}{2}\left({}^{RL}_{g}I^{\nu}_{((m+n)/2)-}f\right)(m)+\frac{\eta}{2}\left(f\left(\frac{t}{2}m+\frac{2-t}{2}n\right),f\left(\frac{2-t}{2}m+\frac{t}{2}n\right)\right). \tag{23}$$

Lemma 8. *[31] Let $0\leq m<n$ and $f:[m,n]\longrightarrow\mathbb{R}$ be a continuous with a derivative $f'\in L^1[m,n]$ such that $f(x)=f(m)+\int_m^x f'(t)dt$ and let $g:[m,n]\longrightarrow\mathbb{R}$ be an integrable, positive, and weighted symmetric function with respect to $(m+n)/2$. If ϕ is a continuous increasing mapping form the interval $[m,n]$ onto itself with a derivative $\phi'(x)$ which is continuous on (m,n), then for $\nu>0$, the following equality is valid:*

$$f\left(\frac{m+n}{2}\right)\left[\left({}_{\phi^{-1}(m+n/2)+}I^{\nu:\phi}(go\phi)\right)\left(\phi^{-1}(n)\right)+\left(I^{\nu:\phi}_{\phi^{-1}((m+n)/2)-}(go\phi)\right)\left(\phi^{-1}(m)\right)\right]-\left[g(n)\left({}_{\phi^{-1}(m+n/2)+}I^{\nu:\phi}_{go\phi}(fo\phi)\right)\left(\phi^{-1}(n)\right)+g(m)\left({}_{go\phi}I^{\nu:\phi}_{\phi^{-1}((m+n)/2)-}(fo\phi)\right)\left(\phi^{-1}(m)\right)\right]=\frac{1}{\Gamma(\nu)}\int_{\phi^{-1}(m)}^{\phi^{-1}((m+n)/2)}\cdot\left[\int_{\phi^{-1}(m)}^{t}\phi'(x)(\phi(x)-m)^{\nu-1}(go\phi)(x)dx\right]\cdot\left(f'o\phi\right)(t)\phi'(t)dt-\frac{1}{\Gamma(\nu)}\int_{\phi^{-1}((m+n)/2)}^{\phi^{-1}(n)}\cdot\left[\int_{t}^{\phi^{-1}(n)}\phi'(x)(n-\phi(x))^{\nu-1}(go\phi)(x)dx\right]\cdot\left(f'o\phi\right)(t)\phi'(t)dt. \tag{24}$$

Remark 9. From Lemma 8, we obtain the following special case:

If $\phi(x)=x$, then equality (24) becomes

$$f\left(\frac{m+n}{2}\right)\left[{}^{RL}_{(m+n)/2+}I^{\nu}g(n)+{}^{Rl}I^{\nu}_{(m+n)/2-}g(m)\right]-\left[g(n)\left({}^{RL}_{(m+n)/2+}I^{\nu}_{g}f\right)(n)+g(m)\left({}^{RL}_{g}I^{\nu}_{(m+n)/2-}f\right)(m)\right]=\frac{1}{\Gamma(\nu)}\int_{m}^{(m+n)/2}\left[\int_{m}^{t}(x-m)^{\nu-1}g(x)dx\right]f'(t)dt-\frac{1}{\Gamma(\nu)}\int_{(m+n)/2}^{n}\left[\int_{t}^{n}(n-x)^{\nu-1}g(x)dx\right]f'(t)dt. \tag{25}$$

Theorem 10. *Let $0\le m<n$ and $f:[m,n]\subseteq[0,\infty)\longrightarrow\mathbb{R}$ be a differentiable function on the interval $[m,n]$ such that $f(x)=f(m)+\int_m^x f'(t)dt$ and let $g:[m,n]\longrightarrow\mathbb{R}$ be an integrable, positive, and weighted symmetric function with respect to $(m+n)/2$. If, in addition, $|f'|$ is convex on $[m,n]$, and ϕ is an increasing and positive function from $[m,n)$ onto itself such that its derivative $\phi'(x)$ is continuous on (m,n), then for $\nu>0$, the following inequalities hold:*

$$\begin{aligned}|\sigma_1+\sigma_2|&=\left|\frac{1}{\Gamma(\nu)}\int_{\phi^{-1}(m)}^{\phi^{-1}((m+n)/2)}\cdot\left[\int_{\phi^{-1}(m)}^{t}\phi'(x)(\phi(x)-m)^{\nu-1}(go\phi)(x)dx\right]\cdot\left(f'o\phi\right)(t)\phi'(t)dt-\frac{1}{\Gamma(\nu)}\int_{\phi^{-1}((m+n)/2)}^{\phi^{-1}(n)}\cdot\left[\int_{t}^{\phi^{-1}(n)}\phi'(x)(n-\phi(x))^{\nu-1}(go\phi)(x)dx\right]\cdot\left(f'o\phi\right)(t)\phi'(t)dt\right|\\
&\le\frac{(n-m)^{\nu+1}}{\Gamma(\nu)2^{\nu+1}(\nu+1)}\cdot\left[\|g\|_{[m,(m+n)/2],\infty}f'(n)+\|g\|_{[m,(m+n)/2],\infty}f'(n)\right]+\frac{(n-m)^{\nu+1}}{\Gamma(\nu)2^{\nu+2}(\nu+2)}\|g\|_{[m,(m+n)/2],\infty}\eta\cdot\left(|f'(m)|,|f'(n)|\right)\\
&\quad+\frac{(n-m)^{\nu+1}}{\Gamma(\nu)2^{\nu+2}(\nu+1)(\nu+2)}\|g\|_{[m,(m+n)/2],\infty}\eta\cdot\left(|f'(m)|,|f'(n)|\right)\\
&\le 2\frac{(n-m)^{\nu+1}\|g\|_{[m,n],\infty}}{\Gamma(\nu)2^{\nu+1}(\nu+1)}f'(n)+\left[\frac{(n-m)^{\nu+1}\|g\|_{[m,n],\infty}}{\Gamma(\nu)2^{\nu+2}(\nu+1)(\nu+2)}+\frac{(n-m)^{\nu+1}}{\Gamma(\nu)2^{\nu+2}(\nu+2)}\right]\cdot\|g\|_{[m,n],\infty}\eta\left(|f'(m)|,|f'(n)|\right).\end{aligned} \tag{26}$$

Proof. By using Lemma 8 and properties of modulus, we get

$$\begin{aligned}|\sigma_1+\sigma_2|&=\left|\frac{1}{\Gamma(\nu)}\int_{\phi^{-1}(m)}^{\phi^{-1}((m+n)/2)}\cdot\left[\int_{\phi^{-1}(m)}^{t}\phi'(x)(\phi(x)-m)^{\nu-1}(go\phi)(x)dx\right]\cdot\left(f'o\phi\right)(t)\phi'(t)dt+\frac{1}{\Gamma(\nu)}\int_{\phi^{-1}((m+n)/2)}^{\phi^{-1}(n)}\cdot\left[\int_{t}^{\phi^{-1}(n)}\phi'(x)(n-\phi(x))^{\nu-1}(go\phi)(x)dx\right]\cdot\left(f'o\phi\right)(t)\phi'(t)dt\right|\\
&\le\frac{1}{\Gamma(\nu)}\int_{\phi^{-1}(m)}^{\phi^{-1}((m+n)/2)}\cdot\left|\int_{\phi^{-1}(m)}^{t}\phi'(x)(\phi(x)-m)^{\nu-1}(go\phi)(x)\mathrm{d}x\right|\cdot\left|\left(f'o\phi\right)(t)\right|\phi'(t)dt+\frac{1}{\Gamma(\nu)}\int_{\phi^{-1}((m+n)/2)}^{\phi^{-1}(n)}\cdot\left|\int_{t}^{\phi^{-1}(n)}\phi'(x)(n-\phi(x))^{\nu-1}(go\phi)(x)dx\right|\cdot\left|\left(f'o\phi\right)(t)\right|\phi'(t)dt.\end{aligned} \tag{27}$$

Since $|f'|$ is η-convex on $[m,n]$ for $t\in[\phi^{-1}(m),\phi^{-1}(n)]$, so

$$\left|\left(f'o\phi\right)(t)\right|=\left|f'\left(\frac{n-\phi(t)}{n-m}m+\frac{\phi(t)-m}{n-m}n\right)\right|\le f'(n)+\frac{n-\phi(t)}{n-m}\eta\left(f'(m),f'(n)\right). \tag{28}$$

So, using (28), we obtain

$$\begin{aligned}|\sigma_1+\sigma_1|&\le\frac{\|g\|_{[m,(m+n)/2],\infty}}{\Gamma(\nu)}\int_{\phi^{-1}(m)}^{\phi^{-1}((m+n)/2)}\cdot\left|\int_{\phi^{-1}(m)}^{t}\phi'(x)(\phi(x)-m)^{\nu-1}dx\right|\times\left[|f'(n)|+\frac{n-\phi(t)}{n-m}\eta\left(|f'(m)|,|f'(n)|\right)\right]\\
&\quad\cdot\phi'(t)dt+\frac{\|g\|_{[(m+n)/2,n],\infty}}{\Gamma(\nu)}\int_{\phi^{-1}((m+n)/2)}^{\phi^{-1}(n)}\cdot\left|\int_{t}^{\phi^{-1}(n)}\phi'(x)(n-\phi(x))^{\nu-1}dx\right|\times\left[|f'(n)|+\frac{n-\phi(t)}{n-m}\eta\left(|f'(m)|,|f'(n)|\right)\right]\end{aligned}$$

$$
\begin{aligned}
&\cdot \phi'(t)dt \\
&\leq \frac{(n-m)^{\nu+1}}{\Gamma(\nu+1)2^{\nu+1}} \\
&\quad \cdot \left[\|g\|_{[m,(m+n)/2],\infty} f'(n) + \|g\|_{[m,(m+n)/2],\infty} f'(n)\right] \\
&\quad + \frac{(n-m)^{\nu+1}}{\Gamma(\nu+3)2^{\nu+2}} \|g\|_{[m,(m+n)/2],\infty} \\
&\quad \cdot \eta\left(\left|f'(m)\right|, \left|f'(n)\right|\right) \\
&\quad + \frac{(n-m)^{\nu+1}}{\Gamma(\nu+2)2^{\nu+2}(\nu+1)} \|g\|_{[m,(m+n)/2],\infty} \\
&\quad \cdot \eta\left(\left|f'(m)\right|, \left|f'(n)\right|\right) \\
&\leq 2\frac{(n-m)^{\nu+1}\|g\|_{[m,n],\infty}}{\Gamma(\nu+1)2^{\nu+1}} f'(n) \\
&\quad + \left[\frac{(n-m)^{\nu+1}\|g\|_{[m,n],\infty}}{\Gamma(\nu+2)2^{\nu+2}(\nu+1)} + \frac{(n-m)^{\nu+1}\|g\|_{[m,n],\infty}}{\Gamma(\nu+3)2^{\nu+2}}\right] \\
&\quad \cdot \eta\left(\left|f'(m)\right|, \left|f'(n)\right|\right),
\end{aligned} \tag{29}
$$

where

$$
\begin{aligned}
&\int_{\phi^{-1}(m)}^{t} \phi'(x)(\phi(x)-m)^{\nu-1}dx \\
&\quad = \frac{(\phi(t)-m)^{\nu}}{\nu}, \\
&\int_{t}^{\phi^{-1}(n)} \phi'(x)(n-\phi(x))^{\nu-1}dx \\
&\quad = \frac{(n-\phi(t))^{\nu}}{\nu}, \\
&\int_{\phi^{-1}(m)}^{\phi^{-1}((m+n)/2)} (\phi(t)-m)^{\nu}(n-\phi(t))\phi'(t)dt \\
&\quad = \frac{(n-m)^{\nu+2}(\nu+3)}{2^{\nu+2}(\nu+1)(\nu+2)}, \\
&\int_{\phi^{-1}((m+n)/2)}^{\phi^{-1}(n)} (n-\phi(t))^{\nu}(n-\phi(t))\phi'(t)dt \\
&\quad = \frac{(n-m)^{\nu+2}}{2^{\nu+2}(\nu+2)}, \\
&\int_{((m+n)/2)}^{\phi^{-1}(n)} (n-\phi(t))^{\nu}\phi'(t)dt \\
&\quad = \int_{\phi^{-1}(m)}^{\phi^{-1}m+n/2} (\phi(t)-m)^{\nu}\phi'(t)dt \\
&\quad = \frac{(n-m)^{\nu+1}}{2^{\nu+1}(\nu+1)}.
\end{aligned} \tag{30}
$$

This completes our proof. □

Remark 11. From Theorem 10, we can get following inequalities:

(1) If $\phi(x)=x$, then inequality (26) becomes

$$
\begin{aligned}
&\left| f\left(\frac{m+n}{2}\right)\left[{}^{RL}_{(m+n)/2+}I^{\nu} g(n) + {}^{RL}I^{\nu}_{m+n/2-} g(m)\right] \right. \\
&\quad - \left[g(n)\left({}^{RL}_{(m+n)/2+}I^{\nu}_{g} f\right)(n) \right. \\
&\quad \left.\left. + g(m)\left({}^{RL}_{g}I^{\nu}_{m+n/2-} f\right)(m)\right]\right| \\
&\leq \frac{(n-m)^{\nu+1}}{\Gamma(\nu)2^{\nu+1}(\nu+1)} \\
&\quad \cdot \left[\|g\|_{[m,(m+n)/2],\infty} f'(n) + \|g\|_{[m,(m+n)/2],\infty} f'(n)\right] \\
&\quad + \frac{(n-m)^{\nu+1}}{\Gamma(\nu)2^{\nu+2}(\nu+2)} \|g\|_{[m,(m+n)/2],\infty} \\
&\quad \cdot \eta\left(\left|f'(m)\right|, \left|f'(n)\right|\right) \\
&\quad + \frac{(n-m)^{\nu+1}}{\Gamma(\nu)2^{\nu+2}(\nu+1)(\nu+2)} \|g\|_{[m,(m+n)/2],\infty} \\
&\quad \cdot \eta\left(\left|f'(m)\right|, \left|f'(n)\right|\right) \\
&\leq 2\frac{(n-m)^{\nu+1}\|g\|_{[m,n],\infty}}{\Gamma(\nu)2^{\nu+1}(\nu+1)} f'(n) \\
&\quad + \left[\frac{(n-m)^{\nu+1}\|g\|_{[m,n],\infty}}{\Gamma(\nu)2^{\nu+2}(\nu+1)(\nu+2)} \right. \\
&\quad \left. + \frac{(n-m)^{\nu+1}\|g\|_{[m,n],\infty}}{\Gamma(\nu)2^{\nu+2}(\nu+2)}\right] \eta\left(\left|f'(m)\right|, \left|f'(n)\right|\right).
\end{aligned} \tag{31}
$$

(2) If $\phi(x)=x$ and $g(x)=1$, then inequality (26) becomes

$$
\begin{aligned}
&\left| \frac{2^{\nu-1}\Gamma(\nu+1)}{(n-m)^{\nu}} \right. \\
&\quad \left. \cdot \left[{}^{RL}_{(m+n)/2+}I^{\nu} f(n) + {}^{RL}I^{\nu}_{m+n/2-} f(m)\right] - f\left(\frac{m+n}{2}\right) \right| \\
&\leq \frac{(n-m)^{\nu+1}}{\Gamma(\nu)2^{\nu+1}(\nu+1)} \\
&\quad \cdot \left[\|g\|_{[m,(m+n)/2],\infty} f'(n) + \|g\|_{[m,(m+n)/2],\infty} f'(n)\right] \\
&\quad + \frac{(n-m)^{\nu+1}}{\Gamma(\nu)2^{\nu+2}(\nu+2)} \|g\|_{[m,(m+n)/2],\infty} \\
&\quad \cdot \eta\left(\left|f'(m)\right|, \left|f'(n)\right|\right) \\
&\quad + \frac{(n-m)^{\nu+1}}{\Gamma(\nu)2^{\nu+2}(\nu+1)(\nu+2)} \|g\|_{[m,(m+n)/2],\infty} \\
&\quad \cdot \eta\left(\left|f'(m)\right|, \left|f'(n)\right|\right)
\end{aligned}
$$

$$\leq 2\frac{(n-m)^{\nu+1}\|g\|_{[m,n],\infty}}{\Gamma(\nu)2^{\nu+1}(\nu+1)}f'(n) + \left[\frac{(n-m)^{\nu+1}\|g\|_{[m,n],\infty}}{\Gamma(\nu)2^{\nu+2}(\nu+1)(\nu+2)} + \frac{(n-m)^{\nu+1}\|g\|_{[m,n],\infty}}{\Gamma(\nu)2^{\nu+2}(\nu+2)}\right] \cdot \eta\left(|f'(m)|, |f'(n)|\right). \quad (32)$$

(3) If $\phi(x)=x$, $g(x)=1$ and $\nu=1$, then inequality (26) becomes

$$\left|\frac{1}{n-m}\int_m^n f(x)dx - f\left(\frac{m+n}{2}\right)\right| \leq \frac{(n-m)^2}{4}f'(n) + \left[\frac{(n-m)^2+2(n-m)^2}{48}\eta\left(|f'(m)|, |f'(n)|\right)\right]. \quad (33)$$

Theorem 12. *Let $0\leq m\leq n$ and $f:[m,n]\subseteq[0,\infty)\longrightarrow\mathbb{R}$ be a continuously differentiable function on the interval $[m,n]$ such that $f(x)=f(m)+\int_m^x f'(t)dt$, and let $g:[m,m]\longrightarrow\mathbb{R}$ be integrable, positive, and weighted symmetric function with respect to $(m+n)/2$. If, in addition, $|f'|^q$ is convex on $[m,n]$, $q\leq 1$, and ϕ is increasing and positive function from $[m,n]$ onto itself such that its derivative $\phi'(x)$ is continuous on $[m,m]$, then for $\nu>0$, we have:*

$$\begin{aligned}|\sigma_1+\sigma_2| &\leq \frac{(n-m)^{\nu+1}}{\Gamma(\nu+1)2^{\nu+1+(1/q)}}\|g\|_{[m,(m+n)/2],\infty}\left(|f'(n)|^q\right)^{1/q} + \|g\|_{[m,(m+n)/2],\infty}\left(|f'(n)|^q\right)^{1/q} \\ &+ \frac{(n-m)^{\nu+1}}{\Gamma(\nu+3)2^{\nu+2+(1/q)}}\|g\|_{[m,(m+n)/2],\infty} \cdot \left[\eta\left(|f'(m)|^q, |f'(n)|^q\right)\right]^{1/q} \\ &+ \frac{(n-m)^{\nu+1}}{\Gamma(\nu+2)2^{\nu+2+(1/q)}(\nu+1)^{1/q}}\|g\|_{[m,(m+n)/2],\infty} \cdot \left[\eta\left(|f'(m)|^q, |f'(n)|^q\right)\right]^{1/q} \\ &\leq 2\frac{(n-m)^{\nu+1}\|g\|_{[m,n],\infty}}{\Gamma(\nu+1)2^{\nu+1+(1/q)}}\left(|f'(n)|^q\right)^{1/q} \\ &+ \left[\frac{(n-m)^{\nu+1}\|g\|_{[m,n],\infty}}{\Gamma(\nu+2)2^{\nu+2+(1/q)}(\nu+1)^{1/q}} + \frac{(n-m)^{\nu+1}\|g\|_{[m,n],\infty}}{\Gamma(\nu+3)2^{\nu+2+(1/q)}}\right]\left(\eta\left(|f'(m)|, |f'(n)|\right)\right)^{1/q}.\end{aligned} \quad (34)$$

Proof. Since $|f'|^q$ is η-convex on $[m,n]$ for $t\in[\phi^{-1}(m), \phi^{-1}(n)]$, so

$$\left|\left(f'o\phi\right)(t)\right|^q = \left|f'\left(\frac{n-\phi(t)}{n-m}m + \frac{\phi(t)-m}{n-m}n\right)\right|^q \leq |f'(n)|^q + \frac{n-\phi(t)}{n-m}\eta\left(|f'(m)|^q, |f'(n)|^q\right). \quad (35)$$

By using power mean integral, Lemma 8, and η-convexity of $|f'|^q$, we have

$$\begin{aligned}|\sigma_1+\sigma_2| &\leq \frac{1}{\Gamma(\nu)}\int_{\phi^{-1}(m)}^{\phi^{-1}(m+n/2)} \cdot \left|\int_{\phi^{-1}(m)}^{t}\phi'(x)(\phi(x)-m)^{\nu-1}(go\phi)(x)dx\right| \cdot \left(f'o\phi\right)(t)\phi'(t)dt + \frac{1}{\Gamma(\nu)}\int_{\phi^{-1}(m+n/2)}^{\phi^{-1}(n)} \\ &\cdot \left|\int_{t}^{\phi^{-1}(n)}\phi'(x)(n-\phi(x))^{\nu-1}(go\phi)(x)dx\right| \cdot \left(f'o\phi\right)(t)\phi'(t)dt \\ &\leq \frac{1}{\Gamma(\nu)}\left(\int_{\phi^{-1}(m)}^{\phi^{-1}(m+n/2)} \cdot \left|\int_{\phi^{-1}(m)}^{t}\phi'(x)(\phi(x)-m)^{\nu-1}(go\phi)(x)dx\right|\phi'(t)dt\right)^{1} \\ &-1/q\times\left(\int_{\phi^{-1}(m)}^{\phi^{-1}(m+n/2)} \cdot \left|\int_{\phi^{-1}(m)}^{t}\phi'(x)(\phi(x)-m)^{\nu-1}(go\phi)(x)dx\right| \cdot \left|\left(f'o\phi\right)(t)\right|^q\phi'(t)dt\right)^{1/q} \\ &+ \frac{1}{\Gamma(\nu)}\left(\int_{\phi^{-1}(m+n/2)}^{\phi^{-1}(n)} \cdot \left|\int_{t}^{\phi^{-1}(n)}\phi'(x)(n-\phi(x))^{\nu-1}(go\phi)(x)dx\right|\phi'(t)dt\right)^{1-1/q} \\ &\times\left(\int_{\phi^{-1}(m+n/2)}^{\phi^{-1}(n)}\left|\int_{t}^{\phi^{-1}(n)}\phi'(x)(n-\phi(x))^{\nu-1}(go\phi)(x)dx\right| \cdot \left|\left(f'o\phi\right)(t)\right|^q\phi'(t)dt\right)^{1/q} \\ &\leq \frac{\|g\|_{[m,m+n/2],\infty}}{\Gamma(\nu)} \cdot \left(\int_{\phi^{-1}(m)}^{\phi^{-1}(m+n/2)}\left|\int_{\phi^{-1}(m)}^{t}\phi'(x)(\phi(x)-m)^{\nu-1}dx\right|\phi'(t)dt\right)^{1-1/q} \\ &\times\left(\int_{\phi^{-1}(m)}^{\phi^{-1}(m+n/2)}\left|\int_{\phi^{-1}(m)}^{t}\phi'(x)(\phi(x)-m)^{\nu-1}dx\right|\left|f'o\phi\right)(t)\right|^q\end{aligned}$$

$$\cdot \phi'(t)dt)^{1/q} + \frac{\|g\|_{[m,m+n/2],\infty}}{\Gamma(\nu)}$$
$$\cdot \left(\int_{\phi^{-1}(m+n/2)}^{\phi^{-1}(n)} \left|\int_t^{\phi^{-1}(m)} \phi'(x)(n-\phi(x))^{\nu-1}dx\right|\phi'(t)dt\right)^{1-1/q}$$
$$\times \left(\int_{\phi^{-1}(m+n/2)}^{\phi^{-1}(n)} \left|\int_t^{\phi^{-1}(n)} \phi'(x)(n-\phi(x))^{\nu-1}dx\right|\right.$$
$$\left.\cdot \left|f' o\phi)(t)\right|^q \phi'(t)dt\right)^{1/q}$$
$$\leq \frac{\|g\|_{[m,m+n/2],\infty}}{\Gamma(\nu)}$$
$$\cdot \left(\int_{\phi^{-1}(m)}^{\phi^{-1}m+n/2} \left|\int_{\phi^{-1}(m)}^t \phi'(x)(\phi(x)-m)^{\nu-1}dx\right|\phi'(t)dt\right)^{1-1/q}$$
$$\times \left[\int_{\phi^{-1}(m)}^{\phi^{-1}m+n/2} \left|\int_{\phi^{-1}(m)}^t \phi'(x)(\phi(x)-m)^{\nu-1}dx\right|\right.$$
$$\times \left(\left|f'(n)\right|^q + \frac{n-\phi(t)}{n-m}\eta\left(\left|f'(m)\right|^q, \left|f'(n)\right|^q\right)\right.$$
$$\left.\cdot \phi'(t)dt\right]^{1/q} + \frac{\|g\|_{[m,m+n/2],\infty}}{\Gamma(\nu)}$$
$$\times \left(\int_{\phi^{-1}(m+n/2)}^{\phi^{-1}(n)} \left|\int_t^{\phi^{-1}(m)} \phi'(x)(n-\phi(x))^{\nu-1}dx\right|\phi'(t)dt\right)^{1-1/q}$$
$$\times \left[\int_{\phi^{-1}(m+n/2)}^{\phi^{-1}(n)} \left|\int_t^{\phi^{-1}(n)} \phi'(x)(n-\phi(x))^{\nu-1}dx\right|\right.$$
$$\times \left(\left|f'(n)\right|^q + \frac{n-\phi(t)}{n-m}\right.$$
$$\left.\cdot \eta\left(\left|f'(m)\right|^q, \left|f'(n)\right|^q\right)\phi'^{(t)}dt\right]^{1/q}$$
$$= \frac{(n-m)^{\nu+1}}{\Gamma(\nu+1)2^{\nu+1+(1/q)}}\|g\|_{[m,m+n/2],\infty}\left(\left|f'(n)\right|^q\right)^{1/q}$$
$$+ \|g\|_{[m,m+n/2],\infty}\left(\left|f'(n)\right|^q\right)^{1/q}$$
$$+ \frac{(n-m)^{\nu+1}}{\Gamma(\nu+3)2^{\nu+2+(1/q)}}\|g\|_{[m,m+n/2],\infty}$$
$$\cdot \left[\eta\left(\left|f'(m)\right|^q, \left|f'(n)\right|^q\right)\right]^{1/q}$$
$$+ \frac{(n-m)^{\nu+1}}{\Gamma(\nu+2)2^{\nu+2+(1/q)}(\nu+1)^{(1/q)}}\|g\|_{[m,m+n/2],\infty}$$
$$\cdot \left[\eta\left(\left|f'(m)\right|^q, \left|f'(n)\right|^q\right)\right]^{1/q}$$
$$\leq 2\frac{(n-m)^{\nu+1}\|g\|_{[m,n],\infty}}{\Gamma(\nu+1)2^{\nu+1+(1/q)}}\left(\left|f'(n)\right|^q\right)^{1/q}$$
$$+ \left[\frac{(n-m)^{\nu+1}\|g\|_{[m,n],\infty}}{\Gamma(\nu+2)2^{\nu+2+(1/q)}(\nu+1)^{1/q}} + \frac{(n-m)^{\nu+1}\|g\|_{[m,n],\infty}}{\Gamma(\nu+3)2^{\nu+2+(1/q)}}\right]$$
$$\cdot \left(\eta\left(\left|f'(m)\right|, \left|f'(n)\right|\right)\right)^{1/q}, \quad (36)$$

where

$$\int_{\phi^{-1}(m)}^{\phi^{-1}(m+n/2)} \left|\int_{\phi^{-1}(m)}^t (\phi(t)-m)^{\nu-1}(n-\phi(t))\right|\phi'(t)dt = \frac{(n-m)^{\nu+2}(\nu+3)}{2^{\nu+2}(\nu+1)(\nu+2)},$$
$$\int_{\phi^{-1}(m+n/2)}^{\phi^{-1}(n)} |\int_t^{\phi^{-1}(n)} (n-\phi(t))^{\nu}(n-\phi(t))|\phi'(t)dt = \frac{(n-m)^{\nu+2}}{2^{\nu+2}(\nu+2)}. \quad (37)$$

□

Remark 13. From Theorem 12, we can get following special cases:

(1) If $\phi(x)=x$, then inequality (34) becomes

$$\left|f\left(\frac{m+n}{2}\right)\left[{}^{RL}_{m+n/2+}I^{\nu}g(n) + {}^{RL}I^{\nu}_{m+n/2-}g(m)\right] - \left[g(n)\left({}^{RL}_{m+n/2+}I^{\nu}_{g}f\right)(n) + g(m)\left({}^{RL}_{g}I^{\nu}_{m+n/2-}f\right)(m)\right]\right|$$
$$\leq \frac{(n-m)^{\nu+1}}{\Gamma(\nu+1)2^{\nu+1+(1/q)}}\|g\|_{[m,m+n/2],\infty}\left(\left|f'(n)\right|^q\right)^{1/q}$$
$$+ \|g\|_{[m,m+n/2],\infty}\left(\left|f'(n)\right|^q\right)^{1/q}$$
$$+ \frac{(n-m)^{\nu+1}}{\Gamma(\nu+3)2^{\nu+2+(1/q)}}\|g\|_{[m,m+n/2],\infty}$$
$$\cdot \left[\eta\left(\left|f'(m)\right|^q, \left|f'(n)\right|^q\right)\right]^{1/q}$$
$$\cdot \frac{(n-m)^{\nu+1}}{\Gamma(\nu+2)2^{\nu+2+(1/q)}(\nu+1)^{(1/q)}}\|g\|_{[m,m+n/2],\infty}$$
$$\cdot \left[\eta\left(\left|f'(m)\right|^q, \left|f'(n)\right|^q\right)\right]^{1/q}$$
$$\leq 2\frac{(n-m)^{\nu+1}\|g\|_{[m,n],\infty}}{\Gamma(\nu+1)2^{\nu+1+(1/q)}}\left(\left|f'(n)\right|^q\right)^{1/q}$$
$$\cdot \left[\frac{(n-m)^{\nu+1}\|g\|_{[m,n],\infty}}{\Gamma(\nu+2)2^{\nu+2+(1/q)}(\nu+1)^{(1/q)}} + \frac{(n-m)^{\nu+1}\|g\|_{[m,n],\infty}}{\Gamma(\nu+3)2^{\nu+2+(1/q)}}\right]$$
$$\cdot \left(\eta\left(\left|f'(m)\right|, \left|f'(n)\right|\right)\right)^{1/q}. \quad (38)$$

(2) If $\phi(x)=x$ and $g(x)=1$, then inequality (34) becomes

$$\left|\frac{2^{\nu-1}\Gamma(\nu+1)}{(n-m)^{\nu}}\cdot\left[{}^{RL}_{m+n/2+}I^{\nu}f(n)+{}^{RL}I^{\nu}_{m+n/2-}f(m)\right]-f\left(\frac{m+n}{2}\right)\right| \le 2\frac{(n-m)^{\nu+1}\|g\|_{[m,n],\infty}}{\Gamma(\nu+1)2^{\nu+1+(1/q)}}\left(\left|f'(n)\right|^{q}\right)^{1/q} + \left[\frac{(n-m)^{\nu+1}\|g\|_{[m,n],\infty}}{\Gamma(\nu+2)2^{\nu+2+(1/q)}(\nu+1)^{1/q}}\cdot\frac{(n-m)^{\nu+1}\|g\|_{[m,n],\infty}}{\Gamma(\nu+3)2^{\nu+2+(1/q)}}\right]\left(\eta\left(\left|f'(m)\right|,\left|f'(n)\right|\right)\right)^{1/q}. \tag{39}$$

(3) If $\phi(x)=x$, $g(x)=1$ and $\nu=1$, then inequality (34) becomes

$$\left|\frac{1}{n-m}\int_m^n f(x)dx-f\left(\frac{m+n}{2}\right)\right| \le 2\frac{(n-m)^2}{2^{2+(1/q)}}\left(\left|f'(n)\right|^{q}\right)^{1/q} + \frac{(n-m)^2+2^{1/q}(n-m)^2}{2^{1/q}3^{3+(1/q)}}\Bigg]\cdot\left(\eta\left(\left|f'(m)\right|,\left|f'(n)\right|\right)\right)^{1/q}. \tag{40}$$

Theorem 14. *Let $0\le m\le n$ and $f:[m,n]\subseteq[0,\infty)\longrightarrow\mathbb{R}$ be a continuously differentiable function on the interval $[m,n]$ such that $f(x)=f(m)+\int_m^x f'(t)dt$, and let $g:[m,m]\longrightarrow\mathbb{R}$ be integrable, positive and weighted symmetric function with respect to $(m+n)/2$. If, in addition, $|f'|^q$ is convex on $[m,n]$, $q\le 1$, and ϕ is increasing and positive function from $[m,n]$ onto itself such that its derivative $\phi'(x)$ is continuous on $[m,m]$, then for $\nu>0$, we have*

$$|\sigma_1+\sigma_2|\le\frac{(n-m)^{\nu+1}}{\Gamma(\nu+1)2^{\nu+1+(2/q)}}\|g\|_{[m,(m+n)/2],\infty}\left(\left|f'(n)\right|^{q}\right)^{1/q} + \|g\|_{[m,(m+n)/2],\infty}\left(\left|f'(n)\right|^{q}\right)^{1/q} + \frac{(n-m)^{\nu+1}}{\Gamma(\nu+3)2^{\nu+2+(2/q)}}\|g\|_{[m,(m+n)/2],\infty}\cdot\left[\eta\left(\left|f'(m)\right|^{q},\left|f'(n)\right|^{q}\right)\right]^{1/q} + \frac{(n-m)^{\nu+1}}{\Gamma(\nu+2)2^{\nu+2+(2/q)}(p\nu+1)^{1/p}}\|g\|_{[m,(m+n)/2],\infty}\cdot\left[\eta\left(\left|f'(m)\right|^{q},\left|f'(n)\right|^{q}\right)\right]^{1/q}$$

$$\le 2\frac{(n-m)^{\nu+1}\|g\|_{[m,n],\infty}}{\Gamma(\nu+1)2^{\nu+1+(2/q)}}\left(\left|f'(n)\right|^{q}\right)^{1/q} + \left[\frac{(n-m)^{\nu+1}\|g\|_{[m,n],\infty}}{\Gamma(\nu+2)2^{\nu+2+(2/q)}(p\nu+1)^{1/p}}+\frac{(n-m)^{\nu+1}\|g\|_{[m,n],\infty}}{\Gamma(\nu+3)2^{\nu+2+(2/q)}}\right]\cdot\left(\eta\left(\left|f'(m)\right|,\left|f'(n)\right|\right)\right)^{1/q}. \tag{41}$$

Proof. Since $|f'|^q$ is η-convex on $[m,n]$, for $t\in[\phi^{-1}(m),\phi^{-1}(n)]$, we get

$$\left|\left(f'o\phi\right)(t)\right|^{q}=\left|f'\left(\frac{n-\phi(t)}{n-m}m+\frac{\phi(t)-m}{n-m}n\right)\right|^{q} \le\left|f'(n)\right|^{q}+\frac{n-\phi(t)}{n-m}\eta\left(\left|f'(m)\right|^{q},\left|f'(n)\right|^{q}\right). \tag{42}$$

By using Hölder's inequality, Lemma 8, η-convexity of $|f'|^q$, and properties of modulus, we get

$$|\sigma_1+\sigma_2|\le\frac{1}{\Gamma(\nu)}\int_{\phi^{-1}(m)}^{\phi^{-1}((m+n)/2)}\cdot\left|\int_{\phi^{-1}(m)}^{t}\phi'(x)(\phi(x)-m)^{\nu-1}(go\phi)(x)dx\right|\cdot\left(f'o\phi\right)(t)\phi'(t)dt+\frac{1}{\Gamma(\nu)}\int_{\phi^{-1}((m+n)/2)}^{\phi^{-1}(n)}\cdot\left|\int_{t}^{\phi^{-1}(n)}\phi'(x)(n-\phi(x))^{\nu-1}(go\phi)(x)dx\right|\cdot\left(f'o\phi\right)(t)\phi'(t)dt$$

$$\le\frac{1}{\Gamma(\nu)}\left(\int_{\phi^{-1}(m)}^{\phi^{-1}((m+n)/2)}\cdot\left|\int_{\phi^{-1}(m)}^{t}\phi'(x)(\phi(x)-m)^{\nu-1}(go\phi)(x)dx\right|^{p}\phi'(t)dt\right)^{1/p}\times\left(\int_{\phi^{-1}(m)}^{\phi^{-1}((m+n)/2)}\left|\left(f'o\phi\right)(t)\right|^{q}\phi'(t)dt\right)^{1/q}+\frac{1}{\Gamma(\nu)}\left(\int_{\phi^{-1}((m+n)/2)}^{\phi^{-1}(n)}\cdot\left|\int_{t}^{\phi^{-1}(n)}\phi'(x)(n-\phi(x))^{\nu-1}(go\phi)(x)dx\right|^{p}\phi'(t)dt\right)^{1/p}\times\left(\int_{\phi^{-1}((m+n)/2)}^{\phi^{-1}(n)}\left|\left(f'o\phi\right)(t)\right|^{q}\phi'(t)dt\right)^{1/q}$$

$$\le\frac{\|g\|_{[m,(m+n)/2],\infty}}{\Gamma(\nu)}\times\left(\int_{\phi^{-1}(m)}^{\phi^{-1}((m+n)/2)}\left|\int_{\phi^{-1}(m)}^{t}\phi'(x)(\phi(x)-m)^{\nu-1}dx\right|^{p}\phi'(t)dt\right)^{1/p}\times\left(\int_{\phi^{-1}(m)}^{\phi^{-1}((m+n)/2)}\left|f'o\phi\right)(t)\right|^{q}\phi'(t)dt\right)^{1/q}+\frac{\|g\|_{[m,(m+n)/2],\infty}}{\Gamma(\nu)}$$

$$\times\left(\int_{\phi^{-1}((m+n)/2)}^{\phi^{-1}(n)}\left|\int_{t}^{\phi^{-1}(m)}\phi'(x)(n-\phi(x))^{\nu-1}dx\right|^{p}\phi'(t)dt\right)^{1/p}$$
$$\times\left(\int_{\phi^{-1}((m+n)/2)}^{\phi^{-1}(n)}\left|f'o\phi\right)(t)\right|^{q}\phi'(t)dt\right)^{1/q}$$
$$\leq\frac{\|g\|_{[m,(m+n)/2],\infty}}{\Gamma(\nu)}$$
$$\cdot\left(\int_{\phi^{-1}(m)}^{\phi^{-1}(m+n)/2}\left|\int_{\phi^{-1}(m)}^{t}\phi'(x)(\phi(x)-m)^{\nu-1}dx\right|^{p}\phi'(t)dt\right)^{1/p}$$
$$\times\left[\int_{\phi^{-1}(m)}^{\phi^{-1}(m+n)/2}\left(\left|f'(n)\right|^{q}+\frac{n-\phi(t)}{n-m}\right.\right.$$
$$\left.\left.\cdot\eta\left(\left|f'(m)\right|^{q},\left|f'(n)\right|^{q}\right)\phi'(t)dt\right]^{1/q}$$
$$+\frac{\|g\|_{[m,(m+n)/2],\infty}}{\Gamma(\nu)}\times\left(\int_{\phi^{-1}((m+n)/2)}^{\phi^{-1}(n)}\right.$$
$$\left.\cdot\left|\int_{t}^{\phi^{-1}(m)}\phi'(x)(n-\phi(x))^{\nu-1}dx\right|^{p}\phi'(t)dt\right)^{1/p}$$
$$\times\left[\int_{\phi^{-1}((m+n)/2)}^{\phi^{-1}(n)}\left(\left|f'(n)\right|^{q}+\frac{n-\phi(t)}{n-m}\right.\right.$$
$$\left.\left.\cdot\eta\left(\left|f'(m)\right|^{q},\left|f'(n)\right|^{q}\right)\phi'(t)dt\right]^{1/q}$$
$$=\frac{(n-m)^{\nu+1}}{\Gamma(\nu+1)2^{\nu+1+(2/q)}}\|g\|_{[m,(m+n)/2],\infty}\left(\left|f'(n)\right|^{q}\right)^{1/q}$$
$$+\|g\|_{[m,(m+n)/2],\infty}\left(\left|f'(n)\right|^{q}\right)^{1/q}$$
$$+\frac{(n-m)^{\nu+1}}{\Gamma(\nu+3)2^{\nu+2+(2/q)}}\|g\|_{[m,(m+n)/2],\infty}$$
$$\cdot\left[\eta\left(\left|f'(m)\right|^{q},\left|f'(n)\right|^{q}\right)\right]^{1/q}$$
$$+\frac{(n-m)^{\nu+1}}{\Gamma(\nu+2)2^{\nu+2+(2/q)}(p\nu+1)^{1/p}}\|g\|_{[m,(m+n)/2],\infty}$$
$$\cdot\left[\eta\left(\left|f'(m)\right|^{q},\left|f'(n)\right|^{q}\right)\right]^{1/q}$$
$$\leq 2\frac{(n-m)^{\nu+1}\|g\|_{[m,n],\infty}}{\Gamma(\nu+1)2^{\nu+1+(2/q)}}\left(\left|f'(n)\right|^{q}\right)^{1/q}$$
$$+\left[\frac{(n-m)^{\nu+1}\|g\|_{[m,n],\infty}}{\Gamma(\nu+2)2^{\nu+2+(2/q)}(p\nu+1)^{1/p}}+\frac{(n-m)^{\nu+1}\|g\|_{[m,n],\infty}}{\Gamma(\nu+3)2^{\nu+2+(2/q)}}\right]$$
$$\cdot\left(\eta\left(\left|f'(m)\right|,\left|f'(n)\right|\right)\right)^{1/q},\qquad(43)$$

where

$$\int_{\phi^{-1}(m)}^{\phi^{-1}((m+n)/2)}\left|\int_{\phi^{-1}(m)}^{t}(\phi(t)-m)^{\nu-1}(n-\phi(t))\right|^{p}\phi'(t)dt$$
$$=\frac{(n-m)^{p\nu+2}(p\nu+3)}{2^{p\nu+2}(p\nu+1)(p\nu+2)},$$
$$\int_{\phi^{-1}((m+n)/2)}^{\phi^{-1}(n)}\left|\int_{t}^{\phi^{-1}(n)}(n-\phi(t))^{\nu}(n-\phi(t))\right|^{p}\phi'(t)dt$$
$$=\frac{(n-m)^{p\nu+2}}{2^{p\nu+2}(p\nu+2)}.\qquad(44)$$

This completes the proof. □

Remark 15. From Theorem 14, we can obtain following special cases:

(1) If $\phi(x)=x$, then inequality (41) becomes

$$\left|f\left(\frac{m+n}{2}\right)\left[{}^{RL}_{(m+n)/2+}I^{\nu}g(n)+{}^{RL}I^{\nu}_{(m+n)/2-}g(m)\right]\right.$$
$$-\left[g(n)\left({}^{RL}_{(m+n)/2+}I^{\nu}_{g}f\right)(n)\right.$$
$$\left.\left.+g(m)\left({}^{RL}_{g}I^{\nu}_{(m+n)/2-}f\right)(m)\right]\right|$$
$$\leq\frac{(n-m)^{\nu+1}}{\Gamma(\nu+1)2^{\nu+1+(2/q)}}\|g\|_{[m,(m+n)/2],\infty}\left(\left|f'(n)\right|^{q}\right)^{1/q}$$
$$+\|g\|_{[m,(m+n)/2],\infty}\left(\left|f'(n)\right|^{q}\right)^{1/q}$$
$$+\frac{(n-m)^{\nu+1}}{\Gamma(\nu+3)2^{\nu+2+(2/q)}}\|g\|_{[m,(m+n)/2],\infty}$$
$$\cdot\left[\eta\left(\left|f'(m)\right|^{q},\left|f'(n)\right|^{q}\right)\right]^{1/q}$$
$$+\frac{(n-m)^{\nu+1}}{\Gamma(\nu+2)2^{\nu+2+(2/q)}(p\nu+1)^{1/p}}\|g\|_{[m,(m+n)/2],\infty}$$
$$\cdot\left[\eta\left(\left|f'(m)\right|^{q},\left|f'(n)\right|^{q}\right)\right]^{1/q}$$
$$\leq 2\frac{(n-m)^{\nu+1}\|g\|_{[m,n],\infty}}{\Gamma(\nu+1)2^{\nu+1+(2/q)}}\left(\left|f'(n)\right|^{q}\right)^{1/q}$$
$$+\left[\frac{(n-m)^{\nu+1}\|g\|_{[m,n],\infty}}{\Gamma(\nu+2)2^{\nu+2+(2/q)}(p\nu+1)^{1/p}}\right.$$
$$\left.+\frac{(n-m)^{\nu+1}\|g\|_{[m,n],\infty}}{\Gamma(\nu+3)2^{\nu+2+(2/q)}}\right]$$
$$\cdot\left(\eta\left(\left|f'(m)\right|,\left|f'(n)\right|\right)\right)^{1/q}.\qquad(45)$$

(2) If $\phi(x) = x$ and $g(x) = 1$, then inequality (41) becomes

$$\begin{aligned}
&\left| \frac{2^{\nu-1}\Gamma(\nu+1)}{(n-m)^{\nu}} \cdot \left[{}^{RL}_{(m+n)/2+}I^{\nu} f(n) + {}^{RL}I^{\nu}_{m+n/2-} f(m) \right] - f\left(\frac{m+n}{2}\right) \right| \\
&\leq 2 \frac{(n-m)^{\nu+1} \|g\|_{[m,n],\infty}}{\Gamma(\nu+1) 2^{\nu+1+(2/q)}} \left(|f'(n)|^q \right)^{1/q} \\
&\quad + \left[\frac{(n-m)^{\nu+1} \|g\|_{[m,n],\infty}}{\Gamma(\nu+2) 2^{\nu+2+(2/q)} (p\nu+1)^{1/p}} + \frac{(n-m)^{\nu+1} \|g\|_{[m,n],\infty}}{\Gamma(\nu+3) 2^{\nu+2+(2/q)}} \right] \\
&\quad \cdot \left(\eta\left(|f'(m)|, |f'(n)| \right) \right)^{1/q}.
\end{aligned} \tag{46}$$

(3) If $\phi(x) = x$, $g(x) = 1$ and $\nu = 1$, then inequality (41) becomes

$$\begin{aligned}
&\left| \frac{1}{n-m} \int_m^n f(x) dxs - f\left(\frac{m+n}{2}\right) \right| \\
&\leq 2 \frac{(n-m)^2}{2^{3+(2/q)}} \left(|f'(n)|^q \right)^{1/q} \\
&\quad + \frac{(n-m)^2 + (p+1)^{1/p}(n-m)^2}{(p+1)^{1/p} 2^{3+(2/q)}} \Bigg] \\
&\quad \cdot \left(\eta\left(|f'(m)|, |f'(n)| \right) \right)^{1/q}.
\end{aligned} \tag{47}$$

3. Conclusion

In this paper, we established Hermite-Hadamarad Fejér type inequalities for η-convex function by using weighted fractional integrals. Our results are extensions and generalizations of many existing results in the literature.

Data Availability

All data required for this research is included within the paper.

Conflicts of Interest

The authors of this paper declare that they have no competing interests.

Authors' Contributions

Lei Chen analyzed the results, Waqas Nazeer proposed the problem, Farman Ali wrote the final version of the paper, Thongchai Botmart verified the results and arranged funding for this paper, and Sarah Mehfooz wrote the first version of the paper. Lei Chen and Farman Ali contributed equally to this work and are first co-authors.

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