Hindawi
Journal of Function Spaces
Volume 2024, Article ID 9852756, 1 page
https://doi.org/10.1155/2024/9852756

Retraction

Retracted: Applications of Hesitant Interval Neutrosophic Linguistic Schweizer-Sklar Power Aggregation Operators to MADM

Journal of Function Spaces

Received 23 January 2024; Accepted 23 January 2024; Published 24 January 2024

This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

(1) Discrepancies in scope

(2) Discrepancies in the description of the research reported

(3) Discrepancies between the availability of data and the research described

(4) Inappropriate citations

(5) Incoherent, meaningless and/or irrelevant content included in the article

(6) Manipulated or compromised peer review

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

References

[1] Q. Khan, R. A. R. Bantan, and M. Elgarhy, "Applications of Hesitant Interval Neutrosophic Linguistic Schweizer-Sklar Power Aggregation Operators to MADM," *Journal of Function Spaces*, vol. 2022, Article ID 1654820, 30 pages, 2022.

Hindawi
Journal of Function Spaces
Volume 2022, Article ID 1654820, 30 pages
https://doi.org/10.1155/2022/1654820

Research Article

Applications of Hesitant Interval Neutrosophic Linguistic Schweizer-Sklar Power Aggregation Operators to MADM

Qaisar Khan,[1] Rashad A. R. Bantan,[2] and Mohammed Elgarhy[3]

[1]*Department of Mathematics and Statistics, The University of Haripur, Haripur, KPK, Pakistan*
[2]*Department of Marine Geology, Faculty of Marine Science, King AbdulAziz University, Jeddah 21551, Saudi Arabia*
[3]*The Higher Institute of Commercial Sciences, Al Mahalla Al Kubra, Algarbia, 31951, Egypt*

Correspondence should be addressed to Mohammed Elgarhy; m_elgarhy85@sva.edu.eg

Received 13 November 2021; Revised 18 December 2021; Accepted 22 December 2021; Published 19 March 2022

Academic Editor: Ganesh Ghorai

Hesitant interval neutrosophic linguistic sets (HINLSs) are one of the core generalization of various sets, such as neutrosophic set (NS), interval neutrosophic set (INS), and interval neutrosophic linguistic set (INLS). HINLS can represent the uncertainty, inconsistency, and reluctance of assessment specialists by expressing qualitative and quantitative information. The goal of this article is to introduce a novel MADM technique that can account for changes in the semantic environment as well as negative consequences of experts' excessive evaluation values. First, several innovative operational rules based on Schweizer-Sklar (SS) t-norm and t-conorm and a novel comparison procedure for HINLS are established by integrating different linguistic scale functions. This allows for varied semantic settings to be handled. Then, various innovative HINL Schweizer-Sklar power aggregation operators (AOs) are suggested, containing hesitant interval neutrosophic SS power average (HINLSSPA) operator, weighted hesitant interval neutrosophic SS power average (WHINLSSPA) operator, hesitant interval neutrosophic SS power geometric average (HINLSSPGA) operator, weighted hesitant interval neutrosophic SS power geometric average (WHINLSSPGA) operator, some core properties, and various special cases of these AOs are examined. Additionally, based on the initiated AOs, a multiple attribute decision making (MADM) technique with HINL information is anticipated. Finally, a numerical example is illustrated to show the effectiveness and practicality of the anticipated MADM method. A comparison with existing approaches are also discussed.

1. Introduction

The preferred information in actual decision-making situations is frequently imprecise, uncertain, and unpredictable. As a result, fuzzy decision making is a beneficial approach in a variety of fuzzy situations [1–6]. Since Smarandache's neutrosophic set (NS) [7, 8] can adequately define imprecise, ambiguous, and inconsistent data. Several researchers have created a few subclasses of NS that may be used easily in real-world scientific and engineering problems. For instance, Wang et al. [9, 10] defined a single-valued neutrosophic set (SVNS) and an interval neutrosophic set (INS), as well as the set-theoretic operators and characteristics of SVNSs and INSs. After, the introduction of SVNSs and INSs, several researchers anticipated correlation coefficient [11–13], distance measures [14–16], and normalized Bonferroni mean [17–20] and apply these concepts to solve MADM problems under SVN and IN environments. For IN MADM problems, Zhang et al. [21] advanced the score, accuracy, and certainty functions of IN numbers (INNs) and created the INN weighted average (INNWA) operator and INN weighted geometric (INNWG) operator. Khan et al. [22] and Liu et al. [23] anticipated IN Dombi power Bonferroni mean operator, IN power Hamy mean operator and applied it to solve MADM, MAGDM problems under IN information.

In many real-life decision scenarios, however, experts may prefer to convey their alternatives using linguistic information rather than numerical numbers. As a result, linguistic term sets (LTS) [24] are studied extensively and used in the decision-making process to convey expertise' preferred alternatives [25–27]. Motivated by INS and LTS, Ye [28] initiated the concept of IN linguistic sets (INLSs) and then

proposed some basic aggregation operators (AOs) to deal with MAGDM problems under INL information. Further, Ye [29] merged SVNS with LTS and initiated the idea of SVN linguistic sets (SVNLS) and introduces an extended TOPSIS method to handle MADM problems under SVNL information. Ji et al. [30] initiated combined MABAC-ELECTRE for SVNLSs and applied it to solve MADM problems under SVNL information. Wang et al. [31] initiated a series of SVNL generalized Maclaurin symmetric mean (SVNLGMSM) operator and apply these AOs to solve MADM problems. Both the linguistic variable expressed by the decision maker's assessment of the assessed entity and the quantitative performance value expressed by an INN as the credibility of the provided linguistic variable are contained in an INLS. When decision makers offer their judgments on characteristics in the form of INLNs in complicated decision-making situations, though, they might hesitate between a variety of possible interval values. To deal with such scenario, Ye [32] initiated the idea of hesitant interval neutrosophic sets (HINLSs) by merging INLS with HFS [33, 34]. Ye also introduced some basic operational laws and some weighted AOs to deal with MADM problems under HINL information.

AOs are extremely helpful tools for combining expert opinions in order to calculate the total value of each option. The power average (PA) operator, which was first developed by Yager [35], can decrease the detrimental influence of high expert evaluation values on final decision results. The power geometric (PG) operator and its weighted form were created by Xu and Yager [36], who were inspired by the notion of PA operators. Zhou et al. [37] merged PA operator with generalized average operator and initiated a new type of AOs, that is, generalized power average (GPA) operators. After the introduction of PA, PG, and GPA operators, several scholars extended these AOs for different types of fuzzy extensions. However, mostly these AOs are based on traditional operational laws, and it is unable to fulfill the various semantic needs of various experts. They cannot be used to aggregate HINLNs; as a result, the goal of this paper is to offer a number of novel generalized power AOs for integrating HINL data. The processing of language information is an essential topic that requires consideration in the research of linguistic decision-making methods. Several linguistic information processing methods have been suggested thus far, including the membership function transformation method [38, 39], the symbolic calculation method based on the subscripts of linguistic words [40–42], the cloud model transformation method, and the 2-tuple linguistic representation approach [43–45]. As the abovementioned decision making have certain advantages, but it cannot deal all types of decision-making problems. When evaluating an object, decision makers may believe that the semantic divergence between "acceptable" and "somewhat acceptable" is more or smaller than the semantic difference between "acceptable" and "completely acceptable." That is, when the number of linguistic subscripts grows, the semantic divergence between adjacent linguistic words does not necessarily remain constant [46]. Decision makers may have various semantic criteria for established linguistic terms in numerous actual decision-making scenarios. Clearly, the existing linguistic technique fails to handle identical decision-making difficulties in the presence of HINL data. So, to overcome such drawbacks, in this article, we utilized LSF to redefine the operational laws for LTs and Schweizer-Sklar t-norm and t-conorm [47] for HINLNs. Then, we further initiate four generalized HINL PA operators to solve MADM problems.

As a result of the foregoing research inspirations, the following are the article's aims and offerings:

(1) To define some novel operational laws for HINLNs based on Schweizer-Sklar t-norm, Schweizer-Sklar t-conorm and linguistic scale function

(2) Anticipating four types of generalized power aggregation operators based on these novel operational laws for HINLNs

(3) Inspecting core properties and specific cases of these generalized power aggregation operators with respect to generalized parameters

(4) Presenting a MADM technique under HINL environment which can not only remove the bad impact of high assessment values on the decision making results but also adjust to distinct semantic environment, fulfill semantic requirements of distinct experts, and make decision-making process flexible

To do so, the rest of the article is organized as follows: in Section 2, some basic ideas are examined briefly. In Section 3, based on LSF and SS t-norm and t-conorm, some core operational laws are initiated for HINLNs. In Section 4, based on these operational laws, various GPA are developed to aggregate HINNs, and various core properties and special cases are investigated. In Section 5, a MADM model is presented to deal with HINL information. In Section 6, a numerical example is given to show the effectiveness and practicality of the developed MADM approach. Finally, comparison with the existing approach is also discussed.

2. Preliminaries

2.1. The Interval Neutrosophic Linguistic Set

Definition 1 (see [28]). Let Ω be the finite set. An INLS in Ω is identified by

$$INL = \left\{ \bar{\bar{a}}, \left\langle s_{\vartheta(\bar{\bar{a}})}, \left(\bar{\bar{T}}_{INL}(\bar{\bar{a}}), \bar{\bar{I}}_{INL}(\bar{\bar{a}}), \bar{\bar{F}}_{INL}(\bar{\bar{a}}) \right) \right\rangle | \bar{\bar{a}} \in \Omega \right\}, \tag{1}$$

where $s_{\vartheta(\bar{\bar{a}})} \in S$, $\bar{\bar{T}}_{INL}(\bar{\bar{a}}) = [\inf \bar{\bar{T}}_{INL}(\bar{\bar{a}}), \sup \bar{\bar{T}}_{INL}(\bar{\bar{a}})] \subseteq [0, 1]$, $\bar{\bar{I}}_{INL}(\bar{\bar{a}}) = [\inf \bar{\bar{I}}_{INL}(\bar{\bar{a}}), \sup \bar{\bar{I}}_{INL}(\bar{\bar{a}})] \subseteq [0, 1]$ and $\bar{\bar{F}}_{INL}(\bar{\bar{a}}) = [\inf \bar{\bar{F}}_{INL}(\bar{\bar{a}}), \sup \bar{\bar{F}}_{INL}(\bar{\bar{a}})] \subseteq [0, 1]$, respectively, signifying the TMED, IND, and FLMD of an element $\bar{\bar{a}}$ in Ω to the linguistic variable $s_{\vartheta(a)}$ with the constraint $0 \leq Sup\bar{\bar{T}}_{INL}(\bar{\bar{a}}) + Sup\bar{\bar{I}}_{INL}(\bar{\bar{a}}) + Sup\bar{\bar{F}}_{INL}(\bar{\bar{a}}) \leq 3$ for any $\bar{\bar{a}} \in \Omega$.

For simplicity, the INLN is signified by $\overline{\overline{\mathrm{we}}} = \langle s_{\vartheta(\overline{\overline{\mathrm{we}}})}, ([\bar{\bar{T}}^L(\overline{\overline{\mathrm{we}}}), \bar{\bar{T}}^U(\overline{\overline{\mathrm{we}}})], [\bar{\bar{I}}^L(\overline{\overline{\mathrm{we}}}), \bar{\bar{I}}^U(\overline{\overline{\mathrm{we}}})], [\bar{\bar{F}}^L(\overline{\overline{\mathrm{we}}}), \bar{\bar{F}}^U(\overline{\overline{\mathrm{we}}})])\rangle$..

Definition 2 (see [28]). Let $\overline{\overline{\mathrm{we}}} = \langle s_{\vartheta(\overline{\overline{\mathrm{we}}})}, ([\bar{\bar{T}}^L(\overline{\overline{\mathrm{we}}}), \bar{\bar{T}}^U(\overline{\overline{\mathrm{we}}})], [\bar{\bar{I}}^L(\overline{\overline{\mathrm{we}}}), \bar{\bar{I}}^U(\overline{\overline{\mathrm{we}}})], [\bar{\bar{F}}^L(\overline{\overline{\mathrm{we}}}), \bar{\bar{F}}^U(\overline{\overline{\mathrm{we}}})])\rangle$,

$\overline{\overline{\mathrm{we}}}_1 = \langle s_{\vartheta(\overline{\overline{\mathrm{we}}}_1)}, ([\bar{\bar{T}}^L(\overline{\overline{\mathrm{we}}}_1), \bar{\bar{T}}^U(\overline{\overline{\mathrm{we}}}_1)], [\bar{\bar{I}}^L(\overline{\overline{\mathrm{we}}}_1), \bar{\bar{I}}^U(\overline{\overline{\mathrm{we}}}_1)], [\bar{\bar{F}}^L(\overline{\overline{\mathrm{we}}}_1), \bar{\bar{F}}^U(\overline{\overline{\mathrm{we}}}_1)])\rangle$, and $\overline{\overline{\mathrm{we}}}_2 = \langle s_{\vartheta(\overline{\overline{\mathrm{we}}}_2)}, ([\bar{\bar{T}}^L(\overline{\overline{\mathrm{we}}}_2), \bar{\bar{T}}^U(\overline{\overline{\mathrm{we}}}_2)], [\bar{\bar{I}}^L(\overline{\overline{\mathrm{we}}}_2), \bar{\bar{I}}^U(\overline{\overline{\mathrm{we}}}_2)], [\bar{\bar{F}}^L(\overline{\overline{\mathrm{we}}}_2), \bar{\bar{F}}^U(\overline{\overline{\mathrm{we}}}_2)])\rangle$ be three INLNs and $\bar{\bar{\xi}} \geq 0$, then, the core operations for INLNs are defined as follows:

$$
\begin{aligned}
(1)\overline{\overline{\mathrm{we}}}_1 + \overline{\overline{\mathrm{we}}}_2 = \Bigg\langle & s_{\vartheta(\overline{\overline{\mathrm{we}}}_1)} + s_{\vartheta(\overline{\overline{\mathrm{we}}}_1)}, \Bigg(\Bigg[\bar{\bar{T}}^L(\overline{\overline{\mathrm{we}}}_1) + \bar{\bar{T}}^L(\overline{\overline{\mathrm{we}}}_2) \\
& - \bar{\bar{T}}^L(\overline{\overline{\mathrm{we}}}_1)\bar{\bar{T}}^L(\overline{\overline{\mathrm{we}}}_2), \bar{\bar{T}}^U(\overline{\overline{\mathrm{we}}}_1) + \bar{\bar{T}}^U(\overline{\overline{\mathrm{we}}}_2) \\
& - \bar{\bar{T}}^U(\overline{\overline{\mathrm{we}}}_1)\bar{\bar{T}}^U(\overline{\overline{\mathrm{we}}}_2)\Bigg], \Bigg[\bar{\bar{I}}^L(\overline{\overline{\mathrm{we}}}_1)\bar{\bar{I}}^L(\overline{\overline{\mathrm{we}}}_2), \bar{\bar{I}}^U \\
& \cdot (\overline{\overline{\mathrm{we}}}_1)\bar{\bar{I}}^U(\overline{\overline{\mathrm{we}}}_2)\Bigg], \Bigg[\bar{\bar{F}}^L(\overline{\overline{\mathrm{we}}}_1)\bar{\bar{F}}^L(\overline{\overline{\mathrm{we}}}_2), \bar{\bar{F}}^U \\
& \cdot (\overline{\overline{\mathrm{we}}}_1)\bar{\bar{F}}^U(\overline{\overline{\mathrm{we}}}_2)\Bigg] \Bigg\rangle;
\end{aligned}
\tag{2}
$$

$$
\begin{aligned}
(2)\overline{\overline{\mathrm{we}}}_1 \times \overline{\overline{\mathrm{we}}}_2 = \Bigg\langle & s_{\vartheta(\overline{\overline{\mathrm{we}}}_1)\times\vartheta(\overline{\overline{\mathrm{we}}}_1)}, \Bigg(, \Bigg[\bar{\bar{T}}^L(\overline{\overline{\mathrm{we}}}_1)\bar{\bar{T}}^L(\overline{\overline{\mathrm{we}}}_2), \bar{\bar{T}}^U(\overline{\overline{\mathrm{we}}}_1)\bar{\bar{T}}^U(\overline{\overline{\mathrm{we}}}_2)\Bigg] \\
& \cdot \Bigg[\bar{\bar{I}}^L(\overline{\overline{\mathrm{we}}}_1) + \bar{\bar{I}}^L(\overline{\overline{\mathrm{we}}}_2) - \bar{\bar{I}}^L(\overline{\overline{\mathrm{we}}}_1)\bar{\bar{I}}^L(\overline{\overline{\mathrm{we}}}_2), \bar{\bar{I}}^U \\
& \cdot (\overline{\overline{\mathrm{we}}}_1) + \bar{\bar{I}}^U(\overline{\overline{\mathrm{we}}}_2) - \bar{\bar{I}}^U(\overline{\overline{\mathrm{we}}}_1)\bar{\bar{I}}^U(\overline{\overline{\mathrm{we}}}_2)\Bigg], \\
& \cdot \Bigg[\bar{\bar{F}}^L(\overline{\overline{\mathrm{we}}}_1) + \bar{\bar{F}}^L(\overline{\overline{\mathrm{we}}}_2) - \bar{\bar{F}}^L(\overline{\overline{\mathrm{we}}}_1)\bar{\bar{F}}^L(\overline{\overline{\mathrm{we}}}_2), \bar{\bar{F}}^U(\overline{\overline{\mathrm{we}}}_1) \\
& + \bar{\bar{F}}^U(\overline{\overline{\mathrm{we}}}_2) - \bar{\bar{F}}^U(\overline{\overline{\mathrm{we}}}_1)\bar{\bar{F}}^U(\overline{\overline{\mathrm{we}}}_2)\Bigg] \Bigg\rangle;
\end{aligned}
\tag{3}
$$

$$
\begin{aligned}
(3)\bar{\bar{\xi}}\overline{\overline{\mathrm{we}}} = \Bigg\langle & s_{\bar{\bar{\xi}}\vartheta(\overline{\overline{\mathrm{we}}})} \Bigg(\Bigg[1 - \left(1 - \bar{\bar{T}}^L(\overline{\overline{\mathrm{we}}})\right)^{\bar{\bar{\xi}}}, 1 - \left(1 - \bar{\bar{T}}^U(\overline{\overline{\mathrm{we}}})\right)^{\bar{\bar{\xi}}} \Bigg], \\
& \cdot \Bigg[\left(\bar{\bar{I}}^L(\overline{\overline{\mathrm{we}}})\right)^{\bar{\bar{\xi}}}, \left(\bar{\bar{I}}^U(\overline{\overline{\mathrm{we}}})\right)^{\bar{\bar{\xi}}} \Bigg], \Bigg[\left(\bar{\bar{F}}^L(\overline{\overline{\mathrm{we}}})\right)^{\bar{\bar{\xi}}}, \left(\bar{\bar{F}}^U(\overline{\overline{\mathrm{we}}})\right)^{\bar{\bar{\xi}}} \Bigg] \Bigg\rangle;
\end{aligned}
\tag{4}
$$

$$
\begin{aligned}
(4)\overline{\overline{\mathrm{we}}}^{\bar{\bar{\xi}}} = \Bigg\langle & s_{\vartheta^{\bar{\bar{\xi}}}(\overline{\overline{\mathrm{we}}})} \Bigg(\Bigg[\left(\bar{\bar{T}}^L(\overline{\overline{\mathrm{we}}})\right)^{\bar{\bar{\xi}}}, \left(\bar{\bar{T}}^U(\overline{\overline{\mathrm{we}}})\right)^{\bar{\bar{\xi}}} \Bigg], \\
& \cdot \Bigg[1 - \left(1 - \bar{\bar{I}}^L(\overline{\overline{\mathrm{we}}})\right)^{\bar{\bar{\xi}}}, 1 - \left(1 - \bar{\bar{I}}^U(\overline{\overline{\mathrm{we}}})\right)^{\bar{\bar{\xi}}} \Bigg], \\
& \cdot \Bigg[1 - \left(1 - \bar{\bar{F}}^L(\overline{\overline{\mathrm{we}}})\right)^{\bar{\bar{\xi}}}, 1 - \left(1 - \bar{\bar{F}}^U(\overline{\overline{\mathrm{we}}})\right)^{\bar{\bar{\xi}}} \Bigg] \Bigg\rangle.
\end{aligned}
\tag{5}
$$

2.2. Hesitant Fuzzy Set

Definition 3 (see [33, 34]). Let Ω be a fixed set, a HFS HF on Ω is an object of the form

$$
HF = \{\langle v, hf_{HF}(v)\rangle \mid v \in \Omega\}, \tag{6}
$$

where $hf_{HF}(v) = \bigcup_{\Theta_{HF}(v)\in hf_{HF}(v)} \{\Theta_{HF}(v)\}$ is a group of finite values in [0, 1], signifying the possible MED of an element $v \in \Omega$ to HF. For ease, we shall inscribe hf as a replacement for $hf_{HF}(v) = \bigcup_{\Theta_{HF}(v)\in hf_{HF}(v)} \{\Theta_{HF}(v)\}$ and is called a hesitant fuzzy element.

Let hf, hf_1 and hf_2 be HFEs, then, the core operational laws for HFEs are identified below:

$$
\begin{aligned}
hf^{\xi} &= \bigcup_{\Theta\in hf} \left\{\Theta^{\xi}\right\}, \xi > 0; \\
\xi hf &= \bigcup_{\Theta\in hf} \left\{1 - (1 - \Theta)^{\xi}\right\}, \xi > 0; \\
hf_1 \oplus hf_2 &= \bigcup_{\Theta_1\in hf_1, \Theta_2\in hf_2} \{\Theta_1 + \Theta_2 - \Theta_1\Theta_2\}; \\
hf_1 \otimes hf_2 &= \bigcup_{\Theta_1\in hf_1, \Theta_2\in hf_2} \{\Theta_1\Theta_2\}.
\end{aligned}
\tag{7}
$$

2.3. Linguistic Scale Function. Linguistic scale functions (LSFs) apply various semantic values to linguistic scale under different conditions to make data more effective and to describe semantics more flexibly [46]. In practice, these functions are preferred because they are more versatile and may produce more predictable outputs based on varied meanings.

Definition 4 (see [46]). If $\bar{\bar{\vartheta}}_{\bar{\bar{z}}} \in [0, 1]$ is a numeric value, then, the LSF f^* that demeanor the mapping from $S_{\bar{\bar{z}}}$ to $\bar{\bar{\vartheta}}_{\bar{\bar{z}}}(\bar{\bar{z}} = 0, 1, \cdots, 2r)$ can be identified as follows: $f^* : s_{\bar{\bar{z}}} \longrightarrow \bar{\bar{\vartheta}}_{\bar{\bar{z}}}(\bar{\bar{z}} = 1, \cdots, 2r)$,

(i) Consider

$$
f_1^*(s_{\bar{\bar{z}}}) = \vartheta_{\bar{\bar{z}}} = \frac{\bar{\bar{z}}}{2r}. \tag{8}
$$

where $0 \leq \bar{\bar{\vartheta}}_0 < \bar{\bar{\vartheta}}_1 < . \cdots < \bar{\bar{\vartheta}}_{2r}$. Evidently, the symbol $\bar{\bar{\vartheta}}_{\bar{\bar{z}}}$ $(\bar{\bar{z}} = 0, 1, \cdots, 2r)$ reflects the preference of the decision makers while they are using the linguistic term $s_{\bar{\bar{z}}} \in S(\bar{\bar{z}} = 0, 1, \cdots, 2r)$.. Consequently, the function value in fact denotes the semantics of the linguistic terms.

On average, the assessment scale for the linguistic information presented above is divided.

(ii) Consider

$$f_2^*(s_{\bar{\bar{z}}}) = \bar{\bar{\vartheta}}_{\bar{\bar{z}}} = \begin{cases} \dfrac{\bar{\bar{\phi}}^r - \bar{\bar{\phi}}^{r-\bar{\bar{z}}}}{2\bar{\bar{\phi}}^r - 2} \left(\bar{\bar{z}} = 0, 1, \cdots, r\right) \\ \dfrac{\bar{\bar{\phi}}^r - \bar{\bar{\phi}}^{\bar{\bar{z}}-r} - 2}{2\bar{\bar{\phi}}^r - 2} \left(\bar{\bar{z}} = r+1, r+2 \cdots, 2r\right). \end{cases} \tag{9}$$

The absolute deviation between neighboring linguistic subscripts rises as the length of the supplied linguistic term set is extended from the middle to both ends.

(iii) Consider

$$f_3^*(s_{\bar{\bar{z}}}) = \bar{\bar{\vartheta}}_{\bar{\bar{z}}} = \begin{cases} \dfrac{r^\alpha - (r - \bar{\bar{z}})^\alpha}{2r^\alpha} \left(\bar{\bar{z}} = 0, 1, \cdots, r\right) \\ \dfrac{r^\beta - (\bar{\bar{z}} - r)^\beta}{2r^\beta} \left(\bar{\bar{z}} = r+1, r+2 \cdots, 2r\right). \end{cases} \tag{10}$$

The absolute deviation between consecutive linguistic subscripts will decrease when the extension from the centre of the supplied linguistic term to both ends is increased.

The above function may be extended to keep all of the provided data and make the computation easier $f^* : S \longrightarrow R^+ (R^+ = \{c \mid c \geq 0, c \in R\})$, which satisfies $f^*(s_{\bar{\bar{z}}}) = \bar{\bar{\vartheta}}_{\bar{\bar{z}}}$ and is a strictly monotonically increasing and continuous function. Therefore, the mapping from S to R^+ is one-to-one because of its monotonicity, and the inverse function of f^* exists and is denoted by f^{*-1}.

2.4. Hesitant Interval Neutrosophic Linguistic Set

Definition 5 (see [32]). Let Ω be the domain set. Then, a HINLS in Ω is characterized by the following mathematical form:

$$\overline{\overline{HN}} = \left\{ \left\langle v, \overline{\overline{hn}}(v) \right\rangle \mid v \in \Omega \right\}, \tag{11}$$

where $\overline{\overline{hn}}(v) = \bigcup_{\overline{\overline{\mathrm{we}}}(v) \in \overline{\overline{hn}}(v)} \{\overline{\overline{\mathrm{we}}}(v)\}$ is a group of INLNs, representing the possible INLNs of the element $v \in \Omega$ to the set $\overline{\overline{HN}}$ is an INLN. For ease, we shall inscribe $\overline{\overline{hn}} = \bigcup_{\overline{\overline{\mathrm{we}}} \in \overline{\overline{hn}}} \{\overline{\overline{\mathrm{we}}}\}$ as a replacement for $\overline{\overline{hn}}(v) = \bigcup_{\overline{\overline{\mathrm{we}}}(v) \in \overline{\overline{hn}}(v)} \{\overline{\overline{\mathrm{we}}}(v)\}$ in $\overline{\overline{HN}}$. Here we identify $\overline{\overline{hn}}$ a HINULE and $\overline{\overline{\mathrm{we}}} = \langle s_{\vartheta(\overline{\overline{\mathrm{we}}})}, ([\overline{\overline{Tr}}^L(\overline{\overline{\mathrm{we}}}), \overline{\overline{Tr}}^U(\overline{\overline{\mathrm{we}}})], [\overline{\overline{\mathrm{In}}}^L(\overline{\overline{\mathrm{we}}}), \overline{\overline{\mathrm{In}}}^U(\overline{\overline{\mathrm{we}}})], [\overline{\overline{Fl}}^L(\overline{\overline{\mathrm{we}}}), \overline{\overline{Fl}}^U(\overline{\overline{\mathrm{we}}})]) \rangle$ is called an INLN. Then, $\overline{\overline{HN}}$ is the group of all INLNs.

Definition 6 (see [32]). Let $\overline{\overline{hn}}_1, \overline{\overline{hn}}_2$ and $\overline{\overline{hn}}_3$ be any three HINLNs and $\bar{\bar{\xi}} \geq 0$. Then, some core operational rules for HINLNs are described as follows:

$$\begin{aligned} (1)\overline{\overline{hn}}_1 \oplus \overline{\overline{hn}}_2 = \bigcup_{\overline{\overline{\mathrm{we}}}_1 \in \overline{\overline{hn}}_1, \bar{\bar{u}}\mathrm{we} \in \overline{\overline{hn}}_2} \Bigg\langle & s_{\vartheta(\overline{\overline{\mathrm{we}}}_1) + \vartheta(\overline{\overline{\mathrm{we}}}_2)}, \Big(\Big[\bar{\bar{T}}^L(\overline{\overline{\mathrm{we}}}_1) + \bar{\bar{T}}^L(\overline{\overline{\mathrm{we}}}_2) \\ & - \bar{\bar{T}}^L(\overline{\overline{\mathrm{we}}}_1) \bar{\bar{T}}^L(\overline{\overline{\mathrm{we}}}_2), \bar{\bar{T}}^U(\overline{\overline{\mathrm{we}}}_1) + \bar{\bar{T}}^U(\overline{\overline{\mathrm{we}}}_2) \\ & - \bar{\bar{T}}^U(\overline{\overline{\mathrm{we}}}_1) \bar{\bar{T}}^U(\overline{\overline{\mathrm{we}}}_2) \Big], \Big[\bar{\bar{I}}^L(\overline{\overline{\mathrm{we}}}_1) \bar{\bar{I}}^L(\overline{\overline{\mathrm{we}}}_2), \bar{\bar{I}}^U \\ & \cdot (\overline{\overline{\mathrm{we}}}_1) \bar{\bar{I}}^U(\overline{\overline{\mathrm{we}}}_2) \Big], \Big[\bar{\bar{F}}^L(\overline{\overline{\mathrm{we}}}_1) \bar{\bar{F}}^L(\overline{\overline{\mathrm{we}}}_2), \bar{\bar{F}}^U \\ & \cdot (\overline{\overline{\mathrm{we}}}_1) \bar{\bar{F}}^U(\overline{\overline{\mathrm{we}}}_2) \Big] \Big) \Bigg\rangle; \end{aligned} \tag{12}$$

$$\begin{aligned} (2)\overline{\overline{hn}}_1 \otimes \overline{\overline{hn}}_2 = \bigcup_{\overline{\overline{\mathrm{we}}}_1 \in \overline{\overline{hn}}_1, \overline{\overline{\mathrm{we}}}_2 \in \overline{\overline{hn}}_2} \Bigg\langle & s_{\vartheta(\overline{\overline{\mathrm{we}}}_1) \times \vartheta(\overline{\overline{\mathrm{we}}}_2)}, \\ & \cdot \Big(\Big[\bar{\bar{T}}^L(\overline{\overline{\mathrm{we}}}_1) \bar{\bar{T}}^L(\overline{\overline{\mathrm{we}}}_2), \bar{\bar{T}}^U(\overline{\overline{\mathrm{we}}}_1) \bar{\bar{T}}^U(\overline{\overline{\mathrm{we}}}_2) \Big], \\ & \cdot \Big[\bar{\bar{I}}^L(\overline{\overline{\mathrm{we}}}_1) + \bar{\bar{I}}^L(\overline{\overline{\mathrm{we}}}_2) - \bar{\bar{I}}^L(\overline{\overline{\mathrm{we}}}_1) \bar{\bar{I}}^L(\overline{\overline{\mathrm{we}}}_2), \bar{\bar{I}}^U(\overline{\overline{\mathrm{we}}}_1) \\ & + \bar{\bar{I}}^U(\overline{\overline{\mathrm{we}}}_2) - \bar{\bar{I}}^U(\overline{\overline{\mathrm{we}}}_1) \bar{\bar{I}}^U(\overline{\overline{\mathrm{we}}}_2) \Big], \\ & \cdot \Big[\bar{\bar{F}}^L(\overline{\overline{\mathrm{we}}}_1) + \bar{\bar{F}}^L(\overline{\overline{\mathrm{we}}}_2) - \bar{\bar{F}}^L(\overline{\overline{\mathrm{we}}}_1) \bar{\bar{F}}^L(\overline{\overline{\mathrm{we}}}_2), \bar{\bar{F}}^U(\overline{\overline{\mathrm{we}}}_1) \\ & + \bar{\bar{F}}^U(\overline{\overline{\mathrm{we}}}_2) - \bar{\bar{F}}^U(\overline{\overline{\mathrm{we}}}_1) \bar{\bar{F}}^U(\overline{\overline{\mathrm{we}}}_2) \Big] \Big) \Bigg\rangle \end{aligned} \tag{13}$$

$$\begin{aligned} (3)\bar{\bar{\xi}}\overline{\overline{hn}} = \bigcup_{\overline{\overline{\mathrm{we}}} \in \overline{\overline{hn}}} \Bigg\langle & s_{\vartheta(\overline{\overline{\mathrm{we}}})}, \Bigg(\Bigg[1 - \left(1 - \bar{\bar{T}}^L(\overline{\overline{\mathrm{we}}})\right)^{\bar{\bar{\xi}}}, 1 - \left(1 - \bar{\bar{T}}^U(\overline{\overline{\mathrm{we}}})\right)^{\bar{\bar{\xi}}} \Bigg], \\ & \cdot \Bigg[\left(\bar{\bar{I}}^L(\overline{\overline{\mathrm{we}}})\right)^{\bar{\bar{\xi}}}, \left(\bar{\bar{I}}^U(\overline{\overline{\mathrm{we}}})\right)^{\bar{\bar{\xi}}} \Bigg], \Bigg[\left(\bar{\bar{F}}^L(\overline{\overline{\mathrm{we}}})\right)^{\bar{\bar{\xi}}}, \left(\bar{\bar{F}}^U(\overline{\overline{\mathrm{we}}})\right)^{\bar{\bar{\xi}}} \Bigg] \Bigg) \Bigg\rangle; \end{aligned} \tag{14}$$

$$\begin{aligned} (4)\overline{\overline{hn}}^{\bar{\bar{\xi}}} = \bigcup_{\overline{\overline{\mathrm{we}}} \in \overline{\overline{hn}}} \Bigg\langle & s_{\vartheta(\overline{\overline{\mathrm{we}}})}, \Bigg(\Bigg[\left(\bar{\bar{T}}^L(\overline{\overline{\mathrm{we}}})\right)^{\bar{\bar{\xi}}}, \left(\bar{\bar{T}}^U(\overline{\overline{\mathrm{we}}})\right)^{\bar{\bar{\xi}}} \Bigg], \\ & \cdot \Bigg[1 - \left(1 - \bar{\bar{I}}^L(\overline{\overline{\mathrm{we}}})\right)^{\bar{\bar{\xi}}}, 1 - \left(1 - \bar{\bar{I}}^U(\overline{\overline{\mathrm{we}}})\right)^{\bar{\bar{\xi}}} \Bigg], \\ & \cdot \Bigg[1 - \left(1 - \bar{\bar{F}}^L(\overline{\overline{\mathrm{we}}})\right)^{\bar{\bar{\xi}}}, 1 - \left(1 - \bar{\bar{F}}^U(\overline{\overline{\mathrm{we}}})\right)^{\bar{\bar{\xi}}} \Bigg] \Bigg) \Bigg\rangle. \end{aligned} \tag{15}$$

Definition 7 (see [32]). Let $\overline{\overline{hn}}$ be a HINLN. Then, the score function is signified as follows:

$$Sre\left(\overline{\overline{hn}}\right) = \frac{1}{\neq \overline{\overline{hn}}} \sum_{\overline{\overline{\mathrm{we}}} \in \overline{\overline{hn}}} \frac{\left(4 + \bar{\bar{T}}^L(\overline{\overline{\mathrm{we}}}) - \bar{\bar{I}}^L(\overline{\overline{\mathrm{we}}}) - \bar{\bar{F}}^L(\overline{\overline{\mathrm{we}}}) + \bar{\bar{T}}^U(\overline{\overline{\mathrm{we}}}) - \bar{\bar{I}}^U(\overline{\overline{\mathrm{we}}}) - \bar{\bar{F}}^U(\overline{\overline{\mathrm{we}}})\right) \overline{\overline{\mathrm{we}}}(\vartheta)}{6l}, \tag{16}$$

where $\neq \overline{\overline{hn}}$ is the number INLNs in $\overline{\overline{hn}}$, ând $l+1$ is the cardinality of the linguistic term set S.

The score and accuracy function identified by Ye [32] have some limitation in some special cases for comparing two HINLNs, and this can be shown in an example below.

Example 1. Let $S = \{s_0, s_1, s_2, s_3, s_4, s_5, s_6\} = \{$very poor, slightly poor, poor, fair, slightly good, good, very good$\}$,

$\overline{\overline{hn}}_1 = \{\langle s_4, ([0.6, 0.7], [0, 0], [0.4, 0.5])\rangle, \langle s_5, ([0.5, 0.6], [0, 0], [0.3, 0.4])\rangle\}$, and $\overline{\overline{hn}}_2 = \{\langle s_4, ([0.6, 0.7], [0.4, 0.5], [0, 0])\rangle, \langle s_5, ([0.5, 0.6], [0.3, 0.4], ,[0, 0])\rangle\}$ be two HINLNs. Then, by utilizing the above score function defined by Ye [32], we have

$$Sre\left(\overline{\overline{hn}}_1\right) = Sre\left(\overline{\overline{hn}}_2\right) = 0.5500. \tag{17}$$

Which shows that $\overline{\overline{hn}}_1 = \overline{\overline{hn}}_2$. However, $\overline{\overline{hn}}_2$ is greater than $\overline{\overline{hn}}_1$.

To overcome the above existing limitation identified in an example, we signified new score function based on linguistic scale functions for comparing HINLNs.

Definition 8. Let $\overline{\overline{hn}}$ be a HINLN. Then, the improved score function can be signified as

$$Scr\left(\overline{\overline{hn}}\right) = \frac{1}{\neq \overline{\overline{hn}}}\left(\sum_{\overline{\overline{we}}\epsilon\overline{\overline{hn}}} \begin{matrix} \chi f^*\left(s_{\vartheta(u)}\right)\left(0.5\left(\bar{\bar{T}}^U\left(\overline{\overline{we}}\right) + 1 - \bar{\bar{F}}^L\left(\overline{\overline{we}}\right) + \chi\bar{\bar{I}}^U\left(\overline{\overline{we}}\right)\right)\right) + \\ (1-\chi)f^*\left(s_{\vartheta(u)}\right)\left(0.5\left(\bar{\bar{T}}^L\left(\overline{\overline{we}}\right) + 1 - \bar{\bar{F}}^U\left(\overline{\overline{we}}\right)\right) + \chi\bar{\bar{I}}^L\left(\overline{\overline{we}}\right)\right) \end{matrix}\right), \tag{18}$$

where the values of a $\chi \in [0, 1]$ indicate the decision-makers' views, and a $\chi > 0.5$, $\chi = 0.5$, and $\chi < 0.5$ denote the decision-makers' levels of optimist, temperance, and pessimist. Furthermore, by using various linguistic scale functions, alternative scoring functions can be produced.

Definition 9. Let $\overline{\overline{hn}}_1$ and $\overline{\overline{hn}}_2$ be two HINLNs. Then, the comparison rules for comparing two HINLNs are identified as follows:

(1) If $Scr(\overline{\overline{hn}}_1) > Scr(\overline{\overline{hn}}_2)$, then $\overline{\overline{hn}}_1 > \overline{\overline{hn}}_2$

(2) If $Scr(\overline{\overline{hn}}_1) < Scr(\overline{\overline{hn}}_2)$, then $\overline{\overline{hn}}_1 < \overline{\overline{hn}}_2$

(3) If $Scr(\overline{\overline{hn}}_1) = Scr(\overline{\overline{hn}}_2)$, then $\overline{\overline{hn}}_1 = \overline{\overline{hn}}_2$

Now, utilizing the improve score function to solve Example 1 and assume that $f^*(s_\vartheta) = i/2l$, $\chi = 0.5$, we have $Scr(\overline{\overline{hn}}_1) = 0.4500$ and $Scr(\overline{\overline{hn}}_2) = 0.8146$.

From the score values, we can observe that $\overline{\overline{hn}}_2$ is greater than $\overline{\overline{hn}}_1$.

Definition 10. Let $\overline{\overline{hn}}_1$ and $\overline{\overline{hn}}_2$ be any two HINLNs. Then, the Hamming distance between $\overline{\overline{hn}}_1$ and $\overline{\overline{hn}}_2$ can be described as

$$\begin{aligned} \overline{\overline{dis}}\left(\overline{\overline{hn}}_1, \overline{\overline{hn}}_2\right) = \frac{1}{\neq len}\sum_{i=1}^{l}\Bigg(&\left|f^*\left(s_{\vartheta\left(\overline{\overline{we}}_1\right)}\right) - f^*\left(s_{\vartheta\left(\overline{\overline{we}}_2\right)}\right)\right| \\ &+ \left(\frac{1}{6}\left(\left|f^*\left(s_{\vartheta\left(\overline{\overline{we}}_1\right)}\right)\bar{\bar{T}}^L\left(\overline{\overline{we}}_1\right) - f^*\left(s_{\vartheta\left(\overline{\overline{we}}_2\right)}\right)\bar{\bar{T}}^L\left(\overline{\overline{we}}_2\right)\right|\right. \\ &+ \left|f^*\left(s_{\vartheta\left(\overline{\overline{we}}_1\right)}\right)\bar{\bar{T}}^U\left(\overline{\overline{we}}_1\right) - f^*\left(s_{\vartheta\left(\overline{\overline{we}}_2\right)}\right)\bar{\bar{T}}^U\left(\overline{\overline{we}}_2\right)\right| \\ &+ \left|f^*\left(s_{\vartheta\left(\overline{\overline{we}}_1\right)}\right)\left(1 - \bar{\bar{I}}^L\left(\overline{\overline{we}}_1\right)\right) - f^*\left(s_{\vartheta\left(\overline{\overline{we}}_2\right)}\right)\right. \\ &\cdot\left.\left(1 - \bar{\bar{I}}^L\left(\overline{\overline{we}}_2\right)\right)\right| + \left|f^*\left(s_{\vartheta\left(\overline{\overline{we}}_1\right)}\right)\left(1 - \bar{\bar{I}}^U\left(\overline{\overline{we}}_1\right)\right)\right. \\ &- \left.f^*\left(s_{\vartheta\left(\overline{\overline{we}}_2\right)}\right)\left(1 - \bar{\bar{I}}^U\left(\overline{\overline{we}}_2\right)\right)\right| + \left|f^*\left(s_{\vartheta\left(\overline{\overline{we}}_1\right)}\right)\right. \\ &\cdot\left.\left(1 - \bar{\bar{F}}^L\left(\overline{\overline{we}}_1\right)\right) - f^*\left(s_{\vartheta\left(\overline{\overline{we}}_2\right)}\right)\left(1 - \bar{\bar{F}}^L\left(\overline{\overline{we}}_2\right)\right)\right| \\ &+ \left|f^*\left(s_{\vartheta\left(\overline{\overline{we}}_1\right)}\right)\left(1 - \bar{\bar{F}}^U\left(\overline{\overline{we}}_1\right)\right)\right. \\ &\left.\left.\left. - f^*\left(s_{\vartheta\left(\overline{\overline{we}}_2\right)}\right)\left(1 - \bar{\bar{F}}^U\left(\overline{\overline{we}}_2\right)\right)\right|\right)\right)\Bigg) \end{aligned} \tag{19}$$

2.5. The PA Operator. PA operator initiated by Yager [35] is one of the imperative AOs. The PA operator reduces a number of unconstructive influences of unreasonably high or unreasonably low arguments given by DMs. The conservative PA operator can only contract with real numbers and is identified as follows.

Definition 11 (see [35]). Let $\{\overline{\overline{we}}_1, \overline{\overline{we}}_2, \cdots, \overline{\overline{we}}_z\}$ be a group of positive real numbers. A PA operator is classified as follows:

$$\mathrm{PA}\left(\overline{\overline{we}}_1, \overline{\overline{we}}_2, \cdots, \overline{\overline{we}}_z\right) = \frac{\sum_{i=1}^{z}\left(1 + T\left(\overline{\overline{we}}_i\right)\right)\overline{\overline{we}}_i}{\sum_{i=1}^{z}\left(1 + T\left(\overline{\overline{we}}_i\right)\right)}, \tag{20}$$

where $T(\overline{\overline{we}}_i) = \sum_{\substack{i=1 \\ j\neq i}}^{z} Sup(\overline{\overline{we}}_i, \overline{\overline{we}}_j)$, and $Sup(\overline{\overline{we}}_i, \overline{\overline{we}}_j)$ are the support degree (SPD) for $\overline{\overline{we}}_i$ from $\overline{\overline{we}}_j$ satisfying the following axioms:

(1) $Sup(\overline{\overline{we}}_i, \overline{\overline{we}}_j) \in [0,1]$;

(2) $Sup(\overline{\overline{we}}_i, \overline{\overline{we}}_j) = Sup(\overline{\overline{we}}_j, \overline{\overline{we}}_i)$;

(3) $Sup(\overline{\overline{we}}_i, \overline{\overline{we}}_j) \geq Sup(\overline{\overline{we}}_p, \overline{\overline{we}}_q)$, if $|(\overline{\overline{we}}_i, \overline{\overline{we}}_j)| < |(\overline{\overline{we}}_p, \overline{\overline{we}}_q)|$

Definition 12 (see [36]). Let $\{\overline{\overline{we}}_1, \overline{\overline{we}}_2, \cdots, \overline{\overline{we}}_z\}$ be a set of positive real numbers. A PG operator is described as follows:

$$PG\left(\overline{\overline{we}}_1, \overline{\overline{we}}_2, \cdots, \overline{\overline{we}}_z\right) = \prod_{i=1}^{z} \overline{\overline{we}}_i^{\frac{\left(1+T\left(\overline{\overline{we}}_i\right)\right)}{\sum_{i=1}^{z}\left(1+T\left(\overline{\overline{we}}_i\right)\right)}}, \tag{21}$$

where $T(\overline{\overline{we}}_i) = \sum_{\substack{i=1 \\ j\neq i}}^{z} Sup(\overline{\overline{we}}_i, \overline{\overline{we}}_j)$, and $Sup(\overline{\overline{we}}_i, \overline{\overline{we}}_j)$ are the SPD for $\overline{\overline{we}}_i$ from $\overline{\overline{we}}_j$ satisfying the above axioms.

Definition 13 (see [37]). Let $\{\overline{\overline{we}}_1, \overline{\overline{we}}_2, \cdots, \overline{\overline{we}}_Z\}$ be a group of positive real numbers. A WGPA operator is described as follows:

$$\mathrm{WGPA}\left(\overline{\overline{we}}_1, \overline{\overline{we}}_2, \cdots, \overline{\overline{we}}_z\right) = \left(\frac{\sum_{i=1}^{z}\left(1+T\left(\overline{\overline{we}}_i\right)\right)\overline{\overline{we}}_i^{\bar{\bar{\mathfrak{A}}}}}{\sum_{i=1}^{z}\left(1+T\left(\overline{\overline{we}}_i\right)\right)}\right)^{\frac{1}{\bar{\bar{\mathfrak{A}}}}}, \tag{22}$$

where $T(\overline{\overline{we}}_G) = \sum_{\substack{G=1 \\ H\neq G}}^{Z} Sup(\overline{\overline{we}}_G, \overline{\overline{we}}_H)$, and $Sup(\overline{\overline{we}}_G, \overline{\overline{we}}_H)$ are the SPD for $\overline{\overline{we}}_G$ from $\overline{\overline{we}}_H$ satisfying the following axioms.

3. SS Operational Laws for HINLNs

The SS operations [47] contain SS product and SS addition, which are exacting cases of Archemedean t-norm and t-conorm.

The SS t-norm and t-conorm are elucidated as follows:

$$\begin{aligned} \bar{\bar{\Theta}}_{SS,\varsigma}\left(\bar{\bar{\alpha}}, \bar{\bar{\beta}}\right) &= \left(\bar{\bar{\alpha}}^{\varsigma} + \bar{\bar{\beta}}^{\varsigma} - 1\right)^{1/\varsigma}, \\ \bar{\bar{\Theta}}^{*}_{SS,\varsigma}\left(\bar{\bar{\alpha}}, \bar{\bar{\beta}}\right) &= 1 - \left((1-\bar{\bar{\alpha}})^{\varsigma} + \left(1-\bar{\bar{\beta}}\right)^{\varsigma} - 1\right)^{1/\varsigma}, \end{aligned} \tag{23}$$

where $\varsigma < 0, \bar{\bar{\alpha}}, \bar{\bar{\beta}} \in [0,1]$.

Moreover, when $\varsigma = 0$, SS t-norm and SS t-conorm degenerate into algebraic t-normn and t-conorm.

Based on t-norm and t-conorm of SS operations, we can provide the following definition about SS operations for HINLNs.

Definition 14. Let hn, hn_1, hn_2 be any three HINLNS, and $\xi > 0$. Then, we initiate some core operational laws for HINLNs based on Schweizer-Sklar t-norm and t-co-norm.

$$\begin{aligned} (1)\overline{\overline{hn}}_1 \oplus \overline{\overline{hn}}_2 = & \bigcup_{\overline{\overline{we}}_1 \epsilon hn_1, \overline{\overline{we}}_2 \epsilon hn_2} \left\langle f^{*-1}\left(f^{*}\left(s_{\vartheta\left(\overline{\overline{we}}_1\right)}\right) + f^{*}\left(s_{\vartheta\left(\overline{\overline{we}}_2\right)}\right)\right), \right. \\ & \cdot \left(\left[1 - \left(\left(1-\bar{\bar{T}}_1^{L}\right)^{\varsigma} + \left(1-\bar{\bar{T}}_2^{L}\right)^{\varsigma} - 1\right)^{1/\varsigma}, 1 \right.\right. \\ & \left. - \left(\left(1-\bar{\bar{T}}_1^{U}\right)^{\varsigma} + \left(1-\bar{\bar{T}}_2^{U}\right)^{\varsigma} - 1\right)^{1/\varsigma}\right], \\ & \cdot \left[\left(\left(\bar{\bar{I}}_1^{L}\right)^{\varsigma} + \left(\bar{\bar{I}}_2^{L}\right)^{\varsigma} - 1\right)^{1/\varsigma}, \left(\left(\bar{\bar{I}}_1^{U}\right)^{\varsigma} + \left(\bar{\bar{I}}_2^{U}\right)^{\varsigma} - 1\right)^{1/\varsigma}\right], \\ & \left. \cdot \left[\left(\left(\bar{\bar{F}}_1^{L}\right)^{\varsigma} + \left(\bar{\bar{F}}_2^{L}\right)^{\varsigma} - 1\right)^{1/\varsigma}, \left(\left(\bar{\bar{F}}_1^{U}\right)^{\varsigma} + \left(\bar{\bar{F}}_2^{U}\right)^{\varsigma} - 1\right)^{1/\varsigma}\right]\right\rangle; \end{aligned} \tag{24}$$

$$\begin{aligned} (2)\overline{\overline{hn}}_1 \oplus \overline{\overline{hn}}_2 = & \bigcup_{\overline{\overline{we}}_1 \epsilon \overline{\overline{hn}}_1, \overline{\overline{we}}_2 \epsilon \overline{\overline{hn}}_2} \left\langle f^{*-1}\left(f^{*}\left(s_{\vartheta\left(\overline{\overline{we}}_1\right)}\right) + f^{*}\left(s_{\vartheta\left(\overline{\overline{we}}_2\right)}\right)\right), \right. \\ & \cdot \left(\left[\left(\left(\bar{\bar{T}}_1^{L}\right)^{\varsigma} + \left(\bar{\bar{T}}_2^{L}\right)^{\varsigma} - 1\right)^{1/\varsigma}, \left(\left(\bar{\bar{T}}_1^{U}\right)^{\varsigma} + \left(\bar{\bar{T}}_2^{U}\right)^{\varsigma} - 1\right)^{1/\varsigma}\right], \right. \\ & \cdot \left[1 - \left(\left(1-\bar{\bar{I}}_1^{L}\right)^{\varsigma} + \left(1-\bar{\bar{I}}_2^{L}\right)^{\varsigma} - 1\right)^{1/\varsigma} 1 \right. \\ & \left. - \left(\left(1-\bar{\bar{I}}_1^{U}\right)^{\varsigma} + \left(1-\bar{\bar{I}}_2^{U}\right)^{\varsigma} - 1\right)^{1/\varsigma}\right], \\ & \cdot \left[1 - \left(\left(1-\bar{\bar{F}}_1^{L}\right)^{\varsigma} + \left(1-\bar{\bar{F}}_2^{L}\right)^{\varsigma} - 1\right)^{1/\varsigma}, 1 \right. \\ & \left. \left. - \left(\left(1-\bar{\bar{F}}_1^{U}\right)^{\varsigma} + \left(1-\bar{\bar{F}}_2^{U}\right)^{\varsigma} - 1\right)^{1/\varsigma}\right]\right\rangle; \end{aligned} \tag{25}$$

$$\begin{aligned} (3)\xi\overline{\overline{hn}} = & \bigcup_{\overline{\overline{we}} \epsilon \overline{\overline{hn}}} \left\langle f^{*-1}\left(\xi\left(f^{*}\left(s_{\vartheta\left(\overline{\overline{we}}\right)}\right)\right)\right), \right. \\ & \cdot \left[1 - \left(\xi\left(1-\bar{\bar{T}}^{L}\right)^{\varsigma} - (\xi-1)\right)^{1/\varsigma}, 1 - \left(\xi\left(1-\bar{\bar{T}}^{U}\right)^{\varsigma} - (\xi-1)\right)^{1/\varsigma}\right], \\ & \cdot \left[\left(\xi\left(\bar{\bar{I}}^{L}\right)^{\varsigma} - (\xi-1)\right)^{1/\varsigma}, \left(\xi\left(\bar{\bar{I}}^{U}\right)^{\varsigma} - (\xi-1)\right)^{1/\varsigma}\right], \\ & \left. \cdot \left[\left(\xi\left(\bar{\bar{F}}^{L}\right)^{\varsigma} - (\xi-1)\right)^{1/\varsigma}, \left(\xi\left(\bar{\bar{F}}^{U}\right)^{\varsigma} - (\xi-1)\right)^{1/\varsigma}\right]\right\rangle, \xi > 0, \end{aligned} \tag{26}$$

$$\begin{aligned} (4)\overline{\overline{hn}}^{\xi} = & \bigcup_{\overline{\overline{we}} \epsilon \overline{\overline{hn}}} \left\langle f^{*-1}\left(\left(f^{*}\left(s_{\vartheta\left(\overline{\overline{we}}\right)}\right)\right)^{\xi}\right), \right. \\ & \cdot \left[\left(\xi\left(\bar{\bar{T}}^{L}\right)^{\varsigma} - (\xi-1)\right)^{1/\varsigma}, \left(\xi\left(\bar{\bar{T}}^{U}\right)^{\varsigma} - (\xi-1)\right)^{1/\varsigma}\right], \\ & \cdot \left[1 - \left(\xi\left(1-\bar{\bar{I}}^{L}\right)^{\varsigma} - (\xi-1)\right)^{1/\varsigma}, 1 - \left(\xi\left(1-\bar{\bar{I}}^{U}\right)^{\varsigma} - (\xi-1)\right)^{1/\varsigma}\right], \\ & \left. \cdot \left[1 - \left(\xi\left(1-\bar{\bar{F}}^{L}\right)^{\varsigma} - (\xi-1)\right)^{1/\varsigma}, 1 - \left(\xi\left(1-\bar{\bar{F}}^{L}\right)^{\varsigma} - (\xi-1)\right)^{1/\varsigma}\right]\right\rangle, \xi > 0. \end{aligned} \tag{27}$$

Theorem 15. *Let* $\overline{\overline{hn}}, \overline{\overline{hn}}_1$ *and* $\overline{\overline{hn}}_2$ *be any three HINLNs. Then*

$$\begin{aligned}
\overline{\overline{hn}}_1 \oplus_{SS} \overline{\overline{hn}}_2 &= \overline{\overline{hn}}_2 \oplus_{SS} \overline{\overline{hn}}_1,\\
\overline{\overline{hn}}_1 \otimes_{SS} \overline{\overline{hn}}_2 &= \overline{\overline{hn}}_2 \otimes_{SS} \overline{\overline{hn}}_1,\\
\xi\left(\overline{\overline{hn}}_1 \oplus_{SS} \overline{\overline{hn}}_2\right) &= \xi\overline{\overline{hn}}_1 \oplus_{SS} \xi\overline{\overline{hn}}_2, \xi \geq 0,\\
\xi_1\overline{\overline{hn}} \oplus_{SS} \xi_2\overline{\overline{hn}} &= (\xi_1 + \xi_2)\overline{\overline{hn}}, \xi_1, \xi_2 \geq 0,\\
\overline{\overline{hn}}^{\xi_1} \otimes_{SS} \overline{\overline{hn}}^{\xi_2} &= \left(\overline{\overline{hn}}\right)^{\xi_1+\xi_2}, \xi_1, \xi_2 \geq 0,\\
\overline{\overline{hn}}_1^{\xi} \otimes_{SS} \overline{\overline{hn}}_2^{\xi} &= \left(\overline{\overline{hn}}_1 \otimes_{SS} \overline{\overline{hn}}_2\right)^{\xi}, \xi > 0.
\end{aligned} \tag{28}$$

4. Some Generalized Power Aggregation Operators for HINLNs

In this part, we develop some generalized power AOs established on the initiated operational rules for HINLNs.

4.1. Weighted Generalized Hesitant Interval Neutrosophic Linguistic Schweizer-Sklar Power Aggregation Operator. In this subpart, we initiate generalized hesitant interval neutrosophic linguistic Schweizer-Sklar power average (GHINLSSPA) operator, weighted (WGHINLSSPA) operator and examine their enviable properties and various particular cases.

Definition 16. For a collection of HINLNs $\overline{\overline{hn}}_g (g = 1, 2, \cdots, s)$, GHINLSSPA operator is a function $\aleph^s \longrightarrow \aleph$,

$$GHINLSSPA_\zeta\left(\overline{\overline{hn}}_1, \overline{\overline{hn}}_2, \cdots, \overline{\overline{hn}}_s\right) = \left(\frac{\overset{s}{\underset{g=1}{\oplus_{SS}}}\left(\left(1 + T\left(\overline{\overline{hn}}_g\right)\right)\overline{\overline{hn}}_g^{\zeta}\right)}{\overset{s}{\underset{g=1}{\oplus}}\left(1 + T\left(\overline{\overline{hn}}_g\right)\right)}\right)^{1/\zeta}, \tag{29}$$

where $T(\overline{\overline{hn}}_g) = \overset{s}{\underset{\substack{h=1,\\ g\neq h}}{\oplus}} \sup(\overline{\overline{hn}}_g, \overline{\overline{hn}}_h)$, parameter $\zeta \in (0,+\infty)$ and $Sup(\overline{\overline{hn}}_g, \overline{\overline{hn}}_h)$ is the SPD for $\overline{\overline{hn}}_g$ from $\overline{\overline{hn}}_h$ with the following constraint:

(1) $Sup(\overline{\overline{hn}}_g, \overline{\overline{hn}}_h) \in [0, 1]$;

(2) $Sup(\overline{\overline{hn}}_g, \overline{\overline{hn}}_h) = Sup(\overline{\overline{hn}}_h, \overline{\overline{hn}}_g)$;

(3) $Sup(\overline{\overline{hn}}_g, \overline{\overline{if}}_h) \geq Sup(\overline{\overline{hn}}_u, \overline{\overline{hn}}_v)$ if $\overline{\overline{dis}}(\overline{\overline{hn}}_g, \overline{\overline{hn}}_h) < \overline{\overline{ds}}(\overline{\overline{hn}}_u, \overline{\overline{hn}}_v)$, where $\overline{\overline{dis}}$ is the distance measure among two HINLNs.

To write Equation (29) in unsophisticated form, we have

$$\overline{\overline{pw}}_g = \frac{\left(1+T\left(\overline{\overline{hn}}_g\right)\right)}{\oplus^{\sim s}_{g=1}\left(1 + T\left(\overline{\overline{hn}}_g\right)\right)}. \tag{30}$$

So, from Equation (30), Equation (29) becomes

$$GHINLSSPA_\zeta\left(\overline{\overline{hn}}_1, \overline{\overline{hn}}_2, \cdots, \overline{\overline{hn}}_s\right) = \left(\overset{s}{\underset{g=1}{\oplus_{SS}}} \overline{\overline{pw}}_g \overline{\overline{hn}}_g^{\zeta}\right)^{1/\zeta}, \tag{31}$$

Theorem 17. *Let $\overline{\overline{hn}}_g (g = 1, 2, \cdots, s)$ be a set of HINLNs, then the value aggregated utilizing Definition 16 is still HINLN, and we have*

$$\begin{aligned}
&GHINLSSPA_\zeta\left(\overline{\overline{hn}}_1, \overline{\overline{hn}}_2, \cdots, \overline{\overline{hn}}_s\right) = \underset{\overline{\overline{we}}_1 \in \overline{\overline{hn}}_1, \overline{\overline{we}}_2 \in \overline{\overline{hn}}_2, \cdots, \overline{\overline{we}}_s \in \overline{\overline{hn}}_s}{\cup}\\
&\cdot \left\langle f^{*-1}\left(\overset{s}{\underset{g=1}{\oplus}} \overline{\overline{pw}}_g \left(f^*\left(s_{\vartheta}\left(\overline{\overline{we}}_g\right)\right)\right)^{\zeta}\right)^{1/\zeta},\right.\\
&\cdot \left(\left[\left(\frac{1}{\zeta}\left(1 - \left(\overset{s}{\underset{g=1}{\oplus}} \overline{\overline{pw}}_g \left(1 - \left(\zeta\left(\overline{\overline{T}}_g^L\right)^{\varsigma} - (\zeta - 1)\right)^{1/\zeta}\right)^{\varsigma} - \overset{s}{\underset{g=1}{\oplus}} \overline{\overline{pw}}_g + 1\right)^{1/\zeta}\right)^{\varsigma}\right.\right.\\
&- \left(\frac{1}{\zeta} - 1\right)\Bigg)^{1/\zeta}, \left(\frac{1}{\zeta}\left(1 - \left(\overset{s}{\underset{g=1}{\oplus}} \overline{\overline{pw}}_g \left(1 - \left(\zeta\left(\overline{\overline{T}}_g^U\right)^{\varsigma} - (\zeta - 1)\right)^{1/\zeta}\right)^{\varsigma} - \overset{s}{\underset{g=1}{\oplus}} \overline{\overline{pw}}_g + 1\right)^{1/\zeta}\right)^{\varsigma}\right.\\
&\left.\left. - \left(\frac{1}{\zeta} - 1\right)\right)^{1/\zeta}\right], \left[1 - \left(\frac{1}{\zeta}\left(1 - \left(\overset{s}{\underset{g=1}{\oplus}} \overline{\overline{pw}}_g \left(1 - \left(\zeta\left(1 - \overline{\overline{I}}_g^L\right)^{\varsigma} - (\zeta - 1)\right)^{1/\zeta}\right)^{\varsigma} - \overset{s}{\underset{g=1}{\oplus}} \overline{\overline{pw}}_g + 1\right)^{1/\zeta}\right)^{\varsigma}\right.\right.\\
&\left.\left. - \left(\frac{1}{\zeta} - 1\right)\right)^{1/\zeta}, 1 - \left(\frac{1}{\zeta}\left(1 - \left(\overset{s}{\underset{g=1}{\oplus}} \overline{\overline{pw}}_g \left(1 - \left(\zeta\left(1 - \overline{\overline{I}}_g^U\right)^{\varsigma} - (\zeta - 1)\right)^{1/\zeta}\right)^{\varsigma} - \overset{s}{\underset{g=1}{\oplus}} \overline{\overline{pw}}_g + 1\right)^{1/\zeta}\right)^{\varsigma}\right.\right.\\
&\left.\left. - \left(\frac{1}{\zeta} - 1\right)\right)^{1/\zeta}\right], \left[1 - \left(\frac{1}{\zeta}\left(1 - \left(\overset{s}{\underset{g=1}{\oplus}} \overline{\overline{pw}}_g \left(1 - \left(\zeta\left(1 - \overline{\overline{F}}_g^L\right)^{\varsigma} - (\zeta - 1)\right)^{1/\zeta}\right)^{\varsigma} - \overset{s}{\underset{g=1}{\oplus}} \overline{\overline{pw}}_g + 1\right)^{1/\zeta}\right)^{\varsigma}\right.\right.\\
&\left.\left.\left. - \left(\frac{1}{\zeta} - 1\right)\right)^{1/\zeta}, 1 - \left(\frac{1}{\zeta}\left(1 - \left(\overset{s}{\underset{g=1}{\oplus}} \overline{\overline{pw}}_g \left(1 - \left(\zeta\left(1 - \overline{\overline{F}}_g^U\right)^{\varsigma} - (\zeta - 1)\right)^{1/\zeta}\right)^{\varsigma} - \overset{s}{\underset{g=1}{\oplus}} \overline{\overline{pw}}_g + 1\right)^{1/\zeta}\right)^{\varsigma} - \left(\frac{1}{\zeta} - 1\right)\right)^{1/\zeta}\right]\right)\right\rangle.
\end{aligned} \tag{32}$$

Proof. In the following, first, we prove

$$\begin{aligned}
&\overline{\overline{pw}}_1\overline{\overline{hn}}_1^{\zeta} \oplus \overline{\overline{pw}}_2\overline{\overline{hn}}_2^{\zeta} \oplus \cdots \oplus \overline{\overline{pw}}_s\overline{\overline{hn}}_s^{\zeta} = \bigcup_{u_1 \in \overline{\overline{hn}}_1, u_2 \in \overline{\overline{hn}}_2, \ldots \cdots, u_s \in \overline{\overline{hn}}_s} \\
&\cdot \left\langle f^{*-1}\left(\overset{s}{\underset{g=1}{\oplus}} \overline{\overline{pw}}_g \left(f^*\left(s_{\vartheta(u_g)}\right)\right)^{\zeta}\right), \left(\left[1 - \left(\overset{s}{\underset{g=1}{\oplus}} \overline{\overline{pw}}_g \left(1 - \left(\zeta\left(\overline{\overline{T}}_g^L\right)^{\varsigma} - (\zeta - 1)\right)^{1/\varsigma}\right)^{\varsigma} - \overset{s}{\underset{g=1}{\oplus}} \overline{\overline{pw}}_g + 1\right)^{1/\varsigma}, 1\right.\right.\right. \\
&\left. - \left(\overset{s}{\underset{g=1}{\oplus}} \overline{\overline{pw}}_g \left(1 - \left(\zeta\left(\overline{\overline{T}}_g^U\right)^{\varsigma} - (\zeta - 1)\right)^{1/\varsigma}\right)^{\varsigma} - \overset{s}{\underset{g=1}{\oplus}} \overline{\overline{pw}}_g + 1\right)^{1/\varsigma}\right], \left[\left(\overset{s}{\underset{g=1}{\oplus}} \overline{\overline{pw}}_g \left(1 - \left(\zeta\left(1 - \overline{\overline{I}}_g^L\right)^{\varsigma} - (\zeta - 1)\right)^{1/\varsigma}\right)^{\varsigma} - \overset{s}{\underset{g=1}{\oplus}} \overline{\overline{pw}}_g + 1\right)^{1/\varsigma},\right. \\
&\cdot \left.\left(\overset{s}{\underset{g=1}{\oplus}} \overline{\overline{pw}}_g \left(1 - \left(\zeta\left(1 - \overline{\overline{I}}_g^U\right)^{\varsigma} - (\zeta - 1)\right)^{1/\varsigma}\right)^{\varsigma} - \overset{s}{\underset{g=1}{\oplus}} \overline{\overline{pw}}_g + 1\right)^{1/\varsigma}\right], \left[\left(\overset{s}{\underset{g=1}{\oplus}} \overline{\overline{pw}}_g \left(1 - \left(\zeta\left(1 - \overline{\overline{F}}_g^L\right)^{\varsigma} - (\zeta - 1)\right)^{1/\varsigma}\right)^{\varsigma} - \overset{s}{\underset{g=1}{\oplus}} \overline{\overline{pw}}_g + 1\right)^{1/\varsigma},\right. \\
&\cdot \left.\left.\left.\left(\overset{s}{\underset{g=1}{\oplus}} \overline{\overline{pw}}_g \left(1 - \left(\zeta\left(1 - \overline{\overline{F}}_g^U\right)^{\varsigma} - (\zeta - 1)\right)^{1/\varsigma}\right)^{\varsigma} - \overset{s}{\underset{g=1}{\oplus}} \overline{\overline{pw}}_g + 1\right)^{1/\varsigma}\right]\right)\right\rangle,
\end{aligned} \tag{33}$$

by exploiting mathematical induction on s. □

From the operational laws explained for HINLNs in Definition 14, we have

$$\begin{aligned}
\overline{\overline{hn}}_1^{\zeta} = \bigcup_{u_1 \in \overline{\overline{hn}}_1} &\left\langle f^{*-1}\left(f^*\left(s_{\vartheta(u_1)}\right)\right)^{\zeta}, \left(\left[\left(\zeta\left(\overline{\overline{T}}_1^L\right)^{\varsigma} - (\zeta - 1)\right)^{1/\varsigma}, \left(\zeta\left(\overline{\overline{T}}_1^U\right)^{\varsigma} - (\zeta - 1)\right)^{1/\varsigma}\right]\right.\right. \\
&\cdot \left[1 - \left(\zeta\left(1 - \overline{\overline{I}}_1^L\right)^{\varsigma} - (\zeta - 1)\right)^{1/\varsigma}, 1 - \left(\zeta\left(1 - \overline{\overline{I}}_1^U\right)^{\varsigma} - (\zeta - 1)\right)^{1/\varsigma}\right], \\
&\cdot \left.\left.\left[1 - \left(\zeta\left(1 - \overline{\overline{F}}_1^L\right)^{\varsigma} - (\zeta - 1)\right)^{1/\varsigma}, 1 - \left(\zeta\left(1 - \overline{\overline{F}}_1^U\right)^{\varsigma} - (\zeta - 1)\right)^{1/\varsigma}\right]\right)\right\rangle,
\end{aligned} \tag{34}$$

and

$$\begin{aligned}
\overline{\overline{pw}}_1\overline{\overline{hn}}_1^{\zeta} = \bigcup_{\overline{\overline{we}}_1 \in \overline{\overline{hn}}_1} &\left\langle f^{*-1}\left(\overline{\overline{\Omega}}_1\left(f^*\left(s_{\vartheta(\overline{\overline{we}}_1)}\right)\right)^{\zeta}\right), \left(\left[1 - \left(\overline{\overline{pw}}_1\left(1 - \left(\zeta\left(\overline{\overline{T}}_1^L\right)^{\varsigma} - (\zeta - 1)\right)^{1/\varsigma}\right)^{\varsigma} - \left(\overline{\overline{pw}}_1 - 1\right)\right)^{1/\varsigma}, 1\right.\right.\right. \\
&\left. - \left(\overline{\overline{pw}}_1\left(1 - \left(\zeta\left(\overline{\overline{T}}_1^U\right)^{\varsigma} - (\zeta - 1)\right)^{1/\varsigma}\right)^{\varsigma} - \left(\overline{\overline{pw}}_1 - 1\right)\right)^{1/\varsigma}\right], \\
&\cdot \left[\left(\overline{\overline{pw}}_1\left(1 - \left(\zeta\left(1 - \overline{\overline{I}}_1^L\right)^{\varsigma} - (\zeta - 1)\right)^{1/\varsigma}\right)^{\varsigma} - \left(\overline{\overline{pw}}_1 - 1\right)\right)^{1/\varsigma},\right. \\
&\cdot \left.\left(\overline{\overline{pw}}_1\left(1 - \left(\zeta\left(1 - \overline{\overline{I}}_1^U\right)^{\varsigma} - (\zeta - 1)\right)^{1/\varsigma}\right)^{\varsigma} - \left(\overline{\overline{pw}}_1 - 1\right)\right)^{1/\varsigma}\right], \\
&\cdot \left[\left(\overline{\overline{pw}}_1\left(1 - \left(\zeta\left(1 - \overline{\overline{F}}_1^L\right)^{\varsigma} - (\zeta - 1)\right)^{1/\varsigma}\right)^{\varsigma} - \left(\overline{\overline{pw}}_1 - 1\right)\right)^{1/\varsigma},\right. \\
&\cdot \left.\left.\left(\overline{\overline{pw}}_1\left(1 - \left(\zeta\left(1 - \overline{\overline{F}}_1^U\right)^{\varsigma} - (\zeta - 1)\right)^{1/\varsigma}\right)^{\varsigma} - \left(\overline{\overline{pw}}_1 - 1\right)\right)^{1/\varsigma}\right]\right\rangle,
\end{aligned} \tag{35}$$

Similarly,

$$\overline{\overline{pw}}_2\overline{\overline{hn}}_2^{\zeta}=\bigcup_{\overline{\overline{we}}_2\in\overline{\overline{hn}}_2}\left\langle f^{*-1}\left(\overline{\overline{pw}}_2\left(f^*\left(s_{\vartheta\left(\overline{\overline{we}}_2\right)}\right)\right)^{\zeta}\right),\left(\left[1-\left(\overline{\overline{pw}}_2\left(1-\left(\zeta\left(\bar{\bar{T}}_2^L\right)^{\varsigma}-(\zeta-1)\right)^{1/\varsigma}\right)^{\varsigma}-\left(\overline{\overline{pw}}_2-1\right)\right)^{1/\varsigma},1\right.\right.\right.$$
$$\left.-\left(\overline{\overline{pw}}_2\left(1-\left(\zeta\left(\bar{\bar{T}}_2^U\right)^{\varsigma}-(\zeta-1)\right)^{1/\varsigma}\right)^{\varsigma}-\left(\overline{\overline{pw}}_2-1\right)\right)^{1/\varsigma}\right],$$
$$\cdot\left[\left(\overline{\overline{pw}}_2\left(1-\left(\zeta\left(1-\bar{\bar{I}}_2^L\right)^{\varsigma}-(\zeta-1)\right)^{1/\varsigma}\right)^{\varsigma}-\left(\overline{\overline{pw}}_2-1\right)\right)^{1/\varsigma},\left(\overline{\overline{pw}}_2\left(1-\left(\zeta\left(1-\bar{\bar{I}}_2^U\right)^{\varsigma}-(\zeta-1)\right)^{1/\varsigma}\right)^{\varsigma}-\left(\overline{\overline{pw}}_2-1\right)\right)^{1/\varsigma}\right],$$
$$\left.\left.\cdot\left[\left(\overline{\overline{pw}}_2\left(1-\left(\zeta\left(1-\bar{\bar{F}}_2^L\right)^{\varsigma}-(\zeta-1)\right)^{1/\varsigma}\right)^{\varsigma}-\left(\overline{\overline{pw}}_2-1\right)\right)^{1/\varsigma},\left(\overline{\overline{pw}}_2\left(1-\left(\zeta\left(1-\bar{\bar{F}}_2^U\right)^{\varsigma}-(\zeta-1)\right)^{1/\varsigma}\right)^{\varsigma}-\left(\overline{\overline{pw}}_2-1\right)\right)^{1/\varsigma}\right]\right)\right\rangle. \tag{36}$$

Then,

$$\overline{\overline{pw}}_1\overline{\overline{hn}}_1^{\zeta}\oplus\overline{\overline{pw}}_2\overline{\overline{hn}}_2^{\zeta}=\bigcup_{\overline{\overline{we}}_1\in\overline{\overline{hn}}_1,\overline{\overline{we}}_2\in\overline{\overline{hn}}_2}\left\langle f^{*-1}\left(\overline{\overline{pw}}_1\left(f^*\left(s_{\vartheta\left(\overline{\overline{we}}_1\right)}\right)\right)^{\zeta}\oplus\overline{\overline{pw}}_1\left(f^*\left(s_{\vartheta\left(\overline{\overline{we}}_1\right)}\right)\right)^{\zeta}\right),\right.$$
$$\cdot\left(\left[1-\left(\left(1-\left(\overline{\overline{pw}}_1\left(1-\left(\zeta\left(\bar{\bar{T}}_1^L\right)^{\varsigma}-(\zeta-1)\right)^{1/\varsigma}\right)^{\varsigma}-\left(\overline{\overline{pw}}_1-1\right)\right)^{1/\varsigma}\right)^{\varsigma}+\left(1-\left(1-\left(\overline{\overline{pw}}_2\left(1-\left(\zeta\left(\bar{\bar{T}}_2^L\right)^{\varsigma}-(\zeta-1)\right)^{1/\varsigma}\right)^{\varsigma}-\left(\overline{\overline{pw}}_2-1\right)\right)^{1/\varsigma}\right)\right)^{\varsigma}-1\right)^{1/\varsigma},1\right.$$
$$\left.-\left(\left(1-\left(1-\left(\overline{\overline{pw}}_1\left(1-\left(\zeta\left(\bar{\bar{T}}_1^L\right)^{\varsigma}-(\zeta-1)\right)^{1/\varsigma}\right)^{\varsigma}-\left(\overline{\overline{pw}}_1-1\right)\right)^{1/\varsigma}\right)^{\varsigma}+\left(1-\left(1-\left(\overline{\overline{pw}}_2\left(1-\left(\zeta\left(\bar{\bar{T}}_2^L\right)^{\varsigma}-(\zeta-1)\right)^{1/\varsigma}\right)^{\varsigma}-\left(\overline{\overline{pw}}_2-1\right)\right)^{1/\varsigma}\right)\right)^{\varsigma}-1\right)^{1/\varsigma}\right],$$
$$\cdot\left[\left(\left(\left(\overline{\overline{pw}}_1\left(1-\left(\zeta\left(1-\bar{\bar{I}}_1^L\right)^{\varsigma}-(\zeta-1)\right)^{1/\varsigma}\right)^{\varsigma}-\left(\overline{\overline{pw}}_1-1\right)\right)^{1/\varsigma}\right)^{\varsigma}+\left(\left(\overline{\overline{pw}}_2\left(1-\left(\zeta\left(1-\bar{\bar{I}}_2^L\right)^{\varsigma}-(\zeta-1)\right)^{1/\varsigma}\right)^{\varsigma}-\left(\overline{\overline{pw}}_2-1\right)\right)^{1/\varsigma}\right)^{\varsigma}-1\right)^{1/\varsigma},\right.$$
$$\left.\cdot\left(\left(\left(\overline{\overline{pw}}_1\left(1-\left(\zeta\left(1-\bar{\bar{I}}_1^U\right)^{\varsigma}-(\zeta-1)\right)^{1/\varsigma}\right)^{\varsigma}-\left(\overline{\overline{pw}}_1-1\right)\right)^{1/\varsigma}\right)^{\varsigma}+\left(\left(\overline{\overline{pw}}_2\left(1-\left(\zeta\left(1-\bar{\bar{I}}_2^U\right)^{\varsigma}-(\zeta-1)\right)^{1/\varsigma}\right)^{\varsigma}-\left(\overline{\overline{pw}}_2-1\right)\right)^{1/\varsigma}\right)^{\varsigma}-1\right)^{1/\varsigma}\right],$$
$$\cdot\left[\left(\left(\left(\overline{\overline{pw}}_1\left(1-\left(\zeta\left(1-\bar{\bar{F}}_1^L\right)^{\varsigma}-(\zeta-1)\right)^{1/\varsigma}\right)^{\varsigma}-\left(\overline{\overline{pw}}_1-1\right)\right)^{1/\varsigma}\right)^{\varsigma}+\left(\left(\overline{\overline{pw}}_2\left(1-\left(\zeta\left(1-\bar{\bar{F}}_2^L\right)^{\varsigma}-(\zeta-1)\right)^{1/\varsigma}\right)^{\varsigma}-\left(\overline{\overline{pw}}_2-1\right)\right)^{1/\varsigma}\right)^{\varsigma}-1\right)^{1/\varsigma},\right.$$
$$\left.\left.\cdot\left(\left(\left(\overline{\overline{pw}}_1\left(1-\left(\zeta\left(1-\bar{\bar{F}}_1^U\right)^{\varsigma}-(\zeta-1)\right)^{1/\varsigma}\right)^{\varsigma}-\left(\overline{\overline{pw}}_1-1\right)\right)^{1/\varsigma}\right)^{\varsigma}+\left(\left(\overline{\overline{pw}}_2\left(1-\left(\zeta\left(1-\bar{\bar{F}}_2^U\right)^{\varsigma}-(\zeta-1)\right)^{1/\varsigma}\right)^{\varsigma}-\left(\overline{\overline{pw}}_2-1\right)\right)^{1/\varsigma}\right)^{\varsigma}-1\right)^{1/\varsigma}\right]\right)\right\rangle;$$
$$=\bigcup_{\overline{\overline{we}}_1\in\overline{\overline{hn}}_1,\overline{\overline{we}}_2\in\overline{\overline{hn}}_2}\left\langle f^{*-1}\left(\overset{2}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g\left(f^*\left(s_{\vartheta\left(\overline{\overline{we}}_g\right)}\right)\right)^{\zeta}\right),\left(\left[1-\left(\left(\overline{\overline{pw}}_1\left(1-\left(\zeta\left(\bar{\bar{T}}_1^L\right)^{\varsigma}-(\zeta-1)\right)^{1/\varsigma}\right)^{\varsigma}-\left(\overline{\overline{pw}}_1-1\right)\right)^{1/\varsigma}\right)^{\varsigma}\right.\right.\right.$$
$$\left.+\left(\left(\overline{\overline{pw}}_2\left(1-\left(\zeta\left(\bar{\bar{T}}_2^L\right)^{\varsigma}-(\zeta-1)\right)^{1/\varsigma}\right)^{\varsigma}-\left(\overline{\overline{pw}}_2-1\right)\right)^{1/\varsigma}\right)^{\varsigma}-1\right)^{1/\varsigma},1-\left(\left(\left(\overline{\overline{pw}}_1\left(1-\left(\zeta\left(\bar{\bar{T}}_1^L\right)^{\varsigma}-(\zeta-1)\right)^{1/\varsigma}\right)^{\varsigma}-\left(\overline{\overline{pw}}_1-1\right)\right)^{1/\varsigma}\right)^{\varsigma}\right.$$
$$\left.+\left(\left(\overline{\overline{pw}}_2\left(1-\left(\zeta\left(\bar{\bar{T}}_2^L\right)^{\varsigma}-(\zeta-1)\right)^{1/\varsigma}\right)^{\varsigma}-\left(\overline{\overline{pw}}_2-1\right)\right)^{1/\varsigma}\right)^{\varsigma}-1\right)^{1/\varsigma}\right],$$
$$\cdot\left[\left(\left(\overline{\overline{pw}}_1\left(1-\left(\zeta\left(1-\bar{\bar{I}}_1^L\right)^{\varsigma}-(\zeta-1)\right)^{1/\varsigma}\right)^{\varsigma}-\left(\overline{\overline{pw}}_1-1\right)\right)+\left(\overline{\overline{pw}}_2\left(1-\left(\zeta\left(1-\bar{\bar{I}}_2^L\right)^{\varsigma}-(\zeta-1)\right)^{1/\varsigma}\right)^{\varsigma}-\left(\overline{\overline{pw}}_2-1\right)\right)-1\right)^{1/\varsigma},\right.$$
$$\cdot\left(\left(\overline{\overline{pw}}_1\left(1-\left(\zeta\left(1-\bar{\bar{I}}_1^U\right)^{\varsigma}-(\zeta-1)\right)^{1/\varsigma}\right)^{\varsigma}-\left(\overline{\overline{pw}}_1-1\right)\right)+\left(\overline{\overline{pw}}_2\left(1-\left(\zeta\left(1-\bar{\bar{I}}_2^U\right)^{\varsigma}-(\zeta-1)\right)^{1/\varsigma}\right)^{\varsigma}-\left(\overline{\overline{pw}}_2-1\right)\right)-1\right)^{1/\varsigma}\right],$$
$$\cdot\left[\left(\left(\overline{\overline{pw}}_1\left(1-\left(\zeta\left(1-\bar{\bar{F}}_1^L\right)^{\varsigma}-(\zeta-1)\right)^{1/\varsigma}\right)^{\varsigma}-\left(\overline{\overline{pw}}_1-1\right)\right)+\left(\overline{\overline{pw}}_2\left(1-\left(\zeta\left(1-\bar{\bar{F}}_2^L\right)^{\varsigma}-(\zeta-1)\right)^{1/\varsigma}\right)^{\varsigma}-\left(\overline{\overline{pw}}_2-1\right)\right)-1\right)^{1/\varsigma},\right.$$
$$\left.\left.\cdot\left(\left(\overline{\overline{pw}}_1\left(1-\left(\zeta\left(1-\bar{\bar{F}}_1^U\right)^{\varsigma}-(\zeta-1)\right)^{1/\varsigma}\right)^{\varsigma}-\left(\overline{\overline{pw}}_1-1\right)\right)+\left(\overline{\overline{pw}}_2\left(1-\left(\zeta\left(1-\bar{\bar{F}}_2^U\right)^{\varsigma}-(\zeta-1)\right)^{1/\varsigma}\right)^{\varsigma}-\left(\overline{\overline{pw}}_2-1\right)\right)-1\right)^{1/\varsigma}\right]\right)\right\rangle;$$

$$\overline{\overline{pw}}_1\overline{\overline{hn}}_1^{\zeta}\oplus\overline{\overline{pw}}_2\overline{\overline{hn}}_2^{\zeta}=\bigcup_{\overline{\overline{we}}_1\in\overline{\overline{hn}}_1,\overline{\overline{we}}_2\in\overline{\overline{hn}}_2}\left\langle f^{*-1}\left(\overset{2}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g\left(f^*\left(s_{\vartheta\left(\overline{\overline{we}}_g\right)}\right)\right)^{\zeta}\right),\left(\left[1-\left(\overset{2}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g\left(1-\left(\zeta\left(\bar{\bar{T}}_g^L\right)^{\varsigma}-(\zeta-1)\right)^{1/\varsigma}\right)^{\varsigma}-\overset{2}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g+1\right)^{1/\varsigma},1\right.\right.\right.$$
$$\left.-\left(\overset{2}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g\left(1-\left(\zeta\left(\bar{\bar{T}}_g^U\right)^{\varsigma}-(\zeta-1)\right)^{1/\varsigma}\right)^{\varsigma}-\overset{2}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g+1\right)^{1/\varsigma}\right],\left[\left(\overset{2}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g\left(1-\left(\zeta\left(1-\bar{\bar{I}}_g^L\right)^{\varsigma}-(\zeta-1)\right)^{1/\varsigma}\right)^{\varsigma}-\overset{2}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g+1\right)^{1/\varsigma},\right.$$
$$\cdot\left(\overset{2}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g\left(1-\left(\zeta\left(1-\bar{\bar{I}}_g^U\right)^{\varsigma}-(\zeta-1)\right)^{1/\varsigma}\right)^{\varsigma}-\overset{2}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g+1\right)^{1/\varsigma}\right]$$
$$\left.\cdot\left[\left(\overset{2}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g\left(1-\left(\zeta\left(1-\bar{\bar{F}}_g^L\right)^{\varsigma}-(\zeta-1)\right)^{1/\varsigma}\right)^{\varsigma}-\overset{2}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g+1\right)^{1/\varsigma},\left(\overset{2}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g\left(1-\left(\zeta\left(1-\bar{\bar{F}}_g^U\right)^{\varsigma}-(\zeta-1)\right)^{1/\varsigma}\right)^{\varsigma}-\overset{2}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g+1\right)^{1/\varsigma}\right]\right)\right\rangle. \tag{37}$$

If Equation (33) holds for $s = m$

$$\overline{\overline{pw}}_1\overline{\overline{hn}}_1^{\zeta} \oplus \overline{\overline{pw}}_2\overline{\overline{hn}}_2^{\zeta} \oplus \cdots \oplus \overline{\overline{pw}}_m\overline{\overline{hn}}_m^{\zeta} = \bigcup_{\overline{\overline{we}}_1 \in \overline{\overline{hn}}_1, \overline{\overline{we}}_2 \in \overline{\overline{hn}}_2, \cdots, \overline{\overline{we}}_s \in \overline{\overline{hn}}_m} \left\langle f^{*-1}\left(\overset{m}{\underset{g=1}{\oplus}} \overline{\overline{pw}}_g \left(f^*\left(s_{\vartheta\left(\overline{\overline{we}}_g\right)}\right)\right)^{\zeta}\right), \right.$$
$$\cdot \left(\left[1 - \left(\overset{m}{\underset{g=1}{\oplus}} \overline{\overline{pw}}_g \left(1 - \left(\zeta\left(\overline{\overline{T}}_g^L\right)^{\varsigma} - (\zeta - 1)\right)^{1/\varsigma}\right)^{\varsigma} - \overset{m}{\underset{g=1}{\oplus}} \overline{\overline{pw}}_g + 1\right)^{1/\varsigma}, 1 - \left(\overset{m}{\underset{g=1}{\oplus}} \overline{\overline{pw}}_g \left(1 - \left(\zeta\left(\overline{\overline{T}}_g^U\right)^{\varsigma} - (\zeta - 1)\right)^{1/\varsigma}\right)^{\varsigma} - \overset{m}{\underset{g=1}{\oplus}} \overline{\overline{pw}}_g + 1\right)^{1/\varsigma}\right],\right.$$
$$\cdot \left[\left(\overset{m}{\underset{g=1}{\oplus}} \overline{\overline{pw}}_g \left(1 - \left(\zeta\left(1 - \overline{\overline{I}}_g^L\right)^{\varsigma} - (\zeta - 1)\right)^{1/\varsigma}\right)^{\varsigma} - \overset{m}{\underset{g=1}{\oplus}} \overline{\overline{pw}}_g + 1\right)^{1/\varsigma}, \left(\overset{m}{\underset{g=1}{\oplus}} \overline{\overline{pw}}_g \left(1 - \left(\zeta\left(1 - \overline{\overline{I}}_g^U\right)^{\varsigma} - (\zeta - 1)\right)^{1/\varsigma}\right)^{\varsigma} - \overset{m}{\underset{g=1}{\oplus}} \overline{\overline{pw}}_g + 1\right)^{1/\varsigma}\right],$$
$$\left.\cdot \left[\left(\overset{m}{\underset{g=1}{\oplus}} \overline{\overline{pw}}_g \left(1 - \left(\zeta\left(1 - \overline{\overline{F}}_g^L\right)^{\varsigma} - (\zeta - 1)\right)^{1/\varsigma}\right)^{\varsigma} - \overset{m}{\underset{g=1}{\oplus}} \overline{\overline{pw}}_g + 1\right)^{1/\varsigma}, \left(\overset{m}{\underset{g=1}{\oplus}} \overline{\overline{pw}}_g \left(1 - \left(\zeta\left(1 - \overline{\overline{F}}_g^U\right)^{\varsigma} - (\zeta - 1)\right)^{1/\varsigma}\right)^{\varsigma} - \overset{m}{\underset{g=1}{\oplus}} \overline{\overline{pw}}_g + 1\right)^{1/\varsigma}\right]\right\rangle. \tag{38}$$

Then, when $s = m + 1$, by the operational laws explained in Definition 14, we have

$$\overline{\overline{pw}}_1\overline{\overline{hn}}_1^{\zeta} \oplus \overline{\overline{pw}}_2\overline{\overline{hn}}_2^{\zeta} \oplus \cdots \oplus \overline{\overline{pw}}_{m+1}\overline{\overline{hn}}_{m+1}^{\zeta}$$
$$= \overline{\overline{pw}}_1\overline{\overline{hn}}_1^{\zeta} \oplus \overline{\overline{pw}}_2\overline{\overline{hn}}_2^{\zeta} \oplus \cdots \oplus \overline{\overline{pw}}_m\overline{\overline{hn}}_m^{\zeta} \oplus \overline{\overline{pw}}_{m+1}\overline{\overline{hn}}_{m+1}^{\zeta}$$
$$= \bigcup_{\overline{\overline{we}}_1 \in \overline{\overline{hn}}_1, \overline{\overline{we}}_2 \in \overline{\overline{hn}}_2, \cdots, \overline{\overline{we}}_s \in \overline{\overline{hn}}_m} \left\langle f^{*-1}\left(\overset{m}{\underset{g=1}{\oplus}} \overline{\overline{pw}}_g \left(f^*\left(s_{\vartheta\left(\overline{\overline{we}}_g\right)}\right)\right)^{\zeta}\right), \right.$$
$$\left(\left[1 - \left(\overset{m}{\underset{g=1}{\oplus}} \overline{\overline{pw}}_g \left(1 - \left(\zeta\left(\overline{\overline{T}}_g^L\right)^{\varsigma} - (\zeta - 1)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma} - \overset{m}{\underset{g=1}{\oplus}} \overline{\overline{pw}}_g + 1\right)^{\frac{1}{\varsigma}},\right.\right.$$
$$\left.1 - \left(\overset{m}{\underset{g=1}{\oplus}} \overline{\overline{pw}}_g \left(1 - \left(\zeta\left(\overline{\overline{T}}_g^U\right)^{\varsigma} - (\zeta - 1)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma} - \overset{m}{\underset{g=1}{\oplus}} \overline{\overline{pw}}_g + 1\right)^{\frac{1}{\varsigma}}\right],$$
$$\left[\left(\overset{m}{\underset{g=1}{\oplus}} \overline{\overline{pw}}_g \left(1 - \left(\zeta\left(1 - \overline{\overline{I}}_g^L\right)^{\varsigma} - (\zeta - 1)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma} - \overset{m}{\underset{g=1}{\oplus}} \overline{\overline{pw}}_g + 1\right)^{\frac{1}{\varsigma}},\right.$$
$$\left.\left(\overset{m}{\underset{g=1}{\oplus}} \overline{\overline{pw}}_g \left(1 - \left(\zeta\left(1 - \overline{\overline{I}}_g^U\right)^{\varsigma} - (\zeta - 1)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma} - \overset{m}{\underset{g=1}{\oplus}} \overline{\overline{pw}}_g + 1\right)^{\frac{1}{\varsigma}}\right],$$
$$\left[\left(\overset{m}{\underset{g=1}{\oplus}} \overline{\overline{pw}}_g \left(1 - \left(\zeta\left(1 - \overline{\overline{F}}_g^L\right)^{\varsigma} - (\zeta - 1)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma} - \overset{m}{\underset{g=1}{\oplus}} \overline{\overline{pw}}_g + 1\right)^{\frac{1}{\varsigma}},\right.$$
$$\left.\left.\left(\overset{m}{\underset{g=1}{\oplus}} \overline{\overline{pw}}_g \left(1 - \left(\zeta\left(1 - \overline{\overline{F}}_g^U\right)^{\varsigma} - (\zeta - 1)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma} - \overset{m}{\underset{g=1}{\oplus}} \overline{\overline{pw}}_g + 1\right)^{\frac{1}{\varsigma}}\right]\right\rangle$$

$$\oplus \bigcup_{\overline{\overline{we}}_{m+1} \in \overline{\overline{hn}}_{m+1}} \left\langle f^*\left(s_{\vartheta\left(\overline{\overline{we}}_{m+1}\right)}\right), \right.$$
$$\left(\left[1 - \left(\overline{\overline{pw}}_{m+1}\left(1 - \left(\zeta\left(\overline{\overline{T}}_{m+1}^L\right)^{\varsigma} - (\zeta - 1)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma} - \left(\overline{\overline{pw}}_{m+1} - 1\right)\right)^{\frac{1}{\varsigma}},\right.\right.$$
$$\left.1 - \left(\overline{\overline{pw}}_{m+1}\left(1 - \left(\zeta\left(\overline{\overline{T}}_{m+1}^U\right)^{\varsigma} - (\zeta - 1)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma} - \left(\overline{\overline{pw}}_{m+1} - 1\right)\right)^{\frac{1}{\varsigma}}\right],$$
$$\left[\left(\overline{\overline{pw}}_{m+1}\left(1 - \left(\zeta\left(1 - \overline{\overline{I}}_{m+1}^L\right)^{\varsigma} - (\zeta - 1)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma} - \left(\overline{\overline{pw}}_{m+1} - 1\right)\right)^{\frac{1}{\varsigma}},\right.$$
$$\left.\left(\overline{\overline{pw}}_{m+1}\left(1 - \left(\zeta\left(1 - \overline{\overline{I}}_{m+1}^U\right)^{\varsigma} - (\zeta - 1)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma} - \left(\overline{\overline{pw}}_{m+1} - 1\right)\right)^{\frac{1}{\varsigma}}\right],$$
$$\left[\left(\overline{\overline{pw}}_{m+1}\left(1 - \left(\zeta\left(1 - \overline{\overline{F}}_{m+1}^L\right)^{\varsigma} - (\zeta - 1)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma} - \left(\overline{\overline{pw}}_{m+1} - 1\right)\right)^{\frac{1}{\varsigma}},\right.$$
$$\left.\left.\left(\overline{\overline{pw}}_{m+1}\left(1 - \left(\zeta\left(1 - \overline{\overline{F}}_{m+1}^U\right)^{\varsigma} - (\zeta - 1)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma} - \left(\overline{\overline{pw}}_{m+1} - 1\right)\right)^{\frac{1}{\varsigma}}\right]\right\rangle,$$

$$\overline{\overline{pw}}_1\overline{\overline{hn}}_1^{\zeta} \oplus \overline{\overline{pw}}_2\overline{\overline{hn}}_2^{\zeta} \oplus \cdots \oplus \overline{\overline{pw}}_{m+1}\overline{\overline{hn}}_{m+1}^{\zeta}$$
$$= \bigcup_{\overline{\overline{we}}_1 \in \overline{\overline{hn}}_1, \overline{\overline{we}}_2 \in \overline{\overline{hn}}_2, \cdots, \overline{\overline{we}}_{m+1} \in \overline{\overline{hn}}_{m+1}} \left\langle f^{*-1}\left(\overset{m+1}{\underset{g=1}{\oplus}} \overline{\overline{pw}}_g \left(f^*\left(s_{\vartheta\left(\overline{\overline{we}}_g\right)}\right)\right)^{\zeta}\right), \right.$$
$$\left(\left[1 - \left(\overset{m+1}{\underset{g=1}{\oplus}} \overline{\overline{pw}}_g \left(1 - \left(\zeta\left(\overline{\overline{T}}_g^L\right)^{\varsigma} - (\zeta - 1)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma} - \overset{m+1}{\underset{g=1}{\oplus}} \overline{\overline{pw}}_g + 1\right)^{\frac{1}{\varsigma}},\right.\right.$$
$$\left.1 - \left(\overset{m+1}{\underset{g=1}{\oplus}} \overline{\overline{pw}}_g \left(1 - \left(\zeta\left(\overline{\overline{T}}_g^U\right)^{\varsigma} - (\zeta - 1)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma} - \overset{m+1}{\underset{g=1}{\oplus}} \overline{\overline{pw}}_g + 1\right)^{\frac{1}{\varsigma}}\right],$$
$$\left[\left(\overset{m+1}{\underset{g=1}{\oplus}} \overline{\overline{pw}}_g \left(1 - \left(\zeta\left(1 - \overline{\overline{I}}_g^L\right)^{\varsigma} - (\zeta - 1)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma} - \overset{m+1}{\underset{g=1}{\oplus}} \overline{\overline{pw}}_g + 1\right)^{\frac{1}{\varsigma}},\right.$$
$$\left.\left(\overset{m+1}{\underset{g=1}{\oplus}} \overline{\overline{pw}}_g \left(1 - \left(\zeta\left(1 - \overline{\overline{I}}_g^U\right)^{\varsigma} - (\zeta - 1)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma} - \overset{m+1}{\underset{g=1}{\oplus}} \overline{\overline{pw}}_g + 1\right)^{\frac{1}{\varsigma}}\right],$$
$$\left[\left(\overset{m+1}{\underset{g=1}{\oplus}} \overline{\overline{pw}}_g \left(1 - \left(\zeta\left(1 - \overline{\overline{F}}_g^L\right)^{\varsigma} - (\zeta - 1)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma} - \overset{m+1}{\underset{g=1}{\oplus}} \overline{\overline{pw}}_g + 1\right)^{\frac{1}{\varsigma}},\right.$$
$$\left.\left.\left(\overset{m+1}{\underset{g=1}{\oplus}} \overline{\overline{pw}}_g \left(1 - \left(\zeta\left(1 - \overline{\overline{F}}_g^U\right)^{\varsigma} - (\zeta - 1)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma} - \overset{m+1}{\underset{g=1}{\oplus}} \overline{\overline{pw}}_g + 1\right)^{\frac{1}{\varsigma}}\right]\right)\right\rangle. \tag{39}$$

That is, Equation (33) is true for $g = m + 1$. So Equation (33) is true for all g. Then,

$$\left(\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g\overline{\overline{hn}}_g^{\zeta}\right)^{\frac{1}{\zeta}} = \bigcup_{\overline{\overline{we}}_1\in\overline{\overline{hn}}_1,\overline{\overline{we}}_2\in\overline{\overline{hn}}_2,\cdots,\overline{\overline{we}}_{m+1}\in\overline{\overline{hn}}_{m+1}}\left\langle f^{*-1}\left(\left(\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g\left(f^*\left(s_{\vartheta\left(\overline{\overline{we}}_g\right)}\right)\right)^{\zeta}\right)^{\frac{1}{\zeta}}\right),\right.$$
$$\cdot\left(\left[\left(\frac{1}{\zeta}\left(1-\left(\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g\left(1-\left(\zeta\left(\overline{\overline{T}}_g^{L}\right)^{\varsigma}-(\zeta-1)\right)^{\frac{1}{\zeta}}\right)^{\varsigma}-\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g+1\right)^{\frac{1}{\zeta}}\right)^{\varsigma}-(\zeta-1)\right)^{\frac{1}{\zeta}},\right.$$
$$\left.\cdot\left(\frac{1}{\zeta}\left(1-\left(\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g\left(1-\left(\zeta\left(\overline{\overline{T}}_g^{U}\right)^{\varsigma}-(\zeta-1)\right)^{\frac{1}{\zeta}}\right)^{\varsigma}-\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g+1\right)^{\frac{1}{\zeta}}\right)^{\varsigma}-(\zeta-1)\right)^{\frac{1}{\zeta}}\right],$$
$$\cdot\left[1-\left(\frac{1}{\zeta}\left(1-\left(\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g\left(1-\left(\zeta\left(1-\overline{\overline{I}}_g^{L}\right)^{\varsigma}-(\zeta-1)\right)^{\frac{1}{\zeta}}\right)^{\varsigma}-\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g+1\right)^{\frac{1}{\zeta}}\right)^{\varsigma}-\left(\frac{1}{\zeta}-1\right)\right)^{\frac{1}{\zeta}},1\right.\tag{40}$$
$$\left.-\left(\frac{1}{\zeta}\left(1-\left(\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g\left(1-\left(\zeta\left(1-\overline{\overline{I}}_g^{U}\right)^{\varsigma}-(\zeta-1)\right)^{\frac{1}{\zeta}}\right)^{\varsigma}-\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g+1\right)^{\frac{1}{\zeta}}\right)^{\varsigma}-\left(\frac{1}{\zeta}-1\right)\right)^{\frac{1}{\zeta}}\right],$$
$$\cdot\left[1-\left(\frac{1}{\zeta}\left(1-\left(\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g\left(1-\left(\zeta\left(1-\overline{\overline{F}}_g^{L}\right)^{\varsigma}-(\zeta-1)\right)^{\frac{1}{\zeta}}\right)^{\varsigma}-\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g+1\right)^{\frac{1}{\zeta}}\right)^{\varsigma}-\left(\frac{1}{\zeta}-1\right)\right)^{\frac{1}{\zeta}},1\right.$$
$$\left.\left.-\left(\frac{1}{\zeta}\left(1-\left(\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g\left(1-\left(\zeta\left(1-\overline{\overline{F}}_g^{U}\right)^{\varsigma}-(\zeta-1)\right)^{\frac{1}{\zeta}}\right)^{\varsigma}-\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g+1\right)^{\frac{1}{\zeta}}\right)^{\varsigma}-\left(\frac{1}{\zeta}-1\right)\right)^{\frac{1}{\zeta}}\right]\right)\right\rangle.$$

Therefore,

$$GHINLSSPA_{\zeta}\left(\overline{\overline{hn}}_1,\overline{\overline{hn}}_2,\cdots,\overline{\overline{hn}}_s\right) = \bigcup_{\overline{\overline{we}}_1\in\overline{\overline{hn}}_1,\overline{\overline{we}}_2\in\overline{\overline{hn}}_2,\cdots,\overline{\overline{we}}_{m+1}\in\overline{\overline{hn}}_{m+1}}\left\langle f^{*-1}\left(\left(\oplus^{s}_{g=1}\overline{\overline{pw}}_g\left(f^*\left(s_{\vartheta\left(\overline{\overline{we}}_g\right)}\right)\right)^{\zeta}\right)^{\frac{1}{\zeta}}\right),\right.$$
$$\cdot\left(\left[\left(\frac{1}{\zeta}\left(1-\left(\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g\left(1-\left(\zeta\left(\overline{\overline{T}}_g^{L}\right)^{\varsigma}-(\zeta-1)\right)^{\frac{1}{\zeta}}\right)^{\varsigma}-\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g+1\right)^{\frac{1}{\zeta}}\right)^{\varsigma}-(\zeta-1)\right)^{\frac{1}{\zeta}},\right.$$
$$\left.\cdot\left(\frac{1}{\zeta}\left(1-\left(\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g\left(1-\left(\zeta\left(\overline{\overline{T}}_g^{U}\right)^{\varsigma}-(\zeta-1)\right)^{\frac{1}{\zeta}}\right)^{\varsigma}-\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g+1\right)^{\frac{1}{\zeta}}\right)^{\varsigma}-(\zeta-1)\right)^{\frac{1}{\zeta}}\right],$$
$$\cdot\left[1-\left(\frac{1}{\zeta}\left(1-\left(\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g\left(1-\left(\zeta\left(1-\overline{\overline{I}}_g^{L}\right)^{\varsigma}-(\zeta-1)\right)^{\frac{1}{\zeta}}\right)^{\varsigma}-\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g+1\right)^{\frac{1}{\zeta}}\right)^{\varsigma}-\left(\frac{1}{\zeta}-1\right)\right)^{\frac{1}{\zeta}},1\right.$$
$$\left.-\left(\frac{1}{\zeta}\left(1-\left(\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g\left(1-\left(\zeta\left(1-\overline{\overline{I}}_g^{U}\right)^{\varsigma}-(\zeta-1)\right)^{\frac{1}{\zeta}}\right)^{\varsigma}-\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g+1\right)^{\frac{1}{\zeta}}\right)^{\varsigma}-\left(\frac{1}{\zeta}-1\right)\right)^{\frac{1}{\zeta}}\right],$$
$$\cdot\left[1-\left(\frac{1}{\zeta}\left(1-\left(\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g\left(1-\left(\zeta\left(1-\overline{\overline{F}}_g^{L}\right)^{\varsigma}-(\zeta-1)\right)^{\frac{1}{\zeta}}\right)^{\varsigma}-\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g+1\right)^{\frac{1}{\zeta}}\right)^{\varsigma}-\left(\frac{1}{\zeta}-1\right)\right)^{\frac{1}{\zeta}},1\right.$$
$$\left.\left.-\left(\frac{1}{\zeta}\left(1-\left(\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g\left(1-\left(\zeta\left(1-\overline{\overline{F}}_g^{U}\right)^{\varsigma}-(\zeta-1)\right)^{\frac{1}{\zeta}}\right)^{\varsigma}-\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g+1\right)^{\frac{1}{\zeta}}\right)^{\varsigma}-\left(\frac{1}{\zeta}-1\right)\right)^{\frac{1}{\zeta}}\right]\right)\right\rangle.\tag{41}$$

Which completes the proof of the Theorem 17.

Theorem 18. *Commutativity: let* $(\overline{\overline{hn}}_1^*, \overline{\overline{hn}}_2^*, \cdots, \overline{\overline{hn}}_s^*)$ *be any permutation of* $(\overline{\overline{hn}}_1, \overline{\overline{hn}}_2, \cdots, \overline{\overline{hn}}_s)$*, then*

$$GHINLSSPA_\zeta\left(\overline{\overline{hn}}_1^*, \overline{\overline{hn}}_2^*, \cdots, \overline{\overline{hn}}_s^*\right) = GHINLSSPA_\zeta\left(\overline{\overline{hn}}_1, \overline{\overline{hn}}_2, \cdots, \overline{\overline{hn}}_s\right). \tag{42}$$

Proof. Suppose that $(\overline{\overline{hn}}_1^*, \overline{\overline{hn}}_2^*, \cdots, \overline{\overline{hn}}_s^*)$ is any permutation of $(\overline{\overline{hn}}_1, \overline{\overline{hn}}_2, \cdots, \overline{\overline{hn}}_s)$, then for each $\overline{\overline{hn}}_g$, there exists one and only one $\overline{\overline{hn}}_h^*$ such that $\overline{\overline{hn}}_g = \overline{\overline{hn}}_h^*$ and vice versa. And $T(\overline{\overline{hn}}_g) = T(\overline{\overline{hn}}_h^*)$.. Then based on Theorem 17, we have

$$GHINLSSPA_\zeta\left(\overline{\overline{hn}}_1, \overline{\overline{hn}}_2, \cdots, \overline{\overline{hn}}_s\right) = \overset{s}{\underset{g=1}{\oplus}}\left(\overline{\overline{pw}}_g \overline{\overline{hn}}_g^\zeta\right)^{1/\zeta} = \overset{s}{\underset{h=1}{\oplus}}\left(\overline{\overline{pw}}_h^*\left(\overline{\overline{hn}}_h^*\right)^\zeta\right)^{1/\zeta} = GHINLSSPA_\zeta\left(\overline{\overline{hn}}_1^*, \overline{\overline{hn}}_2^*, \cdots, \overline{\overline{hn}}_s^*\right). \tag{43}$$

Note that the GHINLSSPA operators are neither idempotent nor monotonic. □

Further, we shall discuss a few cases of the initiated AOs with respect to the parameter ζ, which are listed below:

(1) If $\zeta = 1$, then, the GHINLSSPA operator trim down to HINLS power average (PA) operator:

$$\begin{aligned} GHINLSSPA\left(\overline{\overline{hn}}_1, \overline{\overline{hn}}_2, \cdots, \overline{\overline{hn}}_s\right) &= \overset{s}{\underset{g=1}{\oplus}} \overline{\overline{pw}}_g \overline{\overline{hn}}_g = \bigcup_{u_1 \in \overline{\overline{hn}}_1, u_2 \in \overline{\overline{hn}}_2, \cdots, u_s \in \overline{\overline{hn}}_s} \\ &\cdot \left\langle f^{*-1}\left(\overset{m}{\underset{g=1}{\oplus}} \overline{\overline{pw}}_g \left(f^*\left(s_{\vartheta(u_g)}\right)\right)\right), \left(\left[1 - \left(\overset{s}{\underset{g=1}{\oplus}} \overline{\overline{pw}}_g \left(1 - \bar{\bar{T}}_g^L\right) - \overset{s}{\underset{g=1}{\oplus}} \overline{\overline{pw}}_g + 1\right)^{\frac{1}{\zeta}}, 1 - \left(\overset{s}{\underset{g=1}{\oplus}} \overline{\overline{pw}}_g \left(1 - \bar{\bar{T}}_g^L\right) - \overset{s}{\underset{g=1}{\oplus}} \overline{\overline{pw}}_g + 1\right)^{\frac{1}{\zeta}}\right], \right. \right. \\ &\cdot \left[\left(\overset{s}{\underset{g=1}{\oplus}} \overline{\overline{pw}}_g \left(\bar{\bar{I}}_g^L\right)^\varsigma - \overset{s}{\underset{g=1}{\oplus}} \overline{\overline{pw}}_g + 1\right)^{\frac{1}{\varsigma}}, \left(\overset{s}{\underset{g=1}{\oplus}} \overline{\overline{pw}}_g \left(\bar{\bar{I}}_g^U\right)^\varsigma - \overset{s}{\underset{g=1}{\oplus}} \overline{\overline{pw}}_g + 1\right)^{\frac{1}{\varsigma}}\right], \\ &\left. \left. \cdot \left[\left(\overset{s}{\underset{g=1}{\oplus}} \overline{\overline{pw}}_g \left(\bar{\bar{F}}_g^L\right)^\varsigma - \overset{s}{\underset{g=1}{\oplus}} \overline{\overline{pw}}_g + 1\right)^{\frac{1}{\varsigma}}, \left(\overset{s}{\underset{g=1}{\oplus}} \overline{\overline{pw}}_g \left(\bar{\bar{F}}_g^U\right)^\varsigma - \overset{s}{\underset{g=1}{\oplus}} \overline{\overline{pw}}_g + 1\right)^{\frac{1}{\varsigma}}\right]\right)\right\rangle. \end{aligned} \tag{44}$$

(2) If $\zeta = 1$ and $\varsigma = 0$, then, the GHINLSSPA operator trim down to HINL PA operator based on algebraic operation. That is

$$\begin{aligned} GHINLSSPA_{\varsigma=0}\left(\overline{\overline{hn}}_1, \overline{\overline{hn}}_2, \cdots, \overline{\overline{hn}}_s\right) &= \overset{s}{\underset{g=1}{\oplus}} \overline{\overline{pw}}_g \overline{\overline{hn}}_g = \bigcup_{u_1 \in \overline{\overline{hn}}_1, u_2 \in \overline{\overline{hn}}_2, \cdots, u_s \in \overline{\overline{hn}}_s} \\ &\cdot \left\langle f^{*-1}\left(\overset{m+1}{\underset{g=1}{\oplus}} \overline{\overline{pw}}_g \left(f^*\left(s_{\vartheta(u_g)}\right)\right)\right), \left(\begin{array}{c} \left[1 - \overset{s}{\underset{g=1}{\otimes}}\left(1 - \bar{\bar{T}}_g^L\right)^{\overline{\overline{pw}}_g}, 1 - \overset{s}{\underset{g=1}{\otimes}}\left(1 - \bar{\bar{T}}_g^U\right)^{\overline{\overline{pw}}_g}\right], \\ \left[\overset{s}{\underset{g=1}{\otimes}}\left(\bar{\bar{I}}_g^L\right)^{\overline{\overline{pw}}_g}, \overset{s}{\underset{g=1}{\otimes}}\left(\bar{\bar{I}}_g^U\right)^{\overline{\overline{pw}}_g}\right], \left[\overset{s}{\underset{g=1}{\otimes}}\left(\bar{\bar{F}}_g^L\right)^{\overline{\overline{pw}}_g}, \overset{s}{\underset{g=1}{\otimes}}\left(\bar{\bar{F}}_g^U\right)^{\overline{\overline{pw}}_g}\right] \end{array}\right)\right\rangle. \end{aligned} \tag{45}$$

(3) If $\zeta = 1$, $\sup(\overline{\overline{hn}}_g, \overline{\overline{hn}}_h) = b$ (i.e., a constant) and $\varsigma = 0$, then, the GHINLSSPA operator trim down to HINL average operator based on algebraic operation. That is

$$WSSHINLPA_{\varsigma\longrightarrow 0}\left(\overline{\overline{hn}}_1, \overline{\overline{hn}}_2, \cdots, \overline{\overline{hn}}_s\right) = \frac{1}{s}\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{hn}}_g = \underset{u_1\in\overline{\overline{hn}}_1, u_2\in\overline{\overline{hn}}_2, \cdots, u_s\in\overline{\overline{hn}}_s}{\cup}\left\langle f^{*-1}\left(\overset{m}{\underset{g=1}{\oplus}}\frac{1}{s}\left(f^*\left(s_{\vartheta(u_g)}\right)\right)\right)\right.$$
$$\left.\cdot\left[1-\overset{s}{\underset{g=1}{\otimes}}\left(1-\bar{\bar{T}}_g^L\right)^{\frac{1}{s}}, 1-\overset{s}{\underset{g=1}{\otimes}}\left(1-\bar{\bar{T}}_g^U\right)^{\frac{1}{s}}\right], \left[\overset{s}{\underset{g=1}{\otimes}}\left(\bar{\bar{I}}_g^L\right)^{\frac{1}{s}}, \overset{s}{\underset{g=1}{\otimes}}\left(\bar{\bar{I}}_g^U\right)^{\frac{1}{s}}\right], \left[\overset{s}{\underset{g=1}{\otimes}}\left(\bar{\bar{F}}_g^L\right)^{\frac{1}{s}}, \overset{s}{\underset{g=1}{\otimes}}\left(\bar{\bar{F}}_g^U\right)^{\frac{1}{s}}\right]\right\rangle. \tag{46}$$

Definition 19. For a group of HINLNs $\overline{\overline{hn}}_g(g = 1, 2, \cdots, s)$, a WGHINLSSPA operator is a function $\aleph^s \longrightarrow \aleph$,

$$WGHINLSSPA_{\phi,\zeta}\left(\overline{\overline{hn}}_1, \overline{\overline{hn}}_2, \cdots, \overline{\overline{hn}}_s\right) = \left(\frac{\overset{s}{\underset{g=1}{\oplus_{SS}}}\left(\overline{\overline{We}}_g\left(1+T\left(\overline{\overline{hn}}_g\right)\right)\overline{\overline{hn}}_g^{\zeta}\right)}{\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{We}}_g\left(1+T\left(\overline{\overline{hn}}_g\right)\right)}\right)^{1/\zeta}, \tag{47}$$

where $T(\overline{\overline{hn}}_g) = \overset{s}{\underset{\substack{h=1,\\ g\neq h}}{\oplus}} \mathrm{Sup}(\overline{\overline{hn}}_g, \overline{\overline{hn}}_h)$, $\overline{\overline{We}} = (\overline{\overline{We}}_1, \overline{\overline{We}}_2, \cdots, \overline{\overline{We}}_s)^T$ is the weight vector for $\overline{\overline{hn}}_g(g = 1, 2, \cdots, s)$ such that $\overline{\overline{We}}_g \in [0, 1]$ and $\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{We}}_g = 1$, parameter $\zeta \in (0,+\infty)$ and $\mathrm{Sup}(\overline{\overline{hn}}_g, \overline{\overline{hn}}_h)$ is the support for $\overline{\overline{hn}}_g$ from $\overline{\overline{hn}}_h$ with the following constraint:

$$\begin{aligned}&\mathrm{Sup}\left(\overline{\overline{hn}}_g, \overline{\overline{hn}}_h\right) \in [0, 1];\\ &\mathrm{Sup}\left(\overline{\overline{hn}}_g, \overline{\overline{hn}}_h\right) = \mathrm{Sup}\left(\overline{\overline{hn}}_h, \overline{\overline{hn}}_g\right);\end{aligned} \tag{48}$$

To write Equation (47) in uncomplicated way, we have

$$\overline{\overline{pow}}_g = \frac{\overline{\overline{We}}_g\left(1+T\left(\overline{\overline{hn}}_g\right)\right)}{\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{We}}_g\left(1+T\left(\overline{\overline{hn}}_g\right)\right)}, \tag{49}$$

so, from Equation (49), Equation (47) becomes

$$WGHILNSSPA_{\phi,\zeta}\left(\overline{\overline{hn}}_1, \overline{\overline{hn}}_2, \cdots, \overline{\overline{hn}}_s\right) = \left(\overset{s}{\underset{g=1}{\oplus_{SS}}}\overline{\overline{pow}}_g\overline{\overline{hn}}_g^{\zeta}\right)^{1/\zeta}, \tag{50}$$

Theorem 20. *Let $\overline{\overline{hn}}_g(g = 1, 2, \cdots, s)$ be a group of HINLNs, then the value aggregated utilizing Definition 19 is still HINLN, and we have*

$$\begin{aligned}&WGHINLSSPA_{\zeta,\phi}\left(\overline{\overline{hn}}_1, \overline{\overline{hn}}_2, \cdots, \overline{\overline{hn}}_s\right) \underset{u_1\in\overline{\overline{hn}}_1, u_2\in\overline{\overline{hn}}_2, \cdots, u_{m+1}\in\overline{\overline{hn}}_s}{\cup}\left\langle f^{*-1}\left(\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pow}}_g\left(f^*\left(s_{\vartheta(u_g)}\right)\right)^{\zeta}\right)^{\frac{1}{\zeta}},\right.\\
&\left(\left[\left(\frac{1}{\zeta}\left(1-\left(\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pow}}_g\left(1-\left(\zeta\left(\bar{\bar{T}}_g^L\right)^{\varsigma}-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}-\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pow}}_g+1\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}-\left(\frac{1}{\zeta}-1\right)\right)^{\frac{1}{\varsigma}},\right.\right.\\
&\left.\left.\left(\frac{1}{\zeta}\left(1-\left(\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pow}}_g\left(1-\left(\zeta\left(\bar{\bar{T}}_g^L\right)^{\varsigma}-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}-\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pow}}_g+1\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}-\left(\frac{1}{\zeta}-1\right)\right)^{\frac{1}{\varsigma}}\right],\right.\\
&\left[1-\left(\frac{1}{\zeta}\left(1-\left(\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pow}}_g\left(1-\left(\zeta\left(1-\bar{\bar{F}}_g^L\right)^{\varsigma}-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}-\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pow}}_g+1\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}-\left(\frac{1}{\zeta}-1\right)\right)^{\frac{1}{\varsigma}},\right.\\
&\left.1-\left(\frac{1}{\zeta}\left(1-\left(\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pow}}_g\left(1-\left(\zeta\left(1-\bar{\bar{F}}_g^U\right)^{\varsigma}-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}-\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pow}}_g+1\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}-\left(\frac{1}{\zeta}-1\right)\right)^{\frac{1}{\varsigma}}\right],\\
&\left[1-\left(\frac{1}{\zeta}\left(1-\left(\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pow}}_g\left(1-\left(\zeta\left(1-F_g^L\right)^{\varsigma}-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}-\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pow}}_g+1\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}-\left(\frac{1}{\zeta}-1\right)\right)^{\frac{1}{\varsigma}},\right.\\
&\left.\left.1-\left(\frac{1}{\zeta}\left(1-\left(\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pow}}_g\left(1-\left(\zeta\left(1-F_g^U\right)^{\varsigma}-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}-\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pow}}_g+1\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}-\left(\frac{1}{\zeta}-1\right)\right)^{\frac{1}{\varsigma}}\right]\right\rangle.\end{aligned} \tag{51}$$

Proof. Proof of the Theorem 20 is same as Theorem 17. □

Theorem 21. *For a group of HINLNs* $\overline{\overline{hn}}_g (g = 1, 2, \cdots, s)$, ζ *is a parameter and* $\zeta > 0$. $\overline{\overline{We}} = (\overline{\overline{We}}_1, \overline{\overline{We}}_2, \cdots, \overline{\overline{We}}_s)^T$ *is weighting vector for* $\overline{\overline{hn}}_g (g = 1, 2, \cdots, s)$, $\overline{\overline{We}}_g \in [0, 1]$, *and* $\overset{s}{\underset{g=1}{\oplus}} \overline{\overline{We}}_g = 1$. *If* $\overline{\overline{We}} = ((1/s), (1/s), \cdots, (1/s))^T$, *then WGHINLSSPA trims down to generalized HINLSS PA operator:*

$$
\begin{aligned}
&WGHINLSSPA_{\zeta}\left(\overline{\overline{hn}}_1, \overline{\overline{hn}}_2, \cdots, \overline{\overline{hn}}_s\right) = \left(\overset{s}{\underset{g=1}{\oplus}} \overline{\overline{pow}}_g \overline{\overline{hn}}_g^{\zeta}\right)^{\frac{1}{\zeta}} = \bigcup_{u_1 \in \overline{\overline{hn}}_1, u_2 \in \overline{\overline{hn}}_2, \cdots, u_{m+1} \in \overline{\overline{hn}}_s} \\
&\cdot \left\langle f^{*-1}\left(\overset{s}{\underset{g=1}{\oplus}} \overline{\overline{pow}}_g \left(f^*\left(s_{\vartheta(u_g)}\right)\right)^{\zeta}\right)^{\frac{1}{\zeta}}, \left(\left[\left(\frac{1}{\zeta}\left(1 - \left(\overset{s}{\underset{g=1}{\oplus}} \overline{\overline{pow}}_g \left(1 - \left(\zeta\left(\bar{\bar{T}}_g^L\right)^{\varsigma} - (\zeta - 1)\right)^{\frac{1}{\zeta}}\right)^{\varsigma} - \overset{s}{\underset{g=1}{\oplus}} \overline{\overline{pow}}_g + 1\right)^{\frac{1}{\zeta}}\right)^{\varsigma}\right.\right.\right. \\
&\left.\left.\left. - \left(\frac{1}{\zeta} - 1\right)\right)^{\frac{1}{\zeta}}, \frac{1}{\zeta}\left(1 - \left(\overset{s}{\underset{g=1}{\oplus}} \overline{\overline{pow}}_g \left(1 - \left(\zeta\left(\bar{\bar{T}}_g^U\right)^{\varsigma} - (\zeta - 1)\right)^{\frac{1}{\zeta}}\right)^{\varsigma} - \overset{s}{\underset{g=1}{\oplus}} \overline{\overline{pow}}_g + 1\right)^{\frac{1}{\zeta}}\right)^{\varsigma} - \left(\frac{1}{\zeta} - 1\right)\right], \right. \\
&\cdot \left[1 - \left(\frac{1}{\zeta}\left(1 - \left(\overset{s}{\underset{g=1}{\oplus}} \overline{\overline{pow}}_g \left(1 - \left(\zeta\left(1 - \bar{\bar{I}}_g^L\right)^{\varsigma} - (\zeta - 1)\right)^{\frac{1}{\zeta}}\right)^{\varsigma} - \overset{s}{\underset{g=1}{\oplus}} \overline{\overline{pow}}_g + 1\right)^{\frac{1}{\zeta}}\right)^{\varsigma} - \left(\frac{1}{\zeta} - 1\right)\right)^{\frac{1}{\zeta}}, 1\right. \\
&\left. - \left(\frac{1}{\zeta}\left(1 - \left(\overset{s}{\underset{g=1}{\oplus}} \overline{\overline{pow}}_g \left(1 - \left(\zeta\left(1 - \bar{\bar{I}}_g^U\right)^{\varsigma} - (\zeta - 1)\right)^{\frac{1}{\zeta}}\right)^{\varsigma} - \overset{s}{\underset{g=1}{\oplus}} \overline{\overline{pow}}_g + 1\right)^{\frac{1}{\zeta}}\right)^{\varsigma} - \left(\frac{1}{\zeta} - 1\right)\right)^{\frac{1}{\zeta}}\right], \\
&\cdot \left[1 - \left(\frac{1}{\zeta}\left(1 - \left(\overset{s}{\underset{g=1}{\oplus}} \overline{\overline{pow}}_g \left(1 - \left(\zeta\left(1 - \bar{\bar{F}}_g^L\right)^{\varsigma} - (\zeta - 1)\right)^{\frac{1}{\zeta}}\right)^{\varsigma} - \overset{s}{\underset{g=1}{\oplus}} \overline{\overline{pow}}_g + 1\right)^{\frac{1}{\zeta}}\right)^{\varsigma} - \left(\frac{1}{\zeta} - 1\right)\right)^{\frac{1}{\zeta}}, 1\right. \\
&\left.\left. - \left(\frac{1}{\zeta}\left(1 - \left(\overset{s}{\underset{g=1}{\oplus}} \overline{\overline{pow}}_g \left(1 - \left(\zeta\left(1 - \bar{\bar{F}}_g^U\right)^{\varsigma} - (\zeta - 1)\right)^{\frac{1}{\zeta}}\right)^{\varsigma} - \overset{s}{\underset{g=1}{\oplus}} \overline{\overline{pow}}_g + 1\right)^{\frac{1}{\zeta}}\right)^{\varsigma} - \left(\frac{1}{\zeta} - 1\right)\right)^{\frac{1}{\zeta}}\right]\right)\right\rangle,
\end{aligned}
\tag{52}
$$

where $\overline{\overline{pow}}_g = (1 + T(\overline{\overline{hn}}_g))/\overset{s}{\underset{g=1}{\oplus}}(1 + T(\overline{\overline{hn}}_g))$ is the power weight vector. Further, we shall discuss a few cases of the initiated AOs with respect to the parameter ζ and $\overline{\overline{We}}$, which are listed below:

(1) If $\zeta = 1$, $\overline{\overline{We}} = ((1/s), (1/s), \cdots, (1/s))^T$, then, the WGHINLSSPA operator reduces into HINLSS power average (PA) operator:

$$
\begin{aligned}
&HINLSSPA\left(\overline{\overline{hn}}_1, \overline{\overline{hn}}_2, \cdots, \overline{\overline{hn}}_s\right) = \overset{s}{\underset{g=1}{\oplus}} \overline{\overline{pow}}_g \overline{\overline{hn}}_g = \bigcup_{u_1 \in \overline{\overline{hn}}_1, u_2 \in \overline{\overline{hn}}_2, \cdots, u_{m+1} \in \overline{\overline{hn}}_s} \\
&\cdot \left\langle f^{*-1}\left(\overset{s}{\underset{g=1}{\oplus}} \overline{\overline{pow}}_g \left(f^*\left(s_{\vartheta(u_g)}\right)\right)\right), \left(\left[1 - \left(\overset{s}{\underset{g=1}{\oplus}} \overline{\overline{pow}}_g \left(1 - \bar{\bar{T}}_g^L\right)^{\varsigma} - \overset{s}{\underset{g=1}{\oplus}} \overline{\overline{pow}}_g + 1\right)^{\frac{1}{\varsigma}}, 1 - \left(\overset{s}{\underset{g=1}{\oplus}} \overline{\overline{pow}}_g \left(1 - \bar{\bar{T}}_g^U\right)^{\varsigma} - \overset{s}{\underset{g=1}{\oplus}} \overline{\overline{pow}}_g + 1\right)^{\frac{1}{\varsigma}}\right]\right. \right. \\
&\cdot \left[\left(\overset{s}{\underset{g=1}{\oplus}} \overline{\overline{pow}}_g \left(\bar{\bar{I}}_g^L\right)^{\varsigma} - \overset{s}{\underset{g=1}{\oplus}} \overline{\overline{pow}}_g + 1\right)^{\frac{1}{\varsigma}}, \left(\overset{s}{\underset{g=1}{\oplus}} \overline{\overline{pow}}_g \left(\bar{\bar{I}}_g^U\right)^{\varsigma} - \overset{s}{\underset{g=1}{\oplus}} \overline{\overline{pow}}_g + 1\right)^{\frac{1}{\varsigma}}\right], \\
&\left.\left. \cdot \left[\left(\overset{s}{\underset{g=1}{\oplus}} \overline{\overline{pow}}_g \left(\bar{\bar{F}}_g^L\right)^{\varsigma} - \overset{s}{\underset{g=1}{\oplus}} \overline{\overline{pow}}_g + 1\right)^{\frac{1}{\varsigma}}, \left(\overset{s}{\underset{g=1}{\oplus}} \overline{\overline{pow}}_g \left(F_g^U\right)^{\varsigma} - \overset{s}{\underset{g=1}{\oplus}} \overline{\overline{pow}}_g + 1\right)^{\frac{1}{\varsigma}}\right]\right\rangle.
\end{aligned}
\tag{53}
$$

(2) If $\zeta = 1$, $\overline{\overline{We}} = ((1/s), (1/s), \cdots, (1/s))^T$, and $\varsigma = 0$, then, the WGHINLSSPA operator trims down to HINL PA operator based on algebraic operation. That is

$$
\begin{aligned}
&WGHINLSSPA_{\varsigma=0}\left(\overline{\overline{hn}}_1, \overline{\overline{hn}}_2, \cdots, \overline{\overline{hn}}_s\right) = \overset{s}{\underset{g=1}{\oplus}} \overline{\overline{pow}}_g \overline{\overline{hn}}_g \\
&= \bigcup_{u_1 \in \overline{\overline{hn}}_1, u_2 \in \overline{\overline{hn}}_2, \cdots, u_{m+1} \in \overline{\overline{hn}}_s} \left\langle f^{*-1}\left(\overset{s}{\underset{g=1}{\oplus}} \overline{\overline{pow}}_g \left(f^*\left(s_{\vartheta(u_g)}\right)\right)\right)\right\rangle, \\
&= \left\langle \left(\left[1 - \overset{s}{\underset{g=1}{\otimes}} \left(1 - \bar{\bar{T}}_g^L\right)^{\overline{\overline{pow}}_g}, 1 - \overset{s}{\underset{g=1}{\otimes}} \left(1 - \bar{\bar{T}}_g^U\right)^{\overline{\overline{pow}}_g}\right]\right.\right. \\
&\quad \cdot \left[\overset{s}{\underset{g=1}{\otimes}} \left(\bar{\bar{I}}_g^L\right)^{\overline{\overline{pow}}_g}, \overset{s}{\underset{g=1}{\otimes}} \left(\bar{\bar{I}}_g^U\right)^{\overline{\overline{pow}}_g}\right], \\
&\quad \left.\left. \cdot \left[\overset{s}{\underset{g=1}{\otimes}} \left(\bar{\bar{F}}_g^L\right)^{\overline{\overline{pow}}_g}, \overset{s}{\underset{g=1}{\otimes}} \left(\bar{\bar{F}}_g^U\right)^{\overline{\overline{pow}}_g}\right]\right)\right\rangle.
\end{aligned}
\tag{54}
$$

(3) If $\zeta = 1$, $\overline{\overline{We}} = ((1/s), (1/s), \cdots, (1/s))^T$, $\sup(\overline{\overline{hn}}_g, \overline{\overline{hn}}_h) = b$ (i.e., a constant) and $\varsigma = 0$, then, the WGHINLSSPA operator reduces into HINL average operator based on algebraic operation. That is

$$
\begin{aligned}
&WGHINLSSPA_{\varsigma=0}\left(\overline{\overline{hn}}_1, \overline{\overline{hn}}_2, \cdots, \overline{\overline{hn}}_s\right) = \frac{1}{s} \overset{s}{\underset{g=1}{\oplus}} \overline{\overline{hn}}_g = \bigcup_{u_1 \in \overline{\overline{hn}}_1, u_2 \in \overline{\overline{hn}}_2, \cdots, u_{m+1} \in \overline{\overline{hn}}_s} \\
&\cdot \left\langle f^{*-1}\left(\frac{1}{s} \overset{s}{\underset{g=1}{\oplus}} \left(f^*\left(s_{\vartheta(u_g)}\right)\right)\right)\right\rangle, \\
&\cdot \left(\left[1 - \overset{s}{\underset{g=1}{\otimes}} \left(1 - \bar{\bar{T}}_g^L\right)^{\frac{1}{s}}, 1 - \overset{s}{\underset{g=1}{\otimes}} \left(1 - \bar{\bar{T}}_g^U\right)^{\frac{1}{s}}\right]\right. \\
&\cdot \left.\left[\overset{s}{\underset{g=1}{\otimes}} \left(\bar{\bar{I}}_g^L\right)^{\frac{1}{s}}, \overset{s}{\underset{g=1}{\otimes}} \left(\bar{\bar{I}}_g^U\right)^{\frac{1}{s}}\right], \left[\overset{s}{\underset{g=1}{\otimes}} \left(\bar{\bar{F}}_g^L\right)^{\frac{1}{s}}, \overset{s}{\underset{g=1}{\otimes}} \left(\bar{\bar{F}}_g^U\right)^{\frac{1}{s}}\right]\right)\right\rangle.
\end{aligned}
\tag{55}
$$

4.2. Some Generalized Hesitant Interval Neutrosophic Schweizer-Sklar Power Geometric Aggregation Operators. In this subpart, we initiate generalized hesitant interval neutrosophic linguistic Schweizer-Sklar power geometric average (GHINLSSPWGA) operator, weighted GHINLSSPGA, discuss their desirable properties and some particular cases.

Definition 22. For a group of HINLNs $\overline{\overline{hn}}_g (g = 1, 2, \cdots, s)$, GHINLSSPGA operator is a function $\aleph^s \longrightarrow \aleph$,

$$
GHINLSSPGA_\zeta\left(\overline{\overline{hn}}_1, \overline{\overline{hn}}_2, \cdots, \overline{\overline{hn}}_s\right) = \frac{1}{\zeta}\left(\otimes^{s}{}_{g=1}\left(\zeta \overline{\overline{hn}}_g\right)^{\frac{\left(1+T\left(\overline{\overline{hn}}_g\right)\right)}{\overset{s}{\underset{h=1}{\oplus}}\left(1+T\left(\overline{\overline{hn}}_h\right)\right)}}\right),
\tag{56}
$$

where $T(\overline{\overline{hn}}_g) = \overset{s}{\underset{\substack{h=1, \\ g \neq h}}{\oplus}} \mathrm{Sup}(\overline{\overline{hn}}_g, \overline{\overline{hn}}_h)$, parameter $\zeta \in (0, +\infty)$ and $\mathrm{Sup}(\overline{\overline{hn}}_g, \overline{\overline{hn}}_h)$ is the SPD for $\overline{\overline{hn}}_g$ from $\overline{\overline{hn}}_h$ with the following constraint:

(1) $\mathrm{Sup}(\overline{\overline{hn}}_g, \overline{\overline{hn}}_h) \geq \mathrm{Sup}(\overline{\overline{hn}}_u, \overline{\overline{hn}}_v)$ if $\overline{\overline{dis}}(\overline{\overline{hn}}_g, \overline{\overline{hn}}_h) < \overline{\overline{dis}}(\overline{\overline{hn}}_u, \overline{\overline{hn}}_v)$, where $\overline{\overline{dis}}$ is the distance measure among two HINLNs

$$
\begin{aligned}
&\mathrm{Sup}\left(\overline{\overline{hn}}_g, \overline{\overline{hn}}_h\right) \in [0, 1]; \\
&\mathrm{Sup}\left(\overline{\overline{hn}}_g, \overline{\overline{hn}}_h\right) = \mathrm{Sup}\left(\overline{\overline{hn}}_h, \overline{\overline{hn}}_g\right);
\end{aligned}
\tag{57}
$$

To write Equation (56) in an uncomplicated way, we have

$$
\overline{\overline{pw}}_g = \frac{\left(1 + T\left(\overline{\overline{hn}}_g\right)\right)}{\overset{s}{\underset{g=1}{\oplus}}\left(1 + T\left(\overline{\overline{hn}}_g\right)\right)}.
\tag{58}
$$

So, from Equation (58), Equation (56) becomes

$$
GHINLSSPGA_\zeta\left(\overline{\overline{hn}}_1, \overline{\overline{hn}}_2, \cdots, \overline{\overline{hn}}_s\right) = \frac{1}{\zeta}\left(\overset{s}{\underset{g=1}{\otimes}} \left(\zeta \overline{\overline{hn}}_g\right)^{\overline{\overline{pw}}_g}\right).
\tag{59}
$$

Theorem 23. *Let $\overline{\overline{hn}}_g (g = 1, 2, \cdots, s)$ be a group of HINLNs, then, the value aggregated utilizing Definition 22 is still HINLN, and we have*

$$
\begin{aligned}
&GHINLSSPG_\zeta\left(\overline{\overline{hn}}_1, \overline{\overline{hn}}_2, \cdots, \overline{\overline{hn}}_s\right) = \bigcup_{\overline{\overline{we}}_1 \in \overline{\overline{hn}}_1, \overline{\overline{we}}_2 \in \overline{\overline{hn}}_2, \cdots, \overline{\overline{we}}_s \in \overline{\overline{hn}}_s} \\
&\cdot \left\langle f^{*-1}\left(\frac{1}{\overline{\zeta}} \overset{s}{\underset{g=1}{\otimes}} \left(\zeta f^*\left(s_{\vartheta\left(\overline{\overline{we}}_g\right)}\right)\right)^{\overline{\overline{pw}}_g}\right)\right. \\
&\cdot \left(\left[1 - \left(\frac{1}{\overline{\zeta}}\left(1 - \left(\overset{s}{\underset{g=1}{\oplus}} \overline{\overline{pw}}_g \left(1 - \left(\zeta\left(1 - \bar{\bar{T}}_g^L\right)^\varsigma - (\zeta - 1)\right)^{\frac{1}{\varsigma}}\right)^\varsigma\right.\right.\right.\right.\right. \\
&\quad \left.\left.\left.\left.\left. - \overset{s}{\underset{g=1}{\oplus}} \overline{\overline{pw}}_g + 1\right)^{\frac{1}{\varsigma}}\right)^\varsigma - \left(\frac{1}{\overline{\zeta}} - 1\right)\right)^{\frac{1}{\varsigma}}, 1\right. \\
&\quad - \left(\frac{1}{\overline{\zeta}}\left(1 - \left(\overset{s}{\underset{g=1}{\oplus}} \overline{\overline{pw}}_g \left(1 - \left(\zeta\left(1 - \bar{\bar{T}}_g^U\right)^\varsigma - (\zeta - 1)\right)^{\frac{1}{\varsigma}}\right)^\varsigma\right.\right.\right. \\
&\quad \left.\left.\left. - \overset{s}{\underset{g=1}{\oplus}} \overline{\overline{pw}}_g + 1\right)^{\frac{1}{\varsigma}}\right)^\varsigma - \left(\frac{1}{\overline{\zeta}} - 1\right)\right)^{\frac{1}{\varsigma}}\Bigg], \\
&\quad \cdot \left[\left(\frac{1}{\overline{\zeta}}\left(1 - \left(\overset{s}{\underset{g=1}{\oplus}} \overline{\overline{pw}}_g \left(1 - \left(\zeta\left(\bar{\bar{I}}_g^L\right)^\varsigma - (\zeta - 1)\right)^{\frac{1}{\varsigma}}\right)^\varsigma\right.\right.\right.\right. \\
&\quad \left.\left.\left.\left. - \overset{s}{\underset{g=1}{\oplus}} \overline{\overline{pw}}_g + 1\right)^{\frac{1}{\varsigma}}\right)^\varsigma - \left(\frac{1}{\overline{\zeta}} - 1\right)\right)^{\frac{1}{\varsigma}},
\end{aligned}
$$

$$\cdot\left(\frac{1}{\zeta}\left(1-\left(\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g\left(1-\left(\zeta\left(\bar{\bar{I}}_g^U\right)^\varsigma-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right)^\varsigma-\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g+1\right)^{\frac{1}{\varsigma}}\right)^\varsigma-\left(\frac{1}{\zeta}-1\right)\right)^{\frac{1}{\varsigma}}\right],$$

$$\cdot\left[\left(\frac{1}{\zeta}\left(1-\left(\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g\left(1-\left(\zeta\left(\bar{\bar{F}}_g^L\right)^\varsigma-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right)^\varsigma-\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g+1\right)^{\frac{1}{\varsigma}}\right)^\varsigma-\left(\frac{1}{\zeta}-1\right)\right)^{\frac{1}{\varsigma}},\right.$$

$$\left.\cdot\left(\frac{1}{\zeta}\left(1-\left(\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g\left(1-\left(\zeta\left(\bar{\bar{F}}_g^U\right)^\varsigma-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right)^\varsigma-\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g+1\right)^{\frac{1}{\varsigma}}\right)^\varsigma-\left(\frac{1}{\zeta}-1\right)\right)^{\frac{1}{\varsigma}}\right]\right)\right\rangle. \tag{60}$$

Proof. In the following, first, we prove

$$\begin{aligned}
&\overline{\overline{\zeta hn}}_1^{\overline{\overline{pw}}_1}\times\overline{\overline{\zeta hn}}_2^{\overline{\overline{pw}}_2}\times\cdots\times\overline{\overline{\zeta hn}}_s^{\overline{\overline{pw}}_s}\\
&=\underset{\overline{\overline{we}}_1\in\overline{\overline{hn}}_1,\overline{\overline{we}}_2\in\overline{\overline{hn}}_2,\cdots,\overline{\overline{we}}_s\in\overline{\overline{hn}}_s}{\cup}\left\langle f^{*-1}\left(\overset{s}{\underset{g=1}{\otimes}}\left(\zeta f^*\left(s_{\vartheta\left(\overline{\overline{we}}_g\right)}\right)\right)^{\overline{\overline{pw}}_g}\right)\right.\\
&\cdot\left(\left[\left(\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g\left(1-\left(\zeta\left(1-\bar{\bar{T}}_g^L\right)^\varsigma-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right)^\varsigma-\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g+1\right)^{\frac{1}{\varsigma}},\right.\right.\\
&\left.\cdot\left(\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g\left(1-\left(\zeta\left(1-\bar{\bar{T}}_g^U\right)^\varsigma-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right)^\varsigma-\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g+1\right)^{\frac{1}{\varsigma}}\right],\\
&\cdot\left[1-\left(\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g\left(1-\left(\zeta\left(\bar{\bar{I}}_g^L\right)^\varsigma-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right)^\varsigma-\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g+1\right)^{\frac{1}{\varsigma}},\right.\\
&\left.1-\left(\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g\left(1-\left(\zeta\left(I_g^U\right)^\varsigma-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right)^\varsigma-\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g+1\right)^{\frac{1}{\varsigma}}\right],\\
&\cdot\left[1-\left(\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g\left(1-\left(\zeta\left(\bar{\bar{F}}_g^L\right)^\varsigma-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right)^\varsigma-\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g+1\right)^{\frac{1}{\varsigma}},\right.\\
&\left.\left.\left.1-\left(\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g\left(1-\left(\zeta\left(\bar{\bar{F}}_g^U\right)^\varsigma-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right)^\varsigma-\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g+1\right)^{\frac{1}{\varsigma}}\right]\right)\right\rangle.
\end{aligned}\tag{61}$$

To prove Equation (61), we utilize mathematical induction on s.

For $s=2$.

From the operational rules explained for HINLNs in Definition 14, we have

$$\begin{aligned}
\overline{\overline{\zeta hn}}_1&=\underset{\overline{\overline{we}}_1\in\overline{\overline{hn}}_1}{\cup}\left\langle f^{*-1}\left(\zeta f^*\left(s_{\vartheta\left(\overline{\overline{we}}_g\right)}\right)\right)\right.,\\
&\cdot\left(\left[1-\left(\zeta\left(1-\bar{\bar{T}}_1^L\right)^\varsigma-(\zeta-1)\right)^{\frac{1}{\varsigma}},1-\left(\zeta\left(1-\bar{\bar{T}}_1^U\right)^\varsigma-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right],\right.\\
&\cdot\left[\left(\zeta\left(\bar{\bar{I}}_1^L\right)^\varsigma-(\zeta-1)\right)^{\frac{1}{\varsigma}},\left(\zeta\left(\bar{\bar{I}}_1^L\right)^\varsigma-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right],\\
&\left.\cdot\left[\left(\zeta\left(\bar{\bar{F}}_1^L\right)^\varsigma-(\zeta-1)\right)^{\frac{1}{\varsigma}},\left(\zeta\left(\bar{\bar{F}}_1^U\right)^\varsigma-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right]\right\rangle
\end{aligned}$$

$$\begin{aligned}
\left(\overline{\overline{\zeta hn}}_1\right)^{\overline{\overline{pw}}_1}&=\underset{\overline{\overline{we}}_1\in\overline{\overline{hn}}_1}{\cup}\left\langle f^{*-1}\left(\left(\zeta f^*\left(s_{\vartheta\left(\overline{\overline{we}}_2\right)}\right)\right)^{\overline{\overline{pw}}_1}\right)\right.,\\
&\cdot\left(\left[\left(\overline{\overline{pw}}_1\left(1-\left(\zeta\left(1-\bar{\bar{T}}_1^L\right)^\varsigma-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right)^\varsigma-\left(\overline{\overline{pw}}_1-1\right)\right)^{\frac{1}{\varsigma}},\right.\right.\\
&\left.\cdot\left(\overline{\overline{pw}}_1\left(1-\left(\zeta\left(1-\bar{\bar{T}}_1^U\right)^\varsigma-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right)^\varsigma-\left(\overline{\overline{pw}}_1-1\right)\right)^{\frac{1}{\varsigma}}\right],\\
&\cdot\left[1-\left(\overline{\overline{pw}}_1\left(1-\left(\zeta\left(\bar{\bar{I}}_1^L\right)^\varsigma-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right)^\varsigma-\left(\overline{\overline{pw}}_1-1\right)\right)^{\frac{1}{\varsigma}},\right.\\
&\left.\cdot\left(\overline{\overline{pw}}_1\left(1-\left(\zeta\left(\bar{\bar{I}}_1^U\right)^\varsigma-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right)^\varsigma-\left(\overline{\overline{pw}}_1-1\right)\right)^{\frac{1}{\varsigma}}\right],\\
&\cdot\left[1-\left(\overline{\overline{pw}}_1\left(1-\left(\zeta\left(\bar{\bar{F}}_1^L\right)^\varsigma-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right)^\varsigma-\left(\overline{\overline{pw}}_1-1\right)\right)^{\frac{1}{\varsigma}},\right.\\
&\left.\left.\cdot\left(\overline{\overline{pw}}_1\left(1-\left(\zeta\left(\bar{\bar{F}}_1^U\right)^\varsigma-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right)^\varsigma-\left(\overline{\overline{pw}}_1-1\right)\right)^{\frac{1}{\varsigma}}\right]\right)\right\rangle;
\end{aligned}\tag{62}$$

Similarly,

$$\begin{aligned}
\left(\overline{\overline{\zeta hn}}_2\right)^{\overline{\overline{pw}}_2}&=\underset{\overline{\overline{we}}_2\in\overline{\overline{hn}}_2}{\cup}\left\langle f^{*-1}\left(\left(\zeta f^*\left(s_{\vartheta\left(\overline{\overline{we}}_2\right)}\right)\right)^{\overline{\overline{pw}}_2}\right)\right.,\\
&\cdot\left(\left[\left(\overline{\overline{pw}}_2\left(1-\left(\zeta\left(1-\bar{\bar{T}}_2^L\right)^\varsigma-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right)^\varsigma-\left(\overline{\overline{pw}}_2-1\right)\right)^{\frac{1}{\varsigma}},\right.\right.\\
&\left.\cdot\left(\overline{\overline{pw}}_2\left(1-\left(\zeta\left(1-\bar{\bar{T}}_2^U\right)^\varsigma-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right)^\varsigma-\left(\overline{\overline{pw}}_2-1\right)\right)^{\frac{1}{\varsigma}}\right],\\
&\cdot\left[1-\left(\overline{\overline{pw}}_2\left(1-\left(\zeta\left(\bar{\bar{I}}_2^L\right)^\varsigma-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right)^\varsigma-\left(\overline{\overline{pw}}_2-1\right)\right)^{\frac{1}{\varsigma}},\right.\\
&\left.1-\left(\overline{\overline{pw}}_2\left(1-\left(\zeta\left(\bar{\bar{I}}_2^U\right)^\varsigma-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right)^\varsigma-\left(\overline{\overline{pw}}_2-1\right)\right)^{\frac{1}{\varsigma}}\right],\\
&\cdot\left[1-\left(\overline{\overline{pw}}_2\left(1-\left(\zeta\left(\bar{\bar{F}}_2^L\right)^\varsigma-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right)^\varsigma-\left(\overline{\overline{pw}}_2-1\right)\right)^{\frac{1}{\varsigma}},\right.\\
&\left.\left.1-\left(\overline{\overline{pw}}_2\left(1-\left(\zeta\left(\bar{\bar{F}}_2^U\right)^\varsigma-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right)^\varsigma-\left(\overline{\overline{pw}}_2-1\right)\right)^{\frac{1}{\varsigma}}\right]\right)\right\rangle.
\end{aligned}\tag{63}$$

Then,

$$
\begin{aligned}
\left(\zeta\overline{\overline{hn}}_1\right)^{\overline{\overline{pw}}_1}\otimes\left(\zeta\overline{\overline{hn}}_2\right)^{\overline{\overline{pw}}_2} &= \bigcup_{\overline{\overline{we}}_1\in\overline{\overline{hn}}_1,\overline{\overline{we}}_2\in\overline{\overline{hn}}_2}\\
&\cdot\left\langle f^{*-1}\left(\left(\zeta f^*\left(s_{\vartheta}\left(\overline{\overline{we}}_1\right)\right)\right)^{\overline{\overline{pw}}_1}\left(\zeta f^*\left(s_{\vartheta}\left(\overline{\overline{we}}_2\right)\right)\right)^{\overline{\overline{pw}}_2}\right),\left(\left[\left[\left(\left(\left(\overline{\overline{pw}}_1\left(1-\left(\zeta\left(1-\bar{\bar{T}}_1^L\right)^{\varsigma}-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}-\left(\overline{\overline{pw}}_1-1\right)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}\right.\right.\right.\right.\\
&+\left.\left.\left.\left.\left(\left(\overline{\overline{pw}}_2\left(1-\left(\zeta\left(1-\bar{\bar{T}}_2^L\right)^{\varsigma}-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}-\left(\overline{\overline{pw}}_2-1\right)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}-1\right)^{\frac{1}{\varsigma}},\left(\left(\left(\overline{\overline{pw}}_1\left(1-\left(\zeta\left(1-\bar{\bar{T}}_1^L\right)^{\varsigma}-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}-\left(\overline{\overline{pw}}_1-1\right)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}\right.\right.\right.\\
&+\left.\left.\left.\left(\left(\overline{\overline{pw}}_2\left(1-\left(\zeta\left(1-\bar{\bar{T}}_2^L\right)^{\varsigma}-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}-\left(\overline{\overline{pw}}_2-1\right)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}-1\right)^{\frac{1}{\varsigma}}\right],\left[1-\left(\left(1-\left(1-\left(\overline{\overline{pw}}_1\left(1-\left(\zeta\left(\bar{\bar{I}}_1^L\right)^{\varsigma}-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}-\left(\overline{\overline{pw}}_1-1\right)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}\right)\right.\right.\\
&+\left.\left.\left(1-\left(1-\left(\overline{\overline{pw}}_2\left(1-\left(\zeta\left(\bar{\bar{I}}_2^L\right)^{\varsigma}-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}-\left(\overline{\overline{pw}}_2-1\right)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}-1\right)^{\frac{1}{\varsigma}},1-\left(\left(1-\left(1-\left(\overline{\overline{pw}}_1\left(1-\left(\zeta\left(\bar{\bar{I}}_1^U\right)^{\varsigma}-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}-\left(\overline{\overline{pw}}_1-1\right)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}\right)\right.\right.\\
&+\left.\left.\left(1-\left(1-\left(\overline{\overline{pw}}_2\left(1-\left(\zeta\left(\bar{\bar{I}}_2^U\right)^{\varsigma}-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}-\left(\overline{\overline{pw}}_2-1\right)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}-1\right)^{\frac{1}{\varsigma}}\right],1-\left(\left(1-\left(1-\left(\overline{\overline{pw}}_1\left(1-\left(\zeta\left(\bar{\bar{F}}_1^L\right)^{\varsigma}-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}-\left(\overline{\overline{pw}}_1-1\right)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}\right)\right.\right.\\
&+\left.\left.\left(1-\left(1-\left(\overline{\overline{pw}}_2\left(1-\left(\zeta\left(\bar{\bar{F}}_2^L\right)^{\varsigma}-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}-\left(\overline{\overline{pw}}_2-1\right)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}-1\right)^{\frac{1}{\varsigma}},1-\left(\left(1-\left(1-\left(\overline{\overline{pw}}_1\left(1-\left(\zeta\left(\bar{\bar{F}}_1^U\right)^{\varsigma}-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}-\left(\overline{\overline{pw}}_1-1\right)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}\right)\right.\right.\\
&+\left.\left.\left.\left(1-\left(1-\left(\overline{\overline{pw}}_2\left(1-\left(\zeta\left(\bar{\bar{F}}_2^U\right)^{\varsigma}-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}-\left(\overline{\overline{pw}}_2-1\right)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}-1\right)^{\frac{1}{\varsigma}}\right]\right)\right\rangle;\\
&=\bigcup_{\overline{\overline{we}}_1\in\overline{\overline{hn}}_1,\overline{\overline{we}}_2\in\overline{\overline{hn}}_2}\left\langle f^{*-1}\left(\overset{2}{\underset{g=1}{\otimes}}\left(\zeta f^*\left(s_{\vartheta}\left(\overline{\overline{we}}_g\right)\right)\right)^{\overline{\overline{pw}}_g}\right),\left(\left[\left(\left(\overline{\overline{pw}}_1\left(1-\left(\zeta\left(1-\bar{\bar{T}}_1^L\right)^{\varsigma}-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}-\left(\overline{\overline{pw}}_1-1\right)\right)\right.\right.\right.\right.\\
&+\left.\left.\left.\left(\overline{\overline{pw}}_2\left(1-\left(\zeta\left(1-\bar{\bar{T}}_2^L\right)^{\varsigma}-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}-\left(\overline{\overline{pw}}_2-1\right)\right)-1\right)^{\frac{1}{\varsigma}},\left(\left(\overline{\overline{pw}}_1\left(1-\left(\zeta\left(1-\bar{\bar{T}}_1^U\right)^{\varsigma}-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}-\left(\overline{\overline{pw}}_1-1\right)\right)\right.\right.\\
&+\left.\left.\left(\overline{\overline{pw}}_2\left(1-\left(\zeta\left(1-\bar{\bar{T}}_2^U\right)^{\varsigma}-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}-\left(\overline{\overline{pw}}_2-1\right)\right)-1\right)^{\frac{1}{\varsigma}}\right],\\
&\cdot\left[1-\left(\left(\left(\overline{\overline{pw}}_1\left(1-\left(\zeta\left(\bar{\bar{I}}_1^L\right)^{\varsigma}-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}-\left(\overline{\overline{pw}}_1-1\right)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}+\left(\left(\overline{\overline{pw}}_2\left(1-\left(\zeta\left(\bar{\bar{I}}_2^L\right)^{\varsigma}-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}-\left(\overline{\overline{pw}}_2-1\right)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}-1\right)^{\frac{1}{\varsigma}},1\right.\\
&-\left.\left(\left(\left(\overline{\overline{pw}}_1\left(1-\left(\zeta\left(\bar{\bar{I}}_1^U\right)^{\varsigma}-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}-\left(\overline{\overline{pw}}_1-1\right)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}+\left(\left(\overline{\overline{pw}}_2\left(1-\left(\zeta\left(\bar{\bar{I}}_2^U\right)^{\varsigma}-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}-\left(\overline{\overline{pw}}_2-1\right)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}-1\right)^{\frac{1}{\varsigma}}\right],\\
&\cdot\left[1-\left(\left(\left(\overline{\overline{pw}}_1\left(1-\left(\zeta\left(\bar{\bar{F}}_1^L\right)^{\varsigma}-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}-\left(\overline{\overline{pw}}_1-1\right)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}+\left(\left(\overline{\overline{pw}}_2\left(1-\left(\zeta\left(\bar{\bar{F}}_2^L\right)^{\varsigma}-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}-\left(\overline{\overline{pw}}_2-1\right)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}-1\right)^{\frac{1}{\varsigma}},1\right.\\
&-\left.\left.\left(\left(\left(\overline{\overline{pw}}_1\left(1-\left(\zeta\left(\bar{\bar{F}}_1^U\right)^{\varsigma}-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}-\left(\overline{\overline{pw}}_1-1\right)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}+\left(\left(\overline{\overline{pw}}_2\left(1-\left(\zeta\left(\bar{\bar{F}}_2^U\right)^{\varsigma}-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}-\left(\overline{\overline{pw}}_2-1\right)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}-1\right)^{\frac{1}{\varsigma}}\right]\right)\right\rangle;\\
\left(\zeta\overline{\overline{hn}}_1\right)^{\overline{\overline{pw}}_1}\times\left(\zeta\overline{\overline{hn}}_2\right)^{\overline{\overline{pw}}_2} &= \bigcup_{\overline{\overline{we}}_1\in\overline{\overline{hn}}_1,\overline{\overline{we}}_2\in\overline{\overline{hn}}_2}\left\langle f^{*-1}\left(\overset{2}{\underset{g=1}{\otimes}}\left(\zeta f^*\left(s_{\vartheta}\left(\overline{\overline{we}}_g\right)\right)\right)^{\overline{\overline{pw}}_g}\right),\right.\\
&=\left(\left[\left(\overset{2}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g\left(1-\left(\zeta\left(1-\bar{\bar{T}}_g^L\right)^{\varsigma}-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}-\overset{2}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g+1\right)^{\frac{1}{\varsigma}},\left(\overset{2}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g\left(1-\left(\zeta\left(1-\bar{\bar{T}}_g^U\right)^{\varsigma}-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}-\overset{2}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g+1\right)^{\frac{1}{\varsigma}}\right],\right.\\
&\cdot\left[1-\left(\overset{2}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g\left(1-\left(\zeta\left(\bar{\bar{I}}_g^L\right)^{\varsigma}-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}-\overset{2}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g+1\right)^{\frac{1}{\varsigma}},1-\left(\overset{2}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g\left(1-\left(\zeta\left(\bar{\bar{I}}_g^U\right)^{\varsigma}-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}-\overset{2}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g+1\right)^{\frac{1}{\varsigma}}\right],\\
&\cdot\left.\left[1-\left(\overset{2}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g\left(1-\left(\zeta\left(\bar{\bar{F}}_g^L\right)^{\varsigma}-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}-\overset{2}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g+1\right)^{\frac{1}{\varsigma}},1-\left(\overset{2}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g\left(1-\left(\zeta\left(\bar{\bar{F}}_g^U\right)^{\varsigma}-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}-\overset{2}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g+1\right)^{\frac{1}{\varsigma}}\right]\right)\right\rangle;
\end{aligned}
\tag{64}
$$

If Equation (50) holds for $s = m$

$$\left(\zeta\overline{\overline{hn}}_1\right)^{\overline{\overline{pw}}_1} \otimes \left(\zeta\overline{\overline{hn}}_2\right)^{\overline{\overline{pw}}_2} \otimes \cdots \otimes \left(\zeta\overline{\overline{hn}}_m\right)^{\overline{\overline{pw}}_m} = \bigcup_{\overline{\overline{we}}_1 \in \overline{\overline{hn}}_1, \overline{\overline{we}}_2 \in \overline{\overline{hn}}_2, \cdots, \overline{\overline{we}}_s \in \overline{\overline{hn}}_m}$$
$$\cdot \left\langle f^{*-1}\left(\overset{m}{\underset{g=1}{\otimes}}\left(\zeta f^*\left(s_{\vartheta}\left(\overline{\overline{we}}_g\right)\right)\right)^{\overline{\overline{pw}}_g}\right)\left(\left[\left(\overset{m}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g\left(1-\left(\zeta\left(1-\bar{\bar{T}}_g^L\right)^{\varsigma}-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}-\overset{m}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g+1\right)^{\frac{1}{\varsigma}},\right.\right.\right.$$
$$\cdot\left(\overset{m}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g\left(1-\left(\zeta\left(1-\bar{\bar{T}}_g^U\right)^{\varsigma}-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}-\overset{m}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g+1\right)^{\frac{1}{\varsigma}}\Bigg], \left[1-\left(\overset{m}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g\left(1-\left(\zeta\left(\bar{\bar{I}}_g^L\right)^{\varsigma}-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}-\overset{m}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g+1\right)^{\frac{1}{\varsigma}},\right.$$
$$\cdot\left.\overset{m}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g\left(1-\left(\zeta\left(\bar{\bar{I}}_g^U\right)^{\varsigma}-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}-\overset{m}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g+1\right], \left[1-\left(\overset{m}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g\left(1-\left(\zeta\left(\bar{\bar{F}}_g^L\right)^{\varsigma}-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}-\overset{m}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g+1\right)^{\frac{1}{\varsigma}},\right.$$
$$\left.\left.\left.\cdot\overset{m}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g\left(1-\left(\zeta\left(\bar{\bar{F}}_g^U\right)^{\varsigma}-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}-\overset{m}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g+1\right]\right)\right\rangle. \tag{65}$$

Then, when $s = m + 1$, by the operational rules given in Definition 14, we have

$$\left(\zeta\overline{\overline{hn}}_1\right)^{\overline{\overline{pw}}_1} \otimes \left(\zeta\overline{\overline{hn}}_2\right)^{\overline{\overline{pw}}_2} \otimes \cdots \otimes \left(\zeta\overline{\overline{hn}}_{m+1}\right)^{\overline{\overline{pw}}_{m+1}}$$
$$= \left(\zeta\overline{\overline{hn}}_1\right)^{\overline{\overline{pw}}_1} \otimes \left(\zeta\overline{\overline{hn}}_2\right)^{\overline{\overline{pw}}_2} \otimes \cdots \otimes \left(\zeta\overline{\overline{hn}}_m\right)^{\overline{\overline{pw}}_m} \otimes \left(\zeta\overline{\overline{hn}}_{m+1}\right)^{\overline{\overline{pw}}_{m+1}}$$
$$= \bigcup_{\overline{\overline{we}}_1 \in \overline{\overline{hn}}_1, \overline{\overline{we}}_2 \in \overline{\overline{hn}}_2, \cdots, \overline{\overline{we}}_s \in \overline{\overline{hn}}_m} \left\langle f^{*-1}\left(\overset{m}{\underset{g=1}{\otimes}}\left(\zeta f^*\left(s_{\vartheta}\left(\overline{\overline{we}}_g\right)\right)\right)^{\overline{\overline{pw}}_g}\right)\right.$$
$$\cdot\left(\left[\left(\overset{m}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g\left(1-\left(\zeta\left(1-\bar{\bar{T}}_g^L\right)^{\varsigma}-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}-\overset{m}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g+1\right)^{\frac{1}{\varsigma}},\right.\right.$$
$$\cdot\left(\overset{m}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g\left(1-\left(\zeta\left(1-\bar{\bar{T}}_g^U\right)^{\varsigma}-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}-\overset{m}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g+1\right)^{\frac{1}{\varsigma}}\Bigg],$$
$$\cdot\left[1-\left(\overset{m}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g\left(1-\left(\zeta\left(\bar{\bar{I}}_g^L\right)^{\varsigma}-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}-\overset{m}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g+1\right)^{\frac{1}{\varsigma}},1\right.$$
$$\left.-\left(\overset{m}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g\left(1-\left(\zeta\left(\bar{\bar{I}}_g^U\right)^{\varsigma}-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}-\overset{m}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g+1\right)^{\frac{1}{\varsigma}}\right],$$
$$\cdot\left[1-\left(\overset{m}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g\left(1-\left(\zeta\left(\bar{\bar{F}}_g^L\right)^{\varsigma}-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}-\overset{m}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g+1\right)^{\frac{1}{\varsigma}},1\right.$$
$$\left.\left.-\left(\overset{m}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g\left(1-\left(\zeta\left(\bar{\bar{F}}_g^U\right)^{\varsigma}-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}-\overset{m}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g+1\right)^{\frac{1}{\varsigma}}\right]\right\rangle \oplus \bigcup_{\overline{\overline{we}}_{m+1} \in \overline{\overline{hn}}_{m+1}}$$
$$\cdot\left\langle f^{*-1}\left(\left(\zeta f^*\left(s_{\vartheta}\left(\overline{\overline{we}}_{m+1}\right)\right)\right)^{\overline{\overline{pw}}_{m+1}}\right)\right.,$$
$$\cdot\left(\left[\left(\overline{\overline{pw}}_{m+1}\left(1-\left(\zeta\left(1-\bar{\bar{T}}_{m+1}^L\right)^{\varsigma}-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}-\left(\overline{\overline{pw}}_{m+1}-1\right)\right)^{\frac{1}{\varsigma}},\right.\right.$$
$$\cdot\left(\overline{\overline{pw}}_{m+1}\left(1-\left(\zeta\left(1-\bar{\bar{T}}_{m+1}^U\right)^{\varsigma}-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}-\left(\overline{\overline{pw}}_{m+1}-1\right)\right)^{\frac{1}{\varsigma}}\Bigg],$$
$$\cdot\left[1-\left(\overline{\overline{pw}}_{m+1}\left(1-\left(\zeta\left(\bar{\bar{I}}_{m+1}^L\right)^{\varsigma}-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}-\left(\overline{\overline{pw}}_{m+1}-1\right)\right)^{\frac{1}{\varsigma}},1\right.$$
$$\left.-\left(\overline{\overline{pw}}_{m+1}\left(1-\left(\zeta\left(\bar{\bar{I}}_{m+1}^U\right)^{\varsigma}-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}-\left(\overline{\overline{pw}}_{m+1}-1\right)\right)^{\frac{1}{\varsigma}}\right],$$
$$\cdot\left[1-\left(\overline{\overline{pw}}_{m+1}\left(1-\left(\zeta\left(\bar{\bar{F}}_{m+1}^L\right)^{\varsigma}-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}-\left(\overline{\overline{pw}}_{m+1}-1\right)\right)^{\frac{1}{\varsigma}},1\right.$$
$$\left.-\left(\overline{\overline{pw}}_{m+1}\left(1-\left(\zeta\left(\bar{\bar{F}}_{m+1}^U\right)^{\varsigma}-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}-\left(\overline{\overline{pw}}_{m+1}-1\right)\right)^{\frac{1}{\varsigma}}\right]\Bigg\rangle,$$

$$\left(\zeta\overline{\overline{hn}}_1\right)^{\overline{\overline{pw}}_1} \otimes \left(\zeta\overline{\overline{hn}}_2\right)^{\overline{\overline{pw}}_2} \otimes \cdots \otimes \left(\zeta\overline{\overline{hn}}_{m+1}\right)^{\overline{\overline{pw}}_{m+1}}$$
$$= \bigcup_{\overline{\overline{we}}_1 \in \overline{\overline{hn}}_1, \overline{\overline{we}}_2 \in \overline{\overline{hn}}_2, \cdots, \overline{\overline{we}}_s \in \overline{\overline{hn}}_{m+1}} \left\langle f^{*-1}\left(\overset{m+1}{\underset{g=1}{\otimes}}\left(\zeta f^*\left(s_{\vartheta}\left(\overline{\overline{we}}_g\right)\right)\right)^{\overline{\overline{pw}}_g}\right),\right.$$
$$\cdot\left(\left[\left(\overset{m+1}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g\left(1-\left(\zeta\left(1-\bar{\bar{T}}_g^L\right)^{\varsigma}-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}-\overset{m+1}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g+1\right)^{\frac{1}{\varsigma}},\right.\right.$$
$$\cdot\left(\overset{m+1}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g\left(1-\left(\zeta\left(1-\bar{\bar{T}}_g^U\right)^{\varsigma}-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}-\overset{m+1}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g+1\right)^{\frac{1}{\varsigma}}\Bigg],$$
$$\cdot\left[1-\left(\overset{m+1}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g\left(1-\left(\zeta\left(\bar{\bar{I}}_g^L\right)_g^{\varsigma}-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}-\overset{m+1}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g+1\right)^{\frac{1}{\varsigma}},1\right.$$
$$\left.\left.-\left(\overset{m+1}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g\left(1-\left(\zeta\left(\bar{\bar{I}}_g^U\right)_g^{\varsigma}-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}-\overset{m+1}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g+1\right)^{\frac{1}{\varsigma}}\right]\right\rangle. \tag{66}$$

That is, Equation (61) is true for $g = m + 1$. So Equation (61) is true for all g. Then,

$$\frac{1}{\zeta}\left(\overset{s}{\underset{g=1}{\otimes}}\left(\zeta\overline{\overline{hn}}_g\right)^{\overline{\overline{pw}}_g}\right)=\underset{\overline{\overline{we}}_1\epsilon\overline{\overline{hn}}_1,\overline{\overline{we}}_2\epsilon\overline{\overline{hn}}_2,\cdots,\overline{\overline{we}}_s\epsilon\overline{\overline{hn}}_s}{\cup}$$
$$\cdot\left\langle f^{*-1}\left(\frac{1}{\zeta}\overset{s}{\underset{g=1}{\otimes}}\left(\zeta f^*\left(s_{\vartheta\left(\overline{\overline{we}}_g\right)}\right)\right)^{\overline{\overline{pw}}_g}\right)\right.$$
$$\cdot\left(\left[1-\left(\frac{1}{\zeta}\left(1-\left(\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g\left(1-\left(\zeta\left(1-\bar{\bar{T}}_g^L\right)^{\varsigma}-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right)\right.\right.\right.\right.\right.$$
$$\left.\left.\left.\left.\left.-\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g+1\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}-\left(\frac{1}{\zeta}-1\right)\right)^{\frac{1}{\varsigma}},1\right.\right.$$
$$-\left(\frac{1}{\zeta}\left(1-\left(\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g\left(1-\left(\zeta\left(1-\bar{\bar{T}}_g^U\right)^{\varsigma}-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right)\right.\right.\right.\right.$$
$$\left.\left.\left.\left.-\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g+1\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}-\left(\frac{1}{\zeta}-1\right)\right)^{\frac{1}{\varsigma}}\right],$$
$$\cdot\left[\left(\frac{1}{\zeta}\left(1-\left(\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g\left(1-\left(\zeta\left(\bar{\bar{I}}_g^L\right)_g^{\varsigma}-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right)\right.\right.\right.\right.$$
$$\left.\left.\left.\left.-\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g+1\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}-\left(\frac{1}{\zeta}-1\right)\right)^{\frac{1}{\varsigma}},\left(\frac{1}{\zeta}\left(1-\left(\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g(1\right.\right.\right.\right.$$
$$\left.\left.\left.\left.-\left(\zeta\left(\bar{\bar{I}}_g^U\right)_g^{\varsigma}-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}-\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g+1\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}-\left(\frac{1}{\zeta}-1\right)\right)^{\frac{1}{\varsigma}}\right],$$
$$\cdot\left[\left(\frac{1}{\zeta}\left(1-\left(\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g\left(1-\left(\zeta\left(\bar{\bar{F}}_g^L\right)_g^{\varsigma}-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right)\right.\right.\right.\right.$$
$$\left.\left.\left.\left.-\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g+1\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}-\left(\frac{1}{\zeta}-1\right)\right)^{\frac{1}{\varsigma}},\left(\frac{1}{\zeta}\left(1-\left(\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g(1\right.\right.\right.\right.$$
$$\left.\left.\left.\left.\left.-\left(\zeta\left(\bar{\bar{F}}_g^U\right)_g^{\varsigma}-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}-\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g+1\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}-\left(\frac{1}{\zeta}-1\right)\right)^{\frac{1}{\varsigma}}\right]\right)\left.\right\rangle,\tag{67}$$

Therefore,

$$GHINLSSPGA_{\zeta}\left(\overline{\overline{hn}}_1,\overline{\overline{hn}}_2,\cdots,\overline{\overline{hn}}_s\right)=\underset{\overline{\overline{we}}_1\epsilon\overline{\overline{hn}}_1,\overline{\overline{we}}_2\epsilon\overline{\overline{hn}}_2,\cdots,\overline{\overline{we}}_s\epsilon\overline{\overline{hn}}_s}{\cup}$$
$$\cdot\left\langle f^{*-1}\left(\frac{1}{\zeta}\overset{s}{\underset{g=1}{\otimes}}\left(\zeta f^*\left(s_{\vartheta\left(\overline{\overline{we}}_g\right)}\right)\right)^{\overline{\overline{pw}}_g}\right)\right.$$
$$\cdot\left(\left[1-\left(\frac{1}{\zeta}\left(1-\left(\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g\left(1-\left(\zeta\left(1-\bar{\bar{T}}_g^L\right)^{\varsigma}-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right)\right.\right.\right.\right.\right.$$
$$\left.\left.\left.\left.\left.-\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g+1\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}-\left(\frac{1}{\zeta}-1\right)\right)^{\frac{1}{\varsigma}},1\right.\right.$$
$$-\left(\frac{1}{\zeta}\left(1-\left(\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g\left(1-\left(\zeta\left(1-\bar{\bar{T}}_g^U\right)^{\varsigma}-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right)\right.\right.\right.\right.$$
$$\left.\left.\left.\left.-\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g+1\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}-\left(\frac{1}{\zeta}-1\right)\right)^{\frac{1}{\varsigma}}\right],$$
$$\cdot\left[\left(\frac{1}{\zeta}\left(1-\left(\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g\left(1-\left(\zeta\left(\bar{\bar{I}}_g^L\right)_g^{\varsigma}-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right)\right.\right.\right.\right.$$
$$\left.\left.\left.\left.-\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g+1\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}-\left(\frac{1}{\zeta}-1\right)\right)^{\frac{1}{\varsigma}},\right.$$
$$\cdot\left(\frac{1}{\zeta}\left(1-\left(\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g\left(1-\left(\zeta\left(\bar{\bar{I}}_g^U\right)_g^{\varsigma}-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right)\right.\right.\right.\right.$$
$$\left.\left.\left.\left.-\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g+1\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}-\left(\frac{1}{\zeta}-1\right)\right)^{\frac{1}{\varsigma}}\right),$$
$$\cdot\left[\left(\frac{1}{\zeta}\left(1-\left(\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g\left(1-\left(\zeta\left(\bar{\bar{F}}_g^L\right)_g^{\varsigma}-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right)\right.\right.\right.\right.$$
$$\left.\left.\left.\left.-\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g+1\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}-\left(\frac{1}{\zeta}-1\right)\right)^{\frac{1}{\varsigma}},\right.$$
$$\cdot\left(\frac{1}{\zeta}\left(1-\left(\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g\left(1-\left(\zeta\left(\bar{\bar{F}}_g^U\right)_g^{\varsigma}-(\zeta-1)\right)^{\frac{1}{\varsigma}}\right)\right.\right.\right.\right.$$
$$\left.\left.\left.\left.-\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g+1\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}-\left(\frac{1}{\zeta}-1\right)\right)^{\frac{1}{\varsigma}}\right]\left.\right)\left.\right\rangle,\tag{68}$$

This completes the proof of Theorem 23. □

Further, we shall examine a few cases of the initiated AOs with respect to the parameter ζ and $\overline{\overline{pw}}$, which are listed below:

(1) If $\zeta=1$, then, the GHINLSSPGA operator trims down to HINLSS PGA operator:

$$GHINLSSPGA\left(\overline{\overline{hn}}_1,\overline{\overline{hn}}_2,\cdots,\overline{\overline{hn}}_s\right)=\overset{s}{\underset{g=1}{\otimes}}\left(\overline{\overline{hn}}_g\right)^{\overline{\overline{pw}}_g}$$
$$=\underset{\overline{\overline{we}}_1\epsilon\overline{\overline{hn}}_1,\overline{\overline{we}}_2\epsilon\overline{\overline{hn}}_2,\cdots,\overline{\overline{we}}_s\epsilon\overline{\overline{hn}}_s}{\cup}\left\langle f^{*-1}\left(\overset{s}{\underset{g=1}{\otimes}}\left(f^*\left(s_{\vartheta\left(\overline{\overline{we}}_g\right)}\right)\right)^{\overline{\overline{pw}}_g}\right)\right.$$
$$\cdot\left\langle\left[\left(\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g\left(\bar{\bar{T}}_g^L\right)^{\varsigma}-\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g+1\right)^{\frac{1}{\varsigma}},\right.\right.$$
$$\left.\left.\cdot\left(\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g\left(\bar{\bar{T}}_g^U\right)^{\varsigma}-\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g+1\right)^{\frac{1}{\varsigma}}\right],\right.$$
$$\cdot\left[1-\left(\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g\left(1-\bar{\bar{I}}_g^L\right)-\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g+1\right)^{\frac{1}{\varsigma}},1\right.$$
$$\left.-\left(\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g\left(1-\bar{\bar{I}}_g^U\right)-\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g+1\right)^{\frac{1}{\varsigma}}\right],$$
$$\cdot\left[1-\left(\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g\left(1-\bar{\bar{F}}_g^L\right)-\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g+1\right)^{\frac{1}{\varsigma}},1\right.$$
$$\left.\left.-\left(\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g\left(1-\bar{\bar{F}}_g^U\right)-\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pw}}_g+1\right)^{\frac{1}{\varsigma}}\right]\right\rangle.\tag{69}$$

(2) $\zeta=1$ and $\varsigma=0$, then, the GHINLSSPGA operator reduces into HINL PGA operator based on algebraic operation. That is

$$GHINLSSPGA_{\varsigma=0}\left(\overline{\overline{hn}}_1, \overline{\overline{hn}}_2, \cdots, \overline{\overline{hn}}_s\right) = \overset{s}{\underset{g=1}{\otimes}}\left(\overline{\overline{hn}}_g\right)^{\overline{\overline{pw}}_g} \cdot \left\langle f^{*-1}\left(\overset{s}{\underset{g=1}{\otimes}}\left(f^*\left(s_{\vartheta\left(\overline{\overline{we}}_g\right)}\right)\right)^{\overline{\overline{pw}}_g}\right), \cdot\left(\left[\overset{s}{\underset{g=1}{\otimes}}\left(\overline{\overline{T}}_g^L\right)^{\overline{\overline{pw}}_g}, \overset{s}{\underset{g=1}{\otimes}}\left(\overline{\overline{T}}_g^U\right)^{\overline{\overline{pw}}_g}\right], \cdot\left[1-\overset{s}{\underset{g=1}{\otimes}}\left(1-\overline{\overline{I}}_g^L\right)^{\overline{\overline{pw}}_g}, 1-\overset{s}{\underset{g=1}{\otimes}}\left(1-\overline{\overline{I}}_g^U\right)^{\overline{\overline{pw}}_g}\right], \cdot\left[1-\overset{s}{\underset{g=1}{\otimes}}\left(1-\overline{\overline{F}}_g^L\right)^{\overline{\overline{pw}}_g}, 1-\overset{s}{\underset{g=1}{\otimes}}\left(1-\overline{\overline{F}}_g^U\right)^{\overline{\overline{pw}}_g}\right]\right)\right\rangle \quad \overline{\overline{we}}_1\in\overline{\overline{hn}}_1, \overline{\overline{we}}_2\in\overline{\overline{hn}}_2, \cdots, \overline{\overline{we}}_s\in\overline{\overline{hn}}_s \tag{70}$$

(3) If $\zeta = 1$, $\sup\left(\overline{\overline{hn}}_g, \overline{\overline{hn}}_h\right) = b$ (i.e., a constant) and $\varsigma = 0$, then, the GHINLSSPGA operator reduces into HINLGA operator based on algebraic operation. That is

$$GHIINLSSPGA_{\varsigma=0}\left(\overline{\overline{hn}}_1, \overline{\overline{hn}}_2, \cdots, \overline{\overline{hn}}_s\right) = \overset{s}{\underset{g=1}{\otimes}}\left(\overline{\overline{hn}}_g\right)^{\frac{1}{s}} \cdot \left\langle f^{*-1}\left(\overset{s}{\underset{g=1}{\otimes}}\left(f^*\left(s_{\vartheta\left(\overline{\overline{we}}_g\right)}\right)\right)^{\frac{1}{s}}\right), \cdot\left(\left[\overset{s}{\underset{g=1}{\otimes}}\left(\overline{\overline{T}}_g^L\right)^{\frac{1}{s}}, \overset{s}{\underset{g=1}{\otimes}}\left(\overline{\overline{T}}_g^U\right)^{\frac{1}{s}}\right], \cdot\left[1-\overset{s}{\underset{g=1}{\otimes}}\left(1-\overline{\overline{I}}_g^L\right)^{\frac{1}{s}}, 1-\overset{s}{\underset{g=1}{\otimes}}\left(1-\overline{\overline{I}}_g^U\right)^{\frac{1}{s}}\right], \cdot\left[1-\overset{s}{\underset{g=1}{\otimes}}\left(1-\overline{\overline{F}}_g^L\right)^{\frac{1}{s}}, 1-\overset{s}{\underset{g=1}{\otimes}}\left(1-\overline{\overline{F}}_g^U\right)^{\frac{1}{s}}\right]\right)\right\rangle \quad \overline{\overline{we}}_1\in\overline{\overline{hn}}_1, \overline{\overline{we}}_2\in\overline{\overline{hn}}_2, \cdots, \overline{\overline{we}}_s\in\overline{\overline{hn}}_s \tag{71}$$

The GIFSSPGA operator has the property of commutativity.

Definition 24. For a group of HINLNs $\overline{\overline{hn}}_g(g = 1, 2, \cdots, s)$, WGHINLSSPGA operator is a function $\aleph^s \longrightarrow \aleph$,

$$WGHINLSSPGA_{\overline{\overline{wt}},\zeta}\left(\overline{\overline{hn}}_1, \overline{\overline{hn}}_2, \cdots, \overline{\overline{hn}}_s\right) = \frac{1}{\zeta}\left(\overset{s}{\underset{g=1}{\otimes}}\left(\zeta\overline{\overline{hn}}_g\right)^{\frac{\overline{\overline{wt}}_g\left(1+T\left(\overline{\overline{hn}}_g\right)\right)}{\overset{s}{\underset{h=1}{\oplus}}\overline{\overline{wt}}_h\left(1+T\left(\overline{\overline{hn}}_h\right)\right)}}\right), \tag{72}$$

where $T\left(\overline{\overline{if}}_g\right) = \overset{s}{\underset{\substack{h=1,\\ g\neq h}}{\oplus}}\mathrm{Sup}\left(\overline{\overline{hn}}_g, \overline{\overline{hn}}_h\right)$ is the weight vector for $\overline{\overline{hn}}_g(g = 1, 2, \cdots, s)$ such that $\overline{\overline{Wt}}_g \in [0, 1]$ and $\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{Wt}}_g = 1$, parameter $\zeta \in (0,+\infty)$ and $\mathrm{Sup}\left(\overline{\overline{hn}}_g, \overline{\overline{hn}}_h\right)$ is the support for $\overline{\overline{hn}}_g$ from $\overline{\overline{hn}}_h$ with the following constraint:

(1) $\mathrm{Sup}\left(\overline{\overline{hn}}_g, \overline{\overline{hn}}_h\right) \geq \mathrm{Sup}\left(\overline{\overline{hn}}_u, \overline{\overline{hn}}_v\right)$ if $\overline{\overline{dis}}\left(\overline{\overline{hn}}_g, \overline{\overline{hn}}_h\right) < \overline{\overline{dis}}\left(\overline{\overline{hn}}_u, \overline{\overline{hn}}_v\right)$, where $\overline{\overline{dis}}$ is the distance measure among two HINLNs

$$\mathrm{Sup}\left(\overline{\overline{hn}}_g, \overline{\overline{hn}}_h\right) \in [0, 1]; \quad \mathrm{Sup}\left(\overline{\overline{hn}}_g, \overline{\overline{hn}}_h\right) = \mathrm{Sup}\left(\overline{\overline{hn}}_h, \overline{\overline{hn}}_g\right); \tag{73}$$

To write Equation (72) in unsophisticated way, we have

$$\overline{\overline{pow}}_g = \frac{\overline{\overline{Wt}}_g\left(1+T\left(\overline{\overline{hn}}_g\right)\right)}{\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{Wt}}_g\left(1+T\left(\overline{\overline{hn}}_g\right)\right)}, \tag{74}$$

So, from Equation (74), Equation (72) becomes

$$WGHINLSSPGA_{\overline{\overline{wt}},\zeta}\left(\overline{\overline{hn}}_1, \overline{\overline{hn}}_2, \cdots, \overline{\overline{hn}}_s\right) = \frac{1}{\zeta}\left(\overset{s}{\underset{g=1}{\otimes}}\left(\zeta\overline{\overline{hn}}_g\right)^{\overline{\overline{pow}}_g}\right). \tag{75}$$

Theorem 25. *Let $\overline{\overline{hn}}_g(g = 1, 2, \cdots, s)$ be a group of HINLNs, then, the value aggregated utilizing Definition 24 is still HINLN, and we have*

$$WGHINLSSPGA_{\zeta,\overline{\overline{Wt}}}\left(\overline{\overline{hn}}_1, \overline{\overline{hn}}_2, \cdots, \overline{\overline{hn}}_s\right) = \left\langle f^{*-1}\left(\frac{1}{\zeta}\overset{s}{\underset{g=1}{\otimes}}\left(\zeta f^*\left(s_{\vartheta\left(\overline{\overline{we}}_g\right)}\right)\right)^{\overline{\overline{pow}}_g}\right), \cdot\left(\left[1-\left(\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pow}}_g\left(\frac{1}{\zeta}\left(1-\left(1-\overline{\overline{T}}_g^L\right)^{\varsigma}\right)-\left(\frac{1}{\zeta}-1\right)\right)^{\frac{1}{\varsigma}}, 1-\left(\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pow}}_g\left(\frac{1}{\zeta}\left(1-\left(1-\overline{\overline{T}}_g^U\right)^{\varsigma}\right)-\left(\frac{1}{\zeta}-1\right)\right)^{\frac{1}{\varsigma}}\right], \cdot\left[\left(\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pow}}_g\left(\frac{1}{\zeta}\left(1-\left(\overline{\overline{I}}_g^L\right)^{\varsigma}\right)-\left(\frac{1}{\zeta}-1\right)\right)^{\frac{1}{\varsigma}}, \cdot\left(\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pow}}_g\left(\frac{1}{\zeta}\left(1-\left(\overline{\overline{I}}_g^U\right)^{\varsigma}\right)-\left(\frac{1}{\zeta}-1\right)\right)^{\frac{1}{\varsigma}}\right], \cdot\left[\left(\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pow}}_g\left(\frac{1}{\zeta}\left(1-\left(\overline{\overline{F}}_g^L\right)^{\varsigma}\right)-\left(\frac{1}{\zeta}-1\right)\right)^{\frac{1}{\varsigma}}, \cdot\left(\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pow}}_g\left(\frac{1}{\zeta}\left(1-\left(\overline{\overline{F}}_g^U\right)^{\varsigma}\right)-\left(\frac{1}{\zeta}-1\right)\right)^{\frac{1}{\varsigma}}\right]\right)\right\rangle. \tag{76}$$

Further, we shall discuss a few cases of the initiated AO with respect to the parameter ζ and $\overline{\overline{Wt}}$, which are listed below:

(1) If $\zeta = 1$, $\overline{\overline{Wt}} = ((1/s), (1/s), \cdots, (1/s))^T$, then, the WGHINLSSPGA operator trims down to WHINLSS PA operator:

$$WGHINLSSPGA\left(\overline{\overline{hn}}_1, \overline{\overline{hn}}_2, \cdots, \overline{\overline{hn}}_s\right) = \overset{s}{\underset{g=1}{\otimes}}\left(\overline{\overline{hn}}_g\right)^{\overline{\overline{pow}}_g}$$

$$= \bigcup_{\overline{\overline{we}}_1 \in \overline{\overline{hn}}_1, \overline{\overline{we}}_2 \in \overline{\overline{hn}}_2, \cdots, \overline{\overline{we}}_s \in \overline{\overline{hn}}_s} \left\langle f^{*-1}\left(\overset{s}{\underset{g=1}{\otimes}}\left(f^*\left(s_{\vartheta\left(\overline{\overline{we}}_g\right)}\right)\right)^{\overline{\overline{pow}}_g}\right), \right.$$

$$\cdot\left(\left[\left(\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pow}}_g\left(\overline{\overline{T}}_g^L\right)^{\varsigma} - \overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pow}}_g + 1\right)^{\frac{1}{\varsigma}}, \right.\right.$$

$$\left.\cdot\left(\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pow}}_g\left(\overline{\overline{T}}_g^U\right)^{\varsigma} - \overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pow}}_g + 1\right)^{\frac{1}{\varsigma}}\right],$$

$$\cdot\left[1 - \left(\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pow}}_g\left(1 - \overline{\overline{I}}_g^L\right) - \overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pow}}_g + 1\right)^{\frac{1}{\varsigma}}, 1\right.$$

$$\left. - \left(\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pow}}_g\left(1 - \overline{\overline{I}}_g^U\right) - \overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pow}}_g + 1\right)^{\frac{1}{\varsigma}}\right],$$

$$\cdot\left[1 - \left(\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pow}}_g\left(1 - \overline{\overline{F}}_g^L\right) - \overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pow}}_g + 1\right)^{\frac{1}{\varsigma}}, 1\right.$$

$$\left.\left. - \left(\overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pow}}_g\left(1 - \overline{\overline{F}}_g^U\right) - \overset{s}{\underset{g=1}{\oplus}}\overline{\overline{pow}}_g + 1\right)^{\frac{1}{\varsigma}}\right]\right)\right\rangle. \tag{77}$$

(2) If $\zeta = 1$, $\overline{\overline{Wt}} = ((1/s), (1/s), \cdots, (1/s))^T$, and $\varsigma = 0$, then, the WGHINLSSPGA operator reduces into HINL PGA operator based on algebraic operation. That is

$$WGHINLSSPGA_{\varsigma=0}\left(\overline{\overline{hn}}_1, \overline{\overline{hn}}_2, \cdots, \overline{\overline{hn}}_s\right) = \overset{s}{\underset{g=1}{\otimes}}\left(\overline{\overline{hn}}_g\right)^{\overline{\overline{pow}}_g}$$

$$= \bigcup_{\overline{\overline{we}}_1 \in \overline{\overline{hn}}_1, \overline{\overline{we}}_2 \in \overline{\overline{hn}}_2, \cdots, \overline{\overline{we}}_s \in \overline{\overline{hn}}_s} \left\langle f^{*-1}\left(\overset{s}{\underset{g=1}{\otimes}}\left(f^*\left(s_{\vartheta\left(\overline{\overline{we}}_g\right)}\right)\right)^{\overline{\overline{pow}}_g}\right), \right.$$

$$\cdot\left(\left[\overset{s}{\underset{g=1}{\otimes}}\left(\overline{\overline{T}}_g^L\right)^{\overline{\overline{pow}}_g}, \overset{s}{\underset{g=1}{\otimes}}\left(\overline{\overline{T}}_g^U\right)^{\overline{\overline{pow}}_g}\right],\right.$$

$$\cdot\left[1 - \overset{s}{\underset{g=1}{\otimes}}\left(1 - \overline{\overline{I}}_g^L\right)^{\overline{\overline{pow}}_g}, 1 - \overset{s}{\underset{g=1}{\otimes}}\left(1 - \overline{\overline{I}}_g^U\right)^{\overline{\overline{pow}}_g}\right],$$

$$\left.\cdot\left[1 - \overset{s}{\underset{g=1}{\otimes}}\left(1 - \overline{\overline{F}}_g^L\right)^{\overline{\overline{pow}}_g}, 1 - \overset{s}{\underset{g=1}{\otimes}}\left(1 - \overline{\overline{F}}_g^U\right)^{\overline{\overline{pow}}_g}\right]\right)\right\rangle. \tag{78}$$

(3) If $\zeta = 1$, $\overline{\overline{Wt}} = ((1/s), (1/s), \cdots, (1/s))^T$, $\sup\left(\overline{\overline{hn}}_g, \overline{\overline{hn}}_h\right) = b$ (i.e., a constant) and $\varsigma = 0$, then, the WGHINLSSPGA operator trims down to HINL GA operator based on algebraic operation. That is

$$WGHINLPSSGA_{\varsigma=0}\left(\overline{\overline{hn}}_1, \overline{\overline{hn}}_2, \cdots, \overline{\overline{hn}}_s\right)$$

$$= \overset{s}{\underset{g=1}{\otimes}}\left(\overline{\overline{hn}}_g\right)^{\frac{1}{s}} = \bigcup_{\overline{\overline{we}}_1 \in \overline{\overline{hn}}_1, \overline{\overline{we}}_2 \in \overline{\overline{hn}}_2, \cdots, \overline{\overline{we}}_s \in \overline{\overline{hn}}_s}$$

$$\cdot\left\langle f^{*-1}\left(\overset{s}{\underset{g=1}{\otimes}}\left(f^*\left(s_{\vartheta\left(\overline{\overline{we}}_g\right)}\right)\right)^{\frac{1}{s}}\right), \left(\left[\overset{s}{\underset{g=1}{\otimes}}\left(\overline{\overline{T}}_g^L\right)^{\frac{1}{s}}, \overset{s}{\underset{g=1}{\otimes}}\left(\overline{\overline{T}}_g^U\right)^{\frac{1}{s}}\right],\right.\right.$$

$$\cdot\left[1 - \overset{s}{\underset{g=1}{\otimes}}\left(1 - \overline{\overline{I}}_g^L\right)^{\frac{1}{s}}, 1 - \overset{s}{\underset{g=1}{\otimes}}\left(1 - \overline{\overline{I}}_g^U\right)^{\frac{1}{s}}\right],$$

$$\left.\left.\cdot\left[1 - \overset{s}{\underset{g=1}{\otimes}}\left(1 - \overline{\overline{F}}_g^L\right)^{\frac{1}{s}}, 1 - \overset{s}{\underset{g=1}{\otimes}}\left(1 - \overline{\overline{F}}_g^U\right)^{\frac{1}{s}}\right]\right)\right\rangle. \tag{79}$$

5. An Application of Generalized Hesitant Interval Neutrosophic Linguistic Schweizer-Sklar Power Aggregation Operator to Group Decision Making

In this part, we pertain the aforementioned generalized hesitant interval neutrosophic linguistic Schweizer-Sklar power AOs to ascertain productive approaches for MADM under HINL environments. Let $\overline{\overline{Arb}} = \{\overline{\overline{Arb}}_1, \overline{\overline{Arb}}_2, \cdots, \overline{\overline{Arb}}_g\}$ be the group of detached alternatives, the group of attributes is articulated by $\overline{\overline{Cta}} = \{\overline{\overline{Cta}}_1, \overline{\overline{Cta}}_2, \cdots, \overline{\overline{Cta}}_h\}$, and the weight vector of the attributes is symbolized by $\overline{\overline{\mathrm{WE}}} = (\overline{\overline{we}}_1, \overline{\overline{we}}_2, \cdots, \overline{\overline{we}}_h)^T$ such that $\bar{\bar{w}}_e \in [0, 1], \sum_{e=1}^{h} \bar{\bar{w}}_e = 1$. In the process of decision making, the assessment information about the alternative $\overline{\overline{Arb}}_u (u = 1, 2, \cdots, g)$ with respect to the attribute $\overline{\overline{Cta}}_w (\mathrm{w} = 1, 2, \cdots, \mathrm{h})$ is expressed by a HINL decision matrix $\overline{\overline{DT}} = (\overline{\overline{hn}}_{de})_{g\times h}$, where $\overline{\overline{hn}}_{de} = \underset{u_{de} \in \overline{\overline{hn}}_{de}}{\cup}\{u_{de}\}, u_{de} = \langle s_{\vartheta(u_{de})}[T^L(u_{de}), T^U(u_{de})], [I^L(u_{de}), I^U(u_{de})], [F^L(u_{de}), F^U(u_{de})]\rangle$ is a HINLN.

Then, gamble on factual decision situations where the weight vector of attributes is entirely identified in advance. For that reason, we initiate MADM approaches established on the proposed GHINLSSPA operators.

5.1. MADM with Known Weight Vectors of Attributes. In this subsection, to deal with real decision situations in which the weighting vectors of attributes is totally known, we apply WGHINLSSPA operator and WGHINLSSPGA operator to establish the following approach to solve MADM problems under HINL environments. To do so, immediately go behind the steps below.

Step 1. Find out support $Sup(\overline{\overline{hn}}_{de}, \overline{\overline{hn}}_{dx})$ by the following formula;

$$Sup\left(\overline{\overline{hn}}_{de}, \overline{\overline{hn}}_{ex}\right) = 1 - \overline{\overline{dis}}\left(\overline{\overline{hn}}_{de}, \overline{\overline{hn}}_{ex}\right), \cdot (d = 1, 2, \cdots, g, e = 1, 2, \cdots, h, e \neq x), \tag{80}$$

where $\overline{\overline{dis}}(\overline{\overline{hn}}_{de}, \overline{\overline{hn}}_{ex})$ is the distance measure and is calculated by utilizing Equation (19).

Step 2. Find out the support degree $T(\overline{\overline{hn}}_{de})$ that HINLN $\overline{\overline{hn}}_{de}$ collects from other HINLNs $\overline{\overline{hn}}_{dx}(x = 1, 2, \cdots, h\,; e \neq x)$, where

$$T\left(\overline{\overline{hn}}_{de}\right) = \sum_{x=1, x\neq e}^{h} \sup\left(\overline{\overline{hn}}_{de}, \overline{\overline{hn}}_{dx}\right). \tag{81}$$

Step 3. Determine weighting vector $\Upsilon_{de}(d = 1, 2, \cdots, g, e = 1, 2, \cdots, h)$ associated with $\overline{\overline{hn}}_{de}$,

$$\Upsilon_{de} = \frac{\bar{\bar{\omega}}_e\left(1 + T\left(\overline{\overline{hn}}_{de}\right)\right)}{\sum_{e=1}^{h} \bar{\bar{\omega}}_e\left(1 + T\left(\overline{\overline{hn}}_{de}\right)\right)}. \tag{82}$$

Step 4. Utilize WGHINLSSPA or WHINLSSPGA operators to collective all evaluation values $\overline{\overline{hn}}_{de}(d = 1, 2, \cdots, g, e = 1, 2, \cdots, h)$ into overall evaluation value $\overline{\overline{hn}}_d(d = 1, 2, \cdots, g)$ matching to the alternatives $\overline{\overline{Arb}}_d\,(d = 1, 2, \cdots, g)$;

$$\begin{aligned}
\overline{\overline{hn}}_d &= WGHINLSSPA_{\zeta, \overline{\overline{We}}}\left(\overline{\overline{hn}}_{d1}, \overline{\overline{hn}}_{d2}, \cdots, \overline{\overline{hn}}_{ds}\right) \\
&= \bigcup_{u_1 \in \overline{\overline{hn}}_1, u_2 \in \overline{\overline{hn}}_2, \cdots, u_{m+1} \in \overline{\overline{hn}}_s} \left\langle f^{*-1}\left(\bigoplus_{e=1}^{s} \bar{\bar{\Psi}}_g\left(f^*\left(s_{\vartheta(u_{de})}\right)\right)^{\zeta}\right)^{\frac{1}{\zeta}}, \right. \\
&\cdot \left(\left[\left(\frac{1}{\bar{\zeta}}\left(1 - \left(\bigoplus_{e=1}^{s} \bar{\bar{\Psi}}_{de}\left(1 - \left(\zeta\left(T_{de}^{L}\right)^{\varsigma} - (\zeta - 1)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma} - \bigoplus_{e=1}^{s} \bar{\bar{\Psi}}_{de} + 1\right)^{\frac{1}{\varsigma}}\right)^{\varsigma} \right.\right.\right. \\
&- \left.\left.\left(\frac{1}{\bar{\zeta}} - 1\right)\right)^{\frac{1}{\varsigma}}, \frac{1}{\bar{\zeta}}\left(1 - \left(\bigoplus_{e=1}^{s} \bar{\bar{\Psi}}_g\left(1 - \left(\zeta\left(T_{de}^{U}\right)^{\varsigma} - (\zeta - 1)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}\right.\right.\right. \\
&- \left.\left.\left.\bigoplus_{e=1}^{s} \bar{\bar{\Psi}}_{de} + 1\right)^{\frac{1}{\varsigma}}\right)^{\varsigma} - \left(\frac{1}{\bar{\zeta}} - 1\right)\right], \\
&\cdot \left[1 - \left(\frac{1}{\bar{\zeta}}\left(1 - \left(\bigoplus_{e=1}^{s} \bar{\bar{\Psi}}_{de}\left(1 - \left(\zeta\left(1 - I_{de}^{L}\right)^{\varsigma} - (\zeta - 1)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma} - \bigoplus_{e=1}^{s} \bar{\bar{\Psi}}_{de} + 1\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}\right.\right. \\
&- \left.\left.\left(\frac{1}{\bar{\zeta}} - 1\right)\right)^{\frac{1}{\varsigma}}, 1 - \left(\frac{1}{\bar{\zeta}}\left(1 - \left(\bigoplus_{e=1}^{s} \bar{\bar{\Psi}}_{de}\left(1 - \left(\zeta\left(1 - I_{de}^{U}\right)^{\varsigma} - (\zeta - 1)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}\right.\right.\right.\right. \\
&- \left.\left.\left.\left.\bigoplus_{g=1}^{s} \bar{\bar{\Psi}}_{de} + 1\right)^{\frac{1}{\varsigma}}\right)^{\varsigma} - \left(\frac{1}{\bar{\zeta}} - 1\right)\right)^{\frac{1}{\varsigma}}\right], \\
&\cdot \left[1 - \left(\frac{1}{\bar{\zeta}}\left(1 - \left(\bigoplus_{e=1}^{s} \bar{\bar{\Psi}}_{de}\left(1 - \left(\zeta\left(1 - F_{de}^{L}\right)^{\varsigma} - (\zeta - 1)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma}\right.\right.\right.\right. \\
&- \left.\left.\left.\bigoplus_{e=1}^{s} \bar{\bar{\Psi}}_{de} + 1\right)^{\frac{1}{\varsigma}}\right)^{\varsigma} - \left(\frac{1}{\bar{\zeta}} - 1\right)\right)^{\frac{1}{\varsigma}}, 1 - \left(\frac{1}{\bar{\zeta}}\left(1 - \left(\bigoplus_{e=1}^{s} \bar{\bar{\Psi}}_{de}\left(1 - \left(\zeta\left(1 - F_{de}^{U}\right)^{\varsigma}\right.\right.\right.\right.\right. \\
&- \left.\left.\left.\left.\left.(\zeta - 1)\right)^{\frac{1}{\varsigma}}\right)^{\varsigma} - \bigoplus_{e=1}^{s} \bar{\bar{\Psi}}_{de} + 1\right)^{\frac{1}{\varsigma}}\right)^{\varsigma} - \left(\frac{1}{\bar{\zeta}} - 1\right)\right)^{\frac{1}{\varsigma}}\right]\right)\Bigg\rangle.
\end{aligned} \tag{83}$$

Or

$$\begin{aligned}
\overline{\overline{hn}}_d &= WGHINLSSPGA_{\zeta, \overline{\overline{We}}}\left(\overline{\overline{hn}}_{d1}, \overline{\overline{hn}}_{d2}, \cdots, \overline{\overline{hn}}_{ds}\right) \\
&= \bigcup_{u_{d1} \in \overline{\overline{hn}}_{d1}, u_{d2} \in \overline{\overline{hn}}_{d2}, \cdots, u_{ds} \in \overline{\overline{hn}}_{ds}} \left\langle f^{*-1}\left(\frac{1}{\zeta} \overset{s}{\underset{e=1}{\otimes}} \left(\zeta f^*\left(s_{\vartheta(u_{de})}\right)\right)^{\Xi_{de}}\right), \right. \\
&\cdot \left(\left[1 - \left(\bigoplus_{e=1}^{s} \Xi_{de}\left(\frac{1}{\zeta}\left(1 - \left(1 - T_{de}^{L}\right)^{\varsigma}\right)\right) - \left(\frac{1}{\zeta} - 1\right)\right)^{\frac{1}{\varsigma}}, 1 \right.\right. \\
&- \left.\left.\left(\bigoplus_{e=1}^{s} \Xi_{de}\left(\frac{1}{\zeta}\left(1 - \left(1 - T_{de}^{U}\right)^{\varsigma}\right)\right) - \left(\frac{1}{\zeta} - 1\right)\right)^{\frac{1}{\varsigma}}\right], \right. \\
&\cdot \left[\left(\bigoplus_{e=1}^{s} \Xi_{de}\left(\frac{1}{\zeta}\left(1 - \left(I_{de}^{L}\right)^{\varsigma}\right)\right) - \left(\frac{1}{\zeta} - 1\right)\right)^{\frac{1}{\varsigma}}, \right. \\
&\cdot \left.\left(\bigoplus_{e=1}^{s} \Xi_{de}\left(\frac{1}{\zeta}\left(1 - \left(I_{de}^{U}\right)^{\varsigma}\right)\right) - \left(\frac{1}{\zeta} - 1\right)\right)^{\frac{1}{\varsigma}}\right], \\
&\cdot \left[\left(\bigoplus_{e=1}^{s} \Xi_{de}\left(\frac{1}{\zeta}\left(1 - \left(F_{de}^{L}\right)^{\varsigma}\right)\right) - \left(\frac{1}{\zeta} - 1\right)\right)^{\frac{1}{\varsigma}}, \right. \\
&\cdot \left.\left.\left.\left(\bigoplus_{e=1}^{s} \Xi_{de}\left(\frac{1}{\zeta}\left(1 - \left(F_{de}^{U}\right)^{\varsigma}\right)\right) - \left(\frac{1}{\zeta} - 1\right)\right)^{\frac{1}{\varsigma}}\right]\right)\right\rangle.
\end{aligned} \tag{84}$$

Step 5. Find out the scores $\overline{\overline{Scr}}(\overline{\overline{hn}}_d)$ for the overall HINLN of the alternatives $\overline{\overline{Arb}}_d(d = 1, 2, \cdots, g)$ by exploiting Definition 8.

Step 6. Rank all alternatives $\overline{\overline{Arb}}_d(d = 1, 2, \cdots, g)$ and select the best one (s) with the ranking order $\overline{\overline{hn}}_d(d = 1, 2, \cdots, g)$.

6. Illustrative Example

In this section, an example of alternative selection taken from Ye [32] is utilized to demonstrate the usefulness of the anticipated decision-making process under a hesitant interval neutrosophic linguistic environment. An investment firm would like to put money into the best reasonable choice. A panel with four investment options (alternatives) is available to spend the money. The available options are, $\overline{\overline{Arb}}_1$ a car firm, $\overline{\overline{Arb}}_2$ is a food firm, $\overline{\overline{Arb}}_3$ is a computer firm, and $\overline{\overline{Arb}}_4$ is an arms firm. The investment firm must make a decision based on the three attributes, the risk $\overline{\overline{cta}}_1$, the growth $\overline{\overline{cta}}_2$, and the environmental impact $\overline{\overline{cta}}_3$. The weight vector of the attributes is $\overline{\overline{We}} = (0.35, 0.25, 0.4)^T$. The possible four alternatives are assess with respect to three attributes by three decision maker and provide the assessment values in the form of HINLNs under the linguistic term set $S = \{s_0 = \text{extremely poor}, s_1 = \text{very poor}, s_2 = \text{poor}, s_3 = \text{medium}, s_4 = \text{good}, s_5 = \text{very good}, s_6 = \text{extremely good}\}$. Thus, when the possible four alternatives are assessed by three decision makers with respect to the three attributes, and the HINL decision matrix are constructed as given in Table 1.

Table 1: HINL decision matrix $\overline{\overline{DT}}$.

	$\overline{\overline{cta}}_1$	$\overline{\overline{cta}}_2$	$\overline{\overline{cta}}_3$
$\overline{\overline{Arb}}_1$	$\{\langle s_4, ([0.5, 0.6], [0.1, 0.2], [0.2, 0.3])\rangle, \langle s_5, ([0.3, 0.4], [0.2, 0.3], [0.3, 0.4])\rangle\}$	$\{\langle s_5, ([0.5, 0.6], [0.2, 0.3], [0.3, 0.4])\rangle\}$	$\{\langle s_4, ([0.4, 0.5], [0.1, 0.2], [0.3, 0.5])\rangle, \langle s_4, ([0.2, 0.3], [0.1, 0.2], [0.5, 0.6])\rangle\}$
$\overline{\overline{Arb}}_2$	$\{\langle s_4, ([0.7, 0.8], [0.1, 0.2], [0.2, 0.3])\rangle, \langle s_5, ([0.6, 0.7], [0.1, 0.2], [0.1, 0.3])\rangle\}$	$\{\langle s_4, ([0.6, 0.7], [0.01, 0.1], [0.2, 0.3])\rangle\}$	$\{\langle s_3, ([0.7, 0.8], [0.01, 0.1], [0.1, 0.2])\rangle, \langle s_4, ([0.6, 0.7], [0.1, 0.2], [0.1, 0.2])\rangle, \langle s_5, ([0.5, 0.6], [0.1, 0.2], [0.2, 0.3])\rangle\}$
$\overline{\overline{Arb}}_3$	$\{\langle s_4, ([0.7, 0.9], [0.2, 0.4], [0.1, 0.2])\rangle, \langle s_5, ([0.5, 0.6], [0.3, 0.4], [0.2, 0.3])\rangle\}$	$\{\langle s_4, ([0.5, 0.6], [0.2, 0.3], [0.3, 0.4])\rangle, \langle s_5, ([0.3, 0.5], [0.1, 0.2], [0.4, 0.5])\rangle\}$	$\{\langle s_3, ([0.5, 0.6], [0.1, 0.3], [0.2, 0.3])\rangle\}$
$\overline{\overline{Arb}}_4$	$\{\langle s_3, ([0.7, 0.8], [0.01, 0.1], [0.1, 0.2])\rangle\}$	$\{\langle s_3, ([0.6, 0.7], [0.1, 0.2], [0.2, 0.3])\rangle, \langle s_4, ([0.5, 0.6], [0.1, 0.2], [0.3, 0.4])\rangle, \langle s_5, ([0.4, 0.5], [0.2, 0.3], [0.1, 0.2])\rangle\}$	$\{\langle s_4, ([0.5, 0.6], [0.1, 0.2], [0.1, 0.2])\rangle, \langle s_5, ([0.3, 0.5], [0.01, 0.2], [0.1, 0.2])\rangle\}$

Step 1. Find out the supports $Sup(\overline{\overline{hn}}_{de}, \overline{\overline{hn}}_{dx})$ $(d = 1, 2, 3, 4, e, x = 1, 2, 3, e \neq x)$ by utilizing formula (80). For simplicity, we shall denote $(Sup(\overline{\overline{hn}}_{de}, \overline{\overline{hn}}_{dz}))_{4\times 1}$ by S_{ez} which means the supports between the eth and the zth columns of $\overline{\overline{DT}}$.

$$S_{12} = S_{21} = \begin{bmatrix} 0.8569 \\ 0.8361 \\ 0.8194 \\ 0.7544 \end{bmatrix}, S_{13} = S_{31} = \begin{bmatrix} 0.6819 \\ 0.7581 \\ 0.7840 \\ 0.7379 \end{bmatrix}, S_{23} = S_{32} = \begin{bmatrix} 0.5389 \\ 0.8168 \\ 0.7556 \\ 0.7438 \end{bmatrix}. \tag{85}$$

Step 2. Utilizing Equation (81) to find out the weighted support degree $T(\overline{\overline{hn}}_{de})$ that HINLN $\overline{\overline{hn}}_{de}$ collects from other HINLNs $\overline{\overline{hn}}_{dx}(x = 1, 2, 3)$. We express $(T(\overline{\overline{hn}}_{de}))_{4\times 3}$ by T.

$$T = \begin{bmatrix} 1.5389 & 1.3958 & 1.2208 \\ 1.5942 & 1.6529 & 1.5748 \\ 1.6035 & 1.5750 & 1.5396 \\ 1.4922 & 1.4981 & 1.4816 \end{bmatrix}. \tag{86}$$

Step 3. Utilize Equation (82) to get the weights $\Upsilon_{de}(d = 1, 2, 3, 4; e = 1, 2, 3)$ associated with $\overline{\overline{hn}}_{de}$. This is revealed as follows.

$$V = \begin{bmatrix} 0.3740 & 0.2521 & 0.3739 \\ 0.3491 & 0.2550 & 0.3960 \\ 0.3545 & 0.2504 & 0.3951 \\ 0.3504 & 0.2509 & 0.3987 \end{bmatrix}. \tag{87}$$

Step 4. Utilize Equation (83) to amalgamate all the evaluation values $\overline{\overline{Arb}}_{de}(d = 1, 2, 3, 4; e = 1, 2, 3)$ in the dth row of $\overline{\overline{DT}}$ and acquire the inclusive evaluation value $\overline{\overline{Arb}}_d(d = 1, 2, 3, 4)$. This is revealed as follows. (Consider $\varsigma = -3$ and $\zeta = 2$).

$$\begin{aligned} \overline{\overline{Arb}}_1 = \{&\langle s_{3.9562}, ([0.4677, 0.5690], [0.3358, 0.3832], \\ &\cdot [0.3995, 0.4703])\rangle, \langle s_{4.2742}, ([0.4241, 0.5311], \\ &\cdot [0.3359, 0.3832], [0.4136, 0.4796])\rangle, \\ &\cdot \langle s_{4.36093}, ([0.4005, 0.5037], [0.3474, 0.3995], \\ &\cdot [0.4265, 0.5055])\rangle, \langle s_{4.6513}, ([0.3410, 0.4481], \\ &\cdot [0.3474, 0.3995], [0.4534, 0.5200])\rangle\} \end{aligned} \tag{88}$$

$$\begin{aligned} \overline{\overline{Arb}}_2 = \{&\langle s_{3.7636}, ([0.6810, 0.7835], [0.2973, 0.3381], \\ &\cdot [0.3465, 0.3984]), \langle s_4, ([0.6439, 0.7483], [0.3000, 0.3533], \\ &\cdot [0.3465, 0.3984])\rangle, \langle s_{4.4231}, ([0.6200, 0.7296], \\ &\cdot [0.3000, 0.3533], [0.3753, 0.4265])\rangle, \\ &\cdot \langle s_{4.0460}, ([0.6489, 0.7532], [0.2973, 0.3381], \\ &\cdot [0.3359, 0.3984])\rangle, \langle s_{4.3751}, ([0.6000, 0.7000], \\ &\cdot [0.3000, 0.3533], [0.3359, 0.3984])\rangle, \\ &\cdot \langle s_{4.7650}, [0.5670, 0.6690], [0.3000, 0.3533], [0.3485, 0.4265]\rangle\} \end{aligned} \tag{89}$$

$$\begin{aligned} \overline{\overline{Arb}}_3 = \{&\langle s_{3.6379}, ([0.6033, 0.8349], [0.3465, 0.4419], \\ &\cdot [0.3508, 0.4065])\rangle, \langle s_{3.9354}, ([0.5837, 0.8331], \\ &\cdot [0.3383, 0.4168], [0.3519, 0.4098])\rangle, \\ &\cdot \langle s_{4.0527}, ([0.5000, 0.6000], [0.3498, 0.4419], \\ &\cdot [0.3831, 0.4369])\rangle, \langle s_{4.3218}, ([0.4637, 0.5800], \\ &\cdot [0.3404, 0.4168], [0.3868, 0.4431])\rangle\} \end{aligned} \tag{90}$$

$$\begin{aligned}\overline{\overline{Arb}}_4 = \{&\langle s_{3.4338}, [0.6199, 0.7296], [0.2989, 0.3484],\\ &\cdot [0.3358, 0.3831]\rangle, \langle s_{3.6807}, ([0.6024, 0.7156],\\ &\cdot [0.2989, 0.3484], [0.3370, 0.3868])\rangle,\\ &\cdot \langle s_{3.9756}, ([0.5908, 0.7076], [0.2990, 0.3509],\\ &\cdot [0.3310, 0.3753])\rangle, \langle s_{3.9217}, ([0.5911, 0.7183],\\ &\cdot [0.2970, 0.3484], [0.3358, 0.3831])\rangle,\\ &\cdot \langle s_{4.1396}, ([0.5698, 0.7027], [0.2970, 0.3484],\\ &\cdot [0.3370, 0.3868])\rangle, \langle s_{4.4038}, [0.5555, 0.6937],\\ &\cdot [0.2970, 0.3510], [0.3310, 0.3753]\rangle\}\end{aligned} \tag{91}$$

Or utilize Equation (84) to amalgamate all the evaluation values $\overline{\overline{Arb}}_{de}(d = 1, 2, 3, 4; e = 1, 2, 3)$ in the dth row of $\overline{\overline{DT}}$ and acquire the inclusive evaluation value $\overline{\overline{Arb}}_d(d = 1, 2, 3, 4)$. This is revealed as follows. (Consider $\varsigma = -3$ and $\zeta = 2$).

$$\begin{aligned}\overline{\overline{Arb}}_1 = \{&\langle s_{3.7999}, ([0.5220, 0.5951], [0.1278, 0.2282],\\ &\cdot [0.2662, 0.4153])\rangle, \langle s_{4.2315}, [0.4263, 0.4997],\\ &\cdot [0.1278, 0.2282], [0.3711, 0.4789]\rangle,\\ &\cdot \langle s_{4.1307}, ([0.4657, 0.5334], [0.1657, 0.2662],\\ &\cdot [0.3000, 0.4428])\rangle, \langle s_{4.6000}, ([0.4085, 0.4715],\\ &\cdot [0.1657, 0.2662], [0.3948, 0.5000])\rangle\}\end{aligned} \tag{92}$$

$$\begin{aligned}\overline{\overline{Arb}}_2 = \{&\langle s_{3.5694}, ([0.6873, 0.7772], [0.0436, 0.1380],\\ &\cdot [0.1635, 0.2641])\rangle, \langle s_4, ([0.6537, 0.7407],\\ &\cdot [0.0788, 0.1769], [0.1635, 0.2641])\rangle,\\ &\cdot \langle s_{4.3695}, ([0.6129, 0.6976], [0.0788, 0.1769],\\ &\cdot [0.2000, 0.3000])\rangle, \langle s_{3.8585}, ([0.6574, 0.7447],\\ &\cdot [0.0436, 0.1380], [0.1281, 0.2641])\rangle,\\ &\cdot \langle s_{4.3240}, ([0.6287, 0.7126], [0.0788, 0.1769],\\ &\cdot [0.1281, 0.2640])\rangle, \langle s_{4.7235}, ([0.5934, 0.6746],\\ &\cdot [0.0788, 0.1769], [0.1680, 0.3000])\rangle\}\end{aligned} \tag{93}$$

$$\begin{aligned}\overline{\overline{Arb}}_3 = \{&\langle s_{3.5702}, [0.5922, 0.6921], [0.1636, 0.3398],\\ &\cdot [0.1982, 0.2998]\rangle, \langle s_{3.7754}, ([0.5212, 0.6591],\\ &\cdot [0.1385, 0.3205], [0.2372, 0.34203])\rangle,\\ &\cdot \langle s_{3.8640}, [0.5528, 0.6286], [0.2067, 0.3398],\\ &\cdot [0.2281, 0.3287]\rangle, \langle s_{4.0861}, ([0.4990, 0.6054],\\ &\cdot [0.1845, 0.3205], [0.2639, 0.3670])\rangle\}\end{aligned} \tag{94}$$

$$\begin{aligned}\overline{\overline{Arb}}_4 = \{&\langle s_{3.3620}, [0.6128, 0.6974], [0.0706, 0.1679],\\ &\cdot [0.1277, 0.2281]\rangle, \langle s_{3.6131}, ([0.5917, 0.6743],\\ &\cdot [0.0706, 0.1679], [0.1617, 0.2640])\rangle,\\ &\cdot \langle s_{3.8207}, ([0.5626, 0.6440], [0.1002, 0.1986],\\ &\cdot [0.1000, 0.2000])\rangle, \langle s_{3.6725}, ([0.5026, 0.6461],\\ &\cdot [0.0344, 0.1679], [0.1277, 0.2281])\rangle,\\ &\cdot \langle s_{3.9468}, [0.4943, 0.6285], [0.0344, 0.1679],\\ &\cdot [0.1617, 0.2640]\rangle, \langle s_{4.1737}, [0.4820, 0.6052],\\ &\cdot [0.0666, 0.1986], [0.1000, 0.2000]\rangle\}\end{aligned} \tag{95}$$

Step 5. Exploiting Definition 8 to find out the scores $\overline{\overline{Scr}}(\overline{\overline{hn}}_d)(d = 1, 2, 3, 4)$. This is revealed as follows:

$$\begin{aligned}\overline{\overline{Sre}}\left(\overline{\overline{Arb}}_1\right) = 0.4338, \overline{\overline{Sre}}\left(\overline{\overline{Arb}}_2\right) = 0.5276, \overline{\overline{Sre}}\left(\overline{\overline{Arb}}_3\right)\\ = 0.4863, \overline{\overline{Sre}}\left(\overline{\overline{Arb}}_4\right) = 0.4928.\end{aligned} \tag{96}$$

Or

$$\begin{aligned}\overline{\overline{Sre}}\left(\overline{\overline{Arb}}_1\right) = 0.3922, \overline{\overline{Sre}}\left(\overline{\overline{Arb}}_2\right) = 0.5182, \overline{\overline{Sre}}\left(\overline{\overline{Arb}}_3\right)\\ = 0.4447, \overline{\overline{Sre}}\left(\overline{\overline{Arb}}_4\right) = 0.4501.\end{aligned} \tag{97}$$

Step 6. According to the score values the ranking order of the alternatives $\overline{\overline{Arb}}_d(d = 1, 2, 3, 4)$.

$\overline{\overline{Arb}}_2 > \overline{\overline{Arb}}_4 > \overline{\overline{Arb}}_3 > \overline{\overline{Arb}}_1$, or $\overline{\overline{Arb}}_2 > \overline{\overline{Arb}}_4 > \overline{\overline{Arb}}_3 > \overline{\overline{Arb}}_1$.

So, the best alternative is $\overline{\overline{Arb}}_2$, and the worst alternative is $\overline{\overline{Arb}}_1$.

6.1. The Effect of the LSFs on Ranking Results. In this subsection, other different kinds of LSFs are also used to the above-mentioned decision-making process to obtain the ranking results to demonstrate the effect from other LSFs on the ranking results. The score values and final ranking orders are shown in Table 2.

From Table 2, we can observe that when the LSF is utilized the ranking orders gained from both the aggregation operators remain the same as that gained from the first LSF. But when the second LSF is used, the ranking order acquired from the HINLSSPWA operator is the same as the ranking order gained from the first LSF; however, when the HINLSSPWA operator is used, the ranking order is modified. That is, the best alternative remains the same but the worst one is changed, which is $\overline{\overline{Arb}}_3$. The major explanation for this variation is that three distinct forms of LSFs affect three different sorts of semantic circumstances. This might lead to a variety of semantic preferences and semantic

Table 2: Score values and ranking orders of alternatives utilizing different LSFs.

LSF	HINSSPWA operator	HINSSPWGA operator	Ranking order
f^*_2	$\overline{\overline{Sre}}\left(\overline{\overline{Arb}}_1\right) = 0.3959, \overline{\overline{Sre}}\left(\overline{\overline{Arb}}_2\right) = 0.4793,$ $\overline{\overline{Sre}}\left(\overline{\overline{Arb}}_3\right) = 0.4434, \overline{\overline{Sre}}\left(\overline{\overline{Arb}}_4\right) = 0.4496.$	$\overline{\overline{Sre}}\left(\overline{\overline{Arb}}_1\right) = 0.3887, \overline{\overline{Sre}}\left(\overline{\overline{Arb}}_2\right) = 0.4749,$ $\overline{\overline{Sre}}\left(\overline{\overline{Arb}}_3\right) = 0.3847, \overline{\overline{Sre}}\left(\overline{\overline{Arb}}_4\right) = 0.4429.$	$\overline{\overline{Arb}}_2 > \overline{\overline{Arb}}_4 > \overline{\overline{Arb}}_3 > \overline{\overline{Arb}}_1$ or $\overline{\overline{Arb}}_2 > \overline{\overline{Arb}}_4 > \overline{\overline{Arb}}_1 > \overline{\overline{Arb}}_3$
f^*_3	$\overline{\overline{Sre}}\left(\overline{\overline{Arb}}_1\right) = 0.3486, \overline{\overline{Sre}}\left(\overline{\overline{Arb}}_2\right) = 0.4592,$ $\overline{\overline{Sre}}\left(\overline{\overline{Arb}}_3\right) = 0.3902, \overline{\overline{Sre}}\left(\overline{\overline{Arb}}_4\right) = 0.4154.$	$\overline{\overline{Sre}}\left(\overline{\overline{Arb}}_1\right) = 0.3677, \overline{\overline{Sre}}\left(\overline{\overline{Arb}}_2\right) = 0.5022,$ $\overline{\overline{Sre}}\left(\overline{\overline{Arb}}_3\right) = 0.4018, \overline{\overline{Sre}}\left(\overline{\overline{Arb}}_4\right) = 0.4266.$	$\overline{\overline{Arb}}_2 > \overline{\overline{Arb}}_4 > \overline{\overline{Arb}}_3 > \overline{\overline{Arb}}_1$ or $\overline{\overline{Arb}}_2 > \overline{\overline{Arb}}_4 > \overline{\overline{Arb}}_3 > \overline{\overline{Arb}}_1$

Table 3: Effect of the parameter ς on final ranking order.

Parameter	WSSHINPWA operator	WSSHINPWG operator	Ranking order
$\varsigma = -1$	$\overline{\overline{Sre}}\left(\overline{\overline{Arb}}_1\right) = 0.3938, \overline{\overline{Sre}}\left(\overline{\overline{Arb}}_2\right) = 0.4730,$ $\overline{\overline{Sre}}\left(\overline{\overline{Arb}}_3\right) = 0.4266, \overline{\overline{Sre}}\left(\overline{\overline{Arb}}_4\right) = 0.4295.$	$\overline{\overline{Sre}}\left(\overline{\overline{Arb}}_1\right) = 0.4857, \overline{\overline{Sre}}\left(\overline{\overline{Arb}}_2\right) = 0.5579,$ $\overline{\overline{Sre}}\left(\overline{\overline{Arb}}_3\right) = 0.5030, \overline{\overline{Sre}}\left(\overline{\overline{Arb}}_4\right) = 0.5074.$	$\overline{\overline{Arb}}_2 > \overline{\overline{Arb}}_4 > \overline{\overline{Arb}}_3 > \overline{\overline{Arb}}_1$ or $\overline{\overline{Arb}}_2 > \overline{\overline{Arb}}_4 > \overline{\overline{Arb}}_3 > \overline{\overline{Arb}}_1$
$\varsigma = -2$	$\overline{\overline{Sre}}\left(\overline{\overline{Arb}}_1\right) = 0.4246, \overline{\overline{Sre}}\left(\overline{\overline{Arb}}_2\right) = 0.5143,$ $\overline{\overline{Sre}}\left(\overline{\overline{Arb}}_3\right) = 0.4705, \overline{\overline{Sre}}\left(\overline{\overline{Arb}}_4\right) = 0.4751.$	$\overline{\overline{Sre}}\left(\overline{\overline{Arb}}_1\right) = 0.4147, \overline{\overline{Sre}}\left(\overline{\overline{Arb}}_2\right) = 0.5242,$ $\overline{\overline{Sre}}\left(\overline{\overline{Arb}}_3\right) = 0.4570, \overline{\overline{Sre}}\left(\overline{\overline{Arb}}_4\right) = 0.4622.$	$\overline{\overline{Arb}}_2 > \overline{\overline{Arb}}_4 > \overline{\overline{Arb}}_3 > \overline{\overline{Arb}}_1$ or $\overline{\overline{Arb}}_2 > \overline{\overline{Arb}}_4 > \overline{\overline{Arb}}_3 > \overline{\overline{Arb}}_1$
$\varsigma = -7$	$\overline{\overline{Sre}}\left(\overline{\overline{Arb}}_1\right) = 0.4481, \overline{\overline{Sre}}\left(\overline{\overline{Arb}}_2\right) = 0.5409,$ $\overline{\overline{Sre}}\left(\overline{\overline{Arb}}_3\right) = 0.5043, \overline{\overline{Sre}}\left(\overline{\overline{Arb}}_4\right) = 0.5167.$	$\overline{\overline{Sre}}\left(\overline{\overline{Arb}}_1\right) = 0.3649, \overline{\overline{Sre}}\left(\overline{\overline{Arb}}_2\right) = 0.5135,$ $\overline{\overline{Sre}}\left(\overline{\overline{Arb}}_3\right) = 0.4276, \overline{\overline{Sre}}\left(\overline{\overline{Arb}}_4\right) = 0.4342.$	$\overline{\overline{Arb}}_2 > \overline{\overline{Arb}}_4 > \overline{\overline{Arb}}_3 > \overline{\overline{Arb}}_1$ or $\overline{\overline{Arb}}_2 > \overline{\overline{Arb}}_4 > \overline{\overline{Arb}}_3 > \overline{\overline{Arb}}_1$
$\varsigma = -15$	$\overline{\overline{Sre}}\left(\overline{\overline{Arb}}_1\right) = 0.4608, \overline{\overline{Sre}}\left(\overline{\overline{Arb}}_2\right) = 0.5468,$ $\overline{\overline{Sre}}\left(\overline{\overline{Arb}}_3\right) = 0.5113, \overline{\overline{Sre}}\left(\overline{\overline{Arb}}_4\right) = 0.5269.$	$\overline{\overline{Sre}}\left(\overline{\overline{Arb}}_1\right) = 0.3490, \overline{\overline{Sre}}\left(\overline{\overline{Arb}}_2\right) = 0.5083,$ $\overline{\overline{Sre}}\left(\overline{\overline{Arb}}_3\right) = 0.4123, \overline{\overline{Sre}}\left(\overline{\overline{Arb}}_4\right) = 0.4221.$	$\overline{\overline{Arb}}_2 > \overline{\overline{Arb}}_4 > \overline{\overline{Arb}}_3 > \overline{\overline{Arb}}_1$ or $\overline{\overline{Arb}}_2 > \overline{\overline{Arb}}_4 > \overline{\overline{Arb}}_3 > \overline{\overline{Arb}}_1$
$\varsigma = -25$	$\overline{\overline{Sre}}\left(\overline{\overline{Arb}}_1\right) = 0.4665, \overline{\overline{Sre}}\left(\overline{\overline{Arb}}_2\right) = 0.5489,$ $\overline{\overline{Sre}}\left(\overline{\overline{Arb}}_3\right) = 0.5134, \overline{\overline{Sre}}\left(\overline{\overline{Arb}}_4\right) = 0.5296.$	$\overline{\overline{Sre}}\left(\overline{\overline{Arb}}_1\right) = 0.3434, \overline{\overline{Sre}}\left(\overline{\overline{Arb}}_2\right) = 0.5044,$ $\overline{\overline{Sre}}\left(\overline{\overline{Arb}}_3\right) = 0.4049, \overline{\overline{Sre}}\left(\overline{\overline{Arb}}_4\right) = 0.4163.$	$\overline{\overline{Arb}}_2 > \overline{\overline{Arb}}_4 > \overline{\overline{Arb}}_3 > \overline{\overline{Arb}}_1$ or $\overline{\overline{Arb}}_2 > \overline{\overline{Arb}}_4 > \overline{\overline{Arb}}_3 > \overline{\overline{Arb}}_1$
$\varsigma = -50$	$\overline{\overline{Sre}}\left(\overline{\overline{Arb}}_1\right) = 0.4710, \overline{\overline{Sre}}\left(\overline{\overline{Arb}}_2\right) = 0.5502,$ $\overline{\overline{Sre}}\left(\overline{\overline{Arb}}_3\right) = 0.5148, \overline{\overline{Sre}}\left(\overline{\overline{Arb}}_4\right) = 0.5311.$	$\overline{\overline{Sre}}\left(\overline{\overline{Arb}}_1\right) = 0.3400, \overline{\overline{Sre}}\left(\overline{\overline{Arb}}_2\right) = 0.5004,$ $\overline{\overline{Sre}}\left(\overline{\overline{Arb}}_3\right) = 0.3990, \overline{\overline{Sre}}\left(\overline{\overline{Arb}}_4\right) = 0.4119.$	$\overline{\overline{Arb}}_2 > \overline{\overline{Arb}}_4 > \overline{\overline{Arb}}_3 > \overline{\overline{Arb}}_1$ or $\overline{\overline{Arb}}_2 > \overline{\overline{Arb}}_4 > \overline{\overline{Arb}}_3 > \overline{\overline{Arb}}_1$
$\varsigma = -70$	$\overline{\overline{Sre}}\left(\overline{\overline{Arb}}_1\right) = 0.4722, \overline{\overline{Sre}}\left(\overline{\overline{Arb}}_2\right) = 0.5505,$ $\overline{\overline{Sre}}\left(\overline{\overline{Arb}}_3\right) = 0.5152, \overline{\overline{Sre}}\left(\overline{\overline{Arb}}_4\right) = 0.5314.$	$\overline{\overline{Sre}}\left(\overline{\overline{Arb}}_1\right) = 0.3392, \overline{\overline{Sre}}\left(\overline{\overline{Arb}}_2\right) = 0.4992,$ $\overline{\overline{Sre}}\left(\overline{\overline{Arb}}_3\right) = 0.3974, \overline{\overline{Sre}}\left(\overline{\overline{Arb}}_4\right) = 0.4106.$	$\overline{\overline{Arb}}_2 > \overline{\overline{Arb}}_4 > \overline{\overline{Arb}}_3 > \overline{\overline{Arb}}_1$ or $\overline{\overline{Arb}}_2 > \overline{\overline{Arb}}_4 > \overline{\overline{Arb}}_3 > \overline{\overline{Arb}}_1$
$\varsigma = -100$	$\overline{\overline{Sre}}\left(\overline{\overline{Arb}}_1\right) = 0.4731, \overline{\overline{Sre}}\left(\overline{\overline{Arb}}_2\right) = 0.5507,$ $\overline{\overline{Sre}}\left(\overline{\overline{Arb}}_3\right) = 0.5155, \overline{\overline{Sre}}\left(\overline{\overline{Arb}}_4\right) = 0.5317.$	$\overline{\overline{Sre}}\left(\overline{\overline{Arb}}_1\right) = 0.3386, \overline{\overline{Sre}}\left(\overline{\overline{Arb}}_2\right) = 0.4984,$ $\overline{\overline{Sre}}\left(\overline{\overline{Arb}}_3\right) = 0.3961, \overline{\overline{Sre}}\left(\overline{\overline{Arb}}_4\right) = 0.4097.$	$\overline{\overline{Arb}}_2 > \overline{\overline{Arb}}_4 > \overline{\overline{Arb}}_3 > \overline{\overline{Arb}}_1$ or $\overline{\overline{Arb}}_2 > \overline{\overline{Arb}}_4 > \overline{\overline{Arb}}_3 > \overline{\overline{Arb}}_1$

Table 4: Effect of the parameter ζ on decision result.

Parameter	WSSHINPWA operator	WSSHINPGA operator	Ranking order
$\zeta = 1$	$\overline{\overline{Sre}}(\overline{\overline{Arb}}_1) = 0.4309, \overline{\overline{Sre}}(\overline{\overline{Arb}}_2) = 0.5244,$ $\overline{\overline{Sre}}(\overline{\overline{Arb}}_3) = 0.4737, \overline{\overline{Sre}}(\overline{\overline{Arb}}_4) = 0.4817.$	$\overline{\overline{Sre}}(\overline{\overline{Arb}}_1) = 0.3950, \overline{\overline{Sre}}(\overline{\overline{Arb}}_2) = 0.5184,$ $\overline{\overline{Sre}}(\overline{\overline{Arb}}_3) = 0.4437, \overline{\overline{Sre}}(\overline{\overline{Arb}}_4) = 0.4508.$	$\overline{\overline{Arb}}_2 > \overline{\overline{Arb}}_4 > \overline{\overline{Arb}}_3 > \overline{\overline{Arb}}_1$ or $\overline{\overline{Arb}}_2 > \overline{\overline{Arb}}_4 > \overline{\overline{Arb}}_3 > \overline{\overline{Arb}}_1$
$\zeta = 3$	$\overline{\overline{Sre}}(\overline{\overline{Arb}}_1) = 0.4374, \overline{\overline{Sre}}(\overline{\overline{Arb}}_2) = 0.5311,$ $\overline{\overline{Sre}}(\overline{\overline{Arb}}_3) = 0.4957, \overline{\overline{Sre}}(\overline{\overline{Arb}}_4) = 0.5015.$	$\overline{\overline{Sre}}(\overline{\overline{Arb}}_1) = 0.3920, \overline{\overline{Sre}}(\overline{\overline{Arb}}_2) = 0.5183,$ $\overline{\overline{Sre}}(\overline{\overline{Arb}}_3) = 0.4456, \overline{\overline{Sre}}(\overline{\overline{Arb}}_4) = 0.4502.$	$\overline{\overline{Arb}}_2 > \overline{\overline{Arb}}_4 > \overline{\overline{Arb}}_3 > \overline{\overline{Arb}}_1$ or $\overline{\overline{Arb}}_2 > \overline{\overline{Arb}}_4 > \overline{\overline{Arb}}_3 > \overline{\overline{Arb}}_1$
$\zeta = 5$	$\overline{\overline{Sre}}(\overline{\overline{Arb}}_1) = 0.4446, \overline{\overline{Sre}}(\overline{\overline{Arb}}_2) = 0.5379,$ $\overline{\overline{Sre}}(\overline{\overline{Arb}}_3) = 0.5106, \overline{\overline{Sre}}(\overline{\overline{Arb}}_4) = 0.5158.$	$\overline{\overline{Sre}}(\overline{\overline{Arb}}_1) = 0.3924, \overline{\overline{Sre}}(\overline{\overline{Arb}}_2) = 0.5184,$ $\overline{\overline{Sre}}(\overline{\overline{Arb}}_3) = 0.4469, \overline{\overline{Sre}}(\overline{\overline{Arb}}_4) = 0.4506.$	$\overline{\overline{Arb}}_2 > \overline{\overline{Arb}}_4 > \overline{\overline{Arb}}_3 > \overline{\overline{Arb}}_1$ or $\overline{\overline{Arb}}_2 > \overline{\overline{Arb}}_4 > \overline{\overline{Arb}}_3 > \overline{\overline{Arb}}_1$
$\zeta = 10$	$\overline{\overline{Sre}}(\overline{\overline{Arb}}_1) = 0.4584, \overline{\overline{Sre}}(\overline{\overline{Arb}}_2) = 0.5511,$ $\overline{\overline{Sre}}(\overline{\overline{Arb}}_3) = 0.5335, \overline{\overline{Sre}}(\overline{\overline{Arb}}_4) = 0.5382.$	$\overline{\overline{Sre}}(\overline{\overline{Arb}}_1) = 0.3936, \overline{\overline{Sre}}(\overline{\overline{Arb}}_2) = 0.5186,$ $\overline{\overline{Sre}}(\overline{\overline{Arb}}_3) = 0.4485, \overline{\overline{Sre}}(\overline{\overline{Arb}}_4) = 0.4511.$	$\overline{\overline{Arb}}_2 > \overline{\overline{Arb}}_4 > \overline{\overline{Arb}}_3 > \overline{\overline{Arb}}_1$ or $\overline{\overline{Arb}}_2 > \overline{\overline{Arb}}_4 > \overline{\overline{Arb}}_3 > \overline{\overline{Arb}}_1$
$\zeta = 17$	$\overline{\overline{Sre}}(\overline{\overline{Arb}}_1) = 0.4695, \overline{\overline{Sre}}(\overline{\overline{Arb}}_2) = 0.5618,$ $\overline{\overline{Sre}}(\overline{\overline{Arb}}_3) = 0.5484, \overline{\overline{Sre}}(\overline{\overline{Arb}}_4) = 0.5526.$	$\overline{\overline{Sre}}(\overline{\overline{Arb}}_1) = 0.3948, \overline{\overline{Sre}}(\overline{\overline{Arb}}_2) = 0.5188,$ $\overline{\overline{Sre}}(\overline{\overline{Arb}}_3) = 0.4496, \overline{\overline{Sre}}(\overline{\overline{Arb}}_4) = 0.4514.$	$\overline{\overline{Arb}}_2 > \overline{\overline{Arb}}_4 > \overline{\overline{Arb}}_3 > \overline{\overline{Arb}}_1$ or $\overline{\overline{Arb}}_2 > \overline{\overline{Arb}}_4 > \overline{\overline{Arb}}_3 > \overline{\overline{Arb}}_1$
$\zeta = 25$	$\overline{\overline{Sre}}(\overline{\overline{Arb}}_1) = 0.4762, \overline{\overline{Sre}}(\overline{\overline{Arb}}_2) = 0.5681,$ $\overline{\overline{Sre}}(\overline{\overline{Arb}}_3) = 0.5561, \overline{\overline{Sre}}(\overline{\overline{Arb}}_4) = 0.5600$	$\overline{\overline{Sre}}(\overline{\overline{Arb}}_1) = 0.3956, \overline{\overline{Sre}}(\overline{\overline{Arb}}_2) = 0.5190,$ $\overline{\overline{Sre}}(\overline{\overline{Arb}}_3) = 0.4504, \overline{\overline{Sre}}(\overline{\overline{Arb}}_4) = 0.4515.$	$\overline{\overline{Arb}}_2 > \overline{\overline{Arb}}_4 > \overline{\overline{Arb}}_3 > \overline{\overline{Arb}}_1$ or $\overline{\overline{Arb}}_2 > \overline{\overline{Arb}}_4 > \overline{\overline{Arb}}_3 > \overline{\overline{Arb}}_1$
$\zeta = 60$	$\overline{\overline{Sre}}(\overline{\overline{Arb}}_1) = 0.4848, \overline{\overline{Sre}}(\overline{\overline{Arb}}_2) = 0.5766,$ $\overline{\overline{Sre}}(\overline{\overline{Arb}}_3) = 0.5653, \overline{\overline{Sre}}(\overline{\overline{Arb}}_4) = 0.5689.$	$\overline{\overline{Sre}}(\overline{\overline{Arb}}_1) = 0.3971, \overline{\overline{Sre}}(\overline{\overline{Arb}}_2) = 0.5192,$ $\overline{\overline{Sre}}(\overline{\overline{Arb}}_3) = 0.4518, \overline{\overline{Sre}}(\overline{\overline{Arb}}_4) = 0.4515.$	$\overline{\overline{Arb}}_2 > \overline{\overline{Arb}}_4 > \overline{\overline{Arb}}_3 > \overline{\overline{Arb}}_1$ or $\overline{\overline{Arb}}_2 > \overline{\overline{Arb}}_3 > \overline{\overline{Arb}}_4 > \overline{\overline{Arb}}_1$
$\zeta = 100$	$\overline{\overline{Sre}}(\overline{\overline{Arb}}_1) = 0.4870, \overline{\overline{Sre}}(\overline{\overline{Arb}}_2) = 0.5790,$ $\overline{\overline{Sre}}(\overline{\overline{Arb}}_3) = 0.5677, \overline{\overline{Sre}}(\overline{\overline{Arb}}_4) = 0.5712.$	$\overline{\overline{Sre}}(\overline{\overline{Arb}}_1) = 0.3978, \overline{\overline{Sre}}(\overline{\overline{Arb}}_2) = 0.5193,$ $\overline{\overline{Sre}}(\overline{\overline{Arb}}_3) = 0.4525, \overline{\overline{Sre}}(\overline{\overline{Arb}}_4) = 0.4513.$	$\overline{\overline{Arb}}_2 > \overline{\overline{Arb}}_4 > \overline{\overline{Arb}}_3 > \overline{\overline{Arb}}_1$ or $\overline{\overline{Arb}}_2 > \overline{\overline{Arb}}_3 > \overline{\overline{Arb}}_4 > \overline{\overline{Arb}}_1$

deviations, resulting in a variety of ranking results. As a result, one of the benefits of our suggested technique is that it can adapt to various semantic decision-making environments and fulfill the semantic needs of various experts. So, experts can choose the suitable LSF in real-time decision-making based on their linguistic preferences.

6.2. **Effect of the Parameterςon Final Ranking Order**. From Table 3, one can observe that for different values of the parameter ς, different score values are obtained, while utilizing WHINLSSPWA and WHINLSSPGA operators. We can also observe from Table 3, when the values of the parameter ς increases while exploiting WHINLSSPWA operator, the score values of the alternatives increases. Similarly, when utilizing WHINLSSPWG operator, the score values of the alternatives decreases, while the final ranking order remains the same at both the cases. This makes the decision-making process more flexible, so, the decision makers may choose the value of the parameter ς according to the actual need of the situations.

6.3. **Effect of the Parameterζon Decision Result**. From Table 4, we can see that for different values of the parameter ζ different score values are obtained, while utilizing WHINLSSPA and WHINLSSPGA operators. One can also observe from Table 4, when utilizing WHINLSSPA operator the ranking order remains the same, but when we utilized WHINLSSPGA operator different ranking orders are obtained. This makes the decision-making process more flexible, and the makers may use the value of the parameter ζ according to the need of the actual situations.

6.4. Comparison of the Proposed MADM Method with Existing Method. In this subpart, comparison of the anticipated MADM method initiated on the newly proposed AOs with existing method is discussed.

From Table 5, we can see that the ranking order obtained from Ye [32] is the same as the ranking order obtained from the proposed approach. This shows that the initiated approach is valid. The initiated approach has several advantages over the approach developed by Ye [32]. The initiated approach can remove the bad impact of unreasonable data by power weight vector and also make the decision making process more flexible due to general parameters and fit in with distinct semantic scenarios. Therefore, the proposed

Table 5: Comparison with other approach.

Approach	Score values	Ranking order
Ye [32] HINLWA operator	$\overline{\overline{Sre}}\left(\overline{\overline{Arb}}_1\right) = 0.4413, \overline{\overline{Sre}}\left(\overline{\overline{Arb}}_2\right) = 0.5519, \overline{\overline{Sre}}\left(\overline{\overline{Arb}}_3\right) = 0.4549, \overline{\overline{Sre}}\left(\overline{\overline{Arb}}_4\right) = 0.5051.$	$\overline{\overline{Arb}}_2 > \overline{\overline{Arb}}_4 > \overline{\overline{Arb}}_3 > \overline{\overline{Arb}}_1$
Ye [32] HINLWG operator	$\overline{\overline{Sre}}\left(\overline{\overline{Arb}}_1\right) = 0.4231, \overline{\overline{Sre}}\left(\overline{\overline{Arb}}_2\right) = 0.5363, \overline{\overline{Sre}}\left(\overline{\overline{Arb}}_3\right) = 0.4325, \overline{\overline{Sre}}\left(\overline{\overline{Arb}}_4\right) = 0.4774.$	$\overline{\overline{Arb}}_2 > \overline{\overline{Arb}}_4 > \overline{\overline{Arb}}_3 > \overline{\overline{Arb}}_1$
Proposed HINLSSGPWA operator (Consider $\varsigma = -3$ and $\zeta = 2$)	$\overline{\overline{Sre}}\left(\overline{\overline{Arb}}_1\right) = 0.4338, \overline{\overline{Sre}}\left(\overline{\overline{Arb}}_2\right) = 0.5276, \overline{\overline{Sre}}\left(\overline{\overline{Arb}}_3\right) = 0.4863, \overline{\overline{Sre}}\left(\overline{\overline{Arb}}_4\right) = 0.4928.$	$\overline{\overline{Arb}}_2 > \overline{\overline{Arb}}_4 > \overline{\overline{Arb}}_3 > \overline{\overline{Arb}}_1$
Proposed HINLSSGPWA operator (Consider $\varsigma = -3$ and $\zeta = 2$)	$\overline{\overline{Sre}}\left(\overline{\overline{Arb}}_1\right) = 0.3922, \overline{\overline{Sre}}\left(\overline{\overline{Arb}}_2\right) = 0.5182, \overline{\overline{Sre}}\left(\overline{\overline{Arb}}_3\right) = 0.4447, \overline{\overline{Sre}}\left(\overline{\overline{Arb}}_4\right) = 0.4501.$	$\overline{\overline{Arb}}_2 > \overline{\overline{Arb}}_4 > \overline{\overline{Arb}}_3 > \overline{\overline{Arb}}_1$

technique for solving MADM problems is more practical then the existing one.

7. Conclusion

Accessible information is frequently incomplete and incompatible in real decision-making, and the HINLS is a superior tool for indicating such information. In this article, merging LSFs, SS operational laws, and GPA operators, a technique is initiated to deal with HINL MADM problems and fit in with distinct semantic scenarios. Initially, a number of core operational laws for HINLNs are initiated based on LSF, SS t-norm, and SS t-conorm and some of its core properties are investigated. Second, limitations of the existing score function are discussed, and a new score function and distance measure are anticipated based on LSFs. Then, as standard PA operators cannot handle scenarios when expert assessment values are HINLNs, several novel generalized power AOs are proposed to aggregate HINLNs. The most significant characteristic of these operators are that they can also adapt to a variety of semantic situations while also reducing the detrimental impact of unreseasonably high or unreseasonably low evaluation values. Additionally, utilizing the newly instigated AOs, a novel MADM technique is suggested. Lastly, a numerical example is offered to reveal the potency of the initiated technique, along with comprehensive comparison with the existing approaches.

In future, we will explore LSFs and SS operational laws for other generalizations of INL and SVNL sets, such as hesitant bipolar valued neutrosophic sets [48], single valued spherical hesitant neutrosophic sets [49], interval valued neutrosophic vague sets [50], refined single valued neutrosophic sets [51], and initiate different AOs such as MSM operator, Muirhead mean operators, Hamy Mean operators and apply these AOs to solve MADM and MAGDM problems in different fields.

Data Availability

Data sharing does not apply to this article as no data set were generated or analyzed during the current study.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

Acknowledgments

This work was funded by the Deanship of Scientific Research (DSR), King AbdulAziz University, Jeddah, under Grant no. D-714-150-1441.

References

[1] P. Liu, Q. Khan, T. Mahmood, and N. Hassan, "T-spherical fuzzy power Muirhead mean operator based on novel operational laws and their application in multi-attribute group decision making," *IEEE Access*, vol. 7, pp. 22613–22632, 2019.

[2] P. Liu, Q. Khan, T. Mahmood, R. A. Khan, and H. U. Khan, "Some improved pythagorean fuzzy Dombi power aggregation operators with application in multiple-attribute decision making," *Journal of Intelligent & Fuzzy Systems*, vol. 40, no. 5, pp. 9237–9257, 2021.

[3] M. Akram, A. Khan, and A. Borumand Saeid, "Complex Pythagorean Dombi fuzzy operators using aggregation operators and theirdecision-making," *Expert Systems*, vol. 38, no. 2, article e12626, 2021.

[4] J. C. R. Alcantud, A. Z. Khameneh, and A. Kilicman, "Aggregation of infinite chains of intuitionistic fuzzy sets and their application to choices with temporal intuitionistic fuzzy information," *Information Sciences*, vol. 514, pp. 106–117, 2020.

[5] K. Hu, Q. Tan, T. Zhang, and S. Wang, "Assessing technology portfolios of clean energy-driven desalination-irrigation systems with interval-valued intuitionistic fuzzy sets," *Renewable and Sustainable Energy Reviews*, vol. 132, article 109950, 2020.

[6] H. Garg and D. Rani, "Novel similarity measure based on the transformed right-angled triangles between intuitionistic fuzzy sets and its applications," *Cognitive Computation*, vol. 13, no. 2, pp. 447–465, 2021.

[7] F. Smarandache, *A Unifying Field in Logics. Neutrosophy: Neutrosophic Probability, Set and Logic*, American Research Press, Rehoboth, Mass, USA, 1999.

[8] F. Smarandache, "A unifying field in logics neutrosophic logic," in *Neutrosophy, Neutrosophic Set, Neutrosophic Probability*, American Research Press, 2003.

[9] H. Wang, F. Smarandache, Y. Q. Zhang, and R. Sunderraman, "Single valued neutrosophic sets," *Multispace and Multistructure*, vol. 4, pp. 410–413, 2010.

[10] H. Wang, F. Smarandache, Y. Q. Zhang, and R. Sunderraman, *Interval Neutrosophic Sets and Logic: Theory and Applications in Computing*, Hexis, Phoenix, Ariz, USA, 2005.

[11] H. Y. Zhang, P. Ji, J. Q. Wang, and X. H. Chen, "An improved weighted correlation coefficient based on integrated weight for interval neutrosophic sets and its application in multi-criteria decision-making problems," *International Journal of Computational Intelligence Systems*, vol. 8, no. 6, pp. 1027–1043, 2015.

[12] J. Ye, "Improved correlation coefficients of single valued neutrosophic sets and interval neutrosophic sets for multiple attribute decision making," *Journal of Intelligent & Fuzzy Systems*, vol. 27, no. 5, pp. 2453–2462, 2014.

[13] F. Karaaslan, "Correlation coefficient of neutrosophic sets and its applications in decision-making," in *Fuzzy Multi-criteria Decision-Making Using Neutrosophic Sets*, pp. 327–360, Springer, Cham, 2015.

[14] J. S. Chai, G. Selvachandran, F. Smarandache, V. C. Gerogiannis, Q. T. Bui, and B. Vo, "New similarity measures for single-valued neutrosophic sets with applications in pattern recognition and medical diagnosis problems," *Complex & Intelligent Systems*, vol. 7, no. 2, pp. 703–723, 2021.

[15] D. Karabašević, D. Stanujkić, E. K. Zavadskas et al., "A novel extension of the TOPSIS method adapted for the use of single-valued neutrosophic sets and hamming distance for e-commerce development strategies selection," *Symmetry*, vol. 12, no. 8, p. 1263, 2020.

[16] G. Borah and P. Dutta, "Multi-attribute cognitive decision making via convex combination of weighted vector similarity measures for single-valued neutrosophic sets," *Cognitive Computation*, vol. 13, no. 4, pp. 1019–1033, 2021.

[17] P. Ji, J. Q. Wang, and H. Y. Zhang, "Frank prioritized Bonferroni mean operator with single-valued neutrosophic sets and its application in selecting third-party logistics providers," *Neural Computing and Applications*, vol. 30, no. 3, pp. 799–823, 2018.

[18] G. Wei, R. Wang, J. Wang, C. Wei, and Y. Zhang, "Methods for evaluating the technological innovation capability for the high-tech enterprises with generalized interval neutrosophic number Bonferroni mean operators," *Ieee Access*, vol. 7, pp. 86473–86492, 2019.

[19] G. Wei and Z. Zhang, "Some single-valued neutrosophic Bonferroni power aggregation operators in multiple attribute decision making," *Journal of Ambient Intelligence and Humanized Computing*, vol. 10, no. 3, pp. 863–882, 2019.

[20] P. Liu and H. Li, "Multiple attribute decision-making method based on some normal neutrosophic Bonferroni mean operators," *Neural Computing and Applications*, vol. 28, no. 1, pp. 179–194, 2017.

[21] H. Y. Zhang, J. Q. Wang, and X. H. Chen, "Interval neutrosophic sets and their application in multicriteria decision making problems," *The Scientific World Journal*, vol. 2014, Article ID 645953, 15 pages, 2014.

[22] Q. Khan, P. Liu, T. Mahmood, F. Smarandache, and K. Ullah, "Some interval neutrosophic Dombi power Bonferroni mean operators and their application in multi–attribute decision–making," *Symmetry*, vol. 10, no. 10, p. 459, 2018.

[23] P. Liu, Q. Khan, and T. Mahmood, "Application of interval neutrosophic power Hamy mean operators in MAGDM," *Informatica*, vol. 30, no. 2, pp. 293–325, 2019.

[24] L. A. Zadeh, "The concept of a linguistic variable and its application to approximate reasoning–I," *Information Sciences*, vol. 8, no. 3, pp. 199–249, 1975.

[25] F. Herrera, E. Herrera-Viedma, and L. Martinez, "A fusion approach for managing multi-granularity linguistic term sets in decision making," *Fuzzy Sets and Systems*, vol. 114, no. 1, pp. 43–58, 2000.

[26] Z. Fang and J. Ye, "Multiple attribute group decision-making method based on linguistic neutrosophic numbers," *Symmetry*, vol. 9, no. 7, p. 111, 2017.

[27] P. Liu, T. Mahmood, and Q. Khan, "Group decision making based on power Heronian aggregation operators under linguistic neutrosophic environment," *International Journal of Fuzzy Systems*, vol. 20, no. 3, pp. 970–985, 2018.

[28] J. Ye, "Some aggregation operators of interval neutrosophic linguistic numbers for multiple attribute decision making," *Journal of Intelligent & Fuzzy Systems*, vol. 27, no. 5, pp. 2231–2241, 2014.

[29] J. Ye, "An extended TOPSIS method for multiple attribute group decision making based on single valued neutrosophic linguistic numbers," *Journal of Intelligent & Fuzzy Systems*, vol. 28, no. 1, pp. 247–255, 2015.

[30] P. Ji, H. Y. Zhang, and J. Q. Wang, "Selecting an outsourcing provider based on the combined MABAC-ELECTRE method using single-valued neutrosophic linguistic sets," *Computers & Industrial Engineering*, vol. 120, pp. 429–441, 2018.

[31] J. Q. Wang, Y. Yang, and L. Li, "Multi-criteria decision-making method based on single-valued neutrosophic linguistic Maclaurin symmetric mean operators," *Neural Computing and Applications*, vol. 30, no. 5, pp. 1529–1547, 2018.

[32] J. Ye, "Hesitant interval neutrosophic linguistic set and its application in multiple attribute decision making," *International Journal of Machine Learning and Cybernetics*, vol. 10, no. 4, pp. 667–678, 2019.

[33] V. Torra, "Hesitant fuzzy sets," *International Journal of Intelligent Systems*, vol. 25, no. 6, pp. n/a–539, 2010.

[34] Z. Xu and M. Xia, "Distance and similarity measures for hesitant fuzzy sets," *Information Sciences*, vol. 181, no. 11, pp. 2128–2138, 2011.

[35] R. R. Yager, "The power average operator," *IEEE Transactions on Systems, Man, and Cybernetics-Part A: Systems and Humans*, vol. 31, no. 6, pp. 724–731, 2001.

[36] Z. Xu and R. R. Yager, "Power-geometric operators and their use in group decision making," *IEEE Transactions on Fuzzy Systems*, vol. 18, no. 1, pp. 94–105, 2010.

[37] L. Zhou, H. Chen, and J. Liu, "Generalized power aggregation operators and their applications in group decision making," *Computers & Industrial Engineering*, vol. 62, no. 4, pp. 989–999, 2012.

[38] Y. P. Jiang, Z. P. Fan, and J. Ma, "A method for group decision making with multi-granularity linguistic assessment information," *Information Sciences*, vol. 178, no. 4, pp. 1098–1109, 2008.

[39] M. Delgado, J. L. Verdegay, and M. A. Vila, "Linguistic decision-making models," *International Journal of Intelligent Systems*, vol. 7, no. 5, pp. 479–492, 1992.

[40] F. Herrera and E. Herrera-Viedma, "Linguistic decision analysis: steps for solving decision problems under linguistic information," *Fuzzy Sets and Systems*, vol. 115, no. 1, pp. 67–82, 2000.

[41] H. Wang, Z. Xu, and X. J. Zeng, "Hesitant fuzzy linguistic term sets for linguistic decision making: current developments, issues and challenges," *Information Fusion*, vol. 43, pp. 1–12, 2018.

[42] T. H. Hsu, C. H. Chen, and Y. W. Yang, "A decision analysis model for the brand experience of branded apps using consistency fuzzy linguistic preference relations," *Symmetry*, vol. 13, no. 7, p. 1151, 2021.

[43] O. Sohaib, M. Naderpour, W. Hussain, and L. Martinez, "Cloud computing model selection for e-commerce enterprises using a new 2-tuple fuzzy linguistic decision-making method," *Computers & Industrial Engineering*, vol. 132, pp. 47–58, 2019.

[44] T. He, G. Wei, J. Lu, C. Wei, and R. Lin, "Pythagorean 2-tuple linguistic VIKOR method for evaluating human factors in construction project management," *Mathematics*, vol. 7, no. 12, p. 1149, 2019.

[45] T. He, G. Wei, J. Lu, C. Wei, and R. Lin, "Pythagorean 2-tuple linguistic taxonomy method for supplier selection in medical instrument industries," *International Journal of Environmental Research and Public Health*, vol. 16, no. 23, p. 4875, 2019.

[46] J. Q. Wang, J. T. Wu, J. Wang, H. Y. Zhang, and X. H. Chen, "Interval-valued hesitant fuzzy linguistic sets and their applications in multi-criteria decision-making problems," *Information Sciences*, vol. 288, pp. 55–72, 2014.

[47] G. Deschrijver and E. E. Kerre, "A generalization of operators on intuitionistic fuzzy sets using triangular norms and conorms. Notes Intuit," *Fuzzy Sets*, vol. 8, pp. 19–27, 2002.

[48] N. A. Awang, L. Abdullah, and H. Hashim, "A Bonferroni mean considering Shapley fuzzy measure under hesitant bipolar-valued neutrosophic set environment for an investment decision," *Journal of Ambient Intelligence and Humanized Computing*, pp. 1–28, 2021.

[49] A. Saha, D. Dutta, and S. Broumi, "Bulbul disaster assessment using single-valued spherical hesitant neutrosophic Dombi weighted aggregation operators," in *Neutrosophic Operational Research*, pp. 221–243, Springer, Cham, 2021.

[50] H. Hashim, L. Abdullah, A. Al-Quran, and A. Awang, "Entropy measures for interval neutrosophic vague sets and their application in decision making," *Neutrosophic Sets and Systems*, vol. 45, pp. 74–95, 2021.

[51] R. P. Tan and W. D. Zhang, "Decision-making method based on new entropy and refined single-valued neutrosophic sets and its application in typhoon disaster assessment," *Applied Intelligence*, vol. 51, no. 1, pp. 283–307, 2021.