Hindawi
Journal of Function Spaces
Volume 2022, Article ID 1688741, 10 pages
https://doi.org/10.1155/2022/1688741

Research Article

Certain Analytic Functions Defined by Generalized Mittag-Leffler Function Associated with Conic Domain

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Received 8 March 2022; Accepted 21 May 2022; Published 9 June 2022

Academic Editor: Salah Mahmoud Boulaaras

In the present paper, we investigate and introduce several properties of certain families of analytic functions in the open unit disc, which are defined by q-analogue of Mittag-Leffler function associated with conic domain. A number of coefficient estimates of the functions in these classes have been obtained. Sufficient conditions for the functions belong to these classes are also considered.

1. Introduction

Denoted by $\mathbb{A}$, the class of all analytic functions in $\mathbb{D} = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ has the following form:

$$g(\zeta) = \zeta + \sum_{m=2}^{\infty} b_m \zeta^m. \tag{1}$$

For two functions $g_1(\zeta)$ and $g_2(\zeta)$, analytic in $\mathbb{D}$, we say that the function $g_1(\zeta)$ is subordinate to $g_2(\zeta)$, written as $g_1(\zeta) \prec g_2(\zeta)$, if there exists a Schwarz function $w(\zeta)$, which is analytic in $\mathbb{D}$ with $w(0) = 0$ and $|w(\zeta)| < 1$ such that $g_1(\zeta) = g_2(w(\zeta))$. Furthermore, if the function $g_2(\zeta)$ is univalent in $\mathbb{D}$, then we have the following equivalence:

$$\begin{gathered} g_1(z) \prec g_2(z) \Longleftrightarrow g_1(0) = g_2(0), \\ g_1(\mathbb{D}) \subset g_2(\mathbb{D}). \end{gathered} \tag{2}$$

Let $\mathcal{S}$ be the subclass of $\mathbb{A}$ consisting of univalent functions. Also, let $\mathcal{S}^*(\eta)$ and $\mathcal{C}(\eta)$ denote the subclasses of univalent starlike and convex functions of order η $(0 \le \eta < 1)$. A function $g \in \mathcal{S}$ is said to be k-starlike function, written as $g \in \mathcal{ST}(k)$, if

$$\Re\left\{\frac{\zeta g'(\zeta)}{g(\zeta)}\right\} > k\left|\frac{\zeta g'(\zeta)}{g(\zeta)} - 1\right| (0 \le k < \infty; \zeta \in \mathbb{D}). \tag{3}$$

A function $g \in \mathcal{S}$ is said to be k-convex function, written as $g \in \mathcal{UCV}(k)$, if

$$\Re\left\{\frac{(\zeta g(\zeta))'}{g'(\zeta)}\right\} > k\left|\frac{(\zeta g(\zeta))'}{g'(\zeta)} - 1\right| (0 \le k < \infty; \zeta \in \mathbb{D}). \tag{4}$$

The classes $k - \mathcal{ST}$ and $k - \mathcal{UCV}$ were introduced and studied by Kanas and Wiśniowska [1, 2] (see also, [3–8]). In particular, when $k = 1$, we get $\mathcal{ST}(1) \equiv \mathcal{ST}$ and $\mathcal{UCV}(1) \equiv \mathcal{UCV}$, where $\mathcal{ST}$ and $\mathcal{UCV}$ are the familiar classes of uniformly starlike functions and uniformly convex functions in $\mathbb{D}$, respectively (see [9]). A function $g \in \mathbb{A}$ is said to be in the class $\mathcal{ST}(k, \eta)$ that are k-starlike functions of order η with $0 \le \eta < 1$ if

$$\Re\left\{\frac{\zeta g'(\zeta)}{g(\zeta)} - \eta\right\} > k\left|\frac{\zeta g'(\zeta)}{g(\zeta)} - 1\right| (0 \le k < \infty; \zeta \in \mathbb{D}). \tag{5}$$

A function $g \in \mathbb{A}$ is said to be in the class $\mathcal{UCV}(k, \eta)$ that are k-convex functions of order η with $0 \le \eta < 1$ if

$$\Re\left\{\frac{(\zeta g(\zeta))'}{g'(\zeta)}-\eta\right\}>k\left|\frac{(\zeta g(\zeta))'}{g'(\zeta)}-1\right|(0\le k<\infty;\zeta\in\mathbb{D}). \quad (6)$$

The classes $\mathcal{ST}(k,\eta)$ and $\mathcal{UCV}(k,\eta)$ have been studied by Kanas and Răducanu [10]. We note that $\mathcal{ST}(k,0)=k-\mathcal{ST}$ and $\mathcal{UCV}(k,0)=k-\mathcal{UCV}$. The classes $\mathcal{ST}(1,\eta)\equiv\mathcal{ST}(\eta)$ and $\mathcal{UCV}(1,\eta)\equiv\mathcal{UCV}(\eta)$ were investigated in [11–13].

The study of quantum calculus (or q-calculus) attracted the researches due to its applications in various branches of mathematics and physics, for example, in the areas of special functions, q-difference, ordinary fractional calculus, q-integral equations, and in q-transform analysis (see [14–23]).

For $0<q<1$ and $g\in\mathbb{A}$ given by (1), the q-derivative of g is defined by (see [24–29]):

$$D_q g(\zeta)=\begin{cases} g'(0) & \text{if } \zeta=0,\\ \dfrac{g(\zeta)-g(q\zeta)}{(1-q)\zeta} & \text{if } \zeta\neq 0,\end{cases} \quad (7)$$

provided that $g'(0)$ exists. From (1) and (7), we have

$$D_q g(\zeta)=1+\sum_{m=2}^{\infty}[m]_q b_m \zeta^{m-1}\quad(\zeta\neq0), \quad (8)$$

where $[m]_q$ is q-integer number m defined by

$$[m]_q=\frac{1-q^m}{1-q}=1+q+q^2+\cdots+q^{m-1}\quad(0<q<1). \quad (9)$$

We note that

$$\lim_{q\longrightarrow1^-}D_q g(\zeta)=\lim_{q\longrightarrow1^-}\frac{g(\zeta)-g(q\zeta)}{(1-q)\zeta}=g'(\zeta), \quad (10)$$

for a function g which is differentiable in $\mathbb{D}$. Making use of the q-derivative operator D_q given by (7), we introduce the subclasses $\mathcal{ST}_q(k,\eta)$ and $\mathcal{UCV}_q(k,\eta)$ in $\mathbb{D}$ as follows.

Definition 1. For $0<q<1$, $0\le\eta<1$, $0\le k<\infty$, and $\zeta\in\mathbb{D}$, let $\mathcal{ST}_q(k,\eta)$ and $\mathcal{UCV}_q(k,\eta)$ be the subclasses of $\mathbb{A}$ consisting of functions g of the form (1) and satisfy the analytic criterion:

$$\Re\left\{\frac{\zeta D_q g(\zeta)}{g(\zeta)}-\eta\right\}>k\left|\frac{\zeta D_q g(\zeta)}{g(\zeta)}-1\right|, \quad (11)$$

$$\Re\left\{\frac{D_q(\zeta D_q g(\zeta))}{D_q g(\zeta)}-\eta\right\}>k\left|\frac{D_q(\zeta D_q g(\zeta))}{D_q g(\zeta)}-1\right|. \quad (12)$$

From (11) and (12), it follows that

$$g\in\mathcal{UCV}_q(k,\eta)\Longleftrightarrow \zeta D_q g\in\mathcal{ST}_q(k,\eta). \quad (13)$$

The q-shifted factorials, for any complex number α, are defined by

$$(\alpha;q)_n=\begin{cases}1 & (n=0),\\ \prod_{k=0}^{n-1}\left(1-\alpha q^k\right) & (n\in\mathbb{N}\ll\{1,2,3,\cdots\}).\end{cases} \quad (14)$$

The definition (14) remains meaningful for $n=\infty$ as a convergent infinite product

$$(\alpha;q)_\infty=\prod_{j=0}^{\infty}\left(1-\alpha q^j\right)\text{for } |q|<1. \quad (15)$$

Furthermore, in terms of the q-gamma function $\Gamma_q(\zeta)$ defined by

$$\Gamma_q(\zeta)\coloneqq\frac{(q;q)_\infty(1-q)^{1-\zeta}}{(q^\zeta;q)_\infty}(0<q<1;\zeta\in\mathbb{C}), \quad (16)$$

so that

$$\lim_{q\longrightarrow1^-}\left\{\Gamma_q(\zeta)\right\}=\Gamma(\zeta), \quad (17)$$

for the familiar gamma function $\Gamma(\zeta)$, we find from (14) that

$$(q^\alpha;q)_n=\frac{\Gamma_q(\alpha+n)}{\Gamma_q(\alpha)}(1-q)^n(n\in\mathbb{N};\alpha\in\mathbb{C}). \quad (18)$$

We note that

$$\lim_{q\longrightarrow1^-}\frac{(q^\alpha;q)_n}{(1-q)^n}=(\alpha)_n, \quad (19)$$

where

$$(\alpha)_n=\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}=\begin{cases}1 & (n=0),\\ \alpha(\alpha+1)\cdots(\alpha+n-1) & (n\in\mathbb{N}).\end{cases} \quad (20)$$

For $0<q<1$, α, β, $\gamma\in\mathbb{C}$, $\Re(\alpha)>0$, $\Re(\beta)>0$, and $\Re(\gamma)>0$, consider the q-analogue of Mittag-Leffler defined by Sharma and Jain [30], for generalized Mittag-Leffler function (see, e.g., [31–33])

$$E^{\gamma}_{\alpha,\beta}(\zeta;q)=\sum_{m=0}^{\infty}\frac{(q^\gamma;q)_m}{(q;q)_m}\frac{\zeta^m}{\Gamma_q(\alpha m+\beta)}. \quad (21)$$

As $q\longrightarrow1^-$, the operator $E^{\gamma}_{\alpha,\beta}(\zeta;q)$ reduces to $E^{\gamma}_{\alpha,\beta}(\zeta)$ introduced by Prabhakar [34]. Now, let us define

$$\begin{aligned}\mathbb{E}^{\gamma}_{\alpha,\beta}(\zeta;q) &:= \zeta\Gamma_q(\beta)E^{\gamma}_{\alpha,\beta}(\zeta;q)\\ &= \zeta + \sum_{m=2}^{\infty}\frac{(q^{\gamma};q)_{m-1}}{(q;q)_{m-1}}\frac{\Gamma_q(\beta)}{\Gamma_q(\alpha(m-1)+\beta)}\zeta^m.\end{aligned} \tag{22}$$

We remark that:

(i) $\mathbb{E}^{1}_{1,1}(\zeta;q) = \zeta e_q(\zeta)$

(ii) $\mathbb{E}^{1}_{1,2}(\zeta;q) = e_q(\zeta) - 1$

where $e_q(\zeta)$ is one of the q-analogues of the exponential function e^{ζ} given by

$$e_q(\zeta) = \sum_{m=0}^{\infty}\frac{\zeta^m}{\Gamma_q(m+1)} = \sum_{m=0}^{\infty}\frac{(1-q)^m\zeta^m}{(q;q)_m} = \frac{1}{(\zeta;q)_{\infty}}. \tag{23}$$

Using the Hadamard product (or convolution), we define the linear operator $\mathscr{H}^{\gamma}_{q,\alpha,\beta}: \mathbb{A} \longrightarrow \mathbb{A}$ by

$$\begin{aligned}\mathscr{H}^{\gamma}_{q,\alpha,\beta}g(\zeta) &= \mathbb{E}^{\gamma}_{\alpha,\beta}(\zeta;q) * g(\zeta)\\ &= \zeta + \sum_{m=2}^{\infty}\frac{(q^{\gamma};q)_{m-1}}{(q;q)_{m-1}}\frac{\Gamma_q(\beta)}{\Gamma_q(\alpha(m-1)+\beta)}b_m\zeta^m (\zeta \in \mathbb{D}).\end{aligned} \tag{24}$$

We note that (see [35, 36])

$$\begin{aligned}\lim_{q\longrightarrow 1^-}\mathscr{H}^{\gamma}_{q,\alpha,\beta}g(\zeta) &= \mathscr{H}^{\gamma}_{\alpha,\beta}g(\zeta)\\ &= \zeta + \sum_{m=2}^{\infty}\frac{(\gamma)_{m-1}\Gamma(\beta)}{\Gamma(\alpha(m-1)+\beta)(m-1)!}b_m\zeta^m (\zeta \in \mathbb{D}).\end{aligned} \tag{25}$$

Motivated by the works of Kanas and Yaguchi [37] and Kanas and Răducanu [10], we define the following classes of functions with the theory of q-calculus.

Definition 2. For $0 < q < 1$, $\alpha, \beta, \gamma \in \mathbb{C}$, $\mathfrak{R}(\alpha) > 0$, $\mathfrak{R}(\beta) > 0$, $\mathfrak{R}(\gamma) > 0$, $0 \le k < \infty$, and $0 \le \eta < 1$, let

$$\begin{aligned}\mathcal{ST}^{\gamma}_{q,\alpha,\beta}(k,\eta) &= \left\{g \in \mathbb{A} : \mathscr{H}^{\gamma}_{q,\alpha,\beta}g \in \mathcal{ST}_q(k,\eta)\right\},\\ \mathcal{UCV}^{\gamma}_{q,\alpha,\beta}(k,\eta) &= \left\{g \in \mathbb{A} : \mathscr{H}^{\gamma}_{q,\alpha,\beta}g \in \mathcal{UCV}_q(k,\eta)\right\}.\end{aligned} \tag{26}$$

It is easy to check that

$$g \in \mathcal{UCV}^{\gamma}_{q,\alpha,\beta}(k,\eta) \Longleftrightarrow \zeta D_q g \in \mathcal{ST}^{\gamma}_{q,\alpha,\beta}(k,\eta). \tag{27}$$

Taking $q \longrightarrow 1^-$ in Definition 2, we obtain

$$\begin{aligned}\lim_{q\longrightarrow 1^-}\mathcal{ST}^{\gamma}_{q,\alpha,\beta}(k,\eta) &= \mathcal{ST}^{\gamma}_{\alpha,\beta}(k,\eta)\\ &= \left\{g \in \mathbb{A} : \mathfrak{R}\left\{\frac{\zeta\left(\mathscr{H}^{\gamma}_{\alpha,\beta}g(\zeta)\right)'}{\mathscr{H}^{\gamma}_{\alpha,\beta}g(\zeta)} - \eta\right\}\right.\\ &\qquad \left. > k\left|\frac{\zeta\left(\mathscr{H}^{\gamma}_{\alpha,\beta}g(\zeta)\right)'}{\mathscr{H}^{\gamma}_{\alpha,\beta}g(\zeta)} - 1\right|\right\},\\ \mathcal{ST}^{1}_{q,0,\beta}(k,\eta) = \mathcal{ST}_q(k,\eta),\ &\lim_{q\longrightarrow 1^-}\mathcal{ST}^{1}_{q,0,\beta}(k,\eta) = \mathcal{ST}(k,\eta),\end{aligned} \tag{28}$$

$$\begin{aligned}\lim_{q\longrightarrow 1^-}\mathcal{UCV}^{\gamma}_{q,\alpha,\beta}(k,\eta) &= \mathcal{UCV}^{\gamma}_{\alpha,\beta}(k,\eta)\\ &= \left\{g \in \mathbb{A} : \mathfrak{R}\left\{\frac{\left(\zeta\left(\mathscr{H}^{\gamma}_{\alpha,\beta}g(\zeta)\right)'\right)'}{\left(\mathscr{H}^{\gamma}_{\alpha,\beta}g(\zeta)\right)'} - \eta\right\}\right.\\ &\qquad \left. > k\left|\frac{\left(\zeta\left(\mathscr{H}^{\gamma}_{\alpha,\beta}g(\zeta)\right)'\right)'}{\left(\mathscr{H}^{\gamma}_{\alpha,\beta}g(\zeta)\right)'} - 1\right|\right\},\\ \mathcal{UCV}^{1}_{q,0,\beta}(k,\eta) = \mathcal{UCV}_q(k,\eta),\ &\lim_{q\longrightarrow 1^-}\mathcal{UCV}^{1}_{q,0,\beta}(k,\eta) = \mathcal{UCV}(k,\eta).\end{aligned} \tag{29}$$

Motivated by the works mentioned above, in this paper, we will investigate some important properties, coefficient estimates, and the familiar Fekete–Szegö type inequalities for the subclasses $\mathcal{ST}^{\gamma}_{q,\alpha,\beta}(k,\eta)$ and $\mathcal{UCV}^{\gamma}_{q,\alpha,\beta}(k,\eta)$.

2. Some Results of Functions in $\mathcal{ST}^{\gamma}_{q,\alpha,\beta}(k,\eta)$ and $\mathcal{UCV}^{\gamma}_{q,\alpha,\beta}(k,\eta)$

Unless otherwise mentioned, we assume throughout this paper that $0 < q < 1$, $\alpha, \beta, \gamma > 0$, $0 \le k < \infty$, $0 \le \eta < 1$, and $\zeta \in \mathbb{D}$.

Let $g \in \mathcal{ST}^{\gamma}_{q,\alpha,\beta}(k,\eta)$, we have

$$\mathfrak{R}\left\{\frac{\zeta D_q\left(\mathscr{H}^{\gamma}_{q,\alpha,\beta}g(\zeta)\right)}{\mathscr{H}^{\gamma}_{q,\alpha,\beta}g(\zeta)} - \eta\right\} > k\left|\frac{\zeta D_q\left(\mathscr{H}^{\gamma}_{q,\alpha,\beta}g(\zeta)\right)}{\mathscr{H}^{\gamma}_{q,\alpha,\beta}g(\zeta)} - 1\right| (\zeta \in \mathbb{D}). \tag{30}$$

Consider

$$\psi(\zeta) = \frac{\zeta D_q\left(\mathscr{H}^{\gamma}_{q,\alpha,\beta}g(\zeta)\right)}{\mathscr{H}^{\gamma}_{q,\alpha,\beta}g(\zeta)} (\zeta \in \mathbb{D}). \tag{31}$$

The condition (30) may be rewritten into the form

$$\mathfrak{R}\{\psi(\zeta) - \eta\} > k|\psi(\zeta) - 1| (\zeta \in \mathbb{D}). \tag{32}$$

It follows that the range of the expression $\psi(\zeta)$, $\zeta \in \mathbb{D}$, is a conical domain

$$\Lambda_{k,\eta} = \{w \in \mathbb{C} : \Re\{w - \eta\} > k|w - 1|\}, \tag{33}$$

or

$$\Lambda_{k,\eta} = \left\{ w = u + iv \in \mathbb{C} : u - \eta > k\sqrt{(u-1)^2 + v^2} \right\}, \tag{34}$$

where $0 \le k < \infty$ and $0 \le \eta < 1$. Note that $\Lambda_{k,\eta}$ is such that $1 \in \Lambda_{k,\eta}$ and $\partial\Lambda_{k,\eta}$ is a curve defined by

$$\partial\Lambda_{k,\eta} = \left\{ w = u + iv \in \mathbb{C} : (u-\eta)^2 = k^2(u-1)^2 + k^2v^2 \right\}. \tag{35}$$

Any $w = u + iv$ in $\partial\Lambda_{k,\eta}$ is a quadratic equation in two variables u and v that have no uv term; it is well known that it is a symmetrical conic section about the real axis (for more details, see [1]). It follows that the domain $\Lambda_{k,\eta}$ is bounded by an ellipse for $k > 1$, by a parabola for $k = 1$ and by a hyperbola if $0 < k < 1$.

Finally, for $k = 0$, $\Lambda_{k,\eta}$ is the right half plane $\Re\{w\} > \eta$. From (30), we obtain that $g \in \Lambda_{k,\eta}$ if and only if, for $\zeta \in \mathbb{D}$,

$$\frac{\zeta D_q\left(\mathscr{H}^{\gamma}_{q,\alpha,\beta}g(\zeta)\right)}{\mathscr{H}^{\gamma}_{q,\alpha,\beta}g(\zeta)} \in \Lambda_{k,\eta}(\zeta \in \mathbb{D}). \tag{36}$$

Making use of the properties of the domain $\Lambda_{k,\eta}$ and (36), it follows that if $g \in \Lambda_{k,\eta}$, then

$$\Re\left\{\frac{\zeta D_q\left(\mathscr{H}^{\gamma}_{q,\alpha,\beta}g(\zeta)\right)}{\mathscr{H}^{\gamma}_{q,\alpha,\beta}g(\zeta)}\right\} > \frac{k+\eta}{k+1} (\zeta \in \mathbb{D}),$$
$$\left|\arg \frac{\zeta D_q\left(\mathscr{H}^{\gamma}_{q,\alpha,\beta}g(\zeta)\right)}{\mathscr{H}^{\gamma}_{q,\alpha,\beta}g(\zeta)}\right| \le \begin{cases} \arctan \dfrac{1-\eta}{\sqrt{|k^2-\eta^2|}} & (k > 0), \\ \dfrac{\pi}{2} & (k = 0). \end{cases} \tag{37}$$

Denote by $\mathscr{P}$ the class of analytic and normalized Carathéodory functions and by $\psi_{k,\eta} \in \mathscr{P}$, the function such that $\psi_{k,\eta} = \Lambda_{k,\eta}$. Following the notation applied by Ma and Minda [38], for $0 \le k < \infty$ and $0 \le \eta < 1$, let $\mathscr{P}(\psi_{k,\eta})$ denote the following class of functions:

$$\mathscr{P}\left(\psi_{k,\eta}\right) = \{\psi \in \mathscr{P} : \psi(\mathbb{D}) \subset \Lambda_{k,\eta}\} = \left\{\psi \in \mathscr{P} : \psi \prec \psi_{k,\eta} \text{ in } \mathbb{D}\right\}. \tag{38}$$

The functions which play the role of extremal functions for the class $\mathscr{P}(\psi_{k,\eta})$, see [10] (see also [8, 39]) and are defined by

$$\psi_{k,\eta}(\zeta) = \begin{cases} \dfrac{1+(1-2\eta)\zeta}{1-\zeta} & (k=0), \\ 1 + \dfrac{2(1-\eta)}{\pi^2}\left(\log\dfrac{1+\sqrt{\zeta}}{1-\sqrt{\zeta}}\right)^2 & (k=1), \\ \dfrac{1-\eta}{1-k^2}\cos\left(A(k)i\log\dfrac{1+\sqrt{\zeta}}{1-\sqrt{\zeta}}\right) - \dfrac{k^2-\eta}{1-k^2} & (0<k<1), \\ \dfrac{1-\eta}{k^2-1}\sin\left(\dfrac{\pi}{2\kappa(t)}\displaystyle\int_0^{u(\zeta)/\sqrt{t}}\dfrac{dx}{\sqrt{1-x^2}\sqrt{1-t^2x^2}}\right) + \dfrac{k^2-\eta}{k^2-1} & (k>1), \end{cases} \tag{39}$$

with $A(k) = (2/\pi)\arccos(k)$, $u(\zeta) = ((\zeta - \sqrt{t})/(1 - \sqrt{t\zeta}))(0 < t < 1, \zeta \in \mathbb{D})$, where t is so such that $t = \cosh(\pi\kappa'(t)/4\kappa(t))$ and $\kappa(t)$ is Legendre's complete elliptic integral of the first kind and $\kappa'(t)$ the complementary integral of $\kappa(t)$.

Obviously, if $k = 0$, then

$$\psi_{0,\eta}(\zeta) = 1 + 2(1-\eta)\zeta + 2(1-\eta)\zeta^2 + \cdots. \tag{40}$$

For $k = 1$, we have (see [40, 41])

$$\psi_{1,\eta}(\zeta) = 1 + \frac{8}{\pi^2}(1-\eta)\zeta + \frac{16}{3\pi^2}(1-\eta)\zeta^2 + \cdots. \tag{41}$$

Using the Taylor series in [1, 4], for $0 < k < 1$, we have

$$\psi_{k,\eta}(\zeta) = 1 + \frac{1-\eta}{1-k^2}\sum_{m=1}^{\infty}\left[\sum_{l=1}^{2m} 2^l \binom{A(k)}{l}\binom{2m-1}{2m-l}\right]\zeta^m. \tag{42}$$

Finally, when $k > 1$

$$\psi_{k,\eta}(\zeta) = 1 + \frac{\pi^2(1-\eta)}{4\sqrt{t}(k^2-1)\kappa^2(t)(1+t)}\left\{\zeta + \frac{4\kappa^2(t)(t^2+6t+1)-\pi^2}{24\sqrt{t}\kappa^2(t)(1+t)}\zeta^2 + \cdots\right\}, \tag{43}$$

so that, denoting

$$\psi_{k,\eta}(\zeta) = 1 + L_1\zeta + L_2\zeta^2 + \cdots \quad \left(L_j = L_j(k,\eta); j = 1, 2, \cdots\right), \tag{44}$$

we get

$$L_1 = \begin{cases} \dfrac{8(1-\eta)(\arccos k)^2}{\pi^2(1-k^2)} & (0 \le k < 1), \\ \dfrac{8}{\pi^2}(1-\eta) & (k=1), \\ \dfrac{\pi^2(1-\eta)}{4\sqrt{t}(k^2-1)\kappa^2(t)(1+t)} & (k>1). \end{cases} \tag{45}$$

Let $g_{k,\eta}(\zeta) = \zeta + B_2\zeta^2 + B_3\zeta^3 + \cdots$ be the extremal function in the class $\mathscr{ST}^{\gamma}_{q,\alpha,\beta}(k,\eta)$. Then, the relation between the extremal functions in the classes $\mathscr{P}(\psi_{k,\eta})$ and $\mathcal{S}\mathcal{T}^{\gamma}_{q,\alpha,\beta}(k,\eta)$ is given by

$$\psi_{k,\eta}(\zeta)=\frac{\zeta D_q\left(\mathscr{H}^{\gamma}_{q,\alpha,\beta}g_{k,\eta}(\zeta)\right)}{\mathscr{H}^{\gamma}_{q,\alpha,\beta}g_{k,\eta}(\zeta)}\ (\zeta\in\mathbb{D}). \tag{46}$$

Making use of (24), (30), and (46), we obtain the following coefficient relation for $\psi_{k,\eta}(\zeta)$:

$$\frac{\left([m]_q-1\right)(q^{\gamma};q)_{m-1}}{(q;q)_{m-1}\Gamma_q(\alpha(m-1)+\beta)}B_m=\sum_{i=1}^{m-1}\frac{(q^{\gamma};q)_{i-1}}{(q;q)_{i-1}\Gamma_q(\alpha(i-1)+\beta)}B_iL_{m-i},\ B_1=1. \tag{47}$$

In particular, by a direct computation, we have

$$B_2=\frac{\Gamma_q(\alpha+\beta)}{q[\gamma]_q\Gamma_q(\beta)}L_1, \tag{48}$$

$$B_3=\frac{\Gamma_q(2\alpha+\beta)}{q^2[\gamma]_q[\gamma+1]_q\Gamma_q(\beta)}\left(qL_2+L_1^2\right). \tag{49}$$

Since $\alpha,\beta,\gamma>0$, $0<q<1$ and the L_n's are nonnegative, it follows that the B_n's are nonnegative.

Theorem 3. *If g given by (1) belongs to $\mathcal{S}\mathcal{T}^{\gamma}_{q,\alpha,\beta}(k,\eta)$, then*

$$\begin{aligned}|b_2|&\le B_2,\\|b_3|&\le B_3.\end{aligned} \tag{50}$$

Proof. Let

$$\psi(\zeta)=\frac{\zeta D_q\left(\mathscr{H}^{\gamma}_{q,\alpha,\beta}g(\zeta)\right)}{\mathscr{H}^{\gamma}_{q,\alpha,\beta}g(\zeta)}\ (\zeta\in\mathbb{D}). \tag{51}$$

Using the relation (24) for $\psi(\zeta)=1+\rho_1\zeta+\rho_2\zeta^2+\cdots$, we have

$$\frac{\left([m]_q-1\right)(q^{\gamma};q)_{m-1}}{(q;q)_{m-1}\Gamma_q(\alpha(m-1)+\beta)}b_m=\sum_{i=1}^{m-1}\frac{(q^{\gamma};q)_{i-1}}{(q;q)_{i-1}\Gamma_q(\alpha(i-1)+\beta)}b_i\rho_{m-i},\ b_1=1. \tag{52}$$

Since $\psi_{k,\eta}$ is univalent in $\mathbb{D}$, the function

$$p(\zeta)=\frac{1+\psi^{-1}_{k,\eta}(\psi(\zeta))}{1-\psi^{-1}_{k,\eta}(\psi(\zeta))}=1+c_1\zeta+c_2\zeta^2+\cdots, \tag{53}$$

is analytic in $\mathbb{D}$ and $\mathfrak{R}\{q(\zeta)\}>0$. From

$$\psi(\zeta)=\psi_{k,\eta}\left(\frac{p(\zeta)-1}{p(\zeta)+1}\right)=1+\frac{1}{2}c_1L_1\zeta+\left(\frac{1}{2}c_2L_1+\frac{1}{4}c_1^2(L_2-L_1)\right)\zeta^2+\cdots, \tag{54}$$

we have

$$|b_2|=\frac{\Gamma_q(\alpha+\beta)}{q[\gamma]_q\Gamma_q(\beta)}|\rho_1|=\frac{\Gamma_q(\alpha+\beta)}{2q[\gamma]_q\Gamma_q(\beta)}|c_1L_1|\le\frac{\Gamma_q(\alpha+\beta)}{q[\gamma]_q\Gamma_q(\beta)}L_1=B_2, \tag{55}$$

where we used the inequality $|c_n|\le 2$ and equation (48). From this relation $|\rho_1|^2+|\rho_2|\le L_1^2+L_2$ (see [4]) and equation (49), we have

$$\begin{aligned}|b_3|&=\frac{\Gamma_q(2\alpha+\beta)\left|q\rho_2+\rho_1^2\right|}{q^2[\gamma]_q[\gamma+1]_q\Gamma_q(\beta)}\\&\le\frac{\Gamma_q(2\alpha+\beta)\left[q\left(|\rho_2|+|\rho_1|^2\right)+(1-q)|\rho_1|^2\right]}{q^2[\gamma]_q[\gamma+1]_q\Gamma_q(\beta)}\\&\le\frac{\Gamma_q(2\alpha+\beta)\left[q\left(L_2+L_1^2\right)+(1-q)L_1^2\right]}{q^2[\gamma]_q[\gamma+1]_q\Gamma_q(\beta)}\\&=\frac{\Gamma_q(2\alpha+\beta)\left[qL_2+L_1^2\right]}{q^2[\gamma]_q[\gamma+1]_q\Gamma_q(\beta)}=A3.\end{aligned} \tag{56}$$

So, Theorem 3 has been proven. □

Theorem 4. *If g given by (1) belongs to $\mathcal{S}\mathcal{T}^{\gamma}_{q,\alpha,\beta}(k,\eta)$, then*

$$|b_m|\le\frac{\left(\prod_{j=0}^{m-2}\left(L_1+q[j]_q\right)\right)\Gamma_q((m-1)\alpha+\beta)}{q^{m-1}\left(\prod_{j=0}^{m-2}[\gamma+j]_q\right)\Gamma_q(\beta)}\ (m\ge 2). \tag{57}$$

Proof. The result is clearly true for $m=2$. Let m be an integer with $m\ge 2$, and assume that the inequality is true for all $i\le m-1$. Making use of (47), we have

$$\begin{aligned}|b_m|&=\frac{(q;q)_{m-1}\Gamma_q(\alpha(m-1)+\beta)}{\left([m]_q-1\right)(q^{\gamma};q)_{m-1}\Gamma_q(\beta)}\left|\varrho_{m-1}+\sum_{i=2}^{m-1}\frac{(q^{\gamma};q)_{i-1}\Gamma_q(\beta)}{(q;q)_{i-1}\Gamma_q(\alpha(i-1)+\beta)}b_i\varrho_{m-i}\right|\\&\le\frac{\Gamma_q(\alpha(m-1)+\beta)\,[m-2]_q!}{q\Gamma_q(\beta)\left(\prod_{j=0}^{m-2}[\gamma+j]_q\right)}\left[L_1+\sum_{i=2}^{m-1}\frac{\left(\prod_{j=0}^{i-2}[\gamma+j]_q\right)\Gamma_q(\beta)}{[i-1]_q!\Gamma_q(\alpha(i-1)+\beta)}|b_i|L_1\right]\\&\le\frac{\Gamma_q(\alpha(m-1)+\beta)[m-2]_q!}{q\Gamma_q(\beta)\left(\prod_{j=0}^{m-2}[\gamma+j]_q\right)}L_1\\&\quad\cdot\left[1+\sum_{i=2}^{m-1}\frac{\left(\prod_{j=0}^{i-2}[\gamma+j]_q\right)\Gamma_q(\beta)}{[i-1]_q!\Gamma_q(\alpha(i-1)+\beta)}\frac{\left(\prod_{j=0}^{i-2}\left(L_1+q[j]_q\right)\right)\Gamma_q((i-1)\alpha+\beta)}{q^{i-1}\left(\prod_{j=0}^{i-2}[\gamma+j]_q\right)\Gamma_q(\beta)}\right],\end{aligned} \tag{58}$$

where we applied the induction hypothesis to $|b_m|$ and the Rogosinski result $|\varrho_2|\le L_1$ (see [42]). Therefore,

$$|b_m|\le\frac{\Gamma_q(\alpha(m-1)+\beta)[m-2]_q!}{q\Gamma_q(\beta)\prod_{j=0}^{m-2}[\gamma+j]_q}L_1\left[1+\sum_{i=2}^{m-1}\frac{\left(\prod_{j=0}^{i-2}\left(L_1+q[j]_q\right)\right)}{q^{i-1}[i-1]_q!}\right]. \tag{59}$$

Applying the principle of mathematical induction, we find that

$$1+\sum_{i=2}^{m-1}\frac{\left(\prod_{j=0}^{i-2}\left(L_1+q[j]_q\right)\right)}{q^{i-1}[i-1]_q!}=\frac{\prod_{j=1}^{m-2}\left(L_1+q[j]_q\right)}{q^{m-2}[m-2]_q!},\quad (60)$$

from which the inequality (57) follows. □

Similarly, we can prove the following.

Theorem 5. *If g of the form (1) belongs to the class $\mathscr{UCV}^{\gamma}_{q,\alpha,\beta}(k,\eta)$, then*

$$\begin{aligned}|b_2|&\le\frac{B_2}{[2]_q},\\ |b_3|&\le\frac{B_3}{[2]_q}.\end{aligned}\quad (61)$$

Theorem 6. *If g of the form (1) belongs to the class $\mathscr{UCV}^{\gamma}_{q,\alpha,\beta}(k,\eta)$, then*

$$|b_m|\le\frac{\left(\prod_{j=0}^{m-2}\left(L_1+q[j]_q\right)\right)\Gamma_q((m-1)\alpha+\beta)}{q^{m-1}[m]_q\left(\prod_{j=0}^{m-2}[\gamma+j]_q\right)\Gamma_q(\beta)}\,(m\ge 2).\quad (62)$$

Theorem 7. *Let $g\in\mathbb{A}$ be given by (1). If the inequality*

$$\sum_{m=2}^{\infty}\left[(k+1)[m]_q-k-\eta\right]\frac{(q^{\gamma};q)_{m-1}\Gamma_q(\beta)}{(q;q)_{m-1}\Gamma_q(\alpha(m-1)+\beta)}|b_m|<1-\eta,\quad (63)$$

holds true, then $g\in\mathscr{ST}^{\gamma}_{q,\alpha,\beta}(k,\eta)$.

Proof. Making use of the definition (30) it suffices to prove that

$$k\left|\frac{\zeta D_q\left(\mathscr{H}^{\gamma}_{q,\alpha,\beta}g(\zeta)\right)}{\mathscr{H}^{\gamma}_{q,\alpha,\beta}g(\zeta)}-1\right|-\Re\left\{\frac{\zeta D_q\left(\mathscr{H}^{\gamma}_{q,\alpha,\beta}g(\zeta)\right)}{\mathscr{H}^{\gamma}_{q,\alpha,\beta}g(\zeta)}-1\right\}<1-\eta.\quad (64)$$

Observe that

$$\begin{aligned}&k\left|\frac{\zeta D_q\left(\mathscr{H}^{\gamma}_{q,\alpha,\beta}g(\zeta)\right)}{\mathscr{H}^{\gamma}_{q,\alpha,\beta}g(\zeta)}-1\right|-\Re\left\{\frac{\zeta D_q\left(\mathscr{H}^{\gamma}_{q,\alpha,\beta}g(\zeta)\right)}{\mathscr{H}^{\gamma}_{q,\alpha,\beta}g(\zeta)}-1\right\}<(k+1)\left|\frac{\zeta D_q\left(\mathscr{H}^{\gamma}_{q,\alpha,\beta}g(\zeta)\right)}{\mathscr{H}^{\gamma}_{q,\alpha,\beta}g(\zeta)}-1\right|\\ &=(k+1)\left|\frac{\sum_{m=2}^{\infty}\left([m]_q-1\right)\left((q^{\gamma};q)_{m-1}\Gamma_q(\beta)/(q;q)_{m-1}\Gamma_q(\alpha(m-1)+\beta)\right)b_m\zeta^{m-1}}{1+\sum_{m=2}^{\infty}\left((q^{\gamma};q)_{m-1}\Gamma_q(\beta)/(q;q)_{m-1}\Gamma_q(\alpha(m-1)+\beta)\right)b_m\zeta^{m-1}}\right|\\ &\le(k+1)\frac{\sum_{m=2}^{\infty}\left([m]_q-1\right)\left((q^{\gamma};q)_{m-1}\Gamma_q(\beta)/(q;q)_{m-1}\Gamma_q(\alpha(m-1)+\beta)\right)|b_m|}{1-\sum_{m=2}^{\infty}\left((q^{\gamma};q)_{m-1}\Gamma_q(\beta)/(q;q)_{m-1}\Gamma_q(\alpha(m-1)+\beta)\right)|b_m|}.\end{aligned}\quad (65)$$

The last expression is bounded by $1-\eta$ if inequality (63) holds. □

Similarly, we can prove the following.

Theorem 8. *Let $g\in\mathbb{A}$ be given by (1). If the inequality*

$$\sum_{m=2}^{\infty}[m]_q\left[(k+1)[m]_q-k-\eta\right]\frac{(q^{\gamma};q)_{m-1}\,\Gamma_q(\beta)}{(q;q)_{m-1}\,\Gamma_q(\alpha(m-1)+\beta)}|b_m|<1-\eta,\quad (66)$$

holds true, then $g\in\mathscr{UCV}^{\gamma}_{q,\alpha,\beta}(k,\eta)$.

Now, we need the following lemmas.

Lemma 9 (see [38]). *If $\varphi(\zeta)=1+\varkappa_1\zeta+\varkappa_2\zeta^2+\cdots$ is a function with positive real part in $\mathbb{D}$ and ν is a complex number, then*

$$\left|\varkappa_2-\nu\varkappa_1^2\right|\le 2\max\{1,|2\nu-1|\}.\quad (67)$$

The result is sharp for the functions given by $\varphi(\zeta)=(1+\zeta^2)/(1-\zeta^2)$ or $\varphi(\zeta)=(1+\zeta)/(1-\zeta)$.

Lemma 10 (see [38]). *If $\varphi(\zeta) = 1 + \varkappa_1\zeta + \varkappa_2\zeta^2 + \cdots$ is an analytic function with a positive real part in $\mathbb{D}$, then*

$$\left|\varkappa_2 - \nu\varkappa_1^2\right| \le \begin{cases} -4\nu + 2 & \text{if } \nu \le 0, \\ 2 & \text{if } 0 \le \nu \le 1, \\ 4\nu - 2 & \text{if } \nu \ge 1, \end{cases} \quad (68)$$

when $\upsilon < 0$ or $\nu > 1$, the equality holds if and only if $\varphi(\zeta) = (1+\zeta)/(1-\zeta)$ or one of its rotations. If $0 < \nu < 1$, then the equality holds if and only if $\varphi(\zeta) = (1+\zeta^2)/(1-\zeta^2)$ or one of its rotations. If $\nu = 0$, the equality holds if and only if

$$\varphi(\zeta) = \left(\frac{1+\lambda}{2}\right)\frac{1+\zeta}{1-\zeta} + \left(\frac{1-\lambda}{2}\right)\frac{1-\zeta}{1+\zeta}(0 \le \lambda \le 1), \quad (69)$$

or one of its rotations. If $\nu = 1$, the equality holds if and only if φ is the reciprocal of one of the functions such that equality holds in the case of $\nu = 0$.

Also, the above upper bound is sharp, and it can be improved as follows when $0 < \nu < 1$:

$$\begin{aligned} &\left|\varkappa_2 - \nu\varkappa_1^2\right| + \nu|\varkappa_1|^2 \le 2\left(0 \le \nu \le \frac{1}{2}\right), \\ &\left|\varkappa_2 - \nu\varkappa_1^2\right| + (1-\nu)|\varkappa_1|^2 \le 2\left(\frac{1}{2} \le \nu \le 1\right). \end{aligned} \quad (70)$$

Theorem 11. *If g given by (1) belongs to $\mathcal{ST}^{\gamma}_{q,\alpha,\beta}(k,\eta)$, then*

$$\left|b_3 - \mu b_2^2\right| \le \frac{\Gamma_q(2\alpha+\beta)L_1}{q[\gamma]_q[\gamma+1]_q\Gamma_q(\beta)} \max \cdot \left\{1 ; \left|\frac{L_2}{L_1} + \frac{L_1}{q}\left(1 - \frac{[\gamma+1]_q\Gamma_q^2(\alpha+\beta)\mu}{[\gamma]_q\Gamma_q(\beta)\Gamma_q(2\alpha+\beta)}\right)\right|\right\}. \quad (71)$$

Proof. If $f \in \mathcal{ST}^{\gamma}_{q,\alpha,\beta}(k,\eta)$, we have

$$\frac{\zeta D_q\left(\mathcal{H}^{\gamma}_{q,\alpha,\beta}g(\zeta)\right)}{\mathcal{H}^{\gamma}_{q,\alpha,\beta}g(\zeta)} \prec \psi_{k,\eta}(\zeta), \quad (72)$$

where $\psi_{k,q}(z)$ is given by (39). From the definition of subordination, we have

$$\frac{\zeta D_q\left(\mathcal{H}^{\gamma}_{q,\alpha,\beta}g(\zeta)\right)}{\mathcal{H}^{\gamma}_{q,\alpha,\beta}g(\zeta)} = \psi_{k,\eta}(w(\zeta))(z \in \mathbb{D}), \quad (73)$$

where $w(\zeta)$ is a Schwarz function with $w(0) = 0$ and $|w(\zeta)| < 1$. Let $h(\zeta)$ be a function with positive real part in $\mathbb{D}$ defined by

$$h(z) = \frac{1+w(\zeta)}{1-w(\zeta)} = 1 + \varkappa_1\zeta + \varkappa_2\zeta^2 + \cdots (\zeta \in \mathbb{D}). \quad (74)$$

This gives

$$w(\zeta) = \frac{1}{2}\varkappa_1\zeta + \frac{1}{2}\left(\varkappa_2 - \frac{\varkappa_1^2}{2}\right)\zeta^2 + \cdots, \quad (75)$$

$$\psi_{k,\eta}(w(\zeta)) = 1 + \frac{1}{2}\varkappa_1 L_1\zeta + \left(\frac{1}{2}\varkappa_2 L_1 + \frac{1}{4}\varkappa_1^2(L_2 - L_1)\right)\zeta^2 + \cdots. \quad (76)$$

Using (76) in (73), we obtain

$$\begin{aligned} b_2 &= \frac{\Gamma_q(\alpha+\beta)}{2q[\gamma]_q\Gamma_q(\beta)}\varkappa_1 L_1, \\ b_3 &= \frac{\Gamma_q(2\alpha+\beta)}{2q[\gamma]_q[\gamma+1]_q\Gamma_q(\beta)}\left[\varkappa_2 L_1 + \frac{\varkappa_1^2(L_2 - L_1)}{2} + \frac{\varkappa_1^2 L_1^2}{2q}\right]. \end{aligned} \quad (77)$$

For any complex number μ, we have

$$b_3 - \mu b_2^2 = \frac{\Gamma_q(2\alpha+\beta)L_1}{2q[\gamma]_q[\gamma+1]_q\Gamma_q(\beta)}\left\{\varkappa_2 - \nu c_1^2\right\}, \quad (78)$$

where

$$\nu = \frac{1}{2}\left[1 - \frac{L_2}{L_1} - \frac{L_1}{q}\left(1 - \mu\frac{[\gamma+1]_q\Gamma_q^2(\alpha+\beta)}{[\gamma]_q\Gamma_q(\beta)\Gamma_q(2\alpha+\beta)}\right)\right]. \quad (79)$$

Our result now follows by an application of Lemma 9. This completes the proof of Theorem 11. □

Example 1. Taking $q \longrightarrow 1^-$, $\alpha = k = 0$, and $\gamma = 1$ in Theorem 11, we obtain the following result:

If g given by (1) satisfies the following inequality

$$\Re\left\{\frac{\zeta g'(\zeta)}{g(\zeta)}\right\} > \eta, \quad (80)$$

then

$$\left|b_3 - \mu b_2^2\right| \le (1-\eta)\max\{1 ; |1 + 2(1-\eta)(1-2\mu)|\}. \quad (81)$$

Similarly, we can prove the following theorem for the subclass $\mathcal{UCV}^{\gamma}_{q,\alpha,\beta}(k,\eta)$.

Theorem 12. *If g given by (1) belongs to $\mathscr{UCV}^{\gamma}_{q,\alpha,\beta}(k,\eta)$, then*

$$\left|b_3-\mu b_2^2\right| \le \frac{\Gamma_q(2\alpha+\beta)L_1}{q[3]_q[\gamma]_q[\gamma+1]_q\Gamma_q(\beta)}\max \cdot\left\{1;\left|\frac{L_2}{L_1}+\frac{L_1}{q}\left(1-\frac{[3]_q[\gamma+1]_q\Gamma_q^2(\alpha+\beta)\mu}{\left([2]_q\right)^2[\gamma]_q\Gamma_q(\beta)\Gamma_q(2\alpha+\beta)}\right)\right|\right\}. \tag{82}$$

Example 2. Taking $q\longrightarrow 1^-$, $\alpha=k=0$, and $\gamma=1$ in Theorem 12, we get the following result:

If g given by (1) satisfies the following inequality

$$\Re\left\{1+\frac{\zeta g''(\zeta)}{g'(\zeta)}\right\}>\eta, \tag{83}$$

then

$$\left|b_3-\mu b_2^2\right| \le \frac{(1-\eta)}{3}\max\left\{1;\left|1+2(1-\eta)\left(1-\frac{3}{2}\mu\right)\right|\right\}. \tag{84}$$

Theorem 13. *Let*

$$\begin{aligned}\sigma_1&=\frac{[\gamma]_q\Gamma_q(\beta)\Gamma_q(2\alpha+\beta)\left[L_1^2+q(L_2-L_1)\right]}{[\gamma+1]_q\Gamma_q^2(\alpha+\beta)L_1^2},\\ \sigma_2&=\frac{[\gamma]_q\Gamma_q(\beta)\Gamma_q(2\alpha+\beta)\left[L_1^2+q(L_2+L_1)\right]}{[\gamma+1]_q\Gamma_q^2(\alpha+\beta)L_1^2},\\ \sigma_3&=\frac{[\gamma]_q\Gamma_q(\beta)\Gamma_q(2\alpha+\beta)\left[L_1^2+qL_2\right]}{[\gamma+1]_q\Gamma_q^2(\alpha+\beta)L_1^2}.\end{aligned} \tag{85}$$

If g given by (1) belongs to the class $\mathscr{ST}^{\gamma}_{q,\alpha,\beta}(k,\eta)$, then

$$\left|b_3-\mu b_2^2\right| \le \begin{cases}\dfrac{\Gamma_q(2\alpha+\beta)}{q[\gamma]_q[\gamma+1]_q\Gamma_q(\beta)}\left[L_2+\dfrac{L_1^2}{q}\left(1-\dfrac{[\gamma+1]_q\Gamma_q^2(\alpha+\beta)\mu}{[\gamma]_q\Gamma_q(\beta)\Gamma_q(2\alpha+\beta)}\right)\right] & (\mu\le\sigma_1),\\ \dfrac{\Gamma_q(2\alpha+\beta)L_1}{q[\gamma]_q[\gamma+1]_q\Gamma_q(\beta)} & (\sigma_1\le\mu\le\sigma_2),\\ \dfrac{\Gamma_q(2\alpha+\beta)}{q[\gamma]_q[\gamma+1]_q\Gamma_q(\beta)}\left[L_2+\dfrac{L_1^2}{q}\left(1-\dfrac{[3]_q[\gamma+1]_q\Gamma_q^2(\alpha+\beta)\mu}{[\gamma]_q\Gamma_q(\beta)\Gamma_q(2\alpha+\beta)}\right)\right] & (\mu\ge\sigma_2).\end{cases} \tag{86}$$

Further, if $\sigma_1\le\mu\le\sigma_3$, then

$$\left|b_3-\mu b_2^2\right|+\frac{q[\gamma]_q\Gamma_q(\beta)\Gamma_q(2\alpha+\beta)}{[\gamma+1]_q\Gamma_q^2(\alpha+\beta)L_1^2}\cdot\left[L_1-L_2-\frac{L_1^2}{q}\left(1-\frac{[\gamma+1]_q\Gamma_q^2(\alpha+\beta)\mu}{[\gamma]_q\Gamma_q(\beta)\Gamma_q(2\alpha+\beta)}\right)\right]|b_2|^2 \le \frac{\Gamma_q(2\alpha+\beta)L_1}{q[\gamma]_q[\gamma+1]_q\Gamma_q(\beta)}. \tag{87}$$

If $\sigma_3\le\mu\le\sigma_2$, then

$$\left|b_3-\mu b_2^2\right|+\frac{q[\gamma]_q\Gamma_q(\beta)\Gamma_q(2\alpha+\beta)}{[\gamma+1]_q\Gamma_q^2(\alpha+\beta)L_1^2}\cdot\left[L_1+L_2+\frac{L_1^2}{q}\left(1-\frac{[\gamma+1]_q\Gamma_q^2(\alpha+\beta)\mu}{[\gamma]_q\Gamma_q(\beta)\Gamma_q(2\alpha+\beta)}\right)\right]|b_2|^2 \le \frac{\Gamma_q(2\alpha+\beta)L_1}{q[\gamma]_q[\gamma+1]_q\Gamma_q(\beta)}. \tag{88}$$

Proof. Applying Lemma 10 to (78) and (79), we can obtain our results asserted by Theorem 13. □

Similarly, we can prove the following theorem for the class $\mathscr{UCV}^{\gamma}_{q,\alpha,\beta}(k,\eta)$.

Theorem 14. *Let*

$$\begin{aligned}\sigma_4&=\frac{\left([2]_q\right)^2[\gamma]_q\Gamma_q(\beta)\Gamma_q(2\alpha+\beta)\left[L_1^2+q(L_2-L_1)\right]}{[3]_q[\gamma+1]_q\Gamma_q^2(\alpha+\beta)L_1^2},\\ \sigma_5&=\frac{\left([2]_q\right)^2[\gamma]_q\Gamma_q(\beta)\Gamma_q(2\alpha+\beta)\left[L_1^2+q(L_2+L_1)\right]}{[3]_q[\gamma+1]_q\Gamma_q^2(\alpha+\beta)L_1^2},\\ \sigma_6&=\frac{\left([2]_q\right)^2[\gamma]_q\Gamma_q(\beta)\Gamma_q(2\alpha+\beta)\left[L_1^2+qL_2\right]}{[3]_q[\gamma+1]_q\Gamma_q^2(\alpha+\beta)L_1^2}.\end{aligned} \tag{89}$$

If g given by (1) belongs to the class $\mathscr{UCV}^{\gamma}_{q,\alpha,\beta}(k,\eta)$, then

$$
\left|b_3-\mu b_2^2\right| \leq \begin{cases} \dfrac{\Gamma_q(2\alpha+\beta)L_1}{q[3]_q[\gamma]_q[\gamma+1]_q\Gamma_q(\beta)}\left[\dfrac{L_2}{L_1}+\dfrac{L_1}{q}\left(1-\dfrac{[3]_q[\gamma+1]_q\Gamma_q^2(\alpha+\beta)\mu}{\left([2]_q\right)^2[\gamma]_q\Gamma_q(\beta)\Gamma_q(2\alpha+\beta)}\right)\right] & (\mu\leq\sigma_4), \\ \dfrac{\Gamma_q(2\alpha+\beta)L_1}{q[3]_q[\gamma]_q[\gamma+1]_q\Gamma_q(\beta)} & (\sigma_4\leq\mu\leq\sigma_5), \\ \dfrac{\Gamma_q(2\alpha+\beta)L_1}{q[3]_q[\gamma]_q[\gamma+1]_q\Gamma_q(\beta)}\left[\dfrac{L_2}{L_1}+\dfrac{L_1}{q}\left(1-\dfrac{[3]_q[\gamma+1]_q\Gamma_q^2(\alpha+\beta)\mu}{\left([2]_q\right)^2[\gamma]_q\Gamma_q(\beta)\Gamma_q(2\alpha+\beta)}\right)\right] & (\mu\geq\sigma_5). \end{cases} \tag{90}
$$

Further, if $\sigma_4 \leq \mu \leq \sigma_6$, then

$$
\begin{aligned}
&\left|b_3-\mu b_2^2\right| + \frac{q\left([2]_q\right)^2[\gamma]_q\Gamma_q(\beta)\Gamma_q(2\alpha+\beta)}{[3]_q[\gamma+1]_q\Gamma_q^2(\alpha+\beta)L_1^2} \\
&\quad\cdot\left[L_1-L_2-\frac{L_1^2}{q}\left(1-\frac{[3]_q[\gamma+1]_q\Gamma_q^2(\alpha+\beta)\mu}{\left([2]_q\right)^2[\gamma]_q\Gamma_q(\beta)\Gamma_q(2\alpha+\beta)}\right)\right]|b_2|^2 \\
&\leq \frac{\Gamma_q(2\alpha+\beta)L_1}{q[3]_q[\gamma]_q[\gamma+1]_q\Gamma_q(\beta)}.
\end{aligned} \tag{91}
$$

If $\sigma_6 \leq \mu \leq \sigma_5$, then

$$
\begin{aligned}
&\left|b_3-\mu b_2^2\right| + \frac{q\left([2]_q\right)^2[\gamma]_q\Gamma_q(\beta)\Gamma_q(2\alpha+\beta)}{[3]_q[\gamma+1]_q\Gamma_q^2(\alpha+\beta)L_1^2} \\
&\quad\cdot\left[L_1+L_2+\frac{L_1^2}{q}\left(1-\frac{[3]_q[\gamma+1]_q\Gamma_q^2(\alpha+\beta)\,\mu}{\left([2]_q\right)^2[\gamma]_q\Gamma_q(\beta)\Gamma_q(2\alpha+\beta)}\right)\right]|b_2|^2 \\
&\leq \frac{\Gamma_q(2\alpha+\beta)L_1}{q\,[3]_q[\gamma]_q[\gamma+1]_q\Gamma_q(\beta)}.
\end{aligned} \tag{92}
$$

Remark 15. Putting $q \longrightarrow 1^-$ in the above results, we obtain the corresponding results for the classes $\mathcal{ST}^{\gamma}_{\alpha,\beta}(k,\eta)$ and $\mathcal{UCV}^{\gamma}_{\alpha,\beta}(k,\eta)$.

3. Conclusion

By using the concept of the basic (or q-) calculus, we have introduced two subclasses $\mathcal{ST}^{\gamma}_{q,\alpha,\beta}(k,\eta)$ and $\mathcal{UCV}^{\gamma}_{q,\alpha,\beta}(k,\eta)$ of normalized univalent functions which map the open unit disk $\mathbb{D}=\{\zeta\in\mathbb{C}:|\zeta|<1\}$ onto the generalized conic domain. We have obtained a number of important properties including the coefficient estimates, sufficient conditions, and the Fekete-Szegö inequalities for each of these classes. Our results are connected with those in earlier works which are related to the field of geometric function theory.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflict of interest.

Authors' Contributions

The authors contributed equally to the writing of this paper. All authors approved the final version of the manuscript.

Acknowledgments

This research has been funded by the Scientific Research Deanship at University of Ha'il, Saudi Arabia, through project number RG-21050.

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