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Research Article

Darbo Fixed Point Criterion on Solutions of a Hadamard Nonlinear Variable Order Problem and Ulam-Hyers-Rassias Stability

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The existence aspects along with the stability of solutions to a Hadamard variable order fractional boundary value problem are investigated in this research study. Our results are obtained via generalized intervals and piecewise constant functions and the relevant Green function, by converting the existing Hadamard variable order fractional boundary value problem to an equivalent standard Hadamard fractional boundary problem of the fractional constant order. Further, Darbo's fixed point criterion along with Kuratowski's measure of noncompactness is used in this direction. As well as, the Ulam-Hyers-Rassias stability of the proposed Hadamard variable order fractional boundary value problem is established. A numerical example is presented to express our results' validity.

1. Introduction

Fractional calculus is fundamentally established by having arbitrary numbers in the order of derivation operators instead of natural numbers. This idea is considered preliminary and simple. However, it involves remarkable effects and outcomes which describe some physical processes, dynamics, mathematical modelings, control theory, bioengineering, biomedical applications, etc. [1, 2]. The main effectiveness of this field can be found in recent studies. For example, Thabet et al. in [3] simulated a fractional model of pantograph in the Caputo conformable settings. In [4], Khan et al. designed a model of p-Laplacian FBVP in the form of a singular problem, and Matar et al. derived similar results for a new p-Laplacian model via generalized fractional derivative [5]. The fractional Langevin impulsive equations are studied by Rizwan et al. regarding existence property of solutions in [6], and Zada et al. [7] analyzed the Ulam-Hyers stability for an impulsive integro-differential equations. Etemad et al. used a new property entitled approximate endpoint for studying a novel fractional problem via the Caputo-Hadamard operators [8].

Thabet et al. also modeled COVID-19 transmission by Caputo-Fabrizio operators and analyzed its dynamical behavior [9]. In [10], Shah et al. compared the results of two classical and fractional models of COVID-19 and showed the accuracy of fractional operators in simulation of processes. Pratap et al. [11] studied finite-time Mittag-Leffler stability criteria for fractional quaternion-valued memristive neural networks. Along with these, Boulares et al. [12] conducted a theoretical research on the generalized weakly singular integral inequalities and their applications to generalized FBVPs. Naifar et al. [13] studied a global Mittag-Leffler stabilization by output feedback in relation to a class of nonlinear systems of FBVPs.

It is notable that in recent advanced mathematical models, constant fractional orders have not needed effectiveness for describing the specifications of some processes and phenomena, and consequently, some researchers had to model their boundary problems via the fractional operators equipped with orders as a real-valued functions. These operators are known as variable order ones [14, 15]. The investigation of the variable order fractional boundary problems (VOFBVPs) in the field of existence theory is considered as a new and important branch of fractional calculus, which are published limited research works in this regard. In 2018, Yang et al. presented a numerical scheme for a VOFBVP and analyzed their system from numerical point of view [16]. Zhang studied the solutions of a singular two-point VOFBVP for the first time in [17]. In recent years, Zhang et al. [18, 19] derived approximate solutions of two different VOFIVP on the half-axis in 2018 and 2019. In addition, a multiterm FBVP involving the nonlinear fractional differential equation (NFDE) of the variable order type was investigated in detail by Bouazza et al. in [20]. In this paper, motivated by other related works in this regard, we investigate the solutions' existence of the Hadamard nonlinear VOFBVP as follows:

$$ {}^{\mathscr{H}}\mathfrak{D}_{1^+}^{\vartheta(t)} r(t) + m_1(t, r(t)) = 0, t \in \mathscr{U} := [1, \mathscr{T}], \tag{1} $$

via boundary conditions $r(1) = r(\mathscr{T}) = 0$, where $1 < \mathscr{T} < +\infty$, $1 < \vartheta(t) \le 2$, and $m_1 : \mathscr{U} \times \mathscr{S} \longrightarrow \mathscr{S}$.

$\mathscr{S}$ is a continuous function (S is a real (or complex) space) and ${}^{\mathscr{H}}\mathfrak{D}_{1^+}^{\vartheta(t)}$ specifies the Hadamard derivative of variable order $\psi(t)$. For the first time, as the novelty of this research, we here consider a FBVP in the variable order Hadamard settings and establish the existence specifications of solutions to mentioned system on the generalized subintervals by combining the existing notions in relation to the Kuratowski's measure of noncompactness (KMNCS) in the context of Darbo fixed point criterion. The piecewise constant functions will play a vital role in our study for converting the Hadamard VOFBVP (1) to the standard Hadamard FBVP. Lastly, another criterion of the behavior of solutions like the Ulam-Hyers-Rassias stability (UHRS) is analyzed, and a numerical illustrative example will complete the consistency of our findings.

2. Essential Preliminaries

Basic definitions are discussed in this section to be used later. Throughout the paper, the set S stands for the real numbers.

The symbol $C(U, S)$ denotes a set that contains all continuous functions $f : \mathscr{U} \longrightarrow \mathscr{S}$. It is a Banach space by defining

$$ \|f\| = \sup \{ |f(t)|: t \in \mathscr{U} \}, \quad \mathscr{U} := [1, \mathscr{T}]. \tag{2} $$

Definition 1 (see [21, 22]). For $1 \le a_1 < a_2 < +\infty$, we consider the mappings $h_1(t)$: $[a_1, a_2] \longrightarrow (0,+\infty)$ and $q(t)$: $[a_1, a_2] \longrightarrow (n-1, n)$. The Hadamard $(\vartheta(t))^{\text{th}}$ variable order integral of h_1 is

$$ {}^{\mathscr{H}}\mathscr{I}_{a_1^+}^{\vartheta(t)} h_1(t) = \frac{1}{\Gamma(\vartheta(t))} \int_{a_1}^{t} \left(\log \frac{t}{s} \right)^{\vartheta(t)-1} \frac{h_1(s)}{s} ds, \quad t > a_1, \tag{3} $$

and the Hadamard $(q(t))^{\text{th}}$ variable order derivative of h_1 is

$$ \left({}^{\mathscr{H}}\mathfrak{D}_{a_1^+}^{q(t)} h_1 \right)(t) = \frac{1}{\Gamma(n - q(t))} \left(t \frac{d}{dt} \right)^n \cdot \int_{a_1}^{t} \left(\log \frac{t}{s} \right)^{n-q(t)-1} \frac{h_1(s)}{s} ds, \quad t > a_1. \tag{4} $$

Obviously, in case of $\vartheta(t)$ and $q(t)$ are constant, then both above Hadamard variable order operators are in coincidence with the usual Hadamard constant order operators (refer to [1, 21, 22]).

Lemma 2 (see [1]). *Assume that $a_1 > 1$, $\gamma_1, \gamma_2 > 0$, $h_1 \in L(a_1, a_2)$, and ${}^{\mathscr{H}}\mathfrak{D}_{a_1^+}^{\gamma_1} h_1 \in L(a_1, a_2)$. Then, the homogeneous differential equation*

$$ {}^{\mathscr{H}}\mathfrak{D}_{a_1^+}^{\gamma_1} h_1 = 0 \tag{5} $$

admits the unique solution

$$ \begin{aligned} h_1(t) &= \omega_1 \left(\log \frac{t}{a_1} \right)^{\gamma_1 - 1} + \omega_2 \left(\log \frac{t}{a_1} \right)^{\gamma_1 - 2} + \cdots + \omega_n \left(\log \frac{t}{a_1} \right)^{\gamma_1 - n}, \\ {}^{\mathscr{H}}\mathscr{I}_{a_1^+}^{\gamma_1} \left({}^{H}D_{a_1^+}^{\gamma_1} \right) h_1(t) &= h_1(t) + \omega_1 \left(\log \frac{t}{a_1} \right)^{\gamma_1 - 1} + \omega_2 \left(\log \frac{t}{a_1} \right)^{\gamma_1 - 2} + \cdots + \omega_n \left(\log \frac{t}{a_1} \right)^{\gamma_1 - n}, \end{aligned} \tag{6} $$

with $n = [\gamma_1] + 1$, $\omega_j \in \mathbb{R}$, and $j = 1, 2, \cdots, n$.

Moreover, for constants $\alpha_j > 0, j = 1, 2$,

$$ \begin{gathered} {}^{\mathscr{H}}\mathfrak{D}_{a_1^+}^{\alpha_1} \left({}^{\mathscr{H}}\mathscr{I}_{a_1^+}^{\alpha_1} \right) h_1(t) = h_1(t), \\ {}^{\mathscr{H}}\mathscr{I}_{a_1^+}^{\alpha_1} \left({}^{\mathscr{H}}\mathscr{I}_{a_1^+}^{\alpha_2} \right) h_1(t) = {}^{\mathscr{H}}\mathscr{I}_{a_1^+}^{\alpha_2} \left({}^{\mathscr{H}}\mathscr{I}_{a_1^+}^{\alpha_1} \right) h_1(t) = {}^{\mathscr{H}}\mathscr{I}_{a_1^+}^{\alpha_1 + \alpha_1} h_1(t). \end{gathered} \tag{7} $$

Remark 3. The semigroup property is not fulfilled for the functions $\vartheta(t)$ and $q(t)$, i.e.,

$$ {}^{\mathscr{H}}\mathscr{I}_{a_1^+}^{\vartheta(t)} \left({}^{\mathscr{H}}\mathscr{I}_{a_1^+}^{q(t)} \right) h_1(t) \ne {}^{\mathscr{H}}\mathscr{I}_{a_1^+}^{\vartheta(t)+q(t)} h_1(t). \tag{8} $$

Example 1. Let

$$\vartheta(t)=\begin{cases}1, & t\in[1,2],\\ 2, & t\in]2,4],\end{cases}$$
$$q(t)=\begin{cases}3, & t\in[1,2],\\ 4, & t\in]2,4],\end{cases} \tag{9}$$
$$h_1(t)=2t^2,$$
$$t\in[1,4].$$

We obtain

$$\begin{aligned}{}^{\mathscr{H}}\mathscr{I}_{1^+}^{\vartheta(t)}\left({}^{\mathscr{H}}\mathscr{I}_{1^+}^{q(t)}\right)h_1(t) &= \frac{1}{\Gamma(\vartheta(t))}\int_1^t \frac{1}{s}\left(\log\frac{t}{s}\right)^{\Psi(t)-1}\\
&\cdot\left[\frac{1}{\Gamma(q(s))}\int_1^s\left(\log\frac{s}{\tau}\right)^{q(s)-1}\frac{h_1(\tau)}{\tau}d\tau\right]\\
&\cdot ds=\frac{1}{\Gamma(\Psi(t))}\int_1^2\frac{1}{s}\left(\log\frac{t}{s}\right)^{\Psi(t)-1}\left[\frac{1}{\Gamma(q(s))}\int_1^s\left(\log\frac{s}{\tau}\right)^{q(s)-1}\frac{h_1(\tau)}{\tau}d\tau\right]\\
&\cdot ds+\frac{1}{\Gamma(\vartheta(t))}\int_2^t\frac{1}{s}\left(\log\frac{t}{s}\right)^{\vartheta(t)-1}\left[\frac{1}{\Gamma(q(s))}\int_1^s\left(\log\frac{s}{\tau}\right)^{q(s)-1}\frac{h_1(\tau)}{\tau}d\tau\right]\\
&\cdot ds=\frac{1}{\Gamma(1)}\int_1^2\frac{1}{s}\left(\log\frac{t}{s}\right)^0\int_1^s\frac{1}{\Gamma(3)}\left(\log\frac{s}{\tau}\right)^2 2\tau d\tau ds+\frac{1}{\Gamma(2)}\int_2^t\frac{1}{s}\left(\log\frac{t}{s}\right)\\
&\cdot\left[\frac{1}{\Gamma(3)}\int_1^2\left(\log\frac{s}{\tau}\right)^2 2\tau d\tau+\frac{1}{\Gamma(4)}\int_2^s\left(\log\frac{s}{\tau}\right)^3 2\tau d\tau\right]ds,\\
&=\int_1^2\left(\frac{s}{4}-\frac{1}{2s}(\log s)^2-\frac{1}{2s}(\log s)-\frac{1}{4s}\right)ds+\int_2^t\frac{1}{s}\left(\log\frac{t}{s}\right)\\
&\cdot\left[-\frac{2}{3}\left(\log\frac{s}{2}\right)^3+\left(\log\frac{s}{2}\right)^2+\left(\log\frac{s}{2}\right)-\frac{1}{2}(\log s)^2-\frac{1}{2}(\log s)+\frac{1}{8}s^2+\frac{1}{4}\right]ds,\end{aligned} \tag{10}$$

$${}^{\mathscr{H}}\mathscr{I}_{1^+}^{\vartheta(t)+q(t)}h_1(t)=\frac{1}{\Gamma(\vartheta(t)+q(t))}\int_1^t\left(\log\frac{t}{s}\right)^{\vartheta(t)+q(t)-1}\frac{h_1(s)}{s}ds. \tag{11}$$

So,

$$\begin{aligned}{}^{\mathscr{H}}\mathscr{I}_{1^+}^{\vartheta(t)}\left({}^{\mathscr{H}}\mathscr{I}_{1^+}^{q(t)}\right)h_1(t)|_{t=3} &= -\frac{1}{30}\left(\log\frac{3}{2}\right)^5+\frac{1}{24}\left(\log\frac{3}{2}\right)^4\\
&+\frac{1}{12}\left(\log\frac{3}{2}\right)^3+\frac{1}{8}\left(\log\frac{3}{2}\right)^2-\frac{1}{4}\left(\log\frac{3}{2}\right)\\
&-\frac{1}{6}(\log 2)^2\left(\log\frac{3}{2}\right)^2-\frac{1}{6}(\log 2)\left(\log\frac{3}{2}\right)^3-\frac{(\log 2)^3}{6}\\
&-\frac{(\log 2)^2}{4}-\frac{\log 2}{4}-\frac{1}{4}(\log 2)\left(\log\frac{3}{2}\right)^2+\frac{17}{32}\simeq 0.0522.\end{aligned} \tag{12}$$

On the other side,

$$\begin{aligned}{}^{\mathscr{H}}\mathscr{I}_{1^+}^{\vartheta(t)+q(t)}h_1(t)|_{t=3} &= \int_1^2\frac{1}{\Gamma(4)}\left(\log\frac{3}{s}\right)^3 2sds+\int_2^3\frac{1}{\Gamma(6)}\left(\log\frac{3}{s}\right)^5\\
&\cdot 2sds=-\frac{1}{30}\left(\log\frac{3}{2}\right)^5-\frac{1}{12}\left(\log\frac{3}{2}\right)^4+\frac{1}{3}\left(\log\frac{3}{2}\right)^3+\frac{3}{4}\left(\log\frac{3}{2}\right)^2\\
&+\frac{3}{4}\left(\log\frac{3}{2}\right)-\frac{1}{6}(\log 3)^3-\frac{1}{4}(\log 3)^2-\frac{1}{4}(\log 3)+\frac{17}{32}\simeq 0.1809.\end{aligned} \tag{13}$$

Therefore, we obtain

$${}^{\mathscr{H}}\mathscr{I}_{1^+}^{\vartheta(t)}\left({}^{\mathscr{H}}\mathscr{I}_{1^+}^{q(t)}\right)h_1(t)|_{t=3}\neq{}^{\mathscr{H}}\mathscr{I}_{1^+}^{\vartheta(t)+q(t)}h_1(t)|_{t=3}. \tag{14}$$

Lemma 4 (see [23, 24]). *If $\vartheta:\mathscr{U}\longrightarrow(1,2]$ has the continuity property, then for*

$$\begin{aligned}&h_1\in\mathscr{C}_\beta(\mathscr{U},\mathscr{S})\\
&=\left\{h_1(t)\in\mathscr{C}(\mathscr{U},\mathscr{S}),(\log t)^\beta h_1(t)\in\mathscr{C}(\mathscr{U},\mathscr{S})\right\},\quad 0\le\beta\le 1,\end{aligned} \tag{15}$$

the integral ${}^{\mathscr{H}}\mathscr{I}_{1^+}^{\vartheta(t)}h_1(t)$ admits a finite value $\forall t\in\mathscr{U}$.

Lemma 5 (see [23, 24]). *Assume that $\vartheta:\mathscr{U}\longrightarrow(1,2]$ has the continuity property. Then,*

$${}^{\mathscr{H}}\mathscr{I}_{1^+}^{\vartheta(t)}h_1(t)\in\mathscr{C}(\mathscr{U},\mathscr{S})\text{for } h_1\in\mathscr{C}(\mathscr{U},\mathscr{S}). \tag{16}$$

Definition 6 (see [25–27]). $I\subseteq\mathbb{R}$ is termed as a generalized interval if I is either an interval, or $\{a_1\}$, or $\varnothing$. A finite set $\mathscr{F}$ is defined to be a partition of I if every $x\in I$ belongs to exactly one and one generalized interval $\mathbb{I}$ in $\mathscr{F}$. Finally, $w:I\longrightarrow\mathscr{S}$ is piecewise constant w.r.t $\mathscr{F}$ as a partition of I; if $\forall\mathbb{I}\in\mathscr{F}$, w is constant on $\mathbb{I}$.

2.1. Some Properties regarding KMNCS. Here, we regard $\mathscr{S}$ as a Banach space.

Definition 7 (see [28]). Suppose that $\omega_{\mathscr{S}}$ is a bounded set in $\mathscr{S}$. The KMNCS is the function $\Phi:\omega_{\mathscr{S}}\longrightarrow[0,\infty]$ as

$$\Phi(\mathfrak{P})=\inf\left\{\delta>0:\mathfrak{P}\subseteq\cup_{=1}^{n}\mathfrak{P}_j,\boldsymbol{Diam}\left(\mathfrak{P}_j\right)\le\delta,\left(\mathfrak{P}\in\omega_{\mathscr{S}}\right)\right\}, \tag{17}$$

in which

$$\boldsymbol{Diam}\left(\mathfrak{P}_j\right)=\sup\left\{\|x-r\|:x,r\in\mathfrak{P}_j\right\}. \tag{18}$$

The symbol Diam denotes the diameter of the given set. Some valid properties KMNCS are as follows.

Proposition 8 (see [28, 29]). *Let $\mathfrak{P},\mathfrak{P}_1,\mathfrak{P}_2$ be bounded in $\mathscr{S}$. Then,*

(a) *$\mathfrak{P}$ is relatively compact if $\Phi(\mathfrak{P})=0$*

(b) $\Phi(\varnothing)=0$

(c) $\Phi(\mathfrak{P})=\Phi(\bar{\mathfrak{P}})$

(d) $\mathfrak{P}_1\subset\mathfrak{P}_2\Rightarrow\Phi(\mathfrak{P}_1)\le\Phi(\mathfrak{P}_2)$

(e) $\Phi(\mathfrak{P}_1+\mathfrak{P}_2)\le\Phi(\mathfrak{P}_1)+\Phi(\mathfrak{P}_2)$

(f) $\Phi(\lambda\mathfrak{P})=|\lambda|\Phi(\mathfrak{P}),\lambda\in\mathbb{R}$

(g) $\Phi(\mathfrak{P}_1\cup\mathfrak{P}_2)=\max\{\Phi(\mathfrak{P}_1),\Phi(\mathfrak{P}_2)\}$

(h) $\Phi(\mathfrak{P}_1\cap\mathfrak{P}_2)=\min\{\Phi(\mathfrak{P}_1),\Phi(\mathfrak{P}_2)\}$

(i) $\Phi(\mathfrak{P} + x_0) = \Phi(\mathfrak{P})$ *for any* $x_0 \in \mathbb{R}$

Lemma 9 (see [30]). *If the bounded set* $\mathscr{W} \subset \mathscr{C}(\mathscr{U}, \mathscr{S})$ *is equicontinuous, then*

(i) $\Phi(\mathscr{W})$ *has continuity, and*

$$\widehat{\Phi}(\mathscr{W}) = \sup_{t\in\mathscr{U}} \Phi(\mathscr{W}(t)) \tag{19}$$

(ii) $\Phi(\int_0^{\mathscr{T}} r(\theta)d\theta : r \in \mathscr{W}) \leq \int_0^{\mathscr{T}} \Phi(\mathscr{W}(\theta))d\theta,$

where

$$\mathscr{W}(\theta) = \{r(\theta): r \in \mathscr{W}\}, \quad \theta \in \mathscr{U} \tag{20}$$

In the next theorem, we point out the Darbo's fixed point criterion.

Theorem 10 (see [28]). *Consider the closed, convex, and bounded set* $\Lambda \neq \varnothing$ *in* $\mathscr{S}$ *and the continuous map* $\mathrm{F} : \Lambda \longrightarrow \Lambda$ *satisfying (k-set contractive property for F).*

$$\Phi(F(V)) \leq k\Phi(V), \forall \varnothing \neq V \subset \Lambda, k \in [0, 1). \tag{21}$$

Then, F *admits at least a fixed point belonging to* Λ.

3. Existence Criterion of Solutions

To achieve the main purpose of this section, some assumptions are proposed as:

(H1) Consider $\mathscr{F} = \{\mathscr{U}_1 \coloneqq [1, \mathscr{T}_1], \mathscr{U}_2 \coloneqq (\mathscr{T}_1, \mathscr{T}_2], \mathscr{U}_3 \coloneqq (\mathscr{T}_2, \mathscr{T}_3], \cdots, \mathscr{U}_n \coloneqq (\mathscr{T}_{n-1}, \mathscr{T}]\}$ as a partition for the interval $\mathscr{U}$ and $\vartheta(t)$: $\mathscr{U} \longrightarrow (1, 2]$ as a piecewise constant function w.r.t $\mathscr{F}$, i.e.,

$$\vartheta(t) = \sum_{i=1}^{n} \vartheta_\jmath \mathscr{I}_\jmath(t) = \begin{cases} \vartheta_1, & \text{if } t \in \mathscr{U}_1, \quad 1 < \vartheta_1 \leq 2, \\ \vartheta_2, & \text{if } t \in \mathscr{U}_2, \quad 1 < \vartheta_2 \leq 2, \\ . & \\ . & \\ . & \\ \vartheta_n, & \text{if } t \in \mathscr{U}_n, \quad 1 < \vartheta_n \leq 2, \end{cases} \tag{22}$$

in which $J_\jmath$ interprets the indicator of $\mathscr{I}_\jmath$ interprets the indicator of $\mathscr{U}_\jmath \coloneqq (\mathscr{T}_{\jmath-1}, \mathscr{T}_\jmath], \in\mathbb{N}_1^n$, so that $\mathscr{T}_0 = 1$ and $\mathscr{T}_n = \mathscr{T}$, and

$$\mathscr{I}_\jmath(t) = \begin{cases} 1, & \text{for } t \in \mathscr{U}_\jmath, \\ 0, & \text{for } \textbf{\textit{elsewhere}}. \end{cases} \tag{23}$$

(H2) Let $(\log t)^\beta m_1 : \mathscr{U} \times \mathscr{S} \longrightarrow \mathscr{S}$ be continuous, $(0 \leq \beta \leq 1)$, and $\exists K > 0$, such that $(\log t)^\beta \| m_1(t, r) - m_1(t, \bar{r}) \| \leq K \| r - \bar{r} \|$, for any r, $\bar{r} \in \mathscr{S}$ and $t \in \mathscr{U}$.

Remark 11 (see [31]). Note that the inequality

$$\Phi\left((\log t)^\beta \| m_1(t, B_1) \| \right) \leq K\Phi(B_1) \tag{24}$$

is equivalent to (H2) for each $B_1 \subset \mathscr{S}$ and $t \in \mathscr{U}$, where B_1 is bounded.

Further, for a supposed set $\mathscr{W}$ of all mappings $w : \mathscr{U} \longrightarrow \mathscr{S}$, define

$$\mathscr{W}(t) = \{w(t), w \in \mathscr{W}\}, \quad t \in \mathscr{U}, \tag{25}$$

$\mathscr{W}(t) = \{w(t): w \in \mathscr{W}, t \in \mathscr{U}\}$.

Let us now establish the solutions' existence for the Hadamard VOFBVP (1) via KMNCS and Darbo's criterion (Theorem 10).

Here, $\forall \jmath \in \{1, 2, \cdots, n\}$, the symbol $\mathscr{E}_\jmath = \mathscr{C}(\mathscr{U}_\jmath, r)$, indicated as Banach spaces of continuous mappings $r : \mathscr{U}_\jmath \longrightarrow \mathscr{S}$ is furnished with the norm

$$\|r\|_{\mathscr{E}_\jmath} = \sup_{t\in\mathscr{U}_\jmath} |r(t)|, \tag{26}$$

where $\jmath \in \{1, 2, \cdots, n\}$.

First, we analyze the Hadamard VOFBVP defined in (1). In the light of (4), the Hadamard VOFBVP (1) can be rewritten by

$$\frac{1}{\Gamma(2-\vartheta(t))} \left(t\frac{d}{dt}\right)^2 \int_1^t \left(\log\frac{t}{s}\right)^{1-\vartheta(t)} \frac{r(s)}{s} ds + m_1(t, r(t)) = 0, \quad t \in \mathscr{U}. \tag{27}$$

From (H1), equation (27) on the interval $\mathscr{U}_\jmath, \in\mathbb{N}_1^n$, can be expressed as

$$\left(t\frac{d}{dt}\right)^2 \left(\frac{1}{\Gamma(2-\vartheta_1)} \int_1^{\mathscr{T}_1} \left(\log\frac{t}{s}\right)^{1-\vartheta_1} \frac{r(s)}{s} ds + \cdots + \frac{1}{\Gamma(2-\vartheta_\jmath)} \cdot \int_{\mathscr{T}_{\jmath-1}}^t \left(\log\frac{t}{s}\right)^{1-\vartheta_\jmath} \frac{r(s)}{s} ds\right) + m_1(t, r(t)) = 0, \quad t \in \mathscr{U}_\jmath. \tag{28}$$

Definition 12. The Hadamard VOFBVP (1) admits a solution like functions $r_\jmath, \jmath = 1, 2, \cdots, n$, if $r_\jmath \in \mathscr{C}([1, \mathscr{T}_\jmath], \mathscr{S})$ satisfies equation (28) and $r_\jmath(1) = 0 = r_\jmath(\mathscr{T}_\jmath)$.

From the above, the Hadamard VOFBVP (1) written in (27) can be given as (28) on $\mathscr{U}_\jmath, \in\mathbb{N}_1^n$. For $1 \leq t \leq \mathscr{T}_{\jmath-1}$, put $r(t) \equiv 0$. Then, (28) is formulated by

$$D_{\mathscr{T}_{\jmath-1}^+}^{\vartheta_\jmath} r(t) + m_1(t, r(t)) = 0, t \in \mathscr{U}_\jmath. \tag{29}$$

In this case, we follow our study by considering the standard Hadamard constant-order FBVP (COFBVP) as follows:

$$\begin{cases} {}^{\mathscr{H}}\mathfrak{D}^{\vartheta_\jmath}_{\mathscr{T}^+_{\jmath-1}} r(t) + m_1(t, r(t)), & t \in \mathscr{U}_\jmath, \\ r(\mathscr{T}_{\jmath-1}) = 0, & r(\mathscr{T}_\jmath) = 0. \end{cases} \quad (30)$$

The fundamental part of our analysis regarding solutions of the Hadamard COFBVP (30) is discussed below.

Lemma 13. *A function $r \in \mathscr{E}_\jmath$ is a solution of the Hadamard COFBVP (30) if r fulfills the integral equation*

$$r(t) = \int_{\mathscr{T}_{\jmath-1}}^{\mathscr{T}_\jmath} \frac{1}{s} \mathbb{G}_\jmath(t, s) m_1(s, r(s)) ds, \ t \in \mathscr{U}_\jmath, \quad (31)$$

where $\mathbb{G}_\jmath(t, s)$ stands for the Green function formulated by

$$\mathbb{G}_\jmath(t,s) = \frac{1}{\Gamma(\vartheta_\jmath)} \begin{cases} \left(\log \frac{\mathscr{T}_\jmath}{\mathscr{T}_{\jmath-1}}\right)^{1-\vartheta_\jmath} \left[\left(\log \frac{t}{\mathscr{T}_{\jmath-1}}\right)\left(\log \frac{\mathscr{T}_\jmath}{s}\right)\right]^{\vartheta_\jmath-1} - \left(\log \frac{t}{s}\right)^{\vartheta_\jmath-1}, & \mathscr{T}_{-1} \le s \le t \le \mathscr{T}_\jmath, \\ \left(\log \frac{\mathscr{T}_\jmath}{\mathscr{T}_{\jmath-1}}\right)^{1-\vartheta_\jmath} \left[\left(\log \frac{t}{\mathscr{T}_{\jmath-1}}\right)\left(\log \frac{\mathscr{T}_\jmath}{s}\right)\right]^{\vartheta_\jmath-1}, & \mathscr{T}_{-1} \le t \le s \le \mathscr{T}_\jmath, \end{cases} \quad (32)$$

where $\jmath \in \mathbb{N}_1^n$.

Proof. Suppose that $r \in \mathscr{E}_\jmath$ satisfies the Hadamard COFBVP (30). Let us employ the operator ${}^{\mathscr{H}}\mathscr{I}^{\psi_\jmath}_{\mathscr{T}^+_{\jmath-1}}$ on both sides (30) and using Lemma 2, we get

$$r(t) = \omega_1 \left(\log \frac{t}{\tau_{j-1}}\right)^{\vartheta_j-1} + \omega_2 \left(\log \frac{t}{\tau_{j-1}}\right)^{\vartheta_j-2} {}^{\mathscr{H}}\mathscr{I}^{\vartheta_j}_{\tau^+_{j-1}} m_1(t, r(t)), \quad t \in u_j, j \in \mathbb{N}_1^n. \quad (33)$$

From definition of m_1 along with $r(\mathscr{T}_{\jmath-1}) = 0$, we get $\omega_2 = 0$.

Suppose that r satisfies $r(\mathscr{T}_\jmath) = 0$. Hence,

$$\omega_1 = \left(\log \frac{\mathscr{T}_\jmath}{\mathscr{T}_{\jmath-1}}\right)^{1-\vartheta_\jmath} {}^{\mathscr{H}}_{\jmath}\mathscr{I}^{\vartheta_\jmath}_{\mathscr{T}^+_{\jmath-1}} m_1(\mathscr{T}_\jmath, r(\mathscr{T}_\jmath)). \quad (34)$$

Thus,

$$r(t) = \left(\log \frac{\mathscr{T}_\jmath}{\mathscr{T}_{\jmath-1}}\right)^{1-\vartheta_\jmath} \left(\log \frac{t}{\mathscr{T}_{\jmath-1}}\right)^{\vartheta_\jmath-1\mathscr{H}} \cdot \mathscr{I}^{\vartheta_\jmath}_{\mathscr{T}^+_{\jmath-1}} m_1(\mathscr{T}_\jmath, r(\mathscr{T}_\jmath)) - {}^{\mathscr{H}}\mathscr{I}^{\vartheta_\jmath}_{\mathscr{T}^+_{\jmath-1}} m_1(t, r(t)), \quad (35)$$

Then, the solution of the Hadamard COFBVP (30) is given by

$$\begin{aligned} r(t) &= \left(\log \frac{\mathscr{T}_\jmath}{\mathscr{T}_{\jmath-1}}\right)^{1-\vartheta_\jmath} \left(\log \frac{t}{\mathscr{T}_{\jmath-1}}\right)^{\vartheta_\jmath-1} \frac{1}{\Gamma(\vartheta_\jmath)} \\ &\cdot \int_{\mathscr{T}_{\jmath-1}}^{\mathscr{T}_\jmath} \left(\log \frac{\mathscr{T}_\jmath}{s}\right)^{\vartheta_\jmath-1} \frac{m_1(s, r(s))}{s} \\ &\cdot ds - \frac{1}{\Gamma(\vartheta_\jmath)} \int_{\mathscr{T}_{\jmath-1}}^{t} \left(\log \frac{t}{s}\right)^{\vartheta_\jmath-1} \frac{m_1(s, r(s))}{s} \\ &\cdot ds = \frac{1}{\Gamma(\vartheta_\jmath)} \left[\int_{\mathscr{T}_{\jmath-1}}^{t} \left[\left(\log \frac{\mathscr{T}_\jmath}{\mathscr{T}_{\jmath-1}}\right)^{1-\vartheta_\jmath} \left(\log \frac{t}{\mathscr{T}_{\jmath-1}}\right)^{\vartheta_\imath-1} \right.\right. \\ &\cdot \left.\left(\log \frac{\mathscr{T}_\jmath}{s}\right)^{\vartheta_\jmath-1} - \left(\log \frac{t}{s}\right)^{\vartheta_\jmath-1}\right] \frac{m_1(s, r(s))}{s} \\ &\cdot ds + \left.\int_t^{\mathscr{T}_\jmath} \left(\log \frac{\mathscr{T}_\jmath}{\mathscr{T}_{\jmath-1}}\right)^{1-\vartheta_\jmath} \left(\log \frac{t}{\mathscr{T}_{\jmath-1}}\right)^{\vartheta_\jmath-1} \left(\log \frac{\mathscr{T}_\jmath}{s}\right)^{\vartheta_\jmath-1} \frac{m_1(s, r(s))}{s} ds\right], \end{aligned} \quad (36)$$

and the continuity property of the Green function gives

$$r(t) = \int_{\mathscr{T}_{\jmath-1}}^{\mathscr{T}_\jmath} \frac{1}{s} \mathbb{G}_\jmath(t, s) m_1(s, r(s)) ds, \quad t \in \mathscr{U}_\jmath. \quad (37)$$

Conversely, let $r \in \mathscr{E}_\jmath$ be the integral equation's (31) solution. Because of the continuity of $(\log t)^\beta m_1$ and by Lemma 2, it is simply verified that r satisfies the Hadamard COFBVP (30) solution. □

Proposition 14. *Assume that $(\log t)^\beta m_1$, $(0 \le \beta \le 1)$ belongs to $\mathscr{C}(\mathscr{U} \times \mathscr{S}, \mathscr{S})$ and $\vartheta(t)$: $\mathscr{U} \longrightarrow (1, 2]$ satisfies (H1). Then, $\mathbb{G}_\jmath(t, s)$ given by (32) satisfy the following: ($\jmath \in \mathbb{N}_1^n$)*

(1) $0 \le \mathbb{G}_\jmath(t, s), \forall \mathscr{T}_{\jmath-1} \le t, s \le \mathscr{T}_\jmath$

(2) $\max_{t \in \mathscr{U}_\jmath} \mathbb{G}_\jmath(t, s) = \mathbb{G}_\jmath(s, s), s \in \mathscr{U}_\jmath$

(3) $\mathbb{G}_\jmath(s, s)$ has a maximum value uniquely given by

$$\max_{s\in\mathcal{U}_j}\mathbb{G}_j(s,s)=\frac{1}{\Gamma(\vartheta_j)}\left(\frac{\log\mathcal{T}_j-\log\mathcal{T}_{-1}}{4}\right)^{\vartheta_j-1}, \tag{38}$$

where $j=1,2,\cdots,n$

Proof. Let $\varphi(t,s)=(\log(\mathcal{T}_j/\mathcal{T}_{j-1}))^{1-\vartheta_j}[(\log(t/\mathcal{T}_{j-1}))(\log(\mathcal{T}_j/s))]^{\vartheta_j-1}-(\log(t/s))^{\vartheta_j-1}$. We see that

$$\begin{aligned}\varphi_t(t,s)&=\left(\frac{\vartheta_j-1}{t}\right)\left(\log\frac{\mathcal{T}_j}{\mathcal{T}_{j-1}}\right)^{1-\vartheta_j}\left(\log\frac{\mathcal{T}_j}{s}\right)^{\vartheta_j-1}\left(\log\frac{t}{\mathcal{T}_{j-1}}\right)^{\vartheta_j-2}\\&\quad-\left(\frac{\vartheta_j-1}{t}\right)\left(\log\frac{t}{s}\right)^{\vartheta_j-2}\le\left(\frac{\vartheta_j-1}{t}\right)\left(\log\frac{\mathcal{T}_j}{s}\right)^{1-\vartheta_j}\\&\quad\cdot\left(\log\frac{\mathcal{T}_j}{s}\right)^{\vartheta_j-1}\left(\log\frac{t}{s}\right)^{\vartheta_j-2}-\left(\frac{\vartheta_j-1}{t}\right)\left(\log\frac{t}{s}\right)^{\vartheta_j-2}=0,\end{aligned} \tag{39}$$

which means that $\varphi(t,s)$ is nonincreasing w.r.t t, so $\varphi(t,s)\ge\varphi(\mathcal{T}_j,s)=0$, for $\mathcal{T}_{j-1}\le s\le t\le\mathcal{T}_j$. Thus, $0\le\mathbb{G}_j(t,s)$ for any $\mathcal{T}_{j-1}\le t,s\le\mathcal{T}_j, j=1,\cdots,n$.

Since $\varphi(t,s)$ is nonincreasing w.r.t t, then $\varphi(t,s)\le\varphi(s,s)$ for $\mathcal{T}_{j-1}\le s\le t\le\mathcal{T}_j$.

On the other hand, for $\mathcal{T}_{j-1}\le t\le s\le\mathcal{T}_j$, we get

$$\begin{aligned}&\left(\log\left(\frac{\mathcal{T}_j}{\mathcal{T}_{j-1}}\right)\right)^{1-\vartheta_j}\left(\log\left(\frac{t}{\mathcal{T}_{j-1}}\right)\log\left(\frac{\mathcal{T}_j}{s}\right)\right)^{\vartheta_j-1}\\&\le\left(\log\left(\frac{\mathcal{T}_j}{\mathcal{T}_{j-1}}\right)\right)^{1-\vartheta_j}\left(\log\left(\frac{s}{\mathcal{T}_{j-1}}\right)\log\left(\frac{\mathcal{T}_j}{s}\right)\right)^{\vartheta_j-1}.\end{aligned} \tag{40}$$

These confirm that $\max_{t\in[\mathcal{T}_{j-1},\mathcal{T}_j]}\mathbb{G}_j(t,s)=\mathbb{G}_j(s,s), s\in[\mathcal{T}_{j-1},\mathcal{T}_j], j=1,\cdots,n$.

Further, we verify (3) of Proposition 14. Clearly, the maximum points of $\mathbb{G}_j(s,s)$ are not $\mathcal{T}_{j-1}$ and $\mathcal{T}_j$, $j=\in\mathbb{N}_1^n$. For $s\in[\mathcal{T}_{j-1},\mathcal{T}_j], j=1,\cdots,n$, we have

$$\begin{aligned}\frac{d\mathbb{G}_j(s,s)}{ds}&=\left(\frac{\vartheta_j-1}{s}\right)\left(\log\frac{\mathcal{T}_j}{\mathcal{T}_{j-1}}\right)^{1-\vartheta_j}\left(\log\frac{s}{\mathcal{T}_{j-1}}\right)^{\vartheta_j-2}\\&\quad\cdot\left(\log\frac{\mathcal{T}_j}{s}\right)^{\vartheta_j-2}\left[\left(\log\frac{\mathcal{T}_j}{s}\right)-\left(\log\frac{s}{\mathcal{T}_{j-1}}\right)\right],\\&=\left(\frac{\vartheta_j-1}{s}\right)\left(\log\frac{\mathcal{T}_j}{\mathcal{T}_{j-1}}\right)^{1-\vartheta_j}\left(\log\frac{s}{\mathcal{T}_{j-1}}\right)^{\vartheta_j-2}\\&\quad\cdot\left(\log\frac{\mathcal{T}_j}{s}\right)^{\vartheta_j-2}\left[\log\left(\mathcal{T}_j\mathcal{T}_{j-1}\right)-\log\left(s^2\right)\right],\end{aligned} \tag{41}$$

which indicates that the maximum points of $\mathbb{G}_j(s,s)$ is $s=\sqrt{\mathcal{T}_j\mathcal{T}_{j-1}}, j=1,\cdots,n$.

Hence, for $j=1,\cdots,n$,

$$\begin{aligned}\max_{s\in[\mathcal{T}_{j-1},\mathcal{T}_j]}\mathbb{G}_j(s,s)&=\mathbb{G}_j\left(\sqrt{\mathcal{T}_j\mathcal{T}_{j-1}},\sqrt{\mathcal{T}_j\mathcal{T}_{j-1}}\right)\\&=\frac{1}{\Gamma(\vartheta_j)}\left(\frac{1}{4}\log\frac{\mathcal{T}_j}{\mathcal{T}_{j-1}}\right)^{\vartheta_j-1}\\&=\frac{1}{\Gamma(\vartheta_j)}\left(\frac{\log\mathcal{T}_j-\log\mathcal{T}_{j-1}}{4}\right)^{\vartheta_j-1}.\end{aligned} \tag{42}$$

This shows the completion of the proof. □

The existence criterion of solutions for the Hadamard VOFBVP (1) in this work depends on the hypotheses of Theorem 10 which we investigate them in this position.

Theorem 15. *Suppose that both (H1) and (H2) hold, and*

$$\frac{K\left((\log\mathcal{T}_j)^{1-\beta}-(\log\mathcal{T}_{j-1})^{1-\beta}\right)(\log\mathcal{T}_j-\log\mathcal{T}_{j-1})^{\vartheta_j-1}}{4^{\vartheta_j-1}(1-\beta)\Gamma(\vartheta_j)}<1. \tag{43}$$

Then, there is a solution to the Hadamard VOFBVP (1) on $\mathcal{U}$.

Proof. We construct the operator

$$\mathcal{Z}:\mathcal{E}_j\longrightarrow\mathcal{E}_j \tag{44}$$

by

$$\mathcal{Z}r(t)=\int_{\mathcal{T}_{j-1}}^{\mathcal{T}_j}\frac{1}{s}\mathbb{G}_j(t,s)m(s,r(s))ds, t\in\mathcal{U}_j. \tag{45}$$

Some properties of fractional integrals along with the continuity for the function $(\log)^\beta m_1$ imply that the operator $\mathcal{Z}:\mathcal{E}_j\longrightarrow\mathcal{E}_j$ defined in (45) is well-defined.

Let $\exists R_j>0$ so that

$$R_j\ge\frac{m^\star(\log\mathcal{T}_j-\log\mathcal{T}_{j-1})^{\vartheta_j}/4^{\vartheta_j-1}\Gamma(\vartheta_j)}{1-\left(K\left((\log\mathcal{T}_j)^{1-\beta}-(\log\mathcal{T}_{j-1})^{1-\beta}\right)(\log\mathcal{T}_j-\log\mathcal{T}_{j-1})^{\vartheta_j-1}/4^{\vartheta_j-1}(1-\beta)\Gamma(\vartheta_j)\right)}, \tag{46}$$

with

$$m^\star=\sup_{t\in\mathcal{U}_j}\|m_1(t,0)\|. \tag{47}$$

Let us consider the following set:

$$B_{R_j}=\left\{r\in\mathcal{E}_j,\|r\|_{\mathcal{E}_j}\le R_j\right\}. \tag{48}$$

Clearly $B_{R_j} \neq \varnothing$ contains all three properties of the convexity, boundedness, and closedness.

We shall show that $\mathscr{Z}$ satisfies Theorem 10 in four stages.

Step A. Claim: $\mathscr{Z}(B_{R_j}) \subseteq (B_{R_j})$. For $r \in B_{R_j}$, by Proposition 14 and (H2), we get

$$\begin{aligned}
\|\mathscr{Z}r(t)\| &= \left\| \int_{\mathscr{T}_{j-1}}^{\mathscr{T}_j} \frac{1}{s}\mathbb{G}_j(t,x)m_1(s,r(s))ds \right\| \\
&\leq \int_{\mathscr{T}_{j-1}}^{\mathscr{T}_j} \frac{1}{s}\mathbb{G}_j(t,s)\|m_1(s,r(s))\| \\
&\cdot ds \leq \frac{1}{\Gamma(\vartheta_j)}\left(\frac{\log\mathscr{T}_j - \log\mathscr{T}_{j-1}}{4}\right)^{\vartheta_j-1} \\
&\cdot \int_{\mathscr{T}_{j-1}}^{\mathscr{T}_j} \frac{1}{s}\|m_1(s,r(s))\|ds \leq \frac{1}{\Gamma(\vartheta_j)} \\
&\cdot \left(\frac{\log\mathscr{T}_j - \log\mathscr{T}_{j-1}}{4}\right)^{\vartheta_j-1} \int_{\mathscr{T}_{j-1}}^{\mathscr{T}_j} \frac{1}{s}\|m_1(s,r(s)) \\
&- m_1(s,0)\|ds + \frac{1}{\Gamma(\vartheta_j)}\left(\frac{\log\mathscr{T}_j - \log\mathscr{T}_{j-1}}{4}\right)^{\vartheta_j-1} \\
&\cdot \int_{\mathscr{T}_{j-1}}^{\mathscr{T}_j} \frac{1}{s}\|m_1(s,0)\|ds \leq \frac{1}{\Gamma(\vartheta_j)}\left(\frac{\log\mathscr{T}_j - \log\mathscr{T}_{j-1}}{4}\right)^{\vartheta_j-1} \\
&\cdot \int_{\mathscr{T}_{-1}}^{\mathscr{T}_j} \frac{1}{s}(\log s)^{-\beta}(K\|r(s)\|)ds + \frac{m^\star(\log\mathscr{T}_j - \log\mathscr{T}_{j-1})^{\vartheta_j}}{4^{\vartheta_j-1}\Gamma(\vartheta_j)} \\
&\leq \frac{K}{\Gamma(\vartheta_j)}\left(\frac{\log\mathscr{T}_j - \log\mathscr{T}_{j-1}}{4}\right)^{\vartheta_j-1} \|r\|_{\mathscr{E}_j} \int_{\mathscr{T}_{j-1}}^{\mathscr{T}_j} \frac{1}{s}(\log s)^{-\beta} \\
&\cdot ds + \frac{m^\star(\log\mathscr{T}_j - \log\mathscr{T}_{j-1})^{\vartheta_j}}{4^{\vartheta_j-1}\Gamma(\vartheta_j)} \leq \frac{K}{\Gamma(\vartheta_j)}\left(\frac{\log\mathscr{T}_j - \log\mathscr{T}_{j-1}}{4}\right)^{\vartheta_j-1} \\
&\cdot R_j\left(\frac{(\log\mathscr{T}_j)^{1-\beta} - (\log\mathscr{T}_{j-1})^{1-\beta}}{1-\beta}\right) + \frac{m^\star(\log\mathscr{T}_j - \log\mathscr{T}_{j-1})^{\vartheta_j}}{4^{\vartheta_j-1}\Gamma(\vartheta_j)} \\
&\leq \frac{K\left((\log\mathscr{T}_j)^{1-\beta} - (\log\mathscr{T}_{j-1})^{1-\beta}\right)(\log\mathscr{T}_j - \log\mathscr{T}_{j-1})^{\vartheta_j-1}}{4^{\vartheta_j-1}(1-\beta)\Gamma(\vartheta_j)} \\
&\cdot R_j + \frac{m^\star(\log\mathscr{T}_j - \log\mathscr{T}_{j-1})^{\vartheta_j}}{4^{\vartheta_j-1}\Gamma(\vartheta_j)} \leq R_j,
\end{aligned} \tag{49}$$

which means that $\mathscr{Z}(B_{R_j}) \subseteq B_{R_j}$.

Step B. Claim: $\mathscr{Z}$ is continuous.

The sequence (r_n) is supposed to be convergent to r in $\mathscr{E}_j$ and $t \in \mathscr{U}_j$. Then,

$$\begin{aligned}
\|(\mathscr{Z}r_n)(t) - (\mathscr{Z}r)(t)\| &\leq \int_{\mathscr{T}_{j-1}}^{\mathscr{T}_j} \frac{1}{s}\mathbb{G}_j(t,s)\|m_1(s,r_n(s)) - m_1(s,r(s))\|ds \\
&\leq \frac{1}{\Gamma(\vartheta_j)}\left(\frac{\log\mathscr{T}_j - \log\mathscr{T}_{j-1}}{4}\right)^{\vartheta_j-1} \int_{\mathscr{T}_{j-1}}^{\mathscr{T}_j} \frac{1}{s}\|m_1(s,r_n(s)) - m_1(s,r(s))\|ds \\
&\leq \frac{1}{\Gamma(\vartheta_j)}\left(\frac{\log\mathscr{T}_j - \log\mathscr{T}_{j-1}}{4}\right)^{\vartheta_j-1} \int_{\mathscr{T}_{j-1}}^{\mathscr{T}_j} \frac{1}{s}(\log s)^{-\beta}(K\|r_n(s) - r(s)\|)ds \\
&\leq \frac{1}{\Gamma(\vartheta_j)}\left(\frac{\log\mathscr{T}_j - \log\mathscr{T}_{j-1}}{4}\right)^{\vartheta_j-1} \left(K\|r_n - y\|_{\mathscr{E}_j}\right) \int_{\mathscr{T}_{j-1}}^{\mathscr{T}_j} \frac{1}{s}(\log s)^{-\beta}ds \\
&\leq \frac{K\left((\log\mathscr{T}_j)^{1-\beta} - (\log\mathscr{T}_{j-1})^{1-\beta}\right)(\log\mathscr{T}_j - \log\mathscr{T}_{j-1})^{\vartheta_j-1}}{4^{\vartheta_j-1}(1-\beta)\Gamma(\vartheta_j)}\|r_n - r\|_{\mathscr{E}_j},
\end{aligned} \tag{50}$$

i.e., we get

$$\|(\mathscr{Z}r_n) - (\mathscr{Z}r)\|_{\mathscr{E}_j} \longrightarrow 0 \text{ as } n \longrightarrow \infty, \tag{51}$$

and the correctness of the claim in this step is confirmed for Z.

Step C. Claim: $\mathscr{Z}$ is bounded and equicontinuous.

From A, $\mathscr{Z}(B_{R_j}) = \{\mathscr{Z}(r): r \in B_{R_j}\} \subset B_{R_j}$; thus, for each $r \in B_{R_j}$, we get $\|\mathscr{Z}(r)\|_{\mathscr{E}_j} \leq R_j$; in other ways, it means that $\mathscr{Z}(B_{R_j})$ is bounded. It remains to check the equicontinuity of $\mathscr{Z}(B_{R_j})$.

Now, $\forall t_1 < t_2 \in \mathscr{U}_j, t_1 < t_2$ and $r \in B_{R_j}$, we write

$$\begin{aligned}
\|(\mathscr{Z}r)(t_2) - (\mathscr{Z}r)(t_1)\| &= \left\| \int_{\mathscr{T}_{j-1}}^{\mathscr{T}_j} \frac{1}{s}\mathbb{G}_j(t_2,s)m_1(s,r(s)) \right. \\
&\left. \cdot ds - \int_{\mathscr{T}_{j-1}}^{\mathscr{T}_j} \frac{1}{s}\mathbb{G}_j(t_1,s)m_1(s,r(s))ds \right\| \leq \int_{\mathscr{T}_{j-1}}^{\mathscr{T}_j} \\
&\cdot \frac{1}{s}\left\|\left(\mathbb{G}_j(t_2,s) - \mathbb{G}_j(t_1,s)\right)m_1(s,r(s))\right\| \\
&\cdot ds \leq \int_{\mathscr{T}_{j-1}}^{\mathscr{T}_j} \frac{1}{s}\left\|\mathbb{G}_j(t_2,s) - \mathbb{G}_j(t_1,s)\right\|\|m_1(s,r(s))\| \\
&\cdot ds \leq \int_{\mathscr{T}_{j-1}}^{\mathscr{T}_j} \frac{1}{s}\left\|\mathbb{G}_j(t_2,s) - \mathbb{G}_i(t_1,s)\right\|(\|m_1(s,r(s)) - m_1(s,0)\| \\
&+ \|m_1(s,0)\|)ds \leq \int_{\mathscr{T}_{j-1}}^{\mathscr{T}_j} \frac{1}{s}\left\|\mathbb{G}_j(t_2,s) - \mathbb{G}_j(t_1,s)\right\| \\
&\cdot \left[(\log s)^{-\beta}(K\|r(s)\|) + m^\star\right]ds \leq \int_{\mathscr{T}_{j-1}}^{\mathscr{T}_j} \left\|\mathbb{G}_j(t_2,s) - \mathbb{G}_j(t_1,s)\right\| \\
&\cdot \left[\frac{1}{s}(\log s)^{-\beta}\left(K\|r\|_{\mathscr{E}_j}\right) + \frac{1}{s}m^\star\right]ds \leq \frac{K(\log\mathscr{T}_{j-1})^{-\beta}}{\mathscr{T}_{j-1}}\|r\|_{\mathscr{E}_j} \\
&\cdot \int_{\mathscr{T}_{j-1}}^{\mathscr{T}_j} \left\|\mathbb{G}_j(t_2,s) - \mathbb{G}_j(t_1,s)\right\|ds + \frac{m^\star}{\mathscr{T}_{j-1}}\int_{\mathscr{T}_{j-1}}^{\mathscr{T}_j} \left\|\mathbb{G}_j(t_2,s) - \mathbb{G}_j(t_1,s)\right\|ds,
\end{aligned} \tag{52}$$

using the continuity of G. Hence, $\|(\mathscr{Z}r)(t_2) - (\mathscr{Z}r)(t_1)\|_{\mathscr{E}_j} \longrightarrow 0$ as $|t_2 - t_1| \longrightarrow 0$. It yields that $\mathscr{Z}(B_{R_j})$ is equicontinuous.

Step D. Claim: $\mathscr{Z}$ is k-set contraction.

This time, let $\mathscr{W} \in B_{R_j}$ and $t \in \mathscr{U}_j$. So,

$$\begin{aligned}
\Phi(\mathscr{Z}(\mathscr{W})(t)) &= \Phi((\mathscr{Z}r)(t), r \in \mathscr{W}) \\
&\leq \left\{\int_{\mathscr{T}_{j-1}}^{\mathscr{T}_j} \frac{1}{s}\mathbb{G}_j(t,s)\Phi m_1(s,r(s))ds \quad r \in \mathscr{W}\right\}.
\end{aligned} \tag{53}$$

Remark 11 indicates that

$$\begin{aligned}
\Phi(\mathscr{Z}(\mathscr{W})(t)) &\leq \left\{\int_{\mathscr{T}_{j-1}}^{\mathscr{T}_j} \frac{1}{s}\mathbb{G}_j(t,s)[K\Phi(\{r(s), r \in \mathscr{W}\})]\right\} \\
&\leq \left\{\frac{1}{\Gamma(\vartheta_j)}\left(\frac{\log\mathscr{T}_j - \log\mathscr{T}_{j-1}}{4}\right)^{\vartheta_j-1}\left[K\widehat{\Phi}(\mathscr{W})\int_{\mathscr{T}_{j-1}}^{\mathscr{T}_j} \frac{1}{s}(\boldsymbol{logs})^{-\beta}ds\right], r \in \mathscr{W}\right\} \\
&\leq \frac{K\left(\left(\log\mathscr{T}_j^{1-\beta}\right) - \left(\log\mathscr{T}_{j-1}^{1-\beta}\right)\right)(\log\mathscr{T}_j - \log\mathscr{T}_{j-1})^{\vartheta_j-1}}{4^{\vartheta_j-1}(1-\beta)\Gamma(\vartheta_j)}\widehat{\Phi}(\mathscr{W}),
\end{aligned} \tag{54}$$

for any $s \in \mathscr{U}_j$.

Therefore,

$$\widehat{\Phi}(\mathscr{Z}\mathscr{W}) \leq \frac{K\left(\left(\log \mathscr{T}_{\jmath}{}^{1-\beta}\right)-\left(\log \mathscr{T}_{\jmath-1}{}^{1-\beta}\right)\right)\left(\log \mathscr{T}_{\jmath}-\log \mathscr{T}_{\jmath-1}\right)^{\vartheta_{\jmath}-1}}{4^{\vartheta_{\jmath}-1}(1-\beta)\Gamma(\vartheta_{\jmath})} \widehat{\Phi}(\mathscr{W}). \tag{55}$$

Consequently by (43), we deduce that $\mathscr{Z}$ admits a set contraction.

The conclusion of Theorem 10 gives this result that the Hadamard COFBVP (30) involves at least a solution $\widetilde{r_{\jmath}}$ in $B_{R_{\jmath}}$.

Assume that

$$r_{\jmath}=\begin{cases}0, & t\in\left[1,\mathscr{T}_{\jmath-1}\right],\\ \tilde{r}_{\jmath}, & t\in\mathscr{U}_{\jmath}.\end{cases} \tag{56}$$

We know that $r_{\jmath}\in\mathscr{C}([1,\mathscr{T}_{\jmath}],\mathscr{S})$ defined by (56) satisfies equation

$$\frac{d^2}{dt^2}\left(\int_1^{\mathscr{T}_1}\frac{(t-s)^{1-\vartheta_1}}{\Gamma(2-\vartheta_1)}r_{\jmath}(s)ds+\cdots+\int_{\mathscr{T}_{\jmath-1}}^{t}\frac{(t-s)^{1-\vartheta_{\jmath}}}{\Gamma(2-\vartheta_{\jmath})}r_{\jmath}(s)ds\right)+m_1\left(s,r_{\jmath}(s)\right)=0, \tag{57}$$

for $t\in\mathscr{U}_{\jmath}$, which implies that $r_{\jmath}$ is regarded as a solution for (28) along with $r_{\jmath}(1)=0$ and $r_{\jmath}(\mathscr{T}_{\jmath})=\tilde{r}_{\jmath}(\mathscr{T}_{\jmath})=0$.

Then,

$$r(t)\begin{cases} r_1(t), t\in\mathscr{U}_1,\\ r_2(t)=\begin{cases}0, & t\in\mathscr{U}_1,\\ \tilde{x}_2, & t\in\mathscr{U}_{2,}\end{cases}\\ .\\ .\\ .\\ r_{\jmath}(t)=\begin{cases}0, & t\in\left[1,\mathscr{T}_{\jmath-1}\right],\\ \tilde{r}_{\jmath}, & t\in\mathscr{U}_{\jmath}\end{cases}\end{cases} \tag{58}$$

gives the solution for the Hadamard VOFBVP (1) and this completes the argument. □

4. Ulam-Hyers-Rassias Stability

The stability issue has gained substantially important attention in several research fields through applications. There are many kinds of stability; one of them is the stability introduced by Ulam in 1940. Since then, the problem is known as Ulam-Hyers stability or simply Ulam stability. Later, other generalizations of this notion were introduced by other researchers. Its applications for many types of equations have been investigated by many mathematicians. Ben Makhlouf [32] derived sufficient conditions of different types of stability such as uniform stability, Mittag-Leffler stability, and asymptotic uniform stability for a nonlinear Caputo fraction BVP via a method with respect to Lyapunov-like functions. Ahmad et al. [33] investigated the notion of stability for a nonlinear coupled implicit switched singular fractional differential system with p-Laplacian operator. Now, we aim to accomplish an argument regarding the UHRS stability of the given Hadamard VOFBVP (1) in the framework of Theorem 17.

Definition 16 (see [34]). Let $\varrho\in\mathscr{C}(\mathscr{U},\mathscr{S})$. Then, the Hadamard VOFBVP (1) is Ulam-Hyers-Rassias stable (UHRS) w.r.t ϱ if $\exists c_m>0$, so that $\forall\epsilon>0$ and for every solution $\varsigma\in\mathscr{C}(\mathscr{U},\mathscr{S})$ of

$$\left\|{}^{\mathscr{H}}\mathfrak{D}_{1^+}^{\vartheta(t)}\varsigma(t)-(-m_1(t,\varsigma(t)))\right\|\leq\epsilon\varrho(t),\quad t\in\mathscr{U}, \tag{59}$$

$\exists$ a solution $r\in\mathscr{C}(\mathscr{U},\mathscr{S})$ of (1) with

$$\|\varsigma(t)-r(t)\|\leq c_m\epsilon\varrho(t). \tag{60}$$

Theorem 17. *Let both (H1) and (H2) along with (43) hold. Also,*

(H3) Let $\varrho\in\mathscr{C}(\mathscr{U}_{\jmath},\mathscr{S})$ is a increasing function and $\exists$ $\lambda_{\varrho}>0$ provided

$${}^{\mathscr{H}}\mathscr{I}_{\mathscr{T}_{\jmath-1}{}^+}^{\vartheta_{\jmath}}\varrho(t)\leq\lambda_{\varrho(t)}\varrho(t), \textit{for any } t\in\mathscr{U}_{\jmath}. \tag{61}$$

Then, the Hadamard VOFBVP (1) is UHRS stable w.r.t the function ϱ.

Proof. Assume that $\epsilon>0$ is chosen arbitrarily and ς from $\varsigma\in\mathscr{C}(\mathscr{U}_{\jmath},\mathbb{R})$ satisfies (59). Now, $\forall\jmath\in\mathbb{N}_1^n$, the following are defined: $\varsigma_1(t)\equiv\varsigma(t), t\in[1,\mathscr{T}_1]$ and for $\jmath=2,3,\cdots,n$:

$$\varsigma_{\jmath}(t)=\begin{cases}0, & t\in\left[0,\mathscr{T}_{\jmath-1}\right],\\ \varsigma(t), & t\in\mathscr{U}_{\jmath}.\end{cases} \tag{62}$$

Taking ${}^{\mathscr{H}}\mathscr{I}_{\mathscr{T}_{\jmath-1}^+}^{\vartheta_{\jmath}}$ on both sides (59), we get

$$\begin{aligned}&\left\|\varsigma(t)-\left(-\int_{\mathscr{T}_{\mathscr{J}-1}}^{\mathscr{T}_{\mathscr{J}}}\frac{1}{s}\mathbb{G}_{\mathscr{J}}(t,s)m_1(s,\varsigma(s))ds\right)\right\|\\ &\leq\frac{\epsilon}{\Gamma(\vartheta_{\mathscr{J}})}\int_{\mathscr{T}_{\mathscr{J}-1}}^{t}\frac{1}{s}\left(\log\frac{t}{s}\right)^{\vartheta_{\mathscr{J}}-1}\varrho(s)ds\leq\epsilon\lambda_{\varrho(t)}\varrho(t).\end{aligned} \tag{63}$$

According to the argument above, the Hadamard VOFBVP (1) involves a solution $r\in\mathscr{C}(\mathscr{U},\mathbb{R})$ formulated as $r(t)=r_{\mathscr{J}}(t)$ *for* $t\in\mathscr{U}_{\mathscr{J}},\in\mathbb{N}_1^n$, in which

$$r_j=\left\{\begin{matrix}0, & t\in\left[0,\mathscr{T}_{\mathscr{J}-1}\right],\\ \tilde{r}_j, & t\in\mathscr{U}_{\mathscr{J}},\end{matrix}\right\} \tag{64}$$

and $\tilde{r}_{\mathscr{J}}\in\mathscr{E}_{\mathscr{J}}$ is a solution of the Hadamard COFBVP (30).

In accordance with Lemma 13, we have

$$\tilde{r}_{\mathcal{J}}(t) = -\frac{(\mathcal{T}_{\mathcal{J}} - \mathcal{T}_{\mathcal{J}-1})^{-1}(t - \mathcal{T}_{\mathcal{J}-1})}{\Gamma(\vartheta_{\mathcal{J}})} \cdot \int_{\mathcal{T}_{\mathcal{J}-1}}^{\mathcal{T}_{\mathcal{J}}} (\mathcal{T}_{\mathcal{J}} - s)^{\vartheta_{\mathcal{J}-1}} m_1(s, \tilde{r}_{\mathcal{J}}(s))ds + \frac{1}{\Gamma(\vartheta_{\mathcal{J}})} \cdot \int_{\mathcal{T}_{\mathcal{J}-1}}^{t} (t - s)^{\vartheta_{\mathcal{J}-1}} m_1(s, \tilde{r}_{\mathcal{J}}(s))ds. \quad (65)$$

Suppose that $t \in \mathcal{U}_{\mathcal{J}}, = 1, 2, \cdots, n$. Then, by equations (64) and (65), we get

$$\begin{aligned}
\|\varsigma(t) - r(t)\| &= \|\varsigma(t) - r_{\mathcal{J}}(t)\| = \|\varsigma_{\mathcal{J}}(t) - \tilde{r}_{\mathcal{J}}(t)\| \\
&= \left\| \varsigma_{\mathcal{J}}(t) - \int_{\mathcal{T}_{\mathcal{J}-1}}^{\mathcal{T}_{\mathcal{J}}} \frac{1}{s}\mathbb{G}_{\mathcal{J}}(t,s) m_1(s, \tilde{r}_{\mathcal{J}}(s))ds \right\| \le \left\| \varsigma_{\mathcal{J}}(t) - \int_{\mathcal{T}_{\mathcal{J}-1}}^{\mathcal{T}_{\mathcal{J}}} \right. \\
&\quad \cdot \frac{1}{s}\mathbb{G}_{\mathcal{J}}(t,s) m_1(s, \varsigma_{\mathcal{J}}(s))ds \Big\| + \int_{\mathcal{T}_{\mathcal{J}-1}}^{\mathcal{T}_{\mathcal{J}}} \frac{1}{s}\mathbb{G}_{\mathcal{J}}(t,s) \| m_1(s, \varsigma_{\mathcal{J}}(s)) \\
&\quad - m_1(s, \tilde{r}_{\mathcal{J}}(s)) \| ds \le \left\| \varsigma_{\mathcal{J}}(t) + \int_{\mathcal{T}_{-1}}^{\mathcal{T}_{\mathcal{J}}} \frac{1}{s}\mathbb{G}_{\mathcal{J}}(t,s) m_1(s, \varsigma_{\mathcal{J}}(s)) \right. \\
&\quad \cdot ds \Big\| + \int_{\mathcal{T}_{\mathcal{J}-1}}^{\mathcal{T}_{\mathcal{J}}} \frac{1}{s}\mathbb{G}_{\mathcal{J}}(t,s) \left\| m_1(s, \varsigma_{\mathcal{J}}(s)) - m_1\left(s, \tilde{\tilde{r}}_{\mathcal{J}}(s)\right) \right\| \\
&\quad \cdot ds \le \lambda_{\varrho(t)} \epsilon \varrho(t) + \frac{1}{\Gamma(\vartheta_{\mathcal{J}})} \left(\frac{\log \mathcal{T}_{\mathcal{J}} - \log \mathcal{T}_{\mathcal{J}-1}}{4} \right)^{\vartheta_{\mathcal{J}}-1} \\
&\quad \cdot \int_{\mathcal{T}_{\mathcal{J}-1}}^{\mathcal{T}_{\mathcal{J}}} (\log s)^{-\beta} \frac{K\|\varsigma_{\mathcal{J}}(s) - \tilde{r}_{\mathcal{J}}(s)\|}{s} ds \le \lambda_{\varrho(t)} \epsilon \varrho(t) \\
&\quad + \frac{K}{\Gamma(\vartheta_{\mathcal{J}})} \left(\frac{\log \mathcal{T}_{\mathcal{J}} - \log \mathcal{T}_{\mathcal{J}-1}}{4} \right)^{\vartheta_{\mathcal{J}}-1} \|\varsigma_{\mathcal{J}} - \tilde{r}_{\mathcal{J}}\|_{\mathcal{G}_{\mathcal{J}}} \int_{\mathcal{T}_{\mathcal{J}-1}}^{\mathcal{T}_{\mathcal{J}}} \\
&\quad \cdot \frac{1}{s}(\log s)^{-\beta} ds \le \lambda_{\varrho(t)} \epsilon \varrho(t) \\
&\quad + \frac{K\left((\log \mathcal{T}_{\mathcal{J}})^{1-\beta} - (\log \mathcal{T}_{\mathcal{J}-1})^{1-\beta}\right)(\log \mathcal{T}_{\mathcal{J}} - \log \mathcal{T}_{\mathcal{J}-1})^{\vartheta_{\mathcal{J}}-1}}{(1-\beta)4^{\vartheta_{\mathcal{J}}-1}\Gamma(\vartheta_{\mathcal{J}})} \\
&\quad \cdot \|\varsigma_{\mathcal{J}} - \tilde{r}_{\mathcal{J}}\|_{\mathcal{G}_{\mathcal{J}}} \le \lambda_{\varrho(t)} \epsilon \varrho(t) + \mu \|\varsigma - r\|,
\end{aligned} \quad (66)$$

where

$$\mu = \max_{\mathcal{J}=1,2,\cdots,n} \frac{K\left((\log \mathcal{T}_{\mathcal{J}})^{1-\beta} - (\log \mathcal{T}_{\mathcal{J}-1})^{1-\beta}\right)(\log \mathcal{T}_{\mathcal{J}} - \log \mathcal{T}_{\mathcal{J}-1})^{\vartheta_{\mathcal{J}}-1}}{(1-\beta)4^{\vartheta_{\mathcal{J}}-1}\Gamma(\vartheta_{\mathcal{J}})}. \quad (67)$$

Then,

$$\|\varsigma - r\|(1 - \mu) \le \lambda_{\varrho(t)} \epsilon \varrho(t), \quad (68)$$

and so by assuming $c_{m_1} := \lambda_{\varrho(t)}/(1 - \mu)$,

$$\|\varsigma(t) - r(t)\| \le \frac{\lambda_{\varrho(t)}\varrho(t)}{(1-\mu)} \epsilon := c_{m_1} \epsilon \varrho(t). \quad (69)$$

Therefore, the Hadamard VOFBVP (1) is UHRS stable w.r.t ϱ. This result completes the proof. □

5. Numerical Illustrative Example

Example 2. Consider the Hadamard VOFBVP (based on the VOFBVP (1)) as follows:

$$\begin{aligned}
&{}^{\mathscr{H}}\mathfrak{D}_{1^+}^{\vartheta(t)} r(t) + \frac{(\log t)^{\vartheta(t)}}{\sqrt{\pi}} + \frac{1}{4}(\log t)^{-1/4} r(t) = 0, \\
&t \in \mathcal{U} := [1, e], r(1) = 0, r(e) = 0.
\end{aligned} \quad (70)$$

Hence, $\mathcal{T} = e$ and

$$m_1(t, r) = \frac{(\log t)^{\vartheta(t)}}{\sqrt{\pi}} + \frac{1}{4}(\log t)^{-1/4} r(t), \quad (t, r) \in [1, e] \times [0, +\infty), \quad (71)$$

$$\vartheta(t) \begin{cases} 1.2, & t \in \mathcal{U}_1 := [1, 2], \\ 1.6, & t \in \mathcal{U}_2 :=]2, e]. \end{cases} \quad (72)$$

Then, we get

$$\begin{aligned}
&(\log t)^{1/4} |m(t, r) - m(t, \bar{r})| \\
&= \left| (\log t)^{1/4} \frac{(\log t)^{\vartheta(t)}}{\sqrt{\pi}} + \frac{1}{4} r(t) - (\log t)^{1/4} \frac{(\log t)^{\vartheta(t)}}{\sqrt{\pi}} - \frac{1}{4}\bar{r}(t) \right| \\
&\le \frac{1}{4} |r(t) - \bar{r}(t)|.
\end{aligned} \quad (73)$$

(H2) holds with $\beta = 1/4$ and $K = 1/4$.

From (72), the Hadamard VOFBVP (70) is classified into the following:

$$\begin{cases} {}^{\mathscr{H}}\mathfrak{D}_{1^+}^{1.2} r(t) + \frac{(\log t)^{1.2}}{\sqrt{\pi}} + \frac{1}{4}(\log t)^{-1/4} r(t) = 0, & t \in \mathcal{U}_1, \\ {}^{\mathscr{H}}\mathfrak{D}_{2^+}^{1.6} r(t) + \frac{(\log t)^{1.6}}{\sqrt{\pi}} + \frac{1}{4}(\log t)^{-1/4} r(t) = 0, & t \in \mathcal{U}_2. \end{cases} \quad (74)$$

For $t \in \mathcal{U}_1$, the Hadamard VOFBVP (70) is equivalent to the Hadamard COFBVP

$$\begin{cases} {}^{\mathscr{H}}\mathfrak{D}_{1^+}^{1.2} r(t) + \frac{(\log t)^{1.2}}{\sqrt{\pi}} + \frac{1}{4}(\log t)^{-1/4} r(t) = 0, & t \in \mathcal{U}_1, \\ r(1) = 0, & r(2) = 0. \end{cases} \quad (75)$$

Let us now show that condition (43) is satisfied. Clearly, the following value is obtained

$$\frac{K\left((\log \mathcal{T}_1)^{1-\beta} - (\log \mathcal{T}_0)^{1-\beta}\right)(\log \mathcal{T}_1 - \log \mathcal{T}_0)^{\vartheta_1 - 1}}{4^{\vartheta_1 - 1}(1-\beta)\Gamma(\vartheta_1)} = \frac{1/4(\log 2)^{3/4}(\log 2)^{0.2}}{\left(4^{0.2}\right)3/4\Gamma(1.2)} \simeq 0.1941 < 1. \tag{76}$$

On the other side, let $\varrho(t) = (\log t)^{1/2}$. In this case,

$$\begin{aligned} {}^{\mathscr{H}}\mathscr{I}_{1^+}^{\vartheta_1}\varrho(t) &= \frac{1}{\Gamma(1.2)}\int_1^2 \left(\log \frac{t}{s}\right)^{1.2-1} \frac{(\log t)^{1/2}}{s} \\ &\cdot ds \le \frac{(\log t)^{1/2}}{\Gamma(1.2)}\int_1^2 \left(\log \frac{2}{s}\right)^{0.2} \\ &\cdot \frac{ds}{s} \le \frac{(\log 2)^{1.2}}{\Gamma(2.2)}(\log t)^{1/2} \coloneqq \lambda_{\varrho(t)}\varrho(t). \end{aligned} \tag{77}$$

As a result, (H3) is fulfilled with $\varrho(t) = \sqrt{(\log t)}$ and $\lambda_{\varrho(t)} = (\log 2)^{1.2}/\Gamma(2.2) \in \mathbb{R}$.

Theorem 15 guarantees the existence of a solution for the Hadamard COFBVP (75) like $r_1 \in \mathscr{E}_1$, and from Theorem 17, the Hadamard constant-order system (75) is UHRS stable. For $t \in \mathscr{U}_2$, the Hadamard VOFBVP (70) can be written as the following COFBVP, i.e.,

$$\begin{cases} {}^{\mathscr{H}}\mathfrak{D}_{2^+}^{1.6} r(t) + \dfrac{(\log t)^{1.6}}{\sqrt{\pi}} + \dfrac{1}{4}(\log t)^{-1/4} r(t) = 0, & t \in \mathscr{U}_2, \\ r(2) = 0, & r(e) = 0. \end{cases} \tag{78}$$

We see that

$$\frac{K\left((\log \mathcal{T}_2)^{1-\beta} - (\log \mathcal{T}_1)^{1-\beta}\right)(\log \mathcal{T}_2 - \log \mathcal{T}_1)^{\vartheta_2 - 1}}{4^{\vartheta_2 - 1}(1-\beta)\Gamma(\vartheta_2)} = \frac{1/4\left(1 - (\log 2)^{3/4}\right)(1-\log 2)^{0.6}}{\left(4^{0.6}\right)(3/4)\Gamma(1.6)} \simeq 0.0191 < 1. \tag{79}$$

Accordingly, condition (43) is achieved on the subinterval $\mathscr{U}_2$. Further,

$$\begin{aligned} {}^{\mathscr{H}}\mathscr{I}_{2^+}^{\vartheta_1}\varrho(t) &= \frac{1}{\Gamma(1.6)}\int_2^e \left(\log \frac{t}{s}\right)^{1.6-1} \frac{(\log t)^{1/2}}{s} \\ &\cdot ds \le \frac{(\log t)^{1/2}}{\Gamma(1.6)}\int_2^e \left(\log \frac{e}{s}\right)^{0.6} \\ &\cdot \frac{ds}{s} \le \frac{(\log (e/2))^{1.6}}{\Gamma(2.6)}(\log t)^{1/2} \coloneqq \lambda_{\varrho(t)}\varrho(t). \end{aligned} \tag{80}$$

As a result, (H3) is also valid with $\varrho(t) = \sqrt{(\log t)}$ and $\lambda_{\varrho(t)} = (\log (e/2))^{1.6}/\Gamma(2.6) \in \mathbb{R}$.

On account of Theorem 15, the Hadamard COFBVP (78) possesses a solution $\tilde{x}_2 \in \mathscr{E}_2$. Further, Theorem 17 yields that the mentioned Hadamard system (78) is UHRS stable. It is known that

$$r_2(t) = \begin{cases} 0, & t \in \mathscr{U}_1, \\ \tilde{r}_2(t), & t \in \mathscr{U}_2. \end{cases} \tag{81}$$

Consequently, the Hadamard VOFBVP (70) has a solution

$$r(t) = \begin{cases} r_1(t), & t \in \mathscr{U}_1, \\ r_2(t) = \begin{cases} 0, & t \in \mathscr{U}_1, \\ \tilde{r}_2(t), & t \in \mathscr{U}_2. \end{cases} \end{cases} \tag{82}$$

From Theorem 17, the Hadamard VOFBVP given by (70) is UHRS stable.

6. Conclusions

In this paper, the nonlinear Hadamard VOFBVP (1) was considered in which we established some theorems regarding existence and stability of solutions of it by following a new method based on the generalized subintervals and piecewise constant functions. By applying such notions, we converted the given Hadamard VOFBVP (1) to the standard Hadamard COFBVP (30). After investigating some specifications of the Green function, we focused on the solutions' existence via a combined technique in terms of KMNCS in the context of Darbo's fixed point criterion. The UHRS stability of the proposed Hadamard VOFBVP was also studied. At the end, a numerical example has been discussed to validate the applicability of our results. In future works, our results can be extended to other fractional mathematical models equipped with variable orders such as studies on simulations and dynamical behaviors of COVID-19.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

The authors declare that the study was realized in collaboration with equal responsibility. All authors read and approved the final manuscript.

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