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Research Article

On Atangana–Baleanu-Type Nonlocal Boundary Fractional Differential Equations

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This research paper is devoted to investigating two classes of boundary value problems for nonlinear Atangana–Baleanu-type fractional differential equations with Atangana–Baleanu fractional integral conditions. The applied fractional derivatives work as the nonlocal and nonsingular kernel. Upon using Krasnoselskii's and Banach's fixed point techniques, we establish the existence and uniqueness of solutions for proposed problems. Moreover, the Ulam–Hyers stability theory is constructed by using nonlinear analysis. Eventually, we provide two interesting examples to illustrate the effectiveness of our acquired results.

1. Introduction

Fractional calculus and its potential applications have become a powerful tool to model many complex phenomena in apparently wide-ranging fields of science and technology [1–8]. To meet the needs of modeling many practical problems in different fields of science and engineering, some researchers have realized the necessity development of the concept of fractional calculus by searching for new fractional derivatives with different singular or nonsingular kernels. From this perspective, new fractional operators have turned into the best effective tool of numerous specialists and researchers with their contribution to physical phenomena and their performance in applying to real-world problems. Until 2015, all fractional derivatives had only singular kernels. Therefore, it is difficult to use these singularities to simulate physical phenomena. In 2015, Caputo and Fabrizio [9] studied a new kind of FD in the exponential kernel. Some properties of this new type had been discussed by Losada and Nieto in [10]. A new type and interesting FD with Mittag-Leffler kernels has been investigated by Atangana and Baleanu (A-B) in [11]. Abdeljawad in [12] extended the kind investigated by A-B from order between zero and one to higher arbitrary order and formulated their associated integral operators. Also, he proved some properties such as the existence and uniqueness theorems for two classes of fractional derivative, Riemann type (ABR) and Caputo type (ABC), for initial value problems in higher arbitrary order and proved a Lyapunov-type inequality in the frame of Mittag-Leffler kernels for the ABR fractional boundary value problems of order $2 < \alpha \leq 3$. Abdeljawad and Baleanu, in [13, 14], deliberated the discrete versions of those new operators. For some theoretical works on Atangana–Baleanu FDEs, we refer the reader to a series of papers [15–17]. For important applications and mathematical modeling of the ABC fractional operator, see [18–21]. On the contrary, there are some important numerical approaches regarding nonsingular kernels; for example, in [22], via a spectral collocation method based on the shifted Legendre polynomials with extending the unknown functions and their derivatives using the shifted Legendre basis, Tuan et al. solved fractional rheological models and Newell–Whitehead–Segel equations with the nonlocal and nonsingular kernels. Nikan et al. in [23] developed the solution of the two-dimensional time-fractional evolution model using a finite difference scheme derived from the radial basis function (RBF-FD) method. Ganji et al. in [24] studied the multivariable-order differential equations (MVODEs) with the nonlocal and nonsingular kernels in the Atangana and Baleanu sense of variable orders. They used the fifth-kind Chebyshev polynomials as basic functions to obtain operational matrices and used them with the collocation method to transfer the original equations to a system of algebraic equations. Firoozjaee et al. in [25] studied the Fokker–Planck equation (FPE) with CF-FD being considered. They presented a numerical approach that is based on the Ritz method with known basis functions to transform this equation into an optimization problem. Baleanu et al. in [26] showed that four fractional integrodifferential inclusions have solutions. Also, they showed that the dimension of the set of solutions for the second fractional integrodifferential inclusion problem is infinite-dimensional under some different conditions. By using the fractional CF derivative, Aydogan et al. [27] introduced two types of new high-order derivations called CFD and DCF. Also, they studied the existence of solutions for two such types of high-order fractional integrodifferential equations. Guoa et al. in [28, 29] studied the existence and HU stability of the FDEs with impulse and fractional Brownian motion under nonlocal conditions based on the semigroups of operators method and Mönch fixed point method.

There are some researchers in various areas who investigated the existence and Ulam–Hyers (UH) stability of solutions for fractional differential equations (FDEs) of some problems with initial (or boundary, nonlocal) conditions; more details can be found within research series [30–35].

Recently, Abdo et al. [36] considered ABC-type pantograph FDEs with nonlocal conditions:

$$\begin{cases} {}^{\mathrm{ABC}}D_{a^+}^{p}\vartheta(\sigma) = f(\sigma, \vartheta(\sigma), \vartheta(\gamma\sigma)), \sigma \in [a, T], p \in (0, 1], \\ \vartheta(a) = \sum_{k=1}^{m} c_k \vartheta(\varkappa_k), \varkappa_k \in (a, T). \end{cases} \tag{1}$$

In this regard, the ABC-type implicit FDEs

$${}^{\mathrm{ABC}}D_{a^+}^{p}\vartheta(\sigma) = f\left(\sigma, \vartheta(\sigma), {}^{\mathrm{ABC}}D_{a^+}^{p}\vartheta(\sigma)\right), \quad \sigma \in [a, T], \tag{2}$$

with nonlinear integral conditions

$$\begin{aligned} &\vartheta(a) - \vartheta'(a) = \int_a^T g(s, \vartheta(s))ds, \quad \text{when } p \in (0, 1], \\ &\vartheta(a) = 0, \vartheta(T) = \int_a^T g(s, \vartheta(s))ds, \quad \text{when } p \in (1, 2], \end{aligned} \tag{3}$$

had been studied by Abdo et al. [37].

Motivated by the above argumentations, we intend to investigate and develop the sufficient conditions for existence, uniqueness, and UH stability results of two new classes for the ABR and ABC fractional differential problems described as follows:

$$\begin{cases} {}^{\mathrm{ABR}}D_{a^+}^{p}\vartheta(\sigma) = f(\sigma, \vartheta(\sigma)), & \sigma \in [a, b], p \in (2, 3], \\ \vartheta(a) = 0, \vartheta(b) = {}^{\mathrm{AB}}I_{a^+}^{\delta}\vartheta(\zeta), & \zeta \in (a, b), \end{cases} \tag{4}$$

$$\begin{cases} {}^{\mathrm{ABC}}D_{a^+}^{p}\vartheta(\sigma) = f(\sigma, \vartheta(\sigma)), & \sigma \in [a, b], p \in (1, 2], \\ \vartheta(a) = 0, \vartheta(b) = {}^{\mathrm{AB}}I_{a^+}^{\delta}\vartheta(\zeta), & \zeta \in (a, b), \end{cases} \tag{5}$$

where ${}^{\mathrm{ABR}}D_{a^+}^{p}$, ${}^{\mathrm{ABC}}D_{a^+}^{p}$ are the ABR and ABC fractional derivatives of order $p \in (2, 3]$ and $p \in (1, 2]$, respectively, $\delta \in (0, 1]$, $\zeta \in (a, b)$, and $f\colon [a, b] \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function.

To the best of our understanding, this is the first work that transacts with ABC and ABR fractional derivatives of a high order, especially with AB integral conditions. Also, we studied existence and uniqueness without using the semigroup property. In consequence, our results will be a useful contribution to the existing literature on these interesting operators.

The paper is organized as follows. In Section 2, we present some notations and some preliminary facts which are used throughout the paper. In Section 3, we derive the formula of an equivalent integral equation for ABR-type FDEs (4). Section 4 discusses the existence and uniqueness results for ABR-type FDEs (4) and ABC-type FDEs (5). In Section 4, we discuss the stability results in the frame of UH. Section 5 provides two examples to illustrate the validity of our results. Concluding remarks about our results are in the last section.

2. Preliminaries

Let $J = [a, b] \subset \mathbb{R}$ and $C(J, \mathbb{R})$ be the space of all continuous functions $\vartheta\colon J \longrightarrow \mathbb{R}$ with the norm $\|\vartheta\| = \max \{|\vartheta(\sigma)|\colon \sigma \in J\}$. Then, $(C(J, \mathbb{R}), \|\cdot\|)$ is a Banach space.

Definition 1 (see [11]). Let $0 < p \leq 1$. Then, the left-sided ABC and ABR fractional derivatives of order p for a function ϑ are defined by

$$\begin{aligned} {}^{\mathrm{ABC}}D_{a^+}^{p}\vartheta(\sigma) &= \frac{\mathfrak{B}(p)}{1-p}\int_a^{\sigma} E_p\left(\frac{p}{p-1}(\sigma-\theta)^p\right)\vartheta'(\theta)d\theta, \quad \sigma > a, \\ {}^{\mathrm{ABR}}D_{a^+}^{p}\vartheta(\sigma) &= \frac{\mathfrak{B}(p)}{1-p}\frac{d}{d\sigma}\int_a^{\sigma} E_p\left(\frac{p}{p-1}(\sigma-\theta)^p\right)\vartheta'(\theta)d\theta, \quad \sigma > a, \end{aligned} \tag{6}$$

respectively, where $\mathfrak{B}(p)$ is the normalization function such that $\mathfrak{B}(0) = \mathfrak{B}(1) = 1$ and E_p is the Mittag-Leffler function defined by

$$E_p(\vartheta) = \sum_{i=0}^{\infty} \frac{\vartheta^i}{\Gamma(ip+1)}, \quad \mathrm{Re}(p) > 0, \vartheta \in \mathbb{C}. \tag{7}$$

The analogous AB fractional integral is given by

$$ {}^{AB}I^{p}_{a^+}\vartheta(\sigma) = \frac{1-p}{\mathfrak{B}(p)}\vartheta(\sigma) + \frac{p}{\mathfrak{B}(p)\Gamma(p)}\int_a^\sigma (\sigma-s)^{p-1}\vartheta(s)\mathrm{d}s. \tag{8} $$

Lemma 1 (see [13]). *Let $0<p\leq 1$. If the ABC fractional derivative exists, then we have*

$$ {}^{AB}I^{p}_{a^+}\,{}^{ABC}D^{p}_{a^+}\vartheta(\sigma) = \vartheta(\sigma) - \vartheta(a). \tag{9} $$

Definition 2 (see [12]). The relation between the ABR and ABC fractional derivatives is given by

$$ \begin{aligned} {}^{ABC}D^{p}_{a^+}\vartheta(\sigma) = {}&{}^{ABR}D^{p}_{a^+}\vartheta(\sigma) \\ &- \frac{\mathfrak{B}(p)}{1-p}\vartheta(a)E_p\left(\frac{p}{p-1}(\sigma-a)^p\right). \end{aligned} \tag{10} $$

Remark 1. By applying ${}^{AB}I^{p}_{a^+}\vartheta(\sigma)$ on both sides of the equation defined in Definition 2 and using Lemma 1, we get

$$ {}^{AB}I^{p}_{a^+}\,{}^{ABR}D^{p}_{a^+}\vartheta(\sigma) = \vartheta(\sigma). \tag{11} $$

Lemma 2 (see [11]). *Let $\vartheta>0$. Then, ${}^{AB}I^{p}_{a^+}$ is bounded from $C(J,\mathbb{R})$ into $C(J,\mathbb{R})$.*

Definition 3 (see Definition 3.1 of [12]). Let $n<p\leq n+1$ and ϑ be such that $\vartheta^{(n)}\in H^1(a,b)$. Set $\beta=p-n$. Then, $0<\beta\leq 1$, and we define

$$ \begin{aligned} \left({}^{ABC}D^{p}_{a^+}\vartheta\right)(\sigma) &= \left({}^{ABC}D^{\beta}_{a^+}\vartheta^{(n)}\right)(\sigma), \\ \left({}^{ABR}D^{p}_{a^+}\vartheta\right)(\sigma) &= \left({}^{ABR}D^{\beta}_{a^+}\vartheta^{(n)}\right)(\sigma). \end{aligned} \tag{12} $$

The correspondent AB fractional integral is given by

$$ \left({}^{AB}I^{p}_{a^+}\vartheta\right)(\sigma) = \left(I^{n}_{a^+}\,{}^{AB}I^{\beta}_{a^+}\vartheta\right)(\sigma). \tag{13} $$

Lemma 3 (see Proposition 3.1 of [12]). *For $\vartheta(\sigma)$ defined on J and $n<p\leq n+1$, for some $n\in\mathbb{N}_0$, we have*

(i) $({}^{ABC}D^{p}_{a^+}\,{}^{AB}I^{p}_{a^+}\vartheta)(\sigma) = \vartheta(\sigma)$

(ii) $({}^{AB}I^{p}_{a^+}\,{}^{ABR}D^{p}_{a^+}\vartheta)(\sigma) = \vartheta(\sigma) - \sum_{i=0}^{n-1}(\vartheta^{(i)}(a)/i!)\,(\sigma-a)^i$

(iii) $({}^{AB}I^{p}_{a^+}\,{}^{ABC}D^{p}_{a^+}\vartheta)(\sigma) = \vartheta(\sigma) - \sum_{i=0}^{n-1}(\vartheta^{(i)}(a)/i!)(\sigma-a)^i$

Lemma 4. *Let $n<p\leq n+1$. Then, ${}^{ABR}D^{p}_{a^+}(\sigma-a)^m=0$, for $m=1,2,\ldots,n-1$.*

Proof. Let $\vartheta(\sigma)=(\sigma-a)^m$. Then, by Definition 3, we have

$$ \begin{aligned} {}^{ABC}D^{p}_{a^+}\vartheta(\sigma) &= {}^{ABR}D^{\beta}_{a^+}\vartheta^{(n)}(\sigma) \\ &= {}^{ABR}D^{\beta}_{a^+}\left[(\sigma-a)^m\right]^{(n)} \\ &= {}^{ABR}D^{\beta}_{a^+}\left(\frac{\mathrm{d}}{\mathrm{d}\sigma}\right)^{(n)}(\sigma-a)^m. \end{aligned} \tag{14} $$

Since $m<n\in\mathbb{N}$, we have $(\mathrm{d}/\mathrm{d}\sigma)^{(n)}(\sigma-a)^m=0$. Consequently, ${}^{ABR}D^{p}_{a^+}(\sigma-a)^m=0$. □

Theorem 1 (see [38]). *Assume that* **K** *is a closed subspace from a Banach space* **X**, *and* **G** *is a contraction defined as* **Q**: **K** ⟶ **K**, *i.e.,* $\|\mathbf{Q}(x)-\mathbf{Q}(y)\|\leq \mathbf{M}\|x-y\|$ *for some* $0<\mathbf{M}<1$ *and all* $x,y\in\mathbf{K}$. *Then,* **Q** *has a fixed point in* **K**.

Theorem 2 (see [39]). *Let* **X** *be a Banach space, and let a set* $K\subset\mathbf{X}$ *be nonempty, closed, convex, and bounded. If there are two operators* Φ^1,Φ^2 *such that (i)* $\Phi^1u+\Phi^2v\in\mathbf{X}$, *for all,* $u,v\in\mathbf{X}$ Φ^1 *is compact and continuous, and (iii)* Φ^2 *is a contraction mapping, then there exists a function* $z\in K$ *such that* $z=\Phi^1z+\Phi^2z$.

Lemma 5 (see Lemma 5.1 of [12]). *Let $p\in(2,3]$ and $h\in C(J,\mathbb{R})$. Then, the solution of the following linear problem,*

$$ \left\{ {}^{ABR}D^{p}_{a^+}\vartheta(\sigma) = h(\sigma), \vartheta(a)=c_1, \vartheta(b)=c_2, \right. \tag{15} $$

is given by

$$ \begin{aligned} \vartheta(\sigma) &= c_1 + c_2(\sigma-a) + {}^{AB}I^{p}_{a^+}h(\sigma) \\ &= c_1 + c_2(\sigma-a) + I^{2}_{a^+}\,{}^{AB}I^{\beta}_{a^+}h(\sigma), \quad 0<\beta=(p-2)\leq 1, \end{aligned} \tag{16} $$

where

$$ {}^{AB}I^{\beta}_{a^+}h(\sigma) = \frac{1-\beta}{\mathfrak{B}(\beta)}h(\sigma) + \frac{\beta}{\mathfrak{B}(\beta)\Gamma(\beta)}\int_a^\sigma(\sigma-s)^{\beta-1}h(s)\mathrm{d}s = \frac{3-p}{\mathfrak{B}(p-2)}h(\sigma) + \frac{p-2}{\mathfrak{B}(p-2)\Gamma(p-2)}\int_a^\sigma(\sigma-s)^{p-3}h(s)\mathrm{d}s, \tag{17} $$

$$ {}^{AB}I^{p}_{a^+}h(\sigma) = I^{2}_{a^+}\,{}^{AB}I^{\delta}_{a^+}h(\sigma) = \frac{3-p}{\mathfrak{B}(p-2)}\int_a^\sigma(\sigma-s)h(s)\mathrm{d}s + \frac{p-2}{\mathfrak{B}(p-2)\Gamma(p-2)}\int_a^\sigma(\sigma-s)^{p-1}h(s)\mathrm{d}s. \tag{18} $$

Lemma 6 (see Example 3.3 of [12]). *Let $p\in(1,2]$ and $h\in C(J,\mathbb{R})$. Then, the solution of the following linear problem,*

$$ \left\{ {}^{ABC}D^{p}_{a^+}\vartheta(\sigma) = h(\sigma), \vartheta(a)=c_1, \vartheta'(a)=c_2, \right. \tag{19} $$

is given by

$$\vartheta(\sigma) = c_1 + c_2(\sigma - a) + {}^{\mathrm{AB}}I_{a^+}^{p} h(\sigma), \tag{20}$$

where

$$\begin{aligned} {}^{\mathrm{AB}}I_{a^+}^{p} h(\sigma) = {} & \frac{2-p}{\mathfrak{B}(p-1)} \int_a^\sigma h(s)\mathrm{d}s \\ & + \frac{p-1}{\mathfrak{B}(p-1)\Gamma(p)} \int_a^\sigma (\sigma - s)^{p-1} h(s)\mathrm{d}s. \end{aligned} \tag{21}$$

3. Equivalent Integral Equation for Problem (4)

Lemma 7. *Let* $p \in (2,3], \delta \in (0,1]$, $\Lambda = (b-a) - {}^{\mathrm{AB}}I_{a^+}^{\delta}(\zeta - a) \neq 0$, *and* $h_1 \in C(J, \mathbb{R})$. *A function* ϑ *is a solution of the following ABR problem,*

$$\begin{cases} {}^{\mathrm{ABR}}D_{a^+}^{p} \vartheta(\sigma) = h_1(\sigma), & \sigma \in J, \\ \vartheta(a) = 0, \vartheta(b) = {}^{\mathrm{AB}}I_{a^+}^{\delta} \vartheta(\zeta), & \zeta \in (a,b), \end{cases} \tag{22}$$

if and only if ϑ *satisfies the following fractional integral equation:*

$$\vartheta(\sigma) = \frac{(\sigma - a)}{\Lambda} \left[{}^{\mathrm{AB}}I_{a^+}^{\delta} \, {}^{\mathrm{AB}}I_{a^+}^{p} h_1(\zeta) - {}^{\mathrm{AB}}I_{a^+}^{p} h_1(b) \right] + {}^{\mathrm{AB}}I_{a^+}^{p} h_1(\sigma). \tag{23}$$

Proof. Assume that ϑ is the solution to the first equation of (22). Then, via Lemma 5, we have

$$\vartheta(\sigma) = c_1 + c_2(\sigma - a) + {}^{\mathrm{AB}}I_{a^+}^{p} h_1(\sigma). \tag{24}$$

Now, by condition ($\vartheta(a) = 0$), we get $c_1 = 0$, and hence, equation (24) reduces to

$$\vartheta(\sigma) = c_2(\sigma - a) + {}^{\mathrm{AB}}I_{a^+}^{p} h_1(\sigma). \tag{25}$$

Taking the operator ${}^{\mathrm{AB}}I_{a^+}^{\delta}$ on both sides of equation (25), we have

$${}^{\mathrm{AB}}I_{a^+}^{\delta} \vartheta(\sigma) = {}^{\mathrm{AB}}I_{a^+}^{\delta} c_2(\sigma - a) + {}^{\mathrm{AB}}I_{a^+}^{\delta} \, {}^{\mathrm{AB}}I_{a^+}^{p} h_1(\sigma). \tag{26}$$

Now, replacing σ by ζ, we get

$${}^{\mathrm{AB}}I_{a^+}^{\delta} \vartheta(\zeta) = {}^{\mathrm{AB}}I_{a^+}^{\delta} c_2(\zeta - a) + {}^{\mathrm{AB}}I_{a^+}^{\delta} \, {}^{\mathrm{AB}}I_{a^+}^{p} h_1(\zeta). \tag{27}$$

By the second condition ($\vartheta(b) = {}^{\mathrm{AB}}I_{a^+}^{\delta} \vartheta(\zeta)$) together with equations (25) and (27), we get

$$c_2 = \frac{1}{\Lambda} \left[{}^{\mathrm{AB}}I_{a^+}^{\delta} \, {}^{\mathrm{AB}}I_{a^+}^{p} h_1(\zeta) - {}^{\mathrm{AB}}I_{a^+}^{p} h_1(b) \right]. \tag{28}$$

Substituting c_1 and c_2 in (24), we get (23).

Conversely, assume that ϑ satisfies (23). Then, by applying the operator ${}^{\mathrm{ABR}}D_{a^+}^{p}$ on both sides of (23) and using Lemmas 3 and 4, we get

$$\begin{aligned} {}^{\mathrm{ABR}}D_{a^+}^{p} \vartheta(\sigma) &= {}^{\mathrm{ABR}}D_{a^+}^{p} \frac{(\sigma - a)}{\Lambda} \left[{}^{\mathrm{AB}}I_{a^+}^{\delta} \, {}^{\mathrm{AB}}I_{a^+}^{p} h_1(\zeta) - {}^{\mathrm{AB}}I_{a^+}^{p} h_1(b) \right] \\ &\quad + {}^{\mathrm{ABR}}D_{a^+}^{p} {}^{\mathrm{AB}}I_{a^+}^{p} h_1(\sigma) \\ &= h_1(\sigma). \end{aligned} \tag{29}$$

Next, taking $\sigma \longrightarrow a$ in (23), we get $\vartheta(a) = 0$. On the contrary, applying ${}^{\mathrm{AB}}I_{a^+}^{\delta}$ on both sides of (23), we get

$$\begin{aligned} {}^{\mathrm{AB}}I_{a^+}^{\delta} \vartheta(\sigma) &= \frac{{}^{\mathrm{AB}}I_{a^+}^{\delta}(\sigma - a)}{\Lambda} \left[{}^{\mathrm{AB}}I_{a^+}^{\delta} \, {}^{\mathrm{AB}}I_{a^+}^{p} h_1(\zeta) - {}^{\mathrm{AB}}I_{a^+}^{p} h_1(b) \right] \\ &\quad + {}^{\mathrm{AB}}I_{a^+}^{\delta} \, {}^{\mathrm{AB}}I_{a^+}^{p} h_1(\sigma). \end{aligned} \tag{30}$$

Replacing σ by ζ, we get

$$\begin{aligned} {}^{\mathrm{AB}}I_{a^+}^{\delta} \vartheta(\zeta) &= \frac{{}^{\mathrm{AB}}I_{a^+}^{\delta}(\zeta - a)}{\Lambda} \left[{}^{\mathrm{AB}}I_{a^+}^{\delta} \, {}^{\mathrm{AB}}I_{a^+}^{p} h_1(\zeta) - {}^{\mathrm{AB}}I_{a^+}^{p} h_1(b) \right] \\ &\quad + {}^{\mathrm{AB}}I_{a^+}^{\delta} \, {}^{\mathrm{AB}}I_{a^+}^{p} h_1(\zeta). \end{aligned} \tag{31}$$

By the definition of Λ, we can rewrite (31) as

$$\begin{aligned} {}^{\mathrm{AB}}I_{a^+}^{\delta} \vartheta(\zeta) &= \frac{(b-a) - \left[(b-a) - {}^{\mathrm{AB}}I_{a^+}^{\delta}(\zeta - a)\right]}{(b-a) - {}^{\mathrm{AB}}I_{a^+}^{\delta}(\zeta - a)} \\ &\quad \left[{}^{\mathrm{AB}}I_{a^+}^{\delta} \, {}^{\mathrm{AB}}I_{a^+}^{p} h_1(\zeta) - {}^{\mathrm{AB}}I_{a^+}^{p} h_1(b) \right] + {}^{\mathrm{AB}}I_{a^+}^{\delta} \, {}^{\mathrm{AB}}I_{a^+}^{p} h_1(\zeta) \\ &= \frac{(b-a)}{(b-a) - {}^{\mathrm{AB}}I_{a^+}^{\delta}(\zeta - a)} \\ &\quad \left[{}^{\mathrm{AB}}I_{a^+}^{\delta} \, {}^{\mathrm{AB}}I_{a^+}^{p} h_1(\zeta) - {}^{\mathrm{AB}}I_{a^+}^{p} h_1(b) \right] + {}^{\mathrm{AB}}I_{a^+}^{p} h_1(b) \\ &= \frac{(b-a)}{\Lambda} \left[{}^{\mathrm{AB}}I_{a^+}^{\delta} \, {}^{\mathrm{AB}}I_{a^+}^{p} h_1(\zeta) - {}^{\mathrm{AB}}I_{a^+}^{p} h_1(b) \right] + {}^{\mathrm{AB}}I_{a^+}^{p} h_1(b) \\ &= \vartheta(b). \end{aligned} \tag{32}$$

Thus, the Atangana–Baleanu fractional integral conditions are satisfied. □

As a result of Lemma 7, we get the following theorem.

Theorem 3. *Let* $p \in (2,3], \delta \in (0,1], \Lambda = (b-a) - {}^{\mathrm{AB}}I_{a^+}^{\delta}(\zeta - a) \neq 0$, *and* $f : J \times \mathbb{R} \longrightarrow \mathbb{R}$ *be a continuous function. Then, ABR-type FDEs (4) are equivalent to the following FIE:*

$$\vartheta(\sigma)=\frac{(\sigma-a)}{\Lambda}[\mathscr{A}_1\int_a^\zeta(\zeta-s)f(s,\vartheta(s))\mathrm{d}s+\frac{\mathscr{A}_2}{\Gamma(p-2)}\int_a^\zeta(\zeta-s)^{p-1}f(s,\vartheta(s))\mathrm{d}s$$
$$+\frac{\mathscr{A}_3}{\Gamma(\delta)}\int_a^\zeta(\zeta-s)^{\delta-1}\int_a^s(s-\eta)f(\eta,\vartheta(\eta))\mathrm{d}\eta\mathrm{d}s+\frac{\mathscr{A}_4}{\Gamma(\delta+p-2)}\int_a^\zeta(\zeta-s)^{\delta+p-1}f(s,\vartheta(s))\mathrm{d}s$$
$$-\mathscr{A}_5\int_a^b(b-s)f(s,\vartheta(s))\mathrm{d}s-\frac{\mathscr{A}_6}{\Gamma(p-2)}\int_a^b(b-s)^{p-1}f(s,\vartheta(s))\mathrm{d}s]$$
$$+\mathscr{A}_5\int_a^\sigma(\sigma-s)f(s,\vartheta(s))\mathrm{d}s+\frac{\mathscr{A}_6}{\Gamma(p-2)}\int_a^\sigma(\sigma-s)^{p-1}f(s,\vartheta(s))\mathrm{d}s, \tag{33}$$

where

$$\mathscr{A}_1=\frac{(1-\delta)(3-p)}{\mathfrak{B}(\delta)\mathfrak{B}(p-2)},$$
$$\mathscr{A}_2=\frac{(1-\delta)(p-2)}{\mathfrak{B}(\delta)\mathfrak{B}(p-2)},$$
$$\mathscr{A}_3=\frac{\delta(3-p)}{\mathfrak{B}(\delta)\mathfrak{B}(p-2)},$$
$$\mathscr{A}_4=\frac{\delta(p-2)}{\mathfrak{B}(\delta)\mathfrak{B}(p-2)}, \tag{34}$$
$$\mathscr{A}_5=\frac{3-p}{\mathfrak{B}(p-2)},$$
$$\mathscr{A}_6=\frac{p-2}{\mathfrak{B}(p-2)}.$$

Proof. In view of Lemma 7, we have

$$\vartheta(\sigma)=\frac{(\sigma-a)}{\Lambda}\left[{}^{\mathrm{AB}}I_{a^+}^{\delta}\,{}^{\mathrm{AB}}I_{a^+}^{p}f(\zeta,\vartheta(\zeta))-{}^{\mathrm{AB}}I_{a^+}^{p}f(b,\vartheta(b))\right]$$
$$+{}^{\mathrm{AB}}I_{a^+}^{p}f(\sigma,\vartheta(\sigma)). \tag{35}$$

By using definitions of ${}^{\mathrm{AB}}I_{a^+}^{\delta}$ in the case $\delta\in(0,1)$ defined in (8) and ${}^{\mathrm{AB}}I_{a^+}^{p}$ in the case $p\in(2,3]$ defined in (18), we can rewrite equation (35) as follows:

$$\vartheta(\sigma)=\frac{(\sigma-a)}{\Lambda}[\frac{(1-\delta)}{\mathfrak{B}(\delta)}$$
$$\left\{\frac{3-p}{\mathfrak{B}(p-2)}\int_a^\zeta(\zeta-s)f(s,\vartheta(s))\mathrm{d}s+\frac{p-2}{\mathfrak{B}(p-2)\Gamma(p-2)}\int_a^\zeta(\zeta-s)^{p-1}f(s,\vartheta(s))\mathrm{d}s\right\}$$
$$+\frac{\delta}{\mathfrak{B}(\delta)\Gamma(\delta)}\int_a^\zeta(\zeta-s)^{\delta-1}\left\{\frac{3-p}{\mathfrak{B}(p-2)}\int_a^s(s-\eta)f(\eta,\vartheta(\eta))\mathrm{d}\eta+\frac{p-2}{\mathfrak{B}(p-2)\Gamma(p-2)}\int_a^s(s-\eta)^{p-1}f(\eta,\vartheta(\eta))\mathrm{d}\eta\right\}\mathrm{d}s$$
$$-\frac{3-p}{\mathfrak{B}(p-2)}\int_a^b(b-s)f(s,\vartheta(s))\mathrm{d}s-\frac{p-2}{\mathfrak{B}(p-2)\Gamma(p-2)}\int_a^b(b-s)^{p-1}f(s,\vartheta(s))\mathrm{d}s]$$
$$+\frac{3-p}{\mathfrak{B}(p-2)}\int_a^\sigma(\sigma-s)f(s,\vartheta(s))\mathrm{d}s+\frac{p-2}{\mathfrak{B}(p-2)\Gamma(p-2)}\int_a^\sigma(\sigma-s)^{p-1}f(s,\vartheta(s))\mathrm{d}s$$
$$=\frac{(\sigma-a)}{\Lambda}[\frac{(1-\delta)}{\mathfrak{B}(\delta)}\frac{3-p}{\mathfrak{B}(p-2)}\int_a^\zeta(\zeta-s)f(s,\vartheta(s))\mathrm{d}s$$
$$+\frac{(1-\delta)}{\mathfrak{B}(\delta)}\frac{p-2}{\mathfrak{B}(p-2)\Gamma(p-2)}\int_a^\zeta(\zeta-s)^{p-1}f(s,\vartheta(s))\mathrm{d}s$$

$$
\begin{aligned}
&+\frac{3-p}{\mathfrak{B}(p-2)}\frac{\delta}{\mathfrak{B}(\delta)\Gamma(\delta)}\int_a^\zeta(\zeta-s)^{\delta-1}\int_a^s(s-\eta)f(\eta,\vartheta(\eta))\mathrm{d}\eta\mathrm{d}s\\
&+\frac{p-2}{\mathfrak{B}(p-2)\Gamma(p-2)}\frac{\delta}{\mathfrak{B}(\delta)\Gamma(\delta)}\int_a^\zeta(\zeta-s)^{\delta-1}\int_a^s(s-\eta)f(\eta,\vartheta(\eta))\mathrm{d}\eta\mathrm{d}s\\
&-\frac{3-p}{\mathfrak{B}(p-2)}\int_a^b(b-s)f(s,\vartheta(s))\mathrm{d}s-\frac{p-2}{\mathfrak{B}(p-2)\Gamma(p-2)}\int_a^b(b-s)^{p-1}f(s,\vartheta(s))\mathrm{d}s]\\
&+\frac{3-p}{\mathfrak{B}(p-2)}\int_a^\sigma(\sigma-s)f(s,\vartheta(s))\mathrm{d}s+\frac{p-2}{\mathfrak{B}(p-2)\Gamma(p-2)}\int_a^\sigma(\sigma-s)^{p-1}f(s,\vartheta(s))\mathrm{d}s\\
=&\frac{(\sigma-a)}{\Lambda}[\mathscr{A}_1\int_a^\zeta(\zeta-s)f(s,\vartheta(s))\mathrm{d}s+\frac{\mathscr{A}_2}{\Gamma(p-2)}\int_a^\zeta(\zeta-s)^{p-1}f(s,\vartheta(s))\mathrm{d}s\\
&+\frac{\mathscr{A}_3}{\Gamma(\delta)}\int_a^\zeta(\zeta-s)^{\delta-1}\int_a^s(s-\eta)f(\eta,\vartheta(\eta))\mathrm{d}\eta\mathrm{d}s+\frac{\mathscr{A}_4}{\Gamma(\delta+p-2)}\int_a^\zeta(\zeta-s)^{\delta+p-1}f(s,\vartheta(s))\mathrm{d}s\\
&-\mathscr{A}_5\int_a^b(b-s)f(s,\vartheta(s))\mathrm{d}s-\frac{\mathscr{A}_6}{\Gamma(p-2)}\int_a^b(b-s)^{p-1}f(s,\vartheta(s))\mathrm{d}s]\\
&+\mathscr{A}_5\int_a^\sigma(\sigma-s)f(s,\vartheta(s))\mathrm{d}s+\frac{\mathscr{A}_6}{\Gamma(p-2)}\int_a^\sigma(\sigma-s)^{p-1}f(s,\vartheta(s))\mathrm{d}s.
\end{aligned}
\tag{36}
$$

□

For simplicity, we set

$$
\begin{aligned}
\rho_1&=[\frac{\mathscr{A}_1(\zeta-a)^2}{2}+\frac{\mathscr{A}_2(\zeta-a)^p}{\Gamma(p-1)}+\frac{\mathscr{A}_3(\zeta-a)^{\delta+1}}{\Gamma(\delta+1)}\\
&\quad+\frac{\mathscr{A}_4(\zeta-a)^{\delta+p}}{\Gamma(\delta+p-1)}+\frac{\mathscr{A}_5(b-a)^2}{2}+\frac{\mathscr{A}_6(b-a)^p}{\Gamma(p-1)}],\\
\rho_2&=\frac{\mathscr{A}_5(b-a)^2}{2}+\frac{\mathscr{A}_6(b-a)^p}{\Gamma(p-1)},\\
\rho_3&=[\mathscr{A}_7(\zeta-a)+\frac{\mathscr{A}_8(\zeta-a)^p}{\Gamma(p+1)}+\frac{\mathscr{A}_9(\zeta-a)^{\delta+1}}{\Gamma(\delta+1)}\\
&\quad+\frac{\mathscr{A}_{10}(\zeta-a)^{\delta+p}}{\Gamma(\delta+p+1)}+\mathscr{A}_{11}(b-a)+\frac{\mathscr{A}_{12}(b-a)^p}{\Gamma(p+1)}],\\
\rho_4&=\mathscr{A}_{11}(b-a)+\frac{\mathscr{A}_{12}(b-a)^p}{\Gamma(p+1)}.
\end{aligned}
\tag{37}
$$

4. Existence and Uniqueness of Solutions for Problem (4)

In this section, we devote our intention to prove the existence and uniqueness of solutions for ABR-type FDEs (4).

Theorem 4. *Suppose that $f: J\times\mathbb{R}\longrightarrow\mathbb{R}$ is a continuous function. Moreover, we assume that*

$$
|f(\sigma,u)-f(\sigma,\bar{u})|\le L_f|u-\bar{u}|,\quad L_f>0,u,\bar{u}\in\mathbb{R}. \tag{38}
$$

If

$$
\mathscr{Z}=L_f\left(\frac{\rho_1(b-a)}{\Lambda}+\rho_2\right)<1, \tag{39}
$$

then ABR-type FDEs (4) have a unique solution in J.

Proof. Consider the operator $\Xi: C(J,\mathbb{R})\longrightarrow C(J,\mathbb{R})$ defined by

$$
\begin{aligned}
\Xi\vartheta(\sigma)=&\frac{(\sigma-a)}{\Lambda}[\mathscr{A}_1\int_a^\zeta(\zeta-s)f(s,\vartheta(s))\mathrm{d}s+\frac{\mathscr{A}_2}{\Gamma(p-2)}\int_a^\zeta(\zeta-s)^{p-1}f(s,\vartheta(s))\mathrm{d}s\\
&+\frac{\mathscr{A}_3}{\Gamma(\delta)}\int_a^\zeta(\zeta-s)^{\delta-1}\int_a^s(s-\eta)f(\eta,\vartheta(\eta))\mathrm{d}\eta\mathrm{d}s
\end{aligned}
$$

$$+\frac{\mathcal{A}_4}{\Gamma(\delta+p-2)}\int_a^\zeta(\zeta-s)^{\delta+p-1}f(s,\vartheta(s))\mathrm{d}s$$
$$-\mathcal{A}_5\int_a^b(b-s)f(s,\vartheta(s))\mathrm{d}s-\frac{\mathcal{A}_6}{\Gamma(p-2)}\int_a^b(b-s)^{p-1}f(s,\vartheta(s))\mathrm{d}s] \tag{40}$$
$$+\mathcal{A}_5\int_a^\sigma(\sigma-s)f(s,\vartheta(s))\mathrm{d}s+\frac{\mathcal{A}_6}{\Gamma(p-2)}\int_a^\sigma(\sigma-s)^{p-1}f(s,\vartheta(s))\mathrm{d}s.$$

Define a closed ball Π_φ as

$$\Pi_\varphi=\{\vartheta\in C(J,\mathbb{R}):\quad \|\vartheta\|\le\varphi\}, \tag{41}$$

with radius $\varphi\ge(\mathcal{Z}_1/(1-\mathcal{Z}))$, where

$$\mathcal{Z}_1=\left(\frac{(b-a)}{\Lambda}\rho_1+\rho_2\right)\omega_f, \tag{42}$$

where $\omega_f=\sup_{\sigma\in J}|f(\sigma,0)|$. Now, we show that $\Xi\Pi_\varphi\subset\Pi_\varphi$. For all $\vartheta\in\Pi_\varphi$ and $\sigma\in J$, we have

$$|(\Xi\vartheta)(\sigma)|\le\frac{(\sigma-a)}{\Lambda}[\mathcal{A}_1\int_a^\zeta(\zeta-s)|f(s,\vartheta(s))|\mathrm{d}s+\frac{\mathcal{A}_2}{\Gamma(p-2)}\int_a^\zeta(\zeta-s)^{p-1}|f(s,\vartheta(s))|\mathrm{d}s$$
$$+\frac{\mathcal{A}_3}{\Gamma(\delta)}\int_a^\zeta(\zeta-s)^{\delta-1}\int_a^s(s-\eta)|f(\eta,\vartheta(\eta))|\mathrm{d}\eta\mathrm{d}s$$
$$+\frac{\mathcal{A}_4}{\Gamma(\delta+p-2)}\int_a^\zeta(\zeta-s)^{\delta+p-1}|f(s,\vartheta(s))|\mathrm{d}s \tag{43}$$
$$+\mathcal{A}_5\int_a^b(b-s)|f(s,\vartheta(s))|\mathrm{d}s+\frac{\mathcal{A}_6}{\Gamma(p-2)}\int_a^b(b-s)^{p-1}|f(s,\vartheta(s))|\mathrm{d}s]$$
$$+\mathcal{A}_5\int_a^\sigma(\sigma-s)|f(s,\vartheta(s))|\mathrm{d}s+\frac{\mathcal{A}_6}{\Gamma(p-2)}\int_a^\sigma(\sigma-s)^{p-1}|f(s,\vartheta(s))|\mathrm{d}s.$$

By (38), we have

$$\begin{aligned}&|f(\sigma,\vartheta(\sigma))|\\&=|f(\sigma,\vartheta(\sigma))-f(\sigma,0)|+|f(\sigma,0)|\\&\le L_f|\vartheta(\sigma)|+|f(\sigma,0)|\\&\le L_f\varphi+\omega_f.\end{aligned} \tag{44}$$

Hence,

$$\begin{aligned}\|\Xi\vartheta\|&\le\frac{(L_f\varphi+\omega_f)(\sigma-a)}{\Lambda}\rho_1+(L_f\varphi+\omega_f)\rho_2\\&\le L_f\left(\frac{(b-a)}{\Lambda}\rho_1+\rho_2\right)\varphi+\left(\frac{(b-a)}{\Lambda}\rho_1+\rho_2\right)\omega_f\\&=\mathcal{Z}\varphi+\mathcal{Z}_1<\varphi.\end{aligned} \tag{45}$$

Thus, $\Xi\Pi_\varphi\subset\Pi_\varphi$. Next, we will show that Ξ is a contraction mapping. Let $\vartheta,\widehat{\vartheta}\in C(J,\mathbb{R})$ and $\sigma\in J$. Then, we have

$$\begin{aligned}|(\Xi\vartheta)(\sigma)-(\Xi\widehat{\vartheta})(\sigma)|\leq\frac{(\sigma-a)}{\Lambda}&[\mathscr{A}_1\int_a^\zeta(\zeta-s)|f(s,\vartheta(s))-f(s,\widehat{\vartheta}(s))|\mathrm{d}s\\&+\frac{\mathscr{A}_2}{\Gamma(p-2)}\int_a^\zeta(\zeta-s)^{p-1}|f(s,\vartheta(s))-f(s,\widehat{\vartheta}(s))|\mathrm{d}s\\&+\frac{\mathscr{A}_3}{\Gamma(\delta)}\int_a^\zeta(\zeta-s)^{\delta-1}\int_a^s(s-\eta)|f(\eta,\vartheta(\eta))-f(\eta,\widehat{\vartheta}(\eta))|\mathrm{d}\eta\mathrm{d}s\\&+\frac{\mathscr{A}_4}{\Gamma(\delta+p-2)}\int_a^\zeta(\zeta-s)^{\delta+p-1}|f(s,\vartheta(s))-f(s,\widehat{\vartheta}(s))|\mathrm{d}s\\&+\mathscr{A}_5\int_a^b(b-s)|f(s,\vartheta(s))-f(s,\widehat{\vartheta}(s))|\mathrm{d}s\\&+\frac{\mathscr{A}_6}{\Gamma(p-2)}\int_a^b(b-s)^{p-1}|f(s,\vartheta(s))-f(s,\widehat{\vartheta}(s))|\mathrm{d}s]\\&+\mathscr{A}_5\int_a^\sigma(\sigma-s)|f(s,\vartheta(s))-f(s,\widehat{\vartheta}(s))|\mathrm{d}s\\&+\frac{\mathscr{A}_6}{\Gamma(p-2)}\int_a^\sigma(\sigma-s)^{p-1}|f(s,\vartheta(s))-f(s,\widehat{\vartheta}(s))|\mathrm{d}s.\end{aligned}\tag{46}$$

From our assumptions, we obtain

$$\begin{aligned}|f(s,\vartheta(s))-f(s,\widehat{\vartheta}(s))|&\leq L_f|\vartheta(s)-\widehat{\vartheta}(s)|\\&\leq L_f\|\vartheta-\widehat{\vartheta}\|.\end{aligned}\tag{47}$$

Hence,

$$\begin{aligned}\|\Xi\vartheta-\Xi\widehat{\vartheta}\|\leq\frac{L_f(\sigma-a)}{\Lambda}&[\frac{\mathscr{A}_1(\zeta-a)^2}{2}+\frac{\mathscr{A}_2(\zeta-a)^p}{\Gamma(p-1)}\\&+\frac{\mathscr{A}_3(\zeta-a)^{\delta+1}}{\Gamma(\delta+1)}+\frac{\mathscr{A}_4(\zeta-a)^{\delta+p}}{\Gamma(\delta+p-1)}\\&+\frac{\mathscr{A}_5(b-a)^2}{2}+\frac{\mathscr{A}_6(b-a)^p}{\Gamma(p-1)}]\|\vartheta-\widehat{\vartheta}\|\\&+L_f\left(\frac{\mathscr{A}_5(\sigma-a)^2}{2}+\frac{\mathscr{A}_6(\sigma-a)^p}{\Gamma(p-1)}\right)\|\vartheta-\widehat{\vartheta}\|\\&\leq\left(\frac{L_f\rho_1(b-a)}{\Lambda}+L_f\rho_2\right)\|\vartheta-\widehat{\vartheta}\|\\&=\mathscr{Z}\|\vartheta-\widehat{\vartheta}\|.\end{aligned}\tag{48}$$

Due to $\mathscr{Z}<1$, we conclude that Ξ is a contraction operator. Hence, Theorem 1 implies that Ξ has a unique fixed point in J. □

Theorem 5. *Assume that the conditions in Theorem 4 are satisfied. Then, ABR-type FDEs (4) have at least one solution in $C(J,\mathbb{R})$.*

Proof. Let us consider the operator Ξ defined in Theorem 4. Now, we divide the operator Ξ into two operators Ξ_1 and Ξ_2 such that

$$(\Xi\vartheta)(\sigma)=(\Xi_1\vartheta)(\sigma)+(\Xi_2\vartheta)(\sigma),\tag{49}$$

where

$$\begin{aligned}(\Xi_1\vartheta)(\sigma)=\frac{(\sigma-a)}{\Lambda}&[\mathscr{A}_1\int_a^\zeta(\zeta-s)f(s,\vartheta(s))\mathrm{d}s+\frac{\mathscr{A}_2}{\Gamma(p-2)}\int_a^\zeta(\zeta-s)^{p-1}f(s,\vartheta(s))\mathrm{d}s\\&+\frac{\mathscr{A}_3}{\Gamma(\delta)}\int_a^\zeta(\zeta-s)^{\delta-1}\int_a^s(s-\eta)f(\eta,\vartheta(\eta))\mathrm{d}\eta\mathrm{d}s\\&+\frac{\mathscr{A}_4}{\Gamma(\delta+p-2)}\int_a^\zeta(\zeta-s)^{\delta+p-1}f(s,\vartheta(s))\mathrm{d}s\end{aligned}$$

$$
\begin{aligned}
&- \mathscr{A}_5 \int_a^b (b-s) f(s,\vartheta(s)) \mathrm{d}s - \frac{\mathscr{A}_6}{\Gamma(p-2)} \int_a^b (b-s)^{p-1} f(s,\vartheta(s)) \mathrm{d}s], \\
&(\Xi_2 \vartheta)(\sigma) = \mathscr{A}_5 \int_a^\sigma (\sigma - s) f(s,\vartheta(s)) \mathrm{d}s + \frac{\mathscr{A}_6}{\Gamma(p-2)} \int_a^\sigma (\sigma-s)^{p-1} f(s,\vartheta(s)) \mathrm{d}s.
\end{aligned} \tag{50}
$$

Let us consider a closed ball Π_φ defined in Theorem 4. In order to apply Theorem 2, we will divide the proof into the following steps. □

Step 1. We show that $\Xi_1\vartheta + \Xi_2\widetilde{\vartheta} \in \Pi_\varphi$ for all $\vartheta, \widetilde{\vartheta} \in \Pi_\varphi$. For operator Ξ_1, we have

$$
\begin{aligned}
\left|(\Xi_1\vartheta)(\sigma)\right| \le{}& \frac{(\sigma-a)}{\Lambda} [\mathscr{A}_1 \int_a^\zeta (\zeta - s)|f(s,\vartheta(s)) \mathrm{d}s| + \frac{\mathscr{A}_2}{\Gamma(p-2)} \int_a^\zeta (\zeta-s)^{p-1} |f(s,\vartheta(s))| \mathrm{d}s \\
&+ \frac{\mathscr{A}_3}{\Gamma(\delta)} \int_a^\zeta (\zeta-s)^{\delta-1} \int_a^s (s-\eta) |f(\eta,\vartheta(\eta))| \mathrm{d}\eta \mathrm{d}s \\
&+ \frac{\mathscr{A}_4}{\Gamma(\delta+p-2)} \int_a^\zeta (\zeta-s)^{\delta+p-1} |f(s,\vartheta(s))| \mathrm{d}s \\
&+ \mathscr{A}_5 \int_a^b (b-s)|f(s,\vartheta(s))| \mathrm{d}s + \frac{\mathscr{A}_6}{\Gamma(p-2)} \int_a^b (b-s)^{p-1} |f(s,\vartheta(s))| \mathrm{d}s].
\end{aligned} \tag{51}
$$

By (44), we get

$$
\left\|\Xi_1\vartheta\right\| \le \frac{\left(L_f\varphi + \omega_f\right)(b-a)}{\Lambda}\rho_1. \tag{52}
$$

In the same manner, one can prove that

$$
\left\|\Xi_2\widetilde{\vartheta}\right\| = \left(L_f\varphi + \omega_f\right)\rho_2. \tag{53}
$$

Inequalities (52) and (53) give

$$
\begin{aligned}
\left\|\Xi_1\vartheta + \Xi_2\widetilde{\vartheta}\right\| &\le \left\|\Xi_1\vartheta\right\| + \left\|\Xi_2\vartheta\right\| \\
&\le \frac{\left(L_f\varphi + \omega_f\right)(b-a)}{\Lambda}\rho_1 + \left(L_f\varphi + \omega_f\right)\rho_2 \\
&\le L_f\left(\frac{(b-a)}{\Lambda}\rho_1 + \rho_2\right)\varphi + \left(\frac{(b-a)}{\Lambda}\rho_1 + \rho_2\right)\omega_f \\
&= \mathscr{Z}\varphi + \mathscr{Z}_1 < \varphi.
\end{aligned} \tag{54}
$$

Thus, $\Xi_1\vartheta + \Xi_2\widetilde{\vartheta} \in \Pi_\varphi$.

Step 2. Ξ_1 is a contraction map. Due to the operator Ξ being a contraction map, we conclude that Ξ_1 is contraction too.

Step 3. Ξ_2 is continuous and compact. Since f is continuous, Ξ_2 is continuous too. Also, by (53), Ξ_2 is uniformly bounded on Π_φ. Now, we show that $\Xi_2(\Pi_\varphi)$ is equicontinuous. For this purpose, let $\vartheta \in \Pi_\varphi$ and $a \le \sigma_1 < \sigma_2 \le b$.

Then, by using (44), we have

$$
\begin{aligned}
&\left|(\Xi_2\vartheta)(\sigma_2) - (\Xi_2\vartheta)(\sigma_1)\right| \\
&\le \mathscr{A}_5\left[\int_a^{\sigma_1} (\sigma_2 - s) - (\sigma_1 - s)|f(s,\vartheta(s))| \mathrm{d}s + \int_{\sigma_1}^{\sigma_2} (\sigma_2 - s)|f(s,\vartheta(s))| \mathrm{d}s\right] \\
&\quad + \frac{\mathscr{A}_6}{\Gamma(p-2)} \int_a^{\sigma_1} \left[(\sigma_2 - s)^{p-1} - (\sigma_1 - s)^{p-1}\right] |f(s,\vartheta(s))| \mathrm{d}s \\
&\quad + \frac{\mathscr{A}_6}{\Gamma(p-2)} \int_{\sigma_1}^{\sigma_2} (\sigma_2 - s)^{p-1} |f(s,\vartheta(s))| \mathrm{d}s \\
&\le \left(L_f\varphi + \omega_f\right)\left(\frac{\mathscr{A}_5(\sigma_2-\sigma_1)^2}{2} + \frac{\mathscr{A}_6}{\Gamma(p-1)}\left((\sigma_2 - a)^p - (\sigma_1 - a)^p\right)\right).
\end{aligned} \tag{55}
$$

Thus,

$$\left\|(\Xi_2\vartheta)(\sigma_2)-(\Xi_2\vartheta)(\sigma_1)\right\|\longrightarrow 0 \quad \text{as}\, \sigma_2\longrightarrow\sigma_1. \tag{56}$$

According to the above steps and Arzela–Ascoli theorem, we understand that $(\Xi_2\Pi_\varphi)$ is relatively compact. Consequently, Ξ_2 is completely continuous. Thus, by Theorem 2, we infer that ABR-type FDEs (4) have at least one solution in J.

5. Existence of a Unique Solution for Problem (5)

In this section, we devote our intention to proving the existence of a unique solution for ABC-type FDEs (5).

Lemma 8. *Let $p\in(1,2],\delta\in(0,1],\Lambda=(b-a)-{}^{\mathrm{AB}}I^{\delta}_{a^+}(\zeta-a)\neq 0$, and $h_2\in C(J,\mathbb{R})$. A function ϑ is a solution of the following ABC problem,*

$$\begin{cases} {}^{\mathrm{ABC}}D^{p}_{a^+}\vartheta(\sigma)=h_2(\sigma), & \sigma\in[a,b],\\ \vartheta(a)=0,\vartheta(b)={}^{\mathrm{AB}}I^{\delta}_{a^+}\vartheta(\zeta), & \zeta\in(a,b), \end{cases} \tag{57}$$

if and only if ϑ satisfies the following fractional integral equation:

$$\vartheta(\sigma)=\frac{(\sigma-a)}{\Lambda}\left[{}^{\mathrm{AB}}I^{\delta}_{a^+}\,{}^{\mathrm{AB}}I^{p}_{a^+}h_2(\zeta)-{}^{\mathrm{AB}}I^{p}_{a^+}h_2(b)\right]+{}^{\mathrm{AB}}I^{p}_{a^+}h_2(\sigma), \tag{58}$$

where ${}^{\mathrm{AB}}I^{p}_{a^+}$ was defined in (21).

Proof. Assume that ϑ is the solution to the first equation of (57). Then, via Lemma 6 and by the same way in Lemma 7, one can prove that the solution of equation (57) is given as (58), where ${}^{\mathrm{AB}}I^{p}_{a^+}h_2(\sigma)$ is defined in (21). □

Remark 2. The AB fractional integral ${}^{\mathrm{AB}}I^{p}_{a^+}$ used in Lemma 7 was defined in (18), while the AB fractional integral ${}^{\mathrm{AB}}I^{p}_{a^+}$ used in Lemma 7 was defined in (21).

Theorem 6. *Let $p\in(1,2],\delta\in(0,1],\Lambda=(b-a)-{}^{\mathrm{AB}}I^{\delta}_{a^+}(\zeta-a)\neq 0$, and $f\colon J\times\mathbb{R}\longrightarrow\mathbb{R}\longrightarrow$ be a continuous function. Then, in the light of Lemma 6, ABC-type FDEs (5) are equivalent to the following FIE:*

$$\begin{aligned}\vartheta(\sigma)=\frac{(\sigma-a)}{\Lambda}&\Big[\mathscr{A}_7\int_a^\zeta f(s,\vartheta(s))\mathrm{d}s+\frac{\mathscr{A}_8}{\Gamma(p)}\int_a^\zeta(\zeta-s)^{p-1}(s,\vartheta(s))\mathrm{d}s\\&+\frac{\mathscr{A}_9}{\Gamma(\delta)}\int_a^\zeta(\zeta-s)^{\delta-1}\int_a^s f(\eta,\vartheta(\eta))\mathrm{d}\eta\mathrm{d}s+\frac{\mathscr{A}_{10}}{\Gamma(\delta+p)}\int_a^\zeta(\zeta-s)^{\delta+p-1}(s,\vartheta(s))\mathrm{d}s\\&-\mathscr{A}_{11}\int_a^b f(s,\vartheta(s))\mathrm{d}s-\frac{\mathscr{A}_{12}}{\Gamma(p)}\int_a^b(b-s)^{p-1}(s,\vartheta(s))\mathrm{d}s\Big]\\&+\mathscr{A}_{11}\int_a^\sigma f(s,\vartheta(s))\mathrm{d}s+\frac{\mathscr{A}_{12}}{\Gamma(p)}\int_a^\sigma(\sigma-s)^{p-1}(s,\vartheta(s))\mathrm{d}s,\end{aligned} \tag{59}$$

where

$$\mathscr{A}_7=\frac{(1-\delta)(2-p)}{\mathfrak{B}(\delta)\mathfrak{B}(p-1)},$$

$$\mathscr{A}_8=\frac{(1-\delta)(p-1)}{\mathfrak{B}(\delta)\mathfrak{B}(p-1)},$$

$$\mathscr{A}_9=\frac{\delta(2-p)}{\mathfrak{B}(\delta)\mathfrak{B}(p-1)},$$

$$\mathscr{A}_{10}=\frac{\delta(p-1)}{\mathfrak{B}(\delta)\mathfrak{B}(p-1)},$$

$$\mathscr{A}_{11}=\frac{2-p}{\mathfrak{B}(p-1)},$$

$$\mathscr{A}_{12}=\frac{p-1}{\mathfrak{B}(p-1)}. \tag{60}$$

Proof. In view of Lemma 8, we have

$$\vartheta(\sigma)=\frac{(\sigma-a)}{\Lambda}\left[{}^{\mathrm{AB}}I^{\delta}_{a^+}\,{}^{\mathrm{AB}}I^{p}_{a^+}f(\zeta,\vartheta(\zeta))-{}^{\mathrm{AB}}I^{p}_{a^+}f(b,\vartheta(b))\right]+{}^{\mathrm{AB}}I^{p}_{a^+}f(\sigma,\vartheta(\sigma)). \tag{61}$$

By using definitions of ${}^{\mathrm{AB}}I^{\delta}_{a^+}$ in the case $\delta\in(0,1)$ defined in (8) and ${}^{\mathrm{AB}}I^{p}_{a^+}$ in the case $p\in(1,2]$ defined in (21), we can rewrite equation (61) as follows:

$$
\begin{aligned}
\vartheta(\sigma) &= \frac{(\sigma-a)}{\Lambda}\Bigg[\frac{(1-\delta)}{\mathfrak{B}(\delta)} \\
&\left\{\frac{2-p}{\mathfrak{B}(p-1)}\int_a^\zeta f(s,\vartheta(s))\mathrm{d}s+\frac{p-1}{\mathfrak{B}(p-2)\Gamma(p)}\int_a^\zeta (\zeta-s)^{p-1}(s,\vartheta(s))\mathrm{d}s\right\} \\
&+\frac{\delta}{\mathfrak{B}(\delta)\Gamma(\delta)}\int_a^\zeta (\zeta-s)^{\delta-1}\left\{\frac{2-p}{\mathfrak{B}(p-1)}\int_a^s f(\eta,\vartheta(\eta))\mathrm{d}\eta+\frac{p-1}{\mathfrak{B}(p-2)\Gamma(p)}\int_a^s (s-\eta)^{p-1}f(\eta,\vartheta(\eta))\mathrm{d}\eta\right\}\mathrm{d}s \\
&-\frac{2-p}{\mathfrak{B}(p-1)}\int_a^b f(s,\vartheta(s))\mathrm{d}s-\frac{p-1}{\mathfrak{B}(p-2)\Gamma(p)}\int_a^b (b-s)^{p-1}(s,\vartheta(s))\mathrm{d}s\Bigg] \\
&+\frac{2-p}{\mathfrak{B}(p-1)}\int_a^\sigma f(s,\vartheta(s))\mathrm{d}s+\frac{p-1}{\mathfrak{B}(p-2)\Gamma(p)}\int_a^\sigma (\sigma-s)^{p-1}(s,\vartheta(s))\mathrm{d}s \\
&= \frac{(\sigma-a)}{\Lambda}\Bigg[\mathscr{A}_7\int_a^\zeta f(s,\vartheta(s))\mathrm{d}s+\frac{\mathscr{A}_8}{\Gamma(p)}\int_a^\zeta (\zeta-s)^{p-1}(s,\vartheta(s))\mathrm{d}s \\
&+\frac{\mathscr{A}_9}{\Gamma(\delta)}\int_a^\zeta (\zeta-s)^{\delta-1}\int_a^s f(\eta,\vartheta(\eta))\mathrm{d}\eta\mathrm{d}s+\frac{\mathscr{A}_{10}}{\Gamma(\delta+p)}\int_a^\zeta (\zeta-s)^{\delta+p-1}(s,\vartheta(s))\mathrm{d}s \\
&-\mathscr{A}_{11}\int_a^b f(s,\vartheta(s))\mathrm{d}s-\frac{\mathscr{A}_{12}}{\Gamma(p)}\int_a^b (b-s)^{p-1}(s,\vartheta(s))\mathrm{d}s\Bigg] \\
&+\mathscr{A}_{11}\int_a^\sigma f(s,\vartheta(s))\mathrm{d}s+\frac{\mathscr{A}_{12}}{\Gamma(p)}\int_a^\sigma (\sigma-s)^{p-1}(s,\vartheta(s))\mathrm{d}s.
\end{aligned} \tag{62}
$$

□

Theorem 7. *Suppose that $f: J\times\mathbb{R}\longrightarrow\mathbb{R}$ is a continuous function and there exists a constant number $L_f>0$ such that $|f(\sigma,u)-f(\sigma,\bar{u})|\le L_f|u-\bar{u}|$ for any $u,\bar{u}\in J$. If*

$$
\mathscr{X}_3 = L_f\left(\frac{\rho_3(b-a)}{\Lambda}+\rho_4\right)<1, \tag{63}
$$

then ABC-type FDEs (5) have a unique solution in J.

Proof. Define the operator $\mathsf{F}: C(J,\mathbb{R})\longrightarrow C(J,\mathbb{R})$ by

$$
\begin{aligned}
\mathsf{F}\vartheta(\sigma) &= \frac{(\sigma-a)}{\Lambda}\Bigg[\mathscr{A}_7\int_a^\zeta f(s,\vartheta(s))\mathrm{d}s+\frac{\mathscr{A}_8}{\Gamma(p)}\int_a^\zeta (\zeta-s)^{p-1}(s,\vartheta(s))\mathrm{d}s \\
&+\frac{\mathscr{A}_9}{\Gamma(\delta)}\int_a^\zeta (\zeta-s)^{\delta-1}\int_a^s f(\eta,\vartheta(\eta))\mathrm{d}\eta\mathrm{d}s \\
&+\frac{\mathscr{A}_{10}}{\Gamma(\delta+p)}\int_a^\zeta (\zeta-s)^{\delta+p-1}(s,\vartheta(s))\mathrm{d}s \\
&-\mathscr{A}_{11}\int_a^b f(s,\vartheta(s))\mathrm{d}s-\frac{\mathscr{A}_{12}}{\Gamma(p)}\int_a^b (b-s)^{p-1}(s,\vartheta(s))\mathrm{d}s\Bigg] \\
&+\mathscr{A}_{11}\int_a^\sigma f(s,\vartheta(s))\mathrm{d}s+\frac{\mathscr{A}_{12}}{\Gamma(p)}\int_a^\sigma (\sigma-s)^{p-1}(s,\vartheta(s))\mathrm{d}s.
\end{aligned} \tag{64}
$$

Define a closed ball Θ_ϱ as

$$
\Theta_\varrho = \{\vartheta\in C(J,\mathbb{R}):\quad \|\vartheta\|\le\varrho\}, \tag{65}
$$

with radius $\varrho\ge(\mathscr{X}_2/(1-\mathscr{X}_3))$, where

$$
\mathscr{X}_2 = \left(\frac{(b-a)}{\Lambda}\rho_3+\rho_4\right)\omega_f. \tag{66}
$$

Now, we will show that $\Xi\Theta_\varrho\subset\Theta_\varrho$. For all $\vartheta\in\Theta_\varrho$ and $\sigma\in J$ and by (47), we have

$$\begin{aligned}\|\mathsf{F}\vartheta\| &\leq \frac{\left(L_f \varrho + \omega_f\right)(\sigma - a)}{\Lambda}\rho_3 + \left(L_f \varrho + \omega_f\right)\rho_4 \\ &\leq L_f\left(\frac{(b-a)}{\Lambda}\rho_3 + \rho_4\right)\varrho + \left(\frac{(b-a)}{\Lambda}\rho_3 + \rho_4\right)\omega_f \\ &= \mathscr{Z}_3\varrho + \mathscr{Z}_2 < \varrho.\end{aligned} \tag{67}$$

Thus, $\mathsf{F}\Theta_\varrho \subset \Theta_\varrho$. Next, we need to prove that F is a contraction map. Let $\vartheta, \widehat{\vartheta} \in C(J, \mathbb{R})$ and $\sigma \in J$. Then,

$$\begin{aligned}&|(\mathsf{F}\vartheta)(\sigma) - (\mathsf{F}\widehat{\vartheta})(\sigma)| \\ &\leq \frac{(\sigma-a)}{\Lambda}\left[\mathscr{A}_7 \int_a^\zeta |f(s,\vartheta(s)) - f(s,\widehat{\vartheta}(s))|ds\right. \\ &+ \frac{\mathscr{A}_8}{\Gamma(p)}\int_a^\zeta (\zeta - s)^{p-1}|f(s,\vartheta(s)) - f(s,\widehat{\vartheta}(s))|ds \\ &+ \frac{\mathscr{A}_9}{\Gamma(\delta)}\int_a^\zeta (\zeta - s)^{\delta-1}\int_a^s |f(\eta,\vartheta(\eta)) - f(\eta,\widehat{\vartheta}(\eta))|d\eta ds \\ &+ \frac{\mathscr{A}_{10}}{\Gamma(\delta+p)}\int_a^\zeta (\zeta - s)^{\delta+p-1}|f(s,\vartheta(s)) - f(s,\widehat{\vartheta}(s))|ds \\ &- \mathscr{A}_{11}\int_a^b |f(s,\vartheta(s)) - f(s,\widehat{\vartheta}(s))|ds \\ &\left.- \frac{\mathscr{A}_{12}}{\Gamma(p)}\int_a^b (b-s)^{p-1}|f(s,\vartheta(s)) - f(s,\widehat{\vartheta}(s))|ds\right] \\ &+ \mathscr{A}_{11}\int_a^\sigma |f(s,\vartheta(s)) - f(s,\widehat{\vartheta}(s))|ds \\ &+ \frac{\mathscr{A}_{12}}{\Gamma(p)}\int_a^\sigma (\sigma - s)^{p-1}|f(s,\vartheta(s)) - f(s,\widehat{\vartheta}(s))|ds.\end{aligned} \tag{68}$$

From our assumption, we obtain

$$\begin{aligned}|f(s,\vartheta(s)) - f(s,\widehat{\vartheta}(s))| &\leq L_f|\vartheta(s) - \widehat{\vartheta}(s)| \\ &\leq L_f\|\vartheta - \widehat{\vartheta}\|.\end{aligned} \tag{69}$$

Hence,

$$\begin{aligned}\|\mathsf{F}\vartheta - \mathsf{F}\widehat{\vartheta}\| &\leq \frac{L_f(\sigma-a)}{\Lambda}\left[\mathscr{A}_7(\zeta - a) + \frac{\mathscr{A}_8(\zeta-a)^p}{\Gamma(p+1)}\right. \\ &+ \frac{\mathscr{A}_9(\zeta-a)^{\delta+1}}{\Gamma(\delta+1)} + \frac{\mathscr{A}_{10}(\zeta-a)^{\delta+p}}{\Gamma(\delta+p+1)} \\ &\left.- \mathscr{A}_{11}(b-a) + \frac{\mathscr{A}_{12}(b-a)^p}{\Gamma(p+1)}\right]\|\vartheta - \widehat{\vartheta}\| \\ &+ L_f\left(\mathscr{A}_{11}(\sigma - a) + \frac{\mathscr{A}_{12}(\sigma-a)^p}{\Gamma(p+1)}\right)\|\vartheta - \widehat{\vartheta}\| \\ &\leq \left(\frac{L_f\rho_3(b-a)}{\Lambda} + L_f\rho_4\right)\|\vartheta - \widehat{\vartheta}\| \\ &= \mathscr{Z}_3\|\vartheta - \widehat{\vartheta}\|.\end{aligned} \tag{70}$$

Due to $\mathscr{Z}_3 < 1$, we conclude that F is a contraction operator. Hence, by Theorem 1, we conclude that F has a unique fixed point in J. Consequently, ABC-type FDEs (5) have a unique solution in J. □

6. Ulam–Hyers Stability

In this part, we will discuss two kinds of stability results for ABR-type FDEs (4) and ABC-type FDEs (5), namely, UH and GUH stabilities. Before that, for $\varepsilon > 0$, we consider the following inequality:

$$\left|{}^{\mathrm{ABR}}D_{a^+}^{p}\widehat{\vartheta}(\sigma) - F_{\widehat{\vartheta}}(\sigma)\right| \leq \varepsilon, \quad \sigma \in J. \tag{71}$$

Definition 4. The problem ABR-type FDEs (4) is Ulam–Hyers stable if there exists a real number $C_f > 0$ such that, for each $\varepsilon > 0$ and for each solution $\widehat{\vartheta} \in C(J, \mathbb{R})$ of inequality (71), there exists a unique solution $\vartheta \in C(J, \mathbb{R})$ of ABR-type FDEs (4) with

$$|\widehat{\vartheta}(\sigma) - \vartheta(\sigma)| \leq C_f\varepsilon. \tag{72}$$

Also, ABR-type FDEs (4) have generalized Ulam–Hyers stability if we can find $\varphi_f: \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ with $\varphi_f(0) = 0$ such that

$$|\widehat{\vartheta}(\sigma) - \vartheta(\sigma)| \leq \varphi_f\varepsilon. \tag{73}$$

Remark 3. Let $\widehat{\vartheta} \in C(J, \mathbb{R})$ be the solution of inequality (71) if and only if we have a function $h \in C(J, \mathbb{R})$ which depends on ϑ such that

(i) $|h(\sigma)| \leq \varepsilon$ for all $\sigma \in J$

(ii) ${}^{\mathrm{ABR}}D_{a^+}^{p}\widehat{\vartheta}(\sigma) = f(\sigma, \widehat{\vartheta}(\sigma)) + h(\sigma)$, $\sigma \in J$

Lemma 9. *If $\vartheta \in C(J, \mathbb{R})$ is a solution to inequality (71), then ϑ satisfies the following inequality:*

$$\left|\widehat{\vartheta}(\sigma) - \Psi_{\widehat{\vartheta}} - \mathscr{A}_5 \int_a^\sigma (\sigma - s) f(s, \widehat{\vartheta}(s)) \mathrm{d}s - \frac{\mathscr{A}_6}{\Gamma(p-2)} \int_a^\sigma (\sigma - s)^{p-1} f(s, \widehat{\vartheta}(s)) \mathrm{d}s\right| \leq \varepsilon\left(\frac{\rho_1 (b-a)}{\Lambda} + \rho_2\right), \tag{74}$$

where

$$\begin{aligned}\Psi_{\widehat{\vartheta}} = {} & \frac{(\sigma - a)}{\Lambda} \Bigg[\mathscr{A}_1 \int_a^\zeta (\zeta - s) f(s, \widehat{\vartheta}(s)) \mathrm{d}s + \frac{\mathscr{A}_2}{\Gamma(p-2)} \int_a^\zeta (\zeta - s)^{p-1} f(s, \widehat{\vartheta}(s)) \mathrm{d}s \\ & + \frac{\mathscr{A}_3}{\Gamma(\delta)} \int_a^\zeta (\zeta - s)^{\delta-1} \int_a^s (s - \eta) f(\eta, \widehat{\vartheta}(\eta)) \mathrm{d}\eta \mathrm{d}s + \frac{\mathscr{A}_4}{\Gamma(\delta + p - 2)} \int_a^\zeta (\zeta - s)^{\delta+p-1} f(s, \widehat{\vartheta}(s)) \mathrm{d}s \\ & - \mathscr{A}_5 \int_a^b (b - s) f(s, \widehat{\vartheta}(s)) \mathrm{d}s - \frac{\mathscr{A}_6}{\Gamma(p-2)} \int_a^b (b - s)^{p-1} f(s, \widehat{\vartheta}(s)) \mathrm{d}s\Bigg].\end{aligned} \tag{75}$$

Proof. In view of Remark 3, we have

$$\begin{cases} {}^{\mathrm{ABR}}D_{a^+}^{p} \widehat{\vartheta}(\sigma) = f(s, \widehat{\vartheta}(s)) + h(\sigma), \\ \widehat{\vartheta}(a) = 0, \\ \widehat{\vartheta}(b) = {}^{\mathrm{AB}}I_{a^+}^{\delta} \widehat{\vartheta}(\zeta). \end{cases} \tag{76}$$

Then, by Theorem 3, we get

$$\begin{aligned}\widehat{\vartheta}(\sigma) = {} & \frac{(\sigma - a)}{\Lambda} \Bigg[\mathscr{A}_1 \int_a^\zeta (\zeta - s)(f(s, \widehat{\vartheta}(s)) + h(s)) \mathrm{d}s \\ & + \frac{\mathscr{A}_2}{\Gamma(p-2)} \int_a^\zeta (\zeta - s)^{p-1} (f(s, \widehat{\vartheta}(s)) + h(s)) \mathrm{d}s \\ & + \frac{\mathscr{A}_3}{\Gamma(\delta)} \int_a^\zeta (\zeta - s)^{\delta-1} \int_a^s (s - \eta)(f(\eta, \widehat{\vartheta}(\eta)) + h(\eta)) \mathrm{d}\eta \mathrm{d}s \\ & + \frac{\mathscr{A}_4}{\Gamma(\delta + p - 2)} \int_a^\zeta (\zeta - s)^{\delta+p-1} (f(s, \widehat{\vartheta}(s)) + h(s)) \mathrm{d}s \\ & - \mathscr{A}_5 \int_a^b (b - s)(f(s, \widehat{\vartheta}(s)) + h(s)) \mathrm{d}s \\ & - \frac{\mathscr{A}_6}{\Gamma(p-2)} \int_a^b (b - s)^{p-1} (f(s, \widehat{\vartheta}(s)) + h(s)) \mathrm{d}s\Bigg] \\ & + \mathscr{A}_5 \int_a^\sigma (\sigma - s)(f(s, \widehat{\vartheta}(s)) + h(s)) \mathrm{d}s \\ & + \frac{\mathscr{A}_6}{\Gamma(p-2)} \int_a^\sigma (\sigma - s)^{p-1} (f(s, \widehat{\vartheta}(s)) + h(s)) \mathrm{d}s,\end{aligned} \tag{77}$$

which implies

$$
\begin{aligned}
&\left|\widehat{\vartheta}(\sigma)-\Psi_{\widehat{\vartheta}}-\mathscr{A}_5\int_a^\sigma(\sigma-s)f(s,\widehat{\vartheta}(s))\mathrm{d}s-\frac{\mathscr{A}_6}{\Gamma(p-2)}\int_a^\sigma(\sigma-s)^{p-1}f(s,\widehat{\vartheta}(s))\mathrm{d}s\right|\\
&\leq\frac{(\sigma-a)}{\Lambda}\left[\mathscr{A}_1\int_a^\zeta(\zeta-s)|h(s)|\mathrm{d}s+\frac{\mathscr{A}_2}{\Gamma(p-2)}\int_a^\zeta(\zeta-s)^{p-1}|h(s)|\mathrm{d}s\right.\\
&\quad+\frac{\mathscr{A}_3}{\Gamma(\delta)}\int_a^\zeta(\zeta-s)^{\delta-1}\int_a^s(s-\eta)|h(\eta)|\mathrm{d}\eta\mathrm{d}s\\
&\quad+\frac{\mathscr{A}_4}{\Gamma(\delta+p-2)}\int_a^\zeta(\zeta-s)^{\delta+p-1}|h(s)|\mathrm{d}s\\
&\left.+\mathscr{A}_5\int_a^b(b-s)|h(s)|\mathrm{d}s+\frac{\mathscr{A}_6}{\Gamma(p-2)}\int_a^b(b-s)^{p-1}|h(s)|\mathrm{d}s\right]\\
&\quad+\mathscr{A}_5\int_a^\sigma(\sigma-s)|h(s)|\mathrm{d}s+\frac{\mathscr{A}_6}{\Gamma(p-2)}\int_a^\sigma(\sigma-s)^{p-1}|h(s)|\mathrm{d}s\\
&\leq\varepsilon\left(\frac{\rho_1(b-a)}{\Lambda}+\rho_2\right).
\end{aligned}
\tag{78}
$$

□

Theorem 8. *Suppose that (38) holds. If*

$$
L_f\left(\frac{\mathscr{A}_5(b-a)^2}{2}+\frac{\mathscr{A}_6(b-a)^p}{\Gamma(p-1)}\right)<1,
\tag{79}
$$

then ABR-type FDEs (4) are Ulam–Hyers stable.

Proof. Let $\varepsilon>0$ and $\widehat{\vartheta}\in C(J,\mathbb{R})$ be a function which satisfies inequality (71), and let $\vartheta\in C(J,\mathbb{R})$ be a unique solution of the following problem:

$$
\begin{cases}
{}^{\mathrm{ABR}}D_{a^+}^p\widehat{\vartheta}(\sigma)=f(\sigma,\widehat{\vartheta}(\sigma)), & \sigma\in(a,b),\\
p\in(2,3]\vartheta(a)=\widehat{\vartheta}(a)=0,\\
\vartheta(b)=\widehat{\vartheta}(b)={}^{\mathrm{AB}}I_{a^+}^\delta\widehat{\vartheta}(\zeta).
\end{cases}
\tag{80}
$$

Then, by Theorem 3, we get

$$
\begin{aligned}
\widehat{\vartheta}(\sigma)=&\Psi_{\widehat{\vartheta}}+\mathscr{A}_5\int_a^\sigma(\sigma-s)f(s,\widehat{\vartheta}(s))\mathrm{d}s\\
&-\frac{\mathscr{A}_6}{\Gamma(p-2)}\int_a^\sigma(\sigma-s)^{p-1}f(s,\widehat{\vartheta}(s))\mathrm{d}s.
\end{aligned}
\tag{81}
$$

Since $\vartheta(a)=\widehat{\vartheta}(a)=0$ and $\vartheta(b)=\widehat{\vartheta}(b)={}^{\mathrm{AB}}I_{a^+}^\delta\widehat{\vartheta}(\zeta)$, $\Psi_{\widehat{\vartheta}}=\Psi_\vartheta$, and hence, by Lemma 9 and (47), we have

$$
\begin{aligned}
&|\widehat{\vartheta}(\sigma)-\vartheta(\sigma)|\\
&\leq\left|\widehat{\vartheta}(\sigma)-\Psi_{\widehat{\vartheta}}-\mathscr{A}_5\int_a^\sigma(\sigma-s)f(s,\widehat{\vartheta}(s))\mathrm{d}s-\frac{\mathscr{A}_6}{\Gamma(p-2)}\int_a^\sigma(\sigma-s)^{p-1}f(s,\widehat{\vartheta}(s))\mathrm{d}s\right|\\
&\quad+\mathscr{A}_5\int_a^\sigma(\sigma-s)|f(s,\widehat{\vartheta}(s))-f(s,\vartheta(s))|\mathrm{d}s\\
&\quad+\frac{\mathscr{A}_6}{\Gamma(p-2)}\int_a^\sigma(\sigma-s)^{p-1}|f(s,\widehat{\vartheta}(s))-f(s,\widehat{\vartheta}(s))|\mathrm{d}s\\
&\leq\varepsilon\left(\frac{\rho_1(b-a)}{\Lambda}+\rho_2\right)+L_f\left(\frac{\mathscr{A}_5(b-a)^2}{2}+\frac{\mathscr{A}_6(b-a)^p}{\Gamma(p-1)}\right)\|\vartheta-\widehat{\vartheta}\|.
\end{aligned}
\tag{82}
$$

Thus,

$$
\|\vartheta-\widehat{\vartheta}\|\leq C_f\varepsilon,
\tag{83}
$$

where

$$C_f = \frac{\left(\left(\rho_1 (b-a)^2/\Lambda\right) + \rho_2\right)}{1 - L_f\left(\left(\mathscr{A}_5 (b-a)^2/2\right) + \left(\mathscr{A}_6 (b-a)^p/\Gamma(p-1)\right)\right)}. \tag{84}$$

Now, by choosing $\varphi_f(\varepsilon) = C_f\varepsilon$ such that $\varphi_f(0) = 0$, ABR-type FDEs (4) have generalized Ulam–Hyers stability. □

Theorem 9. *Suppose that (38) holds. If*

$$L_f\left(\mathscr{A}_{11}(b-a) + \frac{\mathscr{A}_{12}(b-a)^p}{\Gamma(p+1)}\right) < 1, \tag{85}$$

then ABC-type FDEs (5) are Ulam–Hyers stable.

Proof. By the same technique of Theorem 8, one can prove that

$$\|\vartheta - \widehat{\vartheta}\| \le C_f^*\varepsilon, \tag{86}$$

where

$$C_f^* = \frac{\left(\left(\rho_3 (b-a)/\Lambda\right) + \rho_4\right)}{1 - L_f\left(\mathscr{A}_{11}(b-a) + \left(\mathscr{A}_{12}(b-a)^p/\Gamma(p+1)\right)\right)}. \tag{87}$$

Now, by choosing $\varphi_f(\varepsilon) = C_f^*\varepsilon$ such that $\varphi_f(0) = 0$, ABC-type FDEs (5) have generalized Ulam–Hyers stability. □

7. An Example

In this section, we justify the validity of Theorems 4, 5, and 7–9 through an example.

Example 1. For $p \in (2,3]$, we consider the following ABR fractional problem:

$$\begin{cases} {}^{\mathrm{ABR}}D_{0^+}^{7/2}\vartheta(\sigma) = \dfrac{\sigma^2}{10e^{\sigma-1}}\left(\dfrac{|\vartheta(\sigma)|}{1+|\vartheta(\sigma)|}\right), & \sigma \in (0,1), \\ \vartheta(0) = 0, & \\ \vartheta(1) = {}^{\mathrm{AB}}I_{0^+}^{1/2}\vartheta\left(\dfrac{1}{2}\right). & \end{cases} \tag{88}$$

Here, $p = (7/2) \in (2,3], a = 0, b = 1, \delta = (1/2), \zeta = (1/2)$, and $f(\sigma,\vartheta(\sigma)) = (\sigma^2/10e^{\sigma-1})(|\vartheta(\sigma)|/(1+|\vartheta(\sigma)|))$. Let $\sigma \in [0,1], u, \overline{u} \in \mathbb{R}$. Then,

$$|f(\sigma,u) - f(\sigma,\overline{u})| \le \frac{1}{10}|u-\overline{u}|. \tag{89}$$

Therefore, hypothesis (H_1) holds with $L_f = 1/10$. Also, $\rho_1 \simeq 1.23, \rho_2 \simeq 2, \Lambda \simeq 0.66$, and $\mathscr{Z} \simeq 0.38 < 1$. Then, all conditions in Theorem 4 are satisfied, and hence, ABR-type FDEs (4) have a unique solution. For every $\varepsilon = \max\{\varepsilon_1, \varepsilon_2\} > 0$ and each $\widehat{\vartheta} \in C(J,\mathbb{R})$ satisfying the inequality

$$\left|{}^{\mathrm{ABR}}D_{a^+}^{p}\widehat{\vartheta}(\sigma) - F_{\widehat{\vartheta}}(\sigma)\right| \le \varepsilon, \tag{90}$$

there exists a solution $\vartheta \in C(J,\mathbb{R})$ of ABR-type FDEs (4) with

$$\left\|\widehat{\vartheta} - \vartheta_1\right\| \le C_f\varepsilon, \tag{91}$$

where

$$C_f = \frac{\left(\left(\rho_1 (b-a)/\Lambda\right) + \rho_2\right)}{1 - L_f\left(\left(\mathscr{A}_5 (b-a)^2/2\right) + \left(\mathscr{A}_6 (\sigma-a)^p/\Gamma(p-1)\right)\right)} > 0. \tag{92}$$

Therefore, all conditions in Theorem 9 are satisfied, and hence, ABR-type FDEs (4) are UH stable.

Example 2. For $p \in (1,2]$, we consider the following ABC fractional problem:

$$\begin{cases} {}^{\mathrm{ABC}}D_{0^+}^{3/2}\vartheta(\sigma) = \dfrac{\sigma^2}{10e^{\sigma-1}}\left(\dfrac{|\vartheta(\sigma)|}{1+|\vartheta(\sigma)|}\right), & \sigma \in (0,1), \\ \vartheta(0) = 0, \vartheta(1) = {}^{\mathrm{AB}}I_{0^+}^{1/2}\vartheta\left(\dfrac{1}{2}\right). & \end{cases} \tag{93}$$

Here, $p = (3/2) \in (1,2], a = 0, b = 1, \delta = (1/2), \zeta = (1/2)$, and $f(\sigma,\vartheta(\sigma)) = (\sigma^2/10e^{\sigma-1})(|\vartheta(\sigma)|/(1+|\vartheta(\sigma)|))$. Let $\sigma \in [0,1], u, \overline{u} \in \mathbb{R}$. Then,

$$\begin{aligned} |f(\sigma,u) - f(\sigma,\overline{u})| &= \frac{\sigma^2}{10e^{\sigma-1}}\left(\frac{u}{1+u} - \frac{\overline{u}}{1+\overline{u}}\right) \\ &\le \frac{1}{10}|u-\overline{u}|. \end{aligned} \tag{94}$$

Therefore, hypothesis (H_1) holds with $L_f = 1/10$. Also, $\rho_3 \simeq 1.4, \rho_4 \simeq 2.5, \Lambda \simeq 0.66$, and

$$\mathscr{Z}_3 = \frac{1}{10}\left(\frac{1.4}{0.66} + 2.5\right) \approx 0.47 < 1. \tag{95}$$

Then, all conditions in Theorem 7 are satisfied, and hence, ABC-type FDEs (5) have a unique solution. For every $\varepsilon = \max\{\varepsilon_1, \varepsilon_2\} > 0$ and each $\widehat{\vartheta} \in C(J,\mathbb{R})$ satisfying

$$\left|{}^{\mathrm{ABC}}D_{a^+}^{p}\widehat{\vartheta}(\sigma) - F_{\widehat{\vartheta}}(\sigma)\right| \le \varepsilon, \tag{96}$$

there exists a solution $\vartheta \in C(J,\mathbb{R})$ of ABC-type FDEs (5) with

$$\left\|\widehat{\vartheta} - \vartheta_1\right\| \le C_f^*\varepsilon, \tag{97}$$

where

$$C_f^* = \frac{\left(\left(\rho_3 (b-a)/\Lambda\right) + \rho_4\right)}{1 - L_f\left(\mathscr{A}_{11}(b-a) + \left(\mathscr{A}_{12}(\sigma-a)^p/\Gamma(p+1)\right)\right)} > 0. \tag{98}$$

Therefore, all conditions in Theorem 9 are satisfied, and hence, ABC-type FDEs (5) are UH stable.

8. Conclusion Remarks

In recent interest, the theory of fractional operators in the frame of Atangana and Baleanu is novel and significant; thus, there are some researchers who studied and developed some qualitative properties of solutions of FDEs involving such operators. On the contrary, there are some important numerical approaches regarding nonsingular kernels that are explained in the introduction. For this purpose, the sufficient conditions of the existence and uniqueness of solutions for two classes of nonlinear Atangana–Baleanu FDEs on the interval $[a, b]$, subjected to integral conditions, have been developed and investigated. Furthermore, the stability results through mathematical analysis techniques have been analyzed. Two examples are provided to justify our results.

Observe that our approach used in this work is new because we prove the existence, uniqueness, and UH stability results without using the semigroup property and relies on a minimum number of hypotheses. Our approach was based on the reduction of the proposed problem into the fractional integral equation and using some standard fixed point theorems due to Banach type and Krasnoselskii type. Furthermore, through mathematical analysis techniques, we analyzed the stability results in the UH sense. An example was provided to justify the main results. In fact, our outcomes extended those in [36, 37]. The supposed problem with given integroderivative boundary conditions can describe some mathematical models of real and physical processes in which some parameters are frequently acclimated to appropriate circumstances. So, the value of these parameters can change the impacts of fractional integrals and derivatives. The main results are illustrated with a numerical example. In future works, we study the new numerical results regarding this operator with the higher order.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Disclosure

This work was conducted during the work of the first author at Hajjah University (Yemen).

Conflicts of Interest

All authors declare that they have no conflicts of interest.

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