

Research Article

Existence of Periodic Solutions for a Class of Fourth-Order Difference Equation

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We apply the continuation theorem of Mawhin to ensure that a fourth-order nonlinear difference equation of the form $\Delta^4 u(k-2) - a(k)u^\alpha(k) + b(k)u^\beta(k) = 0$ with periodic boundary conditions possesses at least one nontrivial positive solution, where $\Delta u(k) = u(k+1) - u(k)$ is the forward difference operator, $\alpha > 0, \beta > 0$ and $\alpha \neq \beta$, $a(k), b(k)$ are T -periodic functions and $a(k)b(k) > 0$. As applications, we give some examples to illustrate the application of these theorems.

1. Introduction

In recent years, the theory of nonlinear difference equations has been widely used in the study of discrete models in the fields of economics, neural networks, ecology, etc. For the general background of difference equations, in particular, there are many authors who have discussed the existence and multiplicity of periodic solutions for discrete boundary value problems by exploiting various methods, including the method of upper and lower solutions, Leray-Schauder degree, fixed point theory, critical theory, and variational methods; see Bereanu and Mawhin [1], Cabada and Dimitrov [2], Graef et al. [3, 4], and Cai et al. [5–10] and the references therein.

Let \mathbb{N}^+ , \mathbb{Z} , and \mathbb{R} denote the sets of all positive integers, integers, and real numbers, respectively. This paper considers the following fourth-order nonlinear difference equation:

$$\Delta^4 u(k-2) - a(k)u^\alpha(k) + b(k)u^\beta(k) = 0, \quad (1)$$

where $\alpha > 0, \beta > 0$ and $\alpha \neq \beta$, $a(k), b(k)$ are T -periodic functions and $\Delta u(k) = u(k+1) - u(k)$ is the forward difference operator.

Equation (1) can be considered as a discrete analogue of a special case of the following fourth-order nonlinear differ-

ential equation:

$$u'''' - a(t)u^\alpha + b(t)u^\beta = 0, \quad t \in \mathbb{R}, \quad (2)$$

which has been studied in [11, 12] when $\alpha = 1, \beta = 3$. In [13], Yang and Han proved the existence of a periodic solution to equation (2) when $\alpha = n, \beta = n + 2$, where n is a positive integer.

When $a(k) \equiv 0, \beta = 1$, Peterson and Ridenhour [14] considered the disconjugacy of the following equation:

$$\Delta^4 u(k-2) + b(k)u(k) = 0, \quad k \in \mathbb{Z}. \quad (3)$$

In 2005, Cai et al. [5] studied the fourth-order nonlinear difference equation

$$\Delta^4 u(k-2) + f(k, u(k)) = 0, \quad k \in \mathbb{Z}. \quad (4)$$

By applying the linking theorem, they obtained some criteria for the existence and multiplicity of periodic solutions of equation (4).

In fact, there is a big difference between the continuous case and the discrete case. For example, the basic ideas of calculus are not always applicable when studying difference equations, such as the embedding theorem. Therefore, we need to consider other methods to deal with the difference

problem. The main tool used is the continuation theorem of Mawhin (see [15]).

Motivated by the above works, the main aim of this paper is to investigate the existence of at least one positive T -periodic solution of (1). In order to obtain the main results of (1), we assume that the coefficient functions $a(k)$ and $b(k)$ satisfy the following condition:

F_1 : Suppose $a(k), b(k)$ are T -periodic functions and $a(k)b(k) > 0$ for all $k \in \mathbb{Z}$. Furthermore, we assume that there exist positive constants a, A, b, B such that

$$\begin{aligned} a &= \min_{k \in \mathbb{Z}} |a(k)|, \\ A &= \max_{k \in \mathbb{Z}} |a(k)|, \\ b &= \min_{k \in \mathbb{Z}} |b(k)|, \\ B &= \max_{k \in \mathbb{Z}} |b(k)|. \end{aligned} \quad (5)$$

Let X be all real T -periodic sequences of the form $\mathbf{u} = \{u(k)\}_{k \in \mathbb{Z}}$. Then, X is a Banach space under the norm $\|\mathbf{u}\|$

$$= \max_{k \in [2, T+1]_{\mathbb{Z}}} |u(k)|.$$

The main results in this paper are stated next: Theorems 1 and 2.

Theorem 1. *Let F_1 hold, if $\alpha < \beta$ and the period T satisfies*

$$16 \leq T^4 \leq \frac{8}{AR_1^{\alpha-1} + BR_1^{\beta-1}}, \quad (6)$$

where $R_1 = (A/b)^{1/(\beta-\alpha)} + \rho$ and $\rho > 0$ small enough such that $(a/b)^{1/(\beta-\alpha)} - \rho > 0$; then, equation (1) admits at least one positive T -periodic solution.

Theorem 2. *Let F_1 hold, if $\alpha > \beta$ and the period T satisfies*

$$16 \leq T^4 \leq \frac{8}{AQ_1^{\alpha-1} + BQ_1^{\beta-1}}, \quad (7)$$

where $Q_1 = (B/a)^{1/(\alpha-\beta)} + \tau$ and $\tau > 0$ small enough such that $(b/a)^{1/(\alpha-\beta)} - \tau > 0$; then, equation (1) admits at least one positive T -periodic solution.

Theorem 3. *Suppose $a(k)b(k) \leq 0$ and $a(k), b(k)$ are not identical to zero for all $k \in \mathbb{Z}$; then, equation (1) has no positive solution.*

This paper is organized as follows: in Section 2, we give some lemmas needed to prove the main results. Section 3 contains the proof of Theorem 1. Section 4 contains the proof of Theorem 2. Section 5 contains the proof of Theorem 3.

2. Preliminary Results

In this section, we introduce some notations and well-known results which will be used in the subsequent sections.

Definition 4 (see [7], p. 12, B.1). Let X, Y be real Banach spaces, $L : \text{Dom} L \subset X \rightarrow Y$ be a linear mapping. The mapping L is said to be a Fredholm mapping of index zero if

- (a) $\text{Im } L$ is closed in Y
- (b) $\dim \text{Ker } L = \text{codim } \text{Im } L < +\infty$

If L is a Fredholm mapping of index zero, then there exist continuous projectors $P : X \rightarrow X$ and $Q : Y \rightarrow Y$ such that

$$\begin{aligned} \text{Im } P &= \text{Ker } L, \\ \text{Ker } Q &= \text{Im } L = \text{Im } (I - Q). \end{aligned} \quad (8)$$

It follows that the restriction

$$L_{\text{Dom} L \cap \text{Ker } P} : (I - P)X \rightarrow \text{Im } L \quad (9)$$

has an inverse which is denoted by K_p .

Definition 5 (see [7], pp. 12–13, B.1, B.2). Let $N : X \rightarrow Y$ be a continuous mapping. If Ω is a bounded open subset of X , N is called L -compact on $\bar{\Omega}$ if $QN(\bar{\Omega})$ is bounded and $K_p(I - Q)N : \bar{\Omega} \rightarrow X$ is compact. Since $\text{Im } Q$ is isomorphic to $\text{Ker } L$, there exists an isomorphism $J : \text{Im } Q \rightarrow \text{Ker } L$.

Lemma 6 (Mawhin's continuation theorem, see [7], Theorem IV.1). *Let L be a Fredholm mapping of index zero, $\Omega \subset X$ is an open bounded set, and let N be L -compact on $\bar{\Omega}$. Suppose*

- (1) for each $\lambda \in (0, 1)$, $L\mathbf{u} \neq \lambda N\mathbf{u}$ for all $\mathbf{u} \in \partial\Omega \cap \text{Dom } L$
- (2) $QN\mathbf{u} \neq 0$, for all $\mathbf{u} \in \partial\Omega \cap \text{Ker } L$
- (3) $\deg(JQN, \Omega \cap \text{Ker } L, 0) \neq 0$, where $J : \text{Im } Q \rightarrow \text{Ker } L$ is an isomorphism

Then, the equation $L\mathbf{u} = N\mathbf{u}$ has at least one solution in $\bar{\Omega} \cap \text{Dom } L$.

Define the operator $L : X \rightarrow X$ by setting

$$L\mathbf{u} = \Delta^4 u(k-2), \mathbf{u} \in X. \quad (10)$$

Direct calculation shows that

$$\text{Ker } L = \mathbb{R} \text{ and } \text{Im } L = \left\{ \mathbf{v} \left| \sum_{i=2}^{T+1} v(i) = 0 \right. \right\}. \quad (11)$$

Indeed, if $\mathbf{v} = \{v(k)\}_{k \in \mathbb{Z}} \in \text{Im } L$, then there is $\mathbf{u} =$

$\{u(k)\}_{k \in \mathbb{Z}} \in X$ such that

$$v(k) = \Delta^4 u(k-2), k \in \mathbb{Z}. \tag{12}$$

Thus,

$$\sum_{i=2}^{T+1} v(i) = \sum_{i=2}^{T+1} \Delta^4 u(i-2). \tag{13}$$

Note that $\{u(k)\}_{k \in \mathbb{Z}} \in X$, it follows that

$$\{\Delta^3 u(k)\}_{k \in \mathbb{Z}} \in X. \tag{14}$$

Furthermore, direct calculation shows that

$$\begin{aligned} \sum_{i=2}^{T+1} \Delta^4 u(i-2) &= [\Delta^3 u(1) - \Delta^3 u(0)] \\ &+ [\Delta^3 u(2) - \Delta^3 u(1)] + \dots + [\Delta^3 u(T) - \Delta^3 u(T-1)] \\ &= \Delta^3 u(T) - \Delta^3 u(0) = \Delta^3 u(0) - \Delta^3 u(0) = 0. \end{aligned} \tag{15}$$

By virtue of the above facts, we have

$$\sum_{i=2}^{T+1} v(i) = \sum_{i=2}^{T+1} \Delta^4 u(i-2) = 0. \tag{16}$$

Since $\dim X = T$ and L is a linear mapping, by the knowledge of linear algebra, we know that $\dim \text{Ker } L \oplus \dim \text{Im } L = \dim X$. It is easy to see that $\dim \text{Ker } L = \text{codim Im } L = 1$ and $\dim \text{Im } L = T - 1$. It follows that $\text{Im } L$ is closed in X . Therefore, the operator L is a Fredholm operator with index zero.

Let us define $N : X \rightarrow X$ by

$$(Nu)(k) = a(k)u^\alpha(k) - b(k)u^\beta(k). \tag{17}$$

We define $P : X \rightarrow \text{Ker } L$ and $Q : X \rightarrow X$ as follows:

$$(Pu)(k) = (Qu)(k) = \frac{1}{T} \sum_{i=2}^{T+1} u(i). \tag{18}$$

The operators P and Q are projections. Hence,

$$\text{Im } P = \text{Ker } L, \text{Ker } Q = \text{Im } L. \tag{19}$$

It follows that $L|_{\text{Dom } L \cap \text{Ker } P} : (I - P)X \rightarrow \text{Im } L$ has an inverse which is denoted by K_p .

In view of (17) and (18), for any $\mathbf{u} \in X$, we can see that

$$(QNu)(k) = \frac{1}{T} \sum_{i=2}^{T+1} [a(i)u^\alpha(i) - b(i)u^\beta(i)], \tag{20}$$

$$((I - Q)Nu)(k) = a(k)u^\alpha(k) - b(k)u^\beta(k) - \frac{1}{T} \sum_{i=2}^{T+1} [a(i)u^\alpha(i) - b(i)u^\beta(i)]. \tag{21}$$

Since the Banach space X is finite dimensional, K_p is linear. By virtue of the relations (20) and (21), we see that QN and $K_p(I - Q)N$ are continuous on X . Hence, we know that if Ω is an open and bounded subset of X , then $QN(\bar{\Omega})$ is bounded. It follows that

$$K_p(I - Q)N : \bar{\Omega} \rightarrow X \tag{22}$$

is compact. Therefore, the mapping N is L -compact on $\bar{\Omega}$ with any open and bounded subset $\Omega \subset X$.

Lemma 7 (see [16], Lemma 2.3). *Let $\{u(k)\}_{k \in \mathbb{Z}}$ be a real T -periodic sequence; then,*

$$\max_{2 \leq i, j \leq T+1} |u(i) - u(j)| \leq \frac{T^3}{16} \sum_{k=2}^{T+1} |\Delta^4 u(k-2)|. \tag{23}$$

3. Proof of Theorem 1

Proof. The content of Theorem 1 is as follows: let F_1 hold, if $\alpha < \beta$ and the period T satisfies

$$16 \leq T^4 \leq \frac{8}{AR_1^{\alpha-1} + BR_1^{\beta-1}}, \tag{24}$$

where $R_1 = (A/B)^{1/(\beta-\alpha)} + \rho$ and $\rho > 0$ small enough such that $(a/B)^{1/(\beta-\alpha)} - \rho > 0$; then, equation (1) admits at least one positive T -periodic solution. Now, we prove that the conclusion holds. We assume that $\alpha < \beta$. From condition F_1 , we know that $a(k)b(k) > 0$, which include both positive and negative cases. So we need to classify the cases where both $a(k)$ and $b(k)$ are positive and both negative. \square

Case 1. If coefficient functions $a(k)$ and $b(k)$ are positive T -periodic functions, in view of F_1 , we have that $0 < a \leq a(k) \leq A$ and $0 < b \leq b(k) \leq B$.

Let

$$\Omega_1 = \{\mathbf{u} \in X | H_1 < u(k) < R_1\}, \tag{25}$$

which is an open set in X , where

$$R_1 = R + \rho, R = \left(\frac{A}{b}\right)^{1/(\beta-\alpha)}, \tag{26}$$

$$H_1 = H - \rho, H = \left(\frac{a}{B}\right)^{1/(\beta-\alpha)}, \tag{27}$$

where $\rho > 0$ small enough such that $(a/B)^{1/(\beta-\alpha)} - \rho > 0$. Obviously, H_1 and R_1 are well defined.

By

$$\begin{aligned} \alpha < \beta, 0 < a \leq a(k) \leq A, \\ 0 < b \leq b(k) \leq B, \end{aligned} \quad (28)$$

we obtain

$$0 < H_1 < H \leq \left(\frac{a(k)}{b(k)} \right)^{1/(\beta-\alpha)} \leq R < R_1, \quad (29)$$

uniformly for $k \in Z$.

Furthermore, it follows from (26), (27), and (29) that

$$\begin{aligned} a(k) - b(k)H_1^{\beta-\alpha} &\geq a - BH_1^{\beta-\alpha} > 0, \\ a(k) - b(k)R_1^{\beta-\alpha} &\leq A - bR_1^{\beta-\alpha} < 0. \end{aligned} \quad (30)$$

Therefore,

$$\begin{aligned} a(k) - b(k)H_1^{\beta-\alpha} &> 0, \\ a(k) - b(k)R_1^{\beta-\alpha} &< 0. \end{aligned} \quad (31)$$

We prove that condition (1) of Lemma 6 holds. Let $0 < \lambda < 1$ and \mathbf{u} be such that

$$\Delta^4 u(k-2) - \lambda a(k)u^\alpha(k) + \lambda b(k)u^\beta(k) = 0. \quad (32)$$

Summing from 2 to $T+1$, we can see that

$$\sum_{i=2}^{T+1} \left[\Delta^4 u(i-2) - \lambda a(i)u^\alpha(i) + \lambda b(i)u^\beta(i) \right] = 0. \quad (33)$$

Firstly, we claim that for each $\lambda \in (0, 1)$ and $\mathbf{u} \in \partial\Omega_1 \cap \text{Dom } L$, $L\mathbf{u} \neq \lambda N\mathbf{u}$. In fact, in view of (25), if $\mathbf{u} \in \partial\Omega_1$, then $\|\mathbf{u}\| = H_1$ or $\|\mathbf{u}\| = R_1$. We only prove the case of $\|\mathbf{u}\| = H_1$, similar to the proof of $\|\mathbf{u}\| = R_1$.

When $\|\mathbf{u}\| = H_1$, we can get that $\max_{2 \leq l, j \leq T+1} |u(l) - u(j)| < H_1/2$ or $\max_{2 \leq l, j \leq T+1} |u(l) - u(j)| \geq H_1/2$ holds. If $\max_{2 \leq l, j \leq T+1} |u(l) - u(j)| < H_1/2$, thus $H_1/2 < \mathbf{u} \leq H_1$. According to the above facts, we have

$$\begin{aligned} 0 &= \sum_{i=2}^{T+1} \left[\lambda a(i)u^\alpha(i) - \lambda b(i)u^\beta(i) \right] \\ &= \lambda \sum_{i=2}^{T+1} u^\alpha(i) \left[a(i) - b(i)u^{\beta-\alpha}(i) \right] \\ &\geq \lambda \left(a - BH_1^{\beta-\alpha} \right) \sum_{i=2}^{T+1} u^\alpha(i) > 0. \end{aligned} \quad (34)$$

This is a contradiction.

If $\max_{2 \leq l, j \leq T+1} |u(l) - u(j)| \geq H_1/2$, we see from Lemma 7 that

$$\begin{aligned} 0 &= \left| \sum_{i=2}^{T+1} \left[\Delta^4 u(i-2) - \lambda a(i)u^\alpha(i) + \lambda b(i)u^\beta(i) \right] \right| \\ &\geq \sum_{i=2}^{T+1} \left| \Delta^4 u(i-2) \right| - \sum_{i=2}^{T+1} \left| \lambda a(i)u^\alpha(i) + \lambda b(i)u^\beta(i) \right| \\ &> \sum_{i=2}^{T+1} \left| \Delta^4 u(i-2) \right| - \sum_{i=2}^{T+1} \left| a(i)u^\alpha(i) + b(i)u^\beta(i) \right| \\ &\geq \frac{16}{T^3} \max_{2 \leq l, j \leq T+1} |u(l) - u(j)| - T \left(AH_1^{\alpha-1} + BH_1^{\beta-1} \right) \|\mathbf{u}\| \\ &\geq \frac{8}{T^3} \|\mathbf{u}\| - T \left(AH_1^{\alpha-1} + BH_1^{\beta-1} \right) \|\mathbf{u}\| \\ &= \left[\frac{8}{T^3} - T \left(AH_1^{\alpha-1} + BH_1^{\beta-1} \right) \right] \|\mathbf{u}\| \\ &\geq 0. \end{aligned} \quad (35)$$

This is a contradiction.

When $\|\mathbf{u}\| = R_1$, we can get that $\max_{2 \leq l, j \leq T+1} |u(l) - u(j)| < R_1/2$ or $\max_{2 \leq l, j \leq T+1} |u(l) - u(j)| \geq R_1/2$ holds. If $\max_{2 \leq l, j \leq T+1} |u(l) - u(j)| < R_1/2$, thus $R_1/2 < \mathbf{u} \leq R_1$; further, we have

$$\begin{aligned} 0 &= \sum_{i=2}^{T+1} \left[\lambda a(i)u^\alpha(i) - \lambda b(i)u^\beta(i) \right] \\ &= \lambda \sum_{i=2}^{T+1} u^\alpha(i) \left[a(i) - b(i)u^{\beta-\alpha}(i) \right] \\ &\leq \lambda \left(A - bR_1^{\beta-\alpha} \right) \sum_{i=2}^{T+1} u^\alpha(i) < 0. \end{aligned} \quad (36)$$

This is a contradiction.

If $\max_{2 \leq l, j \leq T+1} |u(l) - u(j)| \geq R_1/2$, from Lemma 7, we have that

$$\begin{aligned} 0 &= \left| \sum_{i=2}^{T+1} \left[\Delta^4 u(i-2) - \lambda a(i)u^\alpha(i) + \lambda b(i)u^\beta(i) \right] \right| \\ &\geq \sum_{i=2}^{T+1} \left| \Delta^4 u(i-2) \right| - \sum_{i=2}^{T+1} \left| \lambda a(i)u^\alpha(i) + \lambda b(i)u^\beta(i) \right| \\ &> \sum_{i=2}^{T+1} \left| \Delta^4 u(i-2) \right| - \sum_{i=2}^{T+1} \left| a(i)u^\alpha(i) + b(i)u^\beta(i) \right| \\ &\geq \frac{16}{T^3} \max_{2 \leq l, j \leq T+1} |u(l) - u(j)| - T \left(AR_1^{\alpha-1} + BR_1^{\beta-1} \right) \|\mathbf{u}\| \\ &\geq \frac{8}{T^3} \|\mathbf{u}\| - T \left(AR_1^{\alpha-1} + BR_1^{\beta-1} \right) \|\mathbf{u}\| \\ &= \left[\frac{8}{T^3} - T \left(AR_1^{\alpha-1} + BR_1^{\beta-1} \right) \right] \|\mathbf{u}\| \\ &\geq 0, \end{aligned} \quad (37)$$

but this is a contradiction. Hence, for $\mathbf{u} \in \partial\Omega_1$ and $\lambda \in (0, 1)$, we have

$$\Delta^4 u(k-2) - \lambda a(k)u^\alpha(k) + \lambda b(k)u^\beta(k) \neq 0. \quad (38)$$

Therefore, we verify that condition (1) of Lemma 6 holds for Ω_1 .

If $\mathbf{u} \in \partial\Omega_1 \cap \text{Ker } L$, then $\mathbf{u} = \{H_1\}_{k \in \mathbb{Z}}$ or $\mathbf{u} = \{R_1\}_{k \in \mathbb{Z}}$. By virtue of (31), we conclude that

$$(QNu)(k) = \frac{1}{T} \sum_{i=2}^{T+1} u^\alpha(i) [a(i) - b(i)u^{\beta-\alpha}(i)] \neq 0. \quad (39)$$

Hence, $QNu \neq 0$ for each $\mathbf{u} \in \partial\Omega_1 \cap \text{Ker } L$.

Next let us consider $(H_1 + R_1)/2$, the arithmetic mean of H_1 and R_1 . We define $G : X \times \mathbb{R} \rightarrow X$ as follows, for all $\mu \in [0, 1]$,

$$G(\mathbf{u}, \mu) = -(1-\mu) \left(\mathbf{u} - \frac{H_1 + R_1}{2} \right) + \mu \frac{1}{T} \sum_{i=2}^{T+1} u^\alpha(i) [a(i) - b(i)u^{\beta-\alpha}(i)]. \quad (40)$$

Clearly, we find that

$$G(\mathbf{u}, \mu) \neq 0, \forall \mathbf{u} \in \partial\Omega_1 \cap \text{Ker } L. \quad (41)$$

By using the homotopy invariance theorem, it is easy to see that

$$\begin{aligned} \deg(QN, \Omega_1 \cap \text{Ker } L, 0) &= \deg(G(\mathbf{u}, 1), \Omega_1 \cap \text{Ker } L, 0) \\ &= \deg(G(\mathbf{u}, 0), \Omega_1 \cap \text{Ker } L, 0) \\ &= -1 \neq 0. \end{aligned} \quad (42)$$

Therefore, conditions (1)–(3) of Lemma 6 hold for Ω_1 .

Furthermore, according to the above reasoning, we deduce that (1) has at least one positive solution in $\bar{\Omega}_1$.

Case 2. If the coefficient functions $a(k), b(k)$ are negative T -periodic functions, in view of F_1 , we have that $-A \leq a(k) \leq -a < 0$ and $-B \leq b(k) \leq -b < 0$.

Let $\tilde{a}(k) = -a(k), \tilde{b}(k) = -b(k)$; then, we see that

$$\begin{aligned} 0 < a \leq \tilde{a}(k) \leq A, \\ 0 < b \leq \tilde{b}(k) \leq B. \end{aligned} \quad (43)$$

It is obvious that (1) is equivalent to the equation

$$\Delta^4 u(k-2) + \tilde{a}(k)u^\alpha(k) - \tilde{b}(k)u^\beta(k) = 0. \quad (44)$$

Let $0 < \lambda < 1$ and \mathbf{u} be such that

$$\Delta^4 u(k-2) + \lambda \tilde{a}(k)u^\alpha(k) - \lambda \tilde{b}(k)u^\beta(k) = 0. \quad (45)$$

Summing from 2 to $T+1$, we can see that

$$\sum_{i=2}^{T+1} [\Delta^4 u(i-2) + \lambda \tilde{a}(i)u^\alpha(i) - \lambda \tilde{b}(i)u^\beta(i)] = 0. \quad (46)$$

Firstly, we claim that for each $\lambda \in (0, 1)$ and $\mathbf{u} \in \partial\Omega_1 \cap \text{Dom } L, L\mathbf{u} \neq \lambda N\mathbf{u}$. In fact, in view of (25), if $\mathbf{u} \in \partial\Omega_1$, then $\|\mathbf{u}\| = H_1$ or $\|\mathbf{u}\| = R_1$. We only prove the case of $\|\mathbf{u}\| = H_1$, similar to the proof of $\|\mathbf{u}\| = R_1$.

When $\|\mathbf{u}\| = H_1$, we can get that

$$\max_{2 \leq l, j \leq T+1} |u(l) - u(j)| < \frac{H_1}{2} \quad (47)$$

or

$$\max_{2 \leq l, j \leq T+1} |u(l) - u(j)| \geq \frac{H_1}{2} \quad (48)$$

holds.

If $\max_{2 \leq l, j \leq T+1} |u(l) - u(j)| < H_1/2$, thus

$$\frac{H_1}{2} < \mathbf{u} \leq H_1. \quad (49)$$

Further, we have

$$\begin{aligned} 0 &= \sum_{i=2}^{T+1} [\lambda \tilde{a}(i)u^\alpha(i) - \lambda \tilde{b}(i)u^\beta(i)] \\ &= \lambda \sum_{i=2}^{T+1} u^\alpha(i) [\tilde{a}(i) - \tilde{b}(i)u^{\beta-\alpha}(i)] \\ &\geq \lambda \left(a - BH_1^{\beta-\alpha} \right) \sum_{i=2}^{T+1} u^\alpha(i) > 0. \end{aligned} \quad (50)$$

This is a contradiction.

If $\max_{2 \leq l, j \leq T+1} |u(l) - u(j)| \geq H_1/2$, we see from Lemma 7 that

$$\begin{aligned} 0 &= \left| \sum_{i=2}^{T+1} [\Delta^4 u(i-2) - \lambda \tilde{a}(i)u^\alpha(i) + \lambda \tilde{b}(i)u^\beta(i)] \right| \\ &\geq \sum_{i=2}^{T+1} |\Delta^4 u(i-2)| - \sum_{i=2}^{T+1} |\lambda \tilde{a}(i)u^\alpha(i) + \lambda \tilde{b}(i)u^\beta(i)| \\ &> \sum_{i=2}^{T+1} |\Delta^4 u(i-2)| - \sum_{i=2}^{T+1} |\tilde{a}(i)u^\alpha(i) + \tilde{b}(i)u^\beta(i)| \\ &\geq \frac{16}{T^3} \max_{2 \leq l, j \leq T+1} |u(l) - u(j)| - T \left(AH_1^{\alpha-1} + BH_1^{\beta-1} \right) \|\mathbf{u}\| \\ &\geq \frac{8}{T^3} \|\mathbf{u}\| - T \left(AH_1^{\alpha-1} + BH_1^{\beta-1} \right) \|\mathbf{u}\| \\ &= \left[\frac{8}{T^3} - T \left(AH_1^{\alpha-1} + BH_1^{\beta-1} \right) \right] \|\mathbf{u}\| \geq 0, \end{aligned} \quad (51)$$

but this is a contradiction. Hence, for all $\mathbf{u} \in \partial\Omega_1$ and $\lambda \in (0, 1)$, we have

$$\Delta^4 u(k-2) + \lambda \tilde{a}(k)u^\alpha(k) - \lambda \tilde{b}(k)u^\beta(k) \neq 0. \quad (52)$$

Therefore, we verify that condition (1) of Lemma 6 holds for Ω_1 .

The remaining proof is similar to the proof of Case 1, and so we omit it. Furthermore, according to the above reasoning, we deduce that (44) has at least one positive solution in $\bar{\Omega}_1$.

4. Proof of Theorem 2

Proof. The content of Theorem 2 is as follows: let F_1 hold, if $\alpha > \beta$ and the period T satisfies

$$16 \leq T^4 \leq \frac{8}{AQ_1^{\alpha-1} + BQ_1^{\beta-1}}, \quad (53)$$

where $Q_1 = (B/a)^{1/(\alpha-\beta)} + \tau$ and $\tau > 0$ small enough such that $(b/A)^{1/(\alpha-\beta)} - \tau > 0$; then, equation (1) admits at least one positive T -periodic solution. Now, we prove that the conclusion holds. Similarly, in the case of $\alpha > \beta$, we need to discuss the case where the coefficient functions $a(k)$ and $b(k)$ are both positive and negative, respectively. \square

Case 1. If coefficient functions $a(k)$ and $b(k)$ are positive T -periodic functions, we have that $0 < a \leq a(k) \leq A$ and $0 < b \leq b(k) \leq B$.

Let

$$\Omega_2 = \{\mathbf{u} \in X \mid P_1 < u(k) < Q_1\}, \quad (54)$$

which is an open set in X , where

$$Q_1 = Q + \tau, Q = \left(\frac{B}{a}\right)^{1/(\alpha-\beta)}, \quad (55)$$

$$P_1 = P - \tau, P = \left(\frac{b}{A}\right)^{1/(\alpha-\beta)}, \quad (56)$$

where $\tau > 0$ small enough such that $(b/A)^{1/(\alpha-\beta)} - \tau > 0$. Obviously, P_1 and Q_1 are well defined.

By $\alpha > \beta$, $0 < a \leq a(k) \leq A$, and $0 < b \leq b(k) \leq B$, we obtain

$$0 < P_1 < P \leq \left(\frac{a(k)}{b(k)}\right)^{1/(\alpha-\beta)} \leq Q < Q_1, \quad (57)$$

uniformly for $k \in Z$.

By virtue of (55) and (56), we obtain

$$\begin{aligned} a(k)P_1^{\alpha-\beta} - b(k) &\leq AP_1^{\alpha-\beta} - b < 0, \\ a(k)Q_1^{\alpha-\beta} - b(k) &\geq aQ_1^{\alpha-\beta} - B > 0. \end{aligned} \quad (58)$$

Therefore,

$$\begin{aligned} a(k)P_1^{\alpha-\beta} - b(k) &< 0, \\ a(k)Q_1^{\alpha-\beta} - b(k) &> 0, \end{aligned} \quad (59)$$

uniformly for $k \in Z$.

The remaining proof is similar to the proof of Theorem 1, and so we omit it. Furthermore, we deduce that (1) has at least one positive T -periodic solution in $\bar{\Omega}_2$.

Case 2. If the coefficient functions $a(k), b(k)$ are negative T -periodic functions, we have that $-A \leq a(k) \leq -a < 0$ and $-B \leq b(k) \leq -b < 0$.

Let $\tilde{a}(k) = -a(k), \tilde{b}(k) = -b(k)$. Then, we can see that

$$\begin{aligned} 0 < a \leq \tilde{a}(k) \leq A, \\ 0 < b \leq \tilde{b}(k) \leq B. \end{aligned} \quad (60)$$

It is obvious that

$$\begin{aligned} -\tilde{a}(k)P_1^{\alpha-\beta} + \tilde{b}(k) &> 0, \\ -\tilde{a}(k)Q_1^{\alpha-\beta} + \tilde{b}(k) &< 0, \end{aligned} \quad (61)$$

uniformly for $k \in Z$.

The remaining proof is similar to the proof of Theorem 1, and so we omit it. Furthermore, we conclude that (1) has at least one positive T -periodic solution in $\bar{\Omega}_2$.

5. Proof of Theorem 3

Proof. The content of Theorem 3 is as follows: suppose $a(k)b(k) \leq 0$ and $a(k), b(k)$ are not identical to zero for all $k \in Z$; then, equation (1) has no positive solution.

Summing equation (1) from 2 to $T+1$, we obtain that

$$\sum_{i=2}^{T+1} [\Delta^4 u(i-2) - a(i)u^\alpha(i) + b(i)u^\beta(i)] = 0. \quad (62)$$

In view of

$$\begin{aligned} \text{Dom } L &= \{\mathbf{u} \mid \mathbf{u} \in X, \Delta u(k+T) = \Delta u(k), \Delta^2 u(k+T) \\ &= \Delta^2 u(k), \Delta^3 u(k+T) = \Delta^3 u(k)\}. \end{aligned} \quad (63)$$

Hence,

$$\sum_{i=2}^{T+1} [-a(i)u^\alpha(i) + b(i)u^\beta(i)] = 0. \quad (64)$$

If $a(k) > 0$ and $b(k) \leq 0$, it follows from (64) that (1) does not have any positive solution. Other cases are similar. \square

6. Example

Example 1. The difference equation

$$\begin{aligned} \Delta^4 u(k-2) - \left(\frac{1}{100} \sin \left(\frac{2k\pi}{T} \right) + \frac{1}{50} \right) u(k) \\ + \left(\frac{1}{200} \left| \cos \left(\frac{2k\pi}{T} \right) \right| + \frac{1}{100} \right) u^3(k) = 0, \end{aligned} \quad (65)$$

is one of the form (1), where $a = 1/100$, $A = 3/100$, $b = 1/100$, $B = 3/200$, $\alpha = 1$, and $\beta = 3$. Let $\rho = \sqrt{6}/6$, we obtain

$$16 < T^4 \leq \frac{3200}{31 + 6\sqrt{2}}. \quad (66)$$

Therefore, we can prove that (65) has at least one positive T -periodic solution in $\bar{\Omega}_1$, where

$$\Omega_1 = \left\{ \mathbf{u} \in X \mid \frac{\sqrt{2}}{6} < u(k) < \frac{6\sqrt{3} + \sqrt{6}}{6} \right\}. \quad (67)$$

Example 2. The difference equation

$$\begin{aligned} \Delta^4 u(k-2) + \left(\frac{1}{5000} \cos \left(\frac{2k\pi}{T} \right) + \frac{1}{2500} \right) u^3(k) \\ - \left(\frac{1}{2000} \sin \left(\frac{2k\pi}{T} \right) + \frac{1}{500} \right) u(k) = 0, \end{aligned} \quad (68)$$

is one of the form (1), where $-a = -1/5000$, $-A = -3/5000$, $-b = -3/2000$, $-B = -1/400$, $\alpha = 3$, and $\beta = 1$. Let $\tau = \sqrt{10}/4$, we obtain

$$16 < T^4 \leq \frac{64000}{83 + 12\sqrt{5}}. \quad (69)$$

Therefore, we can prove that (68) has at least one positive T -periodic solution in $\bar{\Omega}_2$, where

$$\Omega_2 = \left\{ \mathbf{u} \in X \mid \frac{\sqrt{10}}{4} < u(k) < \frac{11\sqrt{10}}{4} \right\}. \quad (70)$$

Data Availability

Data sharing does not apply to this article as no data set was generated or analyzed during the current study.

Conflicts of Interest

The authors declare that they have no competing interests.

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