Hindawi
Journal of Function Spaces
Volume 2022, Article ID 1925612, 12 pages
https://doi.org/10.1155/2022/1925612

Research Article

Existence and Uniqueness of Common Fixed Point on Complex Partial b-Metric Space

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Received 9 November 2021; Revised 14 January 2022; Accepted 15 January 2022; Published 27 February 2022

Academic Editor: Muhammed Cinar

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In this study, we establish the existence and uniqueness of common fixed point on complex partial b-metric space. An example and application to support our result is presented.

The fourth author Y. U. Gaba would like to dedicate this publication to his wife, Clémence A. Epse G., in celebration of her 33rd birthday

1. Introduction

In 1989, Backhtin [1] and Czerwik [2] introduced the concept of b-metric spaces and provided a framework to extend the results in the classical setting of metric spaces which are known already. Azam et al. [3]introduced complex-valued metric spaces in 2011 and proved some common fixed-point theorems under the contraction condition. Then, in 2013, Rao et al. [4]introduced the definition of complex valued b-metric space and provided a method to extend the results. Later, in 2017, the concept of complex partial metric space was introduced by Dhivya and Marudai [5], and they proved common fixed-point theorems. Recently, Gunaseelan [6]introduced the concept of complex partial b-metric space in 2019. Many authors have discussed significant results and application on complex metric spaces [7–23]. In this study, we establish common fixed-point theorems on complex partial b-metric space using continuity property.

2. Preliminaries

Let C be the set of complex numbers and $\zeta_1, \zeta_2, \zeta_3 \in C$. Define a partial order $\precsim$ on C as follows:

$\zeta_1 \precsim \zeta_2$ if and only if $R(\zeta_1) \le R(\zeta_2)$ and $I(\zeta_1) \le I(\zeta_2)$.

Then, $\zeta_1 \precsim \zeta_2$ if one of the following properties is fulfilled:

(i) $R(\zeta_1) = R(\zeta_2)$, $I(\zeta_1) < I(\zeta_2)$
(ii) $R(\zeta_1) < R(\zeta_2)$, $I(\zeta_1) = I(\zeta_2)$
(iii) $R(\zeta_1) < R(\zeta_2)$, $I(\zeta_1) < I(\zeta_2)$
(iv) $R(\zeta_1) = R(\zeta_2)$, $I(\zeta_1) = I(\zeta_2)$

In particular, we write $\zeta_1 \precnsim \zeta_2$ if $\zeta_1 \neq \zeta_2$ and one of (i), (ii), and (iii) is fulfilled, and we write $\zeta_1 \prec \zeta_2$ if only (iii) is fulfilled.

Definition 1 (see [4]). Let H be a nonvoid set and let $s \ge 1$ be a given real number. A function $\ell: H \times H \longrightarrow C$ is called a complex valued b-metric on H if, for all $\phi, \mu, \lambda \in H$, the following conditions are fulfilled:

(i) $0 \leq \ell(\phi, \mu)$ and $\ell(\phi, \mu) = 0$ if and only if $\phi = \mu$

(ii) $\ell(\phi, \mu) = \ell(\mu, \phi)$

(iii) $\ell(\phi, \mu) \leq s[\ell(\phi, \lambda) + \ell(\lambda, \mu)]$

The pair (H, ℓ) is called a complex valued b-metric space.

Definition 2 (see [5]). A complex partial metric on a nonvoid set H is a function $\eta_{c\mathfrak{b}}: H \times H \longrightarrow C^+$ such that, for all $\phi, \mu, \lambda \in H$,

(i) $0 \leq \eta_{c\mathfrak{b}}(\phi, \phi) \leq \eta_{c\mathfrak{b}}(\phi, \mu)$ (small self – distances)

(ii) $\eta_{c\mathfrak{b}}(\phi, \mu) = \eta_{c\mathfrak{b}}(\mu, \phi)$ (symmetry)

(iii) $\eta_{c\mathfrak{b}}(\phi, \phi) = \eta_{c\mathfrak{b}}(\phi, \mu) = \eta_{c\mathfrak{b}}(\mu, \mu)$ *if and only if* $\phi = \mu$ (equality)

(iv)
$\eta_{c\mathfrak{b}}(\phi, \mu) \leq \eta_{c\mathfrak{b}}(\phi, \lambda) + \eta_{c\mathfrak{b}}(\lambda, \mu) - \eta_{c\mathfrak{b}}(\lambda, \lambda)$ (triangularity)

A complex partial metric space is a pair $(H, \eta_{c\mathfrak{b}})$ such that H is a nonvoid set and $\eta_{c\mathfrak{b}}$ is the complex partial metric on H.

Definition 3 (see [6]). A complex partial b-metric on a nonvoid set H is a function $\ell_{c\mathfrak{b}}: H \times H \longrightarrow C^+$ such that, for all $\phi, \mu, \lambda \in H$,

(i) $0 \leq \ell_{c\mathfrak{b}}(\phi, \phi) \leq \ell_{c\mathfrak{b}}(\phi, \mu)$ (small self – distances)

(ii) $\ell_{c\mathfrak{b}}(\phi, \mu) = \ell_{c\mathfrak{b}}(\mu, \phi)$ (symmetry)

(iii) $\ell_{c\mathfrak{b}}(\phi, \phi) = \ell_{c\mathfrak{b}}(\phi, \mu) = \ell_{c\mathfrak{b}}(\mu, \mu) \Leftrightarrow \phi = \mu$ (equality)

(iv) $\exists s \geq 1$ *such that* $\ell_{c\mathfrak{b}}(\phi, \mu) \leq s[\ell_{c\mathfrak{b}}(\phi, \lambda) + \ell_{c\mathfrak{b}}(\lambda, \mu)] - \ell_{c\mathfrak{b}}(\lambda, \lambda)$ (triangularity)

A complex partial b-metric space (b-CPMS) is a pair $(H, \ell_{c\mathfrak{b}})$ such that H is a nonvoid set and $\ell_{c\mathfrak{b}}$ is the complex partial b-metric on H. The number s is called the coefficient of $(H, \ell_{c\mathfrak{b}})$.

Definition 4 (see [6]). Let $(H, \ell_{c\mathfrak{b}})$ be a complex partial b-metric space with coefficient s. Let $\{\phi_\alpha\}$ be any sequence in H and $\phi \in H$. Then,

(i) The sequence $\{\phi_\alpha\}$ is said to be convergent with respect to $\ell_{c\mathfrak{b}}$ and converges to ϕ if $\lim_{\alpha \longrightarrow \infty} \ell_{c\mathfrak{b}}(\phi_\alpha, \phi) = \ell_{c\mathfrak{b}}(\phi, \phi)$

(ii) The sequence $\{\phi_\alpha\}$ is said to be Cauchy sequence in $(H, \ell_{c\mathfrak{b}})$ if
$\lim_{\alpha, m \longrightarrow \infty} \ell_{c\mathfrak{b}}(\phi_\alpha, \phi_m)$ exists and is finite

(iii) $(H, \ell_{c\mathfrak{b}})$ is said to be a complete complex partial b-metric space if, for every Cauchy sequence $\{\phi_\alpha\}$ in H, there exists $\phi \in H$ such that $\lim_{\alpha, m \longrightarrow \infty} \ell_{c\mathfrak{b}}(\phi_\alpha, \phi_m) = \lim_{\alpha \longrightarrow \infty} \ell_{c\mathfrak{b}}(\phi_\alpha, \phi) = \ell_{c\mathfrak{b}}(\phi, \phi)$

In 2019, Gunaseelan [6] proved some fixed-point theorems on complex partial b-metric space as follows.

Theorem 1. *Let $(H, \ell_{c\mathfrak{b}})$ be any complete complex partial b-metric space with coefficient $s \geq 1$ and $\Delta: H \longrightarrow H$ be a mapping satisfying*

$$\ell_{c\mathfrak{b}}(\Delta\phi, \Delta\mu) \leq \vartheta \max\{\ell_{c\mathfrak{b}}(\phi, \mu), \ell_{c\mathfrak{b}}(\phi, \Delta\phi), \ell_{c\mathfrak{b}}(\mu, \Delta\mu)\}, \quad (1)$$

for all $\phi, \mu \in H$, where $\vartheta \in [0, 1/s)$. Then, Δ has a unique fixed point $\phi_* \in H$ and $\ell_{c\mathfrak{b}}(\phi_*, \phi_*) = 0$.

We prove the existence and uniqueness of common fixed point on complex partial b-metric space, inspired by his work.

3. Main Results

Theorem 2. *Let $(H, \ell_{c\mathfrak{b}})$ be a complete b-CPMS with the coefficient $s \geq 1$ and $\Delta, \Omega: H \longrightarrow H$ be two continuous mappings such that*

$$\ell_{c\mathfrak{b}}(\Delta\phi, \Omega\mu) \leq \vartheta \max\left\{\ell_{c\mathfrak{b}}(\phi, \mu), \ell_{c\mathfrak{b}}(\phi, \Delta\phi), \ell_{c\mathfrak{b}}(\mu, \Omega\mu), \frac{1}{2}\left(\ell_{c\mathfrak{b}}(\phi, \Omega\mu) + \ell_{c\mathfrak{b}}(\mu, \Delta\phi)\right)\right\}, \quad (2)$$

for all $\phi, \mu \in H$, where $0 \leq \vartheta < 1/s$. Then, Δ and Ω have a unique common fixed point $\phi^* \in H$ and $\ell_{c\mathfrak{b}}(\phi^*, \phi^*) = 0$.

Proof. Let $\phi_0 \in H$. Define

$$\phi_{2\alpha+1} = \Delta\phi_{2\alpha} \quad \text{and} \quad \phi_{2\alpha+2} = \Omega\phi_{2\alpha+1}, \alpha = 0, 1, 2, \ldots. \quad (3)$$

Then, by (1) and (2), we obtain

$$\begin{aligned}\ell_{cb}(\phi_{2\alpha+1},\phi_{2\alpha+2}) &= \ell_{cb}(\Delta\phi_{2\alpha},\Omega\phi_{2\alpha+1}),\\ &\lesssim \vartheta\max\left\{\ell_{cb}(\phi_{2\alpha},\phi_{2\alpha+1}),\ell_{cb}(\phi_{2\alpha},\Delta\phi_{2\alpha}),\ell_{cb}(\phi_{2\alpha+1},\Omega\phi_{2\alpha+1}),\frac{1}{2}(\ell_{cb}(\phi_{2\alpha},\Omega\phi_{2\alpha+1}))+\ell_{cb}(\phi_{2\alpha+1},\Delta\phi_{2\alpha})\right\},\\ &\lesssim \vartheta\max\left\{\ell_{cb}(\phi_{2\alpha},\phi_{2\alpha+1}),\ell_{cb}(\phi_{2\alpha},\phi_{2\alpha+1}),\ell_{cb}(\phi_{2\alpha+1},\phi_{2\alpha+2}),\frac{1}{2}(\ell_{cb}(\phi_{2\alpha},\phi_{2\alpha+2}))+\ell_{cb}(\phi_{2\alpha+1},\phi_{2\alpha+1})\right\},\\ &\lesssim \vartheta\max\left\{\ell_{cb}(\phi_{2\alpha},\phi_{2\alpha+1}),\ell_{cb}(\phi_{2\alpha+1},\phi_{2\alpha+2}),\ \frac{1}{2}(s(\ell_{cb}(\phi_{2\alpha},\phi_{2\alpha+1}))+\ell_{cb}(\phi_{2\alpha+1},\phi_{2\alpha+2})-\ell_{cb}(\phi_{2\alpha+1},\phi_{2\alpha+1})\right.\\ &\quad\left.+\ell_{cb}(\phi_{2\alpha+1},\phi_{2\alpha+1}))\right\},\\ &= \vartheta\max\left\{\ell_{cb}(\phi_{2\alpha},\phi_{2\alpha+1}),\ell_{cb}(\phi_{2\alpha+1},\phi_{2\alpha+2}),\frac{s}{2}(\ell_{cb}(\phi_{2\alpha},\phi_{2\alpha+1})+\ell_{cb}(\phi_{2\alpha+1},\phi_{2\alpha+2}))\right\}.\end{aligned} \tag{4}$$

□

Case 1. If $\max\{\ell_{cb}(\phi_{2\alpha},\phi_{2\alpha+1}),\ell_{cb}(\phi_{2\alpha+1},\phi_{2\alpha+2}),s/2(\ell_{cb}(\phi_{2\alpha},\phi_{2\alpha+1})+\ell_{cb}(\phi_{2\alpha+1},\phi_{2\alpha+2}))\}=\ell_{cb}(\phi_{2\alpha+1},\phi_{2\alpha+2})$, then we have

$$\ell_{cb}(\phi_{2\alpha+1},\phi_{2\alpha+2})\lesssim\vartheta\ell_{cb}(\phi_{2\alpha+1},\phi_{2\alpha+2}). \tag{5}$$

This implies $\vartheta\geq 1$, which is a reductio ad absurdum.

Case 2. If $\max\{\ell_{cb}(\phi_{2\alpha},\phi_{2\alpha+1}),\ell_{cb}(\phi_{2\alpha+1},\phi_{2\alpha+2}),s/2(\ell_{cb}(\phi_{2\alpha},\phi_{2\alpha+1})+\ell_{cb}(\phi_{2\alpha+1},\phi_{2\alpha+2}))\}=\ell_{cb}(\phi_{2\alpha},\phi_{2\alpha+1})$, then we have

$$\ell_{cb}(\phi_{2\alpha+1},\phi_{2\alpha+2})\lesssim\vartheta\ell_{cb}(\phi_{2\alpha},\phi_{2\alpha+1}). \tag{6}$$

From the next step, we have

$$\ell_{cb}(\phi_{2\alpha+2},\phi_{2\alpha+3})\lesssim\vartheta\max\left\{\ell_{cb}(\phi_{2\alpha+1},\phi_{2\alpha+2}),\ell_{cb}(\phi_{2\alpha+2},\phi_{2\alpha+3}),\frac{s}{2}(\ell_{cb}(\phi_{2\alpha+1},\phi_{2\alpha+2})+\ell_{cb}(\phi_{2\alpha+2},\phi_{2\alpha+3}))\right\}. \tag{7}$$

We consider three cases.

Case 3.

$$\ell_{cb}(\phi_{2\alpha+2},\phi_{2n+3})\lesssim\vartheta\ell_{cb}(\phi_{2\alpha+2},\phi_{2n+3}), \tag{8}$$

which implies $\vartheta\geq 1$, is a reductio ad absurdum.

Case 4.

$$\ell_{cb}(\phi_{2\alpha+2},\phi_{2n+3})\lesssim\vartheta\ell_{cb}(\phi_{2\alpha+1},\phi_{2\alpha+2}). \tag{9}$$

From (6) and (9), $\forall\alpha=0,1,2,\ldots$, we obtain

$$\ell_{cb}(\phi_{\alpha+1},\phi_{\alpha+2})\lesssim\vartheta\ell_{cb}(\phi_{\alpha},\phi_{\alpha+1})\lesssim\cdots\lesssim\vartheta^{\alpha+1}\ell_{cb}(\phi_0,\phi_1). \tag{10}$$

For $\mathfrak{q},\alpha\in\mathbb{N}$, with $\mathfrak{q}>\alpha$, we have

$$\begin{aligned}\ell_{cb}(\phi_\alpha,\phi_{\mathfrak{q}}) &\lesssim s[\ell_{cb}(\phi_\alpha,\phi_{\alpha+1})+\ell_{cb}(\phi_{\alpha+1},\phi_{\mathfrak{q}})-\ell_{cb}(\phi_{\alpha+1},\phi_{\alpha+1})],\\ &\lesssim s[\ell_{cb}(\phi_\alpha,\phi_{\alpha+1})+\ell_{cb}(\phi_{\alpha+1},\phi_{\mathfrak{q}})],\\ &\lesssim s(\ell_{cb}(\phi_\alpha,\phi_{\alpha+1}))+s^2(\ell_{cb}(\phi_{\alpha+1},\phi_{\alpha+2})+\ell_{cb}(\phi_{\alpha+2},\phi_{\mathfrak{q}}))-\ell_{cb}(\phi_{\alpha+2},\phi_{\alpha+2}),\\ &\lesssim s(\ell_{cb}(\phi_\alpha,\phi_{\alpha+1})+s^2[\ell_{cb}(\phi_{\alpha+1},\phi_{\alpha+2})+\ell_{cb}(\phi_{\alpha+2},\phi_{\mathfrak{q}})]\\ &\lesssim s(\ell_{cb}(\phi_\alpha,\phi_{\alpha+1}))+s^2(\ell_{cb}(\phi_{\alpha+1},\phi_{\alpha+2}))+s^3(\ell_{cb}(\phi_{\alpha+2},\phi_{\alpha+3})),\\ &\quad+\cdots+s^{\mathfrak{q}-\alpha-1}(\ell_{cb}(\phi_{\mathfrak{q}-2},\phi_{\mathfrak{q}-1}))+s^{\mathfrak{q}-\alpha}(\ell_{cb}(\phi_{\mathfrak{q}-1},\phi_{\mathfrak{q}})).\end{aligned} \tag{11}$$

Moreover, by using (9), we obtain

$$\begin{aligned}\ell_{cb}(\phi_\alpha,\phi_{\mathfrak{q}}) &\lesssim s\vartheta^{\alpha}(\ell_{cb}(\phi_0,\phi_1))+s^2\vartheta^{\alpha+1}(\ell_{cb}(\phi_0,\phi_1))+s^3\vartheta^{\alpha+2}(\ell_{cb}(\phi_0,\phi_1))\\ &\quad+\cdots+s^{\mathfrak{q}-\alpha-1}\vartheta^{\mathfrak{q}-2}(\ell_{cb}(\phi_0,\phi_1))+s^{\mathfrak{q}-\alpha}\vartheta^{\mathfrak{q}-1}(\ell_{cb}(\phi_0,\phi_1))=\sum_{i=1}^{\mathfrak{q}-\alpha}s^i\vartheta^{i+\alpha-1}(\ell_{cb}(\phi_0,\phi_1)).\end{aligned} \tag{12}$$

Therefore,

$$\left|\ell_{c\mathfrak{b}}\left(\phi_\alpha, \phi_{\mathfrak{q}}\right)\right| \leq \sum_{i=1}^{\mathfrak{q}-\alpha} s^{i+\alpha-1}\vartheta^{i+\alpha-1}\left|\ell_{c\mathfrak{b}}\left(\phi_0, \phi_1\right)\right| = \sum_{t=\alpha}^{\mathfrak{q}-1} s^t\vartheta^t\left|\ell_{c\mathfrak{b}}\left(\phi_0, \phi_1\right)\right|,$$
$$\leq \sum_{i=\alpha}^{\infty} (s\vartheta)^t\left|\ell_{c\mathfrak{b}}\left(\phi_0, \phi_1\right)\right|, \tag{13}$$
$$= \frac{(s\vartheta)^\alpha}{1-s\vartheta}\left|\ell_{c\mathfrak{b}}\left(\phi_0, \phi_1\right)\right|.$$

Then, we have

$$\left|\ell_{c\mathfrak{b}}\left(\phi_\alpha, \phi_{\mathfrak{q}}\right)\right| \leq \frac{(s\vartheta)^\alpha}{1-s\vartheta}\left|\ell_{c\mathfrak{b}}\left(\phi_0, \phi_1\right)\right|| \longrightarrow 0 \quad \text{as} \quad \alpha \longrightarrow \infty. \tag{14}$$

Hence, $\{\phi_\alpha\}$ is a Cauchy sequence in H.

Case 5.

$$\ell_{c\mathfrak{b}}\left(\phi_{2\alpha+2}, \phi_{2\alpha+3}\right) \lesssim \vartheta\frac{s}{2}\left(\ell_{c\mathfrak{b}}\left(\phi_{2\alpha+1}, \phi_{2\alpha+2}\right) + \ell_{c\mathfrak{b}}\left(\phi_{2\alpha+2}, \phi_{2\alpha+3}\right)\right). \tag{15}$$

This implies that

$$\ell_{c\mathfrak{b}}\left(\phi_{2\alpha+2}, \phi_{2n+3}\right) \lesssim \frac{\vartheta s}{(2-\vartheta s)}\ell_{c\mathfrak{b}}\left(\phi_{2\alpha+1}, \phi_{2\alpha+2}\right). \tag{16}$$

Since $a := \vartheta s/(2-\vartheta s) < 1$, we get $\ell_{c\mathfrak{b}}\left(\phi_{\alpha+1}, \phi_{\alpha+2}\right) \lesssim a\ell_{c\mathfrak{b}}\left(\phi_\alpha, \phi_{\alpha+1}\right)$. Therefore, $\{\phi_\alpha\}_{\alpha\in\mathbb{N}}$ is a Cauchy sequence in H.

Case 6. If $\max\{\ell_{c\mathfrak{b}}(\phi_{2\alpha}, \phi_{2\alpha+}\ 1), \ell_{c\mathfrak{b}}(\phi_{2\alpha+1}, \phi_{2\alpha+2}), s/2(\ell_{c\mathfrak{b}}(\phi_{2\alpha}, \phi_{2\alpha+1}) + \ell_{c\mathfrak{b}}(\phi_{2\alpha+1}, \phi_{2\alpha+2}))\} = s/2\,(\ell_{c\mathfrak{b}}(\phi_{2\alpha}, \phi_{2\alpha+1}) + \ell_{c\mathfrak{b}}(\phi_{2\alpha+1}, \phi_{2\alpha+2}))$, then we have

$$\ell_{c\mathfrak{b}}\left(\phi_{2\alpha+1}, \phi_{2\alpha+2}\right) \lesssim \vartheta s/2\left(\ell_{c\mathfrak{b}}\left(\phi_{2\alpha}, \phi_{2\alpha+1}\right) + \ell_{c\mathfrak{b}}\left(\phi_{2\alpha+1}, \phi_{2\alpha+2}\right)\right). \tag{17}$$

Hence,

$$\ell_{c\mathfrak{b}}\left(\phi_{2\alpha+1}, \phi_{2\alpha+2}\right) \lesssim \frac{\vartheta s}{(2-\vartheta s)}\ell_{c\mathfrak{b}}\left(\phi_{2\alpha}, \phi_{2\alpha+1}\right). \tag{18}$$

For the next step, we have

$$\ell_{c\mathfrak{b}}\left(\phi_{2\alpha+2}, \phi_{2\alpha+3}\right) \lesssim \vartheta \max\left\{\ell_{c\mathfrak{b}}\left(\phi_{2\alpha+1}, \phi_{2\alpha+2}\right), \ell_{c\mathfrak{b}}\left(\phi_{2\alpha+2}, \phi_{2\alpha+3}\right), \frac{s}{2}\left(\ell_{c\mathfrak{b}}\left(\phi_{2\alpha+1}, \phi_{2\alpha+2}\right) + \ell_{c\mathfrak{b}}\left(\phi_{2\alpha+2}, \phi_{2\alpha+3}\right)\right)\right\}. \tag{19}$$

Then, we consider three cases.

Case 7.

$$\ell_{c\mathfrak{b}}\left(\phi_{2\alpha+2}, \phi_{2\alpha+3}\right) \lesssim \vartheta\ell_{c\mathfrak{b}}\left(\phi_{2\alpha+2}, \phi_{2\alpha+3}\right), \tag{20}$$

which implies $\vartheta \geq 1$ and is a reductio ad absurdum.

Case 8.

$$\ell_{c\mathfrak{b}}\left(\phi_{2\alpha+2}, \phi_{2n+3}\right) \lesssim \vartheta\ell_{c\mathfrak{b}}\left(\phi_{2\alpha+1}, \phi_{2\alpha+2}\right). \tag{21}$$

Then, by (18) and (21), we get $\ell_{c\mathfrak{b}}\left(\phi_{\alpha+1}, \phi_{\alpha+2}\right) \lesssim \gamma\ell_{c\mathfrak{b}}\left(\phi_\alpha, \phi_{\alpha+1}\right)$, where $\gamma = \max\{\vartheta, \vartheta s/2 - \vartheta s\} < 1$. Hence, $\{\phi_\alpha\}_{\alpha\in N}$ is a Cauchy sequence in H.

Case 9.

$$\ell_{c\mathfrak{b}}\left(\phi_{2\alpha+2}, \phi_{2n+3}\right) \lesssim \frac{s}{2}\left(\ell_{c\mathfrak{b}}\left(\phi_{2\alpha+1}, \phi_{2\alpha+2}\right)\right) + \ell_{c\mathfrak{b}}\left(\phi_{2\alpha+2}, \phi_{2\alpha+3}\right). \tag{22}$$

Hence, we obtain

$$\ell_{c\mathfrak{b}}\left(\phi_{2\alpha+2}, \phi_{2\alpha+3}\right) \lesssim \frac{\vartheta s}{(2-\vartheta s)}\ell_{c\mathfrak{b}}\left(\phi_{2\alpha+1}, \phi_{2\alpha+2}\right). \tag{23}$$

By using (18) and (21) yields

$$\ell_{c\mathfrak{b}}\left(\phi_{\alpha+1}, \phi_{\alpha+2}\right) \lesssim \wr\ell_{c\mathfrak{b}}\left(\phi_\alpha, \phi_{\alpha+1}\right), \tag{24}$$

where $0 \leq \wr = \vartheta s/(2-\vartheta s) < 1$.

Then, $\forall \alpha = 0, 1, 2, \ldots$, we obtain

$$\ell_{c\mathfrak{b}}\left(\phi_{\alpha+1}, \phi_{\alpha+2}\right) \lesssim \wr\ell_{c\mathfrak{b}}\left(\phi_\alpha, \phi_{\alpha+1}\right) \lesssim \cdots \lesssim \wr^{\alpha+1}\ell_{c\mathfrak{b}}\left(\phi_0, \phi_1\right). \tag{25}$$

For $\mathfrak{q}, \alpha \in \mathbb{N}$, with $\mathfrak{q} > \alpha$,

$$\begin{aligned}\ell_{c\mathfrak{b}}(\phi_\alpha,\phi_{\mathfrak{q}}) &\lesssim s[\ell_{c\mathfrak{b}}(\phi_\alpha,\phi_{\alpha+1})+\ell_{c\mathfrak{b}}(\phi_{\alpha+1},\phi_{\mathfrak{q}})-\ell_{c\mathfrak{b}}(\phi_{\alpha+1},\phi_{\alpha+1})],\\ &\lesssim s[\ell_{c\mathfrak{b}}(\phi_\alpha,\phi_{\alpha+1})+\ell_{c\mathfrak{b}}(\phi_{\alpha+1},\phi_{\mathfrak{q}})],\\ &\lesssim s(\ell_{c\mathfrak{b}}(\phi_\alpha,\phi_{\alpha+1}))+s^2(\ell_{c\mathfrak{b}}(\phi_{\alpha+1},\phi_{\alpha+2})+\ell_{c\mathfrak{b}}(\phi_{\alpha+2},\phi_{\mathfrak{q}})-\ell_{c\mathfrak{b}}(\phi_{\alpha+2},\phi_{\alpha+2})),\\ &\lesssim s(\ell_{c\mathfrak{b}}(\phi_\alpha,\phi_{\alpha+1}))+s^2[\ell_{c\mathfrak{b}}(\phi_{\alpha+1},\phi_{\alpha+2})+\ell_{c\mathfrak{b}}(\phi_{\alpha+2},\phi_{\mathfrak{q}})],\\ &\lesssim s(\ell_{c\mathfrak{b}}(\phi_\alpha,\phi_{\alpha+1}))+s^2(\ell_{c\mathfrak{b}}(\phi_{\alpha+1},\phi_{\alpha+2}))+s^3(\ell_{c\mathfrak{b}}(\phi_{\alpha+2},\phi_{\alpha+3})),\\ &\quad+\cdots+s^{\mathfrak{q}-\alpha-1}(\ell_{c\mathfrak{b}}(\phi_{\mathfrak{q}-2},\phi_{\mathfrak{q}-1}))+s^{\mathfrak{q}-\alpha}(\ell_{c\mathfrak{b}}(\phi_{\mathfrak{q}-1},\phi_{\mathfrak{q}})).\end{aligned} \tag{26}$$

Using (24), we obtain

$$\begin{aligned}\ell_{c\mathfrak{b}}(\phi_\alpha,\phi_{\mathfrak{q}}) &\lesssim s\wr^\alpha(\ell_{c\mathfrak{b}}(\phi_0,\phi_1))+s^2\wr^{\alpha+1}(\ell_{c\mathfrak{b}}(\phi_0,\phi_1))+s^3\wr^{\alpha+2}(\ell_{c\mathfrak{b}}(\phi_0,\phi_1))\\ &\quad+\cdots+s^{\mathfrak{q}-\alpha-1}\wr^{\mathfrak{q}-2}(\ell_{c\mathfrak{b}}(\phi_0,\phi_1))+s^{\mathfrak{q}-\alpha}\wr^{\mathfrak{q}-1}(\ell_{c\mathfrak{b}}(\phi_0,\phi_1))\\ &=\sum_{i=1}^{\mathfrak{q}-\alpha}s^i\wr^{i+\alpha-1}(\ell_{c\mathfrak{b}}((\phi_0,\phi_1)).\end{aligned} \tag{27}$$

Therefore,

$$\begin{aligned}\left|\ell_{c\mathfrak{b}}(\phi_\alpha,\phi_{\mathfrak{q}})\right| &\le \sum_{i=1}^{\mathfrak{q}-\alpha}s^{i+\alpha-1}\wr^{i+\alpha-1}\left|\ell_{c\mathfrak{b}}(\phi_0,\phi_1)\right| = \sum_{t=\alpha}^{\mathfrak{q}-1}s^t\wr^t\left|\ell_{c\mathfrak{b}}(\phi_0,\phi_1)\right|\\ &\le \sum_{i=\alpha}^{\infty}(s\wr)^t\left|\ell_{c\mathfrak{b}}(\phi_0,\phi_1)\right|,\\ &=\frac{(s\wr)^\alpha}{1-s\wr}\left|\ell_{c\mathfrak{b}}(\phi_0,\phi_1)\right|.\end{aligned} \tag{28}$$

Hence, we have

$$\left|\ell_{c\mathfrak{b}}(\phi_\alpha,\phi_{\mathfrak{q}})\right| \le \frac{(s\wr)^\alpha}{1-s\wr}\left|\ell_{c\mathfrak{b}}(\phi_0,\phi_1)\right| \longrightarrow 0 \quad \text{as} \quad \alpha\longrightarrow\infty. \tag{29}$$

Hence, $\{\phi_\alpha\}$ is a Cauchy sequence in H. In all cases, $\{\phi_\alpha\}_{\alpha\in\mathbb{N}}$ is a Cauchy sequence. Since H is complete, there exists $\phi^* \in H$ such that $\phi_\alpha \longrightarrow \phi^*$ as $\alpha\longrightarrow\infty$ and

$$\ell_{c\mathfrak{b}}(\phi^*,\phi^*)=\lim_{\alpha\longrightarrow\infty}\ell_{c\mathfrak{b}}(\phi^*,\phi_\alpha)=\lim_{\alpha\longrightarrow\infty}\ell_{c\mathfrak{b}}(\phi_\alpha,\phi_\alpha)=0. \tag{30}$$

By the continuity of Δ, $\phi_{2\alpha+1}=\Delta\phi_{2\alpha}\longrightarrow\Delta\phi^*$ as $\alpha\longrightarrow\infty$:

$$\text{i.e.}\,\ell_{c\mathfrak{b}}(\Delta\phi^*,\Delta\phi^*)=\lim_{\alpha\longrightarrow\infty}\ell_{c\mathfrak{b}}(\Delta\phi^*,\Delta\phi_{2\alpha})=\lim_{\alpha\longrightarrow\infty}\ell_{c\mathfrak{b}}(\Delta\phi_{2\alpha},\Delta\phi_{2\alpha}). \tag{31}$$

However,

$$\ell_{c\mathfrak{b}}(\Delta\phi^*,\Delta\phi^*)=\lim_{\alpha\longrightarrow\infty}\ell_{c\mathfrak{b}}(\Delta\phi_{2\alpha},\Delta\phi_{2\alpha})=\lim_{\alpha\longrightarrow\infty}\ell_{c\mathfrak{b}}(\phi_{2\alpha+1},\phi_{2\alpha+1})=0. \tag{32}$$

Next, we prove that ϕ^* is a fixed point of Δ:

$$\ell_{c\mathfrak{b}}(\Delta\phi^*,\phi^*) \lesssim \ell_{c\mathfrak{b}}(\Delta\phi^*,\Delta\phi_{2\alpha}) + \ell_{c\mathfrak{b}}(\Delta\phi_{2\alpha},\phi^*) - \ell_{c\mathfrak{b}}(\Delta\phi_{2\alpha},\Delta\phi_{2\alpha}). \tag{33}$$

As $\alpha \longrightarrow \infty$, we obtain $|\ell_{c\mathfrak{b}}(\Delta\phi^*,\phi^*)| \leq 0$. Thus, $\ell_{cp}(\Delta\phi^*,\phi^*) = 0$. Hence, $\ell_{c\mathfrak{b}}(\phi^*,\phi^*) = \ell_{c\mathfrak{b}}(\phi^*,\Delta\phi^*) = \ell_{c\mathfrak{b}}(\Delta\phi^*,\Delta\phi^*) = 0$ and $\Delta\phi^* = \phi^*$. In the same way, we have $\phi^* \in H$ such that $\phi_\alpha \longrightarrow \phi^*$ as $\alpha \longrightarrow \infty$ and

$$\ell_{c\mathfrak{b}}(\phi^*,\phi^*) = \lim_{\alpha\longrightarrow\infty}\ell_{c\mathfrak{b}}(\phi^*,\phi_\alpha) = \lim_{\alpha\longrightarrow\infty}\ell_{c\mathfrak{b}}(\phi_\alpha,\phi_\alpha) = 0. \tag{34}$$

By the continuity of $\Delta\,\phi_{2\alpha+2} = \Omega\phi_{2\alpha+1} \longrightarrow \Omega\phi^*$ as $\alpha \longrightarrow \infty$,

$$\begin{aligned}\text{i.e.}\,\ell_{c\mathfrak{b}}(\Omega\phi^*,\Omega\phi^*) &= \lim_{\alpha\longrightarrow\infty}\ell_{c\mathfrak{b}}(\Omega\phi^*,\Omega\phi_{2\alpha+1})\\ &= \lim_{\alpha\longrightarrow\infty}\ell_{c\mathfrak{b}}(\Omega\phi_{2\alpha+1},\Omega\phi_{2\alpha+1}).\end{aligned} \tag{35}$$

However,

$$\begin{aligned}\ell_{c\mathfrak{b}}(\Omega\phi^*,\Omega\phi)^* &= \lim_{\alpha\longrightarrow\infty}\ell_{c\mathfrak{b}}(\Omega\phi_{2\alpha+1},\Omega\phi_{2\alpha+1})\\ &= \lim_{\alpha\longrightarrow\infty}\ell_{c\mathfrak{b}}(\phi_{2\alpha+2},\phi_{2\alpha+2}) = 0.\end{aligned} \tag{36}$$

Next, we prove that ϕ^* is a fixed point of Ω:

$$\begin{aligned}\ell_{c\mathfrak{b}}(\Omega\phi^*,\phi^*) &\leq \ell_{c\mathfrak{b}}(\Omega\phi^*,\Omega\phi_{2\alpha+1}) + \ell_{c\mathfrak{b}}(\Omega\phi_{2\alpha+1},\phi^*)\\ &\quad - \ell_{c\mathfrak{b}}(\Omega\phi_{2\alpha+1},\Delta\phi_{2\alpha+1}).\end{aligned} \tag{37}$$

As $\alpha \longrightarrow \infty$, we obtain $|\ell_{c\mathfrak{b}}(\Omega\phi^*,\phi^*)| \leq 0$. Thus, $\ell_{cp}(\Omega\phi^*,\phi^*) = 0$. Hence, $\ell_{c\mathfrak{b}}(\phi^*,\phi^*) = \ell_{c\mathfrak{b}}(\phi^*,\Omega\phi^*) = \ell_{c\mathfrak{b}}(\Omega\phi^*,\Omega\phi^*) = 0$ and $\Omega\phi^* = \phi^*$. Therefore, Δ and Ω have a common fixed point ϕ^*.

Let $\mu^* \in H$ be another common fixed point for the mappings Δ and Ω. Then,

$$\begin{aligned}\ell_{c\mathfrak{b}}(\phi^*,\mu^*) = \ell_{c\mathfrak{b}}(\Delta\phi^*,\Omega\mu^*) &\lesssim \vartheta\max\Big\{\ell_{c\mathfrak{b}}(\phi^*,\mu^*),\ell_{c\mathfrak{b}}(\phi^*,\Delta\phi^*),\ell_{c\mathfrak{b}}(\mu^*,\Omega\mu^*),\\ &\quad\frac{1}{2}(\ell_{c\mathfrak{b}}(\phi^*,\Omega\mu^*)) + \ell_{c\mathfrak{b}}(\mu^*,\Delta\phi^*)\Big\},\\ &\lesssim \vartheta\max\Big\{\ell_{c\mathfrak{b}}(\phi^*,\mu^*),\ell_{c\mathfrak{b}}(\phi^*,\phi^*),\ell_{c\mathfrak{b}}(\mu^*,\mu^*)\\ &\quad\frac{1}{2}(\ell_{c\mathfrak{b}}(\phi^*,\mu^*)) + \ell_{c\mathfrak{b}}(\mu^*,\phi^*)\Big\}\\ &\lesssim \vartheta\ell_{c\mathfrak{b}}(\phi^*,\mu^*).\end{aligned} \tag{38}$$

This implies that $\phi^* = \mu^*$.

Theorem 3. *Let $(H,\ell_{c\mathfrak{b}})$ be a complete b-CPMS with the coefficient $s \geq 1$ and $\Delta,\Omega\colon H \longrightarrow H$ be two continuous mappings such that*

$$\begin{aligned}\ell_{c\mathfrak{b}}(\Delta\phi,\Omega\mu) &\lesssim \vartheta\max\Big\{\ell_{c\mathfrak{b}}(\phi,\mu),\ell_{c\mathfrak{b}}(\phi,\Delta\phi),\ell_{c\mathfrak{b}}(\mu,\Omega\mu),\\ &\quad\frac{1}{2}(\ell_{c\mathfrak{b}}(\phi,\Omega\mu) + \ell_{c\mathfrak{b}}(\mu,\Delta\phi))\Big\},\end{aligned} \tag{39}$$

for all $\phi,\mu \in H$, where $0 \leq \vartheta < 1/s$. Then, Δ and Ω have a unique common fixed point $\phi^* \in H$ and $\ell_{c\mathfrak{b}}(\phi^*,\phi^*) = 0$.

Proof. Following from Theorem 2, we can easily prove $\{\phi_\alpha\}$ is a Cauchy sequence. Since H is complete, there exists $\phi^* \in H$ such that $\phi_\alpha \longrightarrow \phi^*$ as $\alpha \longrightarrow \infty$.

Suppose that $\ell_{c\mathfrak{b}}(\phi^*,\Delta\phi^*) = \lambda > 0$.

Then, we estimate

$$\begin{aligned}\lambda &= \ell_{c\mathfrak{b}}(\phi^*, \Delta\phi^*),\\ &\precsim s\{\ell_{c\mathfrak{b}}(\phi^*, \phi_{2i+2}) + \ell_{c\mathfrak{b}}(\phi_{2i+2}, \Delta\phi^*) - \ell_{c\mathfrak{b}}(\phi_{2i+2}, \phi_{2i+2})\},\\ &\precsim s\{\ell_{c\mathfrak{b}}(\phi^*, \phi_{2i+2}) + \ell_{c\mathfrak{b}}(\phi_{2i+2}, \Delta\phi^*)\},\\ &\precsim s\ell_{c\mathfrak{b}}(\phi^*, \phi_{2i+2}) + s\ell_{c\mathfrak{b}}(\Omega\phi_{2i+1}, \Delta\phi^*),\\ &\precsim s\ell_{c\mathfrak{b}}(\phi^*, \phi_{2i+2}) + \vartheta s \max\{\ell_{c\mathfrak{b}}(\phi_{2i+1}, \phi^*), \ell_{c\mathfrak{b}}(\phi_{2i+1}, \Omega\phi_{2i+1}), \ell_{c\mathfrak{b}}(\phi^*, \Delta\phi^*)\\ &\quad \frac{1}{2}((\ell_{c\mathfrak{b}}(\phi_{2i+1}, \Delta\phi^*) + \ell_{c\mathfrak{b}}(\phi^*, \Omega\phi_{2i+1}))\Big\},\\ &\precsim s\ell_{c\mathfrak{b}}(\phi^*, \phi_{2i+2}) + \vartheta s \max\{\ell_{c\mathfrak{b}}(\phi_{2i+1}, \phi^*), \ell_{c\mathfrak{b}}(\phi_{2i+1}, \phi_{2i+2}), \ell_{c\mathfrak{b}}(\phi^*, \Delta\phi^*)\\ &\quad \frac{1}{2}((\ell_{c\mathfrak{b}}(\phi_{2i+1}, \Delta\phi^*) + \ell_{c\mathfrak{b}}(\phi^*, \phi_{2i+2}))\Big\},\\ &\precsim s\ell_{c\mathfrak{b}}(\phi^*, \phi_{2i+2}) + s\vartheta\ell_{c\mathfrak{b}}(\phi^*, \Delta\phi^*),\\ &\precsim s\ell_{c\mathfrak{b}}(\phi^*, \phi_{2i+2}) + s\vartheta\lambda.\end{aligned} \tag{40}$$

This yields

$$|\lambda| \le s\left|\ell_{c\mathfrak{b}}(\phi^*, \phi_{2i+2})\right| + s\vartheta|\lambda|. \tag{41}$$

Hence, $\vartheta \ge 1$, which is a reductio ad absurdum. Then, $\phi^* = \Delta\phi^*$. Similarly, we derive that $\phi^* = \Omega\phi^*$. Therefore, Δ and Ω have a common fixed point ϕ^*. Following from Theorem 2, we can easily prove uniqueness part. ☐

Theorem 4. *Let $(H, \ell_{c\mathfrak{b}})$ be a complete b-CPMS with the coefficient $s \ge 1$ and $\Delta, \Omega: H \longrightarrow H$ be two continuous mappings such that*

$$\ell_{c\mathfrak{b}}(\Delta\phi, \Omega\mu) \precsim \vartheta \max\left\{\ell_{c\mathfrak{b}}(\phi, \mu), \frac{\ell_{c\mathfrak{b}}(\phi, \Delta\phi)\ell_{c\mathfrak{b}}(\mu, \Omega\mu)}{1 + \ell_{c\mathfrak{b}}(\phi, \mu)}, \frac{\ell_{c\mathfrak{b}}(\phi, \Delta\phi)\ell_{c\mathfrak{b}}(\Delta\phi, \Omega\mu)}{1 + \ell_{c\mathfrak{b}}(\phi, \mu)}\right\}, \tag{42}$$

for all $\phi, \mu \in H$, where $0 \le \vartheta < 1/s$. Then, Δ and Ω have a unique common fixed point $\phi^* \in H$ and $\ell_{c\mathfrak{b}}(\phi^*, \phi^*) = 0$.

Proof. Let $\phi_0 \in H$. Define

$$\phi_{2\alpha+1} = \Delta\phi_{2\alpha} \quad \text{and} \quad \phi_{2\alpha+2} = \Omega\phi_{2\alpha+1}, \alpha = 0, 1, 2, \ldots. \tag{43}$$

Then, by (42) and (43), we obtain

$$\begin{aligned}\ell_{c\mathfrak{b}}(\phi_{2\alpha+1}, \phi_{2\alpha+2}) &= \ell_{c\mathfrak{b}}(\Delta\phi_{2\alpha}, \Omega\phi_{2\alpha+1}),\\ &\precsim \vartheta \max\left\{\ell_{c\mathfrak{b}}(\phi_{2\alpha}, \phi_{2\alpha+1}), \frac{\ell_{c\mathfrak{b}}(\phi_{2\alpha}, \phi_{2\alpha+1})\ell_{c\mathfrak{b}}(\Omega\phi_{2\alpha+1}, \Delta\phi_{2\alpha})}{1 + \ell_{c\mathfrak{b}}(\phi_{2\alpha}, \phi_{2\alpha+1})},\right.\\ &\quad \left.\frac{\ell_{c\mathfrak{b}}(\phi_{2\alpha}, \Delta\phi_{2\alpha})\ell_{c\mathfrak{b}}(\Delta\phi_{2\alpha}, \Omega\phi_{2\alpha+1})}{1 + \ell_{c\mathfrak{b}}(\phi_{2\alpha}, \phi_{2\alpha+1})}\right\},\\ &\precsim \vartheta \max\left\{\ell_{c\mathfrak{b}}(\phi_{2\alpha}, \phi_{2\alpha+1}), \frac{\ell_{c\mathfrak{b}}(\phi_{2\alpha}, \phi_{2\alpha+1})\ell_{c\mathfrak{b}}(\phi_{2\alpha+1}, \phi_{2\alpha+2})}{1 + \ell_{c\mathfrak{b}}(\phi_{2\alpha}, \phi_{2\alpha+1})},\right.\\ &\quad \left.\frac{\ell_{c\mathfrak{b}}(\phi_{2\alpha}, \phi_{2\alpha+1})\ell_{c\mathfrak{b}}(\phi_{2\alpha+1}, \phi_{2\alpha+2})}{1 + \ell_{c\mathfrak{b}}(\phi_{2\alpha}, \phi_{2\alpha+1})}\right\},\\ &\precsim \vartheta \max\{\ell_{c\mathfrak{b}}(\phi_{2\alpha}, \phi_{2\alpha+1}), \ell_{c\mathfrak{b}}(\phi_{2\alpha+1}, \phi_{2\alpha+2})\}.\end{aligned} \tag{44}$$

If $\max\{\ell_{cb}(\phi_{2\alpha},\phi_{2\alpha+1}),\ell_{cb}(\phi_{2\alpha+1},\phi_{2\alpha+2})\} = \ell_{cb}(\phi_{2\alpha+1},\phi_{2\alpha+2})$, then

$$\ell_{cb}(\phi_{2\alpha+1},\phi_{2\alpha+2}) \lesssim \vartheta \ell_{cb}(\phi_{2\alpha+1},\phi_{2\alpha+2}). \tag{45}$$

This shows that $\vartheta \geq 1$, which is a reductio ad absurdum. Therefore,

$$\ell_{cb}(\phi_{2\alpha+1},\phi_{2\alpha+2}) \lesssim \vartheta \ell_{cb}(\phi_{2\alpha},\phi_{2\alpha+1}). \tag{46}$$

Similarly, we obtain

$$\ell_{cb}(\phi_{2\alpha+2},\phi_{2\alpha+3}) \lesssim \vartheta \ell_{cb}(\phi_{2\alpha+1},\phi_{2\alpha+2}). \tag{47}$$

From (46) and (47), $\forall \alpha = 0, 1, 2, \ldots$, we obtain

$$\ell_{cb}(\phi_{\alpha+1},\phi_{\alpha+2}) \lesssim \vartheta \ell_{cb}(\phi_{\alpha},\phi_{\alpha+1}) \lesssim \cdots \lesssim \vartheta^{\alpha+1} \ell_{cb}(\phi_0,\phi_1). \tag{48}$$

For $\mathfrak{q}, \alpha \in \mathbb{N}$, with $\mathfrak{q} > \alpha$, we have

$$\begin{aligned}
\ell_{cb}(\phi_\alpha,\phi_{\mathfrak{q}}) &\lesssim s\left[\ell_{cb}(\phi_\alpha,\phi_{\alpha+1}) + \ell_{cb}(\phi_{\alpha+1},\phi_{\mathfrak{q}}) - \ell_{cb}(\phi_{\alpha+1},\phi_{\alpha+1})\right],\\
&\lesssim s\left[\ell_{cb}(\phi_\alpha,\phi_{\alpha+1}) + \ell_{cb}(\phi_{\alpha+1},\phi_{\mathfrak{q}})\right],\\
&\lesssim s(\ell_{cb}(\phi_\alpha,\phi_{\alpha+1})) + s^2\left(\ell_{cb}(\phi_{\alpha+1},\phi_{\alpha+2}) + \ell_{cb}(\phi_{\alpha+2},\phi_{\mathfrak{q}}) - \ell_{cb}(\phi_{\alpha+2},\phi_{\alpha+2})\right),\\
&\lesssim s(\ell_{cb}(\phi_\alpha,\phi_{\alpha+1})) + s^2\left[\ell_{cb}(\phi_{\alpha+1},\phi_{\alpha+2}) + \ell_{cb}(\phi_{\alpha+2},\phi_{\mathfrak{q}})\right],\\
&\lesssim s(\ell_{cb}(\phi_\alpha,\phi_{\alpha+1})) + s^2(\ell_{cb}(\phi_{\alpha+1},\phi_{\alpha+2})) + s^3(\ell_{cb}(\phi_{\alpha+2},\phi_{\alpha+3})),\\
&\quad + \cdots + s^{\mathfrak{q}-\alpha-1}\left(\ell_{cb}(\phi_{\mathfrak{q}-2},\phi_{\mathfrak{q}-1})\right) + s^{\mathfrak{q}-\alpha}\left(\ell_{cb}(\phi_{\mathfrak{q}-1},\phi_{\mathfrak{q}})\right).
\end{aligned} \tag{49}$$

By using (48), we obtain

$$\begin{aligned}
\ell_{cb}(\phi_\alpha,\phi_{\mathfrak{q}}) &\lesssim s\vartheta^{\alpha}(\ell_{cb}(\phi_0,\phi_1)) + s^2\vartheta^{\alpha+1}(\ell_{cb}(\phi_0,\phi_1)) + s^3\vartheta^{\alpha+2}(\ell_{cb}(\phi_0,\phi_1))\\
&\quad + \cdots + s^{\mathfrak{q}-\alpha-1}\vartheta^{\mathfrak{q}-2}(\ell_{cb}(\phi_0,\phi_1)) + s^{\mathfrak{q}-\alpha}\vartheta^{\mathfrak{q}-1}(\ell_{cb}(\phi_0,\phi_1)),\\
&= \sum_{i=1}^{\mathfrak{q}-\alpha} s^i \vartheta^{i+\alpha-1}(\ell_{cb}(\phi_0,\phi_1)).
\end{aligned} \tag{50}$$

Therefore,

$$\begin{aligned}
\left|\ell_{cb}(\phi_\alpha,\phi_{\mathfrak{q}})\right| &\leq \sum_{i=1}^{\mathfrak{q}-\alpha} s^{i+\alpha-1}\vartheta^{i+\alpha-1}\left|\ell_{cb}(\phi_0,\phi_1)\right| = \sum_{i=1}^{\mathfrak{q}-\alpha} s^t\vartheta^t\left|\ell_{cb}(\phi_0,\phi_1)\right|\\
&\leq \sum_{i=\alpha}^{\infty}(s\vartheta)^t\left|\ell_{cb}(\phi_0,\phi_1)\right|,\\
&= \frac{(s\vartheta)^\alpha}{1-s\vartheta}\left|\ell_{cb}(\phi_0,\phi_1)\right|.
\end{aligned} \tag{51}$$

Hence, we have

$$\left|\ell_{cb}(\phi_\alpha,\phi_{\mathfrak{q}})\right| \leq \frac{(s\vartheta)^\alpha}{1-s\vartheta}\left|\ell_{cb}(\phi_0,\phi_1)\right| \longrightarrow 0 \quad \text{as} \quad \alpha \longrightarrow \infty. \tag{52}$$

Hence, $\{\phi_\alpha\}$ is a Cauchy sequence in H. Since H is complete, there exists $\phi^* \in H$ such that $\phi_\alpha \longrightarrow \phi^*$ as $\alpha \longrightarrow \infty$ and

$$\ell_{cb}(\phi^*,\phi^*) = \lim_{\alpha\longrightarrow\infty}\ell_{cb}(\phi^*,\phi_\alpha) = \lim_{\alpha\longrightarrow\infty}\ell_{cb}(\phi_\alpha,\phi_\alpha) = 0. \tag{53}$$

Since Ω is continuous, we obtain

$$\phi^* = \lim_{\alpha\longrightarrow\infty}\phi_{2\alpha+2} = \lim_{\alpha\longrightarrow\infty}\Omega\phi_{2\alpha+1} = \Omega\lim_{\alpha\longrightarrow\infty}\phi_{2\alpha+1} = \Omega\phi^*. \tag{54}$$

Similarly, we derive that $\phi^* = \Delta\phi^*$. Then, Δ and Ω have a common fixed point. Let $\mu^* \in H$ be another common fixed point for the mappings Δ and Ω. Then,

$$\begin{aligned}\ell_{c\mathfrak{b}}(\phi^*,\mu^*) &= \ell_{c\mathfrak{b}}(\Delta\phi^*,\Omega\mu^*),\\ &\lesssim \vartheta \max\left\{\ell_{c\mathfrak{b}}(\phi^*,\mu)^*, \frac{\ell_{c\mathfrak{b}}(\phi^*,\Delta\phi^*)\ell_{c\mathfrak{b}}(\mu^*,\Omega\mu^*)}{1+\ell_{c\mathfrak{b}}(\phi^*,\mu^*)},\right.\\ &\left.\frac{\ell_{c\mathfrak{b}}(\phi^*,\Delta\phi^*)\ell_{c\mathfrak{b}}(\Omega\mu^*,\Delta\phi^*)}{1+\ell_{c\mathfrak{b}}(\phi^*,\mu^*)}\right\},\\ &\lesssim \vartheta\ell_{c\mathfrak{b}}(\phi^*,\mu^*).\end{aligned} \tag{55}$$

This implies that $\phi^* = \mu^*$. □

Theorem 5. *Let $(H,\ell_{c\mathfrak{b}})$ be a complete b-CPMS with the coefficient $s\geq 1$ and $\Delta,\Omega\colon H\longrightarrow H$ be two continuous mappings such that*

$$\ell_{c\mathfrak{b}}(\Delta\phi,\Omega\mu)\lesssim\vartheta\max\left\{\ell_{c\mathfrak{b}}(\phi,\mu), \frac{\ell_{c\mathfrak{b}}(\phi,\Delta\phi)\ell_{c\mathfrak{b}}(\mu,\Omega\mu)}{1+\ell_{c\mathfrak{b}}(\phi,\mu)}, \frac{\ell_{c\mathfrak{b}}(\phi,\Delta\phi)\ell_{c\mathfrak{b}}(\Delta\phi,\Omega\mu)}{1+\ell_{c\mathfrak{b}}(\phi,\mu)}\right\}, \tag{56}$$

for all $\phi,\mu\in H$, where $0\leq\vartheta<1/s$. Then, Δ and Ω have a unique common fixed point $\phi^*\in H$ and $\ell_{c\mathfrak{b}}(\phi^*,\phi^*)=0$.

Proof. Following from Theorem 5, we can easily prove $\{\phi_\alpha\}$ is a Cauchy sequence. Since H is complete, there exists $\phi^*\in H$ such that $\phi_\alpha\longrightarrow\phi^*$ as $\alpha\longrightarrow\infty$ and

$$\ell_{c\mathfrak{b}}(\phi^*,\phi^*) = \lim_{\alpha\longrightarrow\infty}\ell_{c\mathfrak{b}}(\phi^*,\phi_\alpha) = \lim_{\alpha\longrightarrow\infty}\ell_{c\mathfrak{b}}(\phi_\alpha,\phi_\alpha) = 0. \tag{57}$$

Suppose that $\ell_{c\mathfrak{b}}(\phi^*,\Delta\phi^*)=\lambda>0$.
Then, we estimate

$$\begin{aligned}\lambda &= \ell_{c\mathfrak{b}}(\phi^*,\Delta\phi^*),\\ &\lesssim s\{\ell_{c\mathfrak{b}}(\phi^*,\phi_{2i+2})+\ell_{c\mathfrak{b}}(\phi_{2i+2},\Delta\phi^*)-\ell_{c\mathfrak{b}}(\phi_{2i+2},\phi_{2i+2})\},\\ &\lesssim s\{\{\ell_{c\mathfrak{b}}(\phi^*,\phi_{2i+2})+\ell_{c\mathfrak{b}}(\Delta\phi^*,\phi_{2i+2})\},\\ &\lesssim s\ell_{c\mathfrak{b}}(\phi^*,\phi_{2i+2})+s\ell_{c\mathfrak{b}}(\Delta\phi^*,\Omega\phi_{2i+1}),\\ &\lesssim s\ell_{c\mathfrak{b}}(\phi^*,\phi_{2i+2})+\vartheta s\max\left\{\ell_{c\mathfrak{b}}(\phi^*,\phi_{2i+1}),\frac{\ell_{c\mathfrak{b}}(\phi^*,\Delta\phi^*)\ell_{c\mathfrak{b}}(\phi_{2i+1},\Omega\phi_{2i+1})}{1+\ell_{c\mathfrak{b}}(\phi^*,\phi_{2i+1})},\right.\\ &\left.\frac{\ell_{c\mathfrak{b}}(\phi^*,\Delta\phi_*)\ell_{c\mathfrak{b}}(\Delta\phi^*,\Omega\phi_{2i+1})}{1+\ell_{c\mathfrak{b}}(\phi^*,\phi_{2i+1})}\right\},\\ &\lesssim s\ell_{c\mathfrak{b}}(\phi^*,\phi_{2i+2})+\vartheta s\max\left\{\ell_{c\mathfrak{b}}(\phi^*,\phi_{2i+1}),\frac{\ell_{c\mathfrak{b}}(\phi^*,\Delta\phi^*)\ell_{c\mathfrak{b}}(\phi_{2i+1},\phi_{2i+2})}{1+\ell_{c\mathfrak{b}}(\phi^*,\phi_{2i+1})}\right\}\\ &\frac{\ell_{c\mathfrak{b}}(\phi^*,\Delta\phi_*)\ell_{c\mathfrak{b}}(\Delta\phi^*,\phi_{2i+2})}{1+\ell_{c\mathfrak{b}}(\phi^*,\phi_{2i+1})},\\ &\lesssim s\ell_{c\mathfrak{b}}(\phi^*,\phi_{2i+2})+s\vartheta\ell_{c\mathfrak{b}}(\phi^*,\Delta\phi^*)^2,\\ &\lesssim s\ell_{c\mathfrak{b}}(\phi^*,\phi_{2i+2})+s\vartheta\lambda^2.\end{aligned} \tag{58}$$

This yields

$$|\lambda|\leq s\left|\ell_{c\mathfrak{b}}(\phi^*,\phi_{2i+2})\right|+s\vartheta|\lambda|^2. \tag{59}$$

Hence, $\vartheta\geq 1$, which is a reductio ad absurdum. Then, $\phi^*=\Delta\phi^*$. Similarly, we derive that $\phi^*=\Omega\phi^*$. Therefore, Δ and Ω have a common fixed point ϕ^*. Following from Theorem 5, we can easily prove the uniqueness part. □

Example 3.5. Let $H=\{1,2,3,4\}$ be endowed with the partial order $\phi\lesssim\mu$ iff $\mu\leq\phi$. We define $\ell_{c\mathfrak{b}}\colon H\times H\longrightarrow C^+$ in Tables 1 and 2.

It is easy to verify that $(H,\ell_{c\mathfrak{b}})$ is a complete b-CPMS with the coefficient $s\geq 1$ for $x\in[0,\pi/2]$. Define $\Delta,\Omega\colon H\longrightarrow H$ by $\Delta\phi=1$:

Table 1: Example of unique common fixed point.

(ϕ, μ)	$\ell_{cb}(\phi, \mu)$
(1,1), (3,3)	0
(1,2), (2,1), (1,3), (3,1), (2,3), (3,2), (2,2)	e^{2ix}
(1,4), (4,1), (2,4), (4,2), (3,4), (4,3), (4,4)	$9e^{2ix}$

Table 2: Example of no common fixed point.

(ϕ, μ)	$\ell_{cb}(\phi, \mu)$
(1,1), (3,3)	0
(1,2), (2,1), (1,3), (3,1), (2,3), (3,2), (2,2)	e^{2ix}
(1,4), (4,1), (2,4), (4,2), (3,4), (4,3), (4,4)	e^{2ix}

$$\Omega((\phi) = \begin{cases} 1 & \text{if } \phi \in \{1,2,3\}, \\ 2 & \text{if } \phi = 4. \end{cases} \quad (60)$$

Clearly, Δ and Ω are continuous functions. Now, for $\vartheta = 1/9$, we consider the following cases:

(A) If $\phi = 1$ and $\mu \in H - \{4\}$, then $\Delta(\phi) = \Omega(\mu) = 1$. Hence, all the conditions of Theorem 2 are fulfilled.

(B) If $\phi = 1$ and $\mu = 4$, then $\Delta\phi = 1$ and $\Omega\mu = 2$:

$$\begin{aligned} \ell_{cb}(\Delta\phi, \Omega\mu) &= e^{2ix} \lesssim 9\vartheta e^{i2x} \\ &= \vartheta \max\left\{9e^{i2x}, 0, 9e^{i2x}, \frac{1}{2}\left(e^{i2x} + 9e^{i2x}\right)\right\} \\ &= \vartheta \max\{\ell_{cb}(\phi,\mu), \ell_{cb}(\phi,\Delta\phi), \ell_{cb}(\mu,\Omega\mu) \\ &\quad \frac{1}{2}\left(\ell_{cb}(\phi,\Omega\mu)\right) + \ell_{cb}(\mu,\Delta\phi)\right\}. \end{aligned} \quad (61)$$

(C) *If* $\phi = 2$ and $\mu = 4$, *then* $\Delta\phi = 1$ and $\Omega\mu = 2$:

$$\begin{aligned} \ell_{cb}(\Delta\phi, \Omega\mu) &= e^{2ix} \lesssim 9\vartheta e^{i2x} \\ &= \vartheta \max\left\{9e^{i2x}, e^{i2x}, 9e^{i2x}, \frac{1}{2}\left(0 + 9e^{i2x}\right)\right\} \\ &= \vartheta \max\{\ell_{cb}(\phi,\mu), \ell_{cb}(\phi,\Delta\phi), \ell_{cb}(\mu,\Omega\mu) \\ &\quad \frac{1}{2}\left(\ell_{cb}(\phi,\Omega\mu)\right) + \ell_{cb}(\mu,\Delta\phi)\right\}. \end{aligned} \quad (62)$$

(D) If $\phi = 3$ and $\mu = 4$, then $\Delta\phi = 1$ and $\Omega\mu = 2$:

$$\begin{aligned} \ell_{cb}(\Delta\phi, \Omega\mu) &= e^{2ix} \lesssim 9\vartheta e^{i2x} \\ &= \vartheta \max\left\{9e^{i2x}, e^{i2x}, 9e^{i2x}, \frac{1}{2}\left(e^{i2x} + 9e^{i2x}\right)\right\} \\ &= \vartheta \max\{\ell_{cb}(\phi,\mu), \ell_{cb}(\phi,\Delta\phi), \ell_{cb}(\mu,\Omega\mu) \\ &\quad \frac{1}{2}\left(\ell_{cb}(\phi,\Omega\mu) + \ell_{cb}(\mu,\Delta\phi)\right)\right\}. \end{aligned} \quad (63)$$

(E) If $\phi = 4$ and $\mu = 4$, then $\Delta\phi = 2$ and $\Omega\mu = 2$:

$$\begin{aligned} \ell_{cb}(\Delta\phi, \Omega\mu) &= e^{2ix} \lesssim 9\vartheta e^{i2x} \\ &= \vartheta \max\left\{9e^{i2x}, 9e^{i2x}, 9e^{i2x}, \frac{1}{2}\left(9e^{i2x} + 9e^{i2x}\right)\right\} \\ &= \vartheta \max\{\ell_{cb}(\phi,\mu), \ell_{cb}(\phi,\Delta\phi), \ell_{cb}(\mu,\Omega\mu) \\ &\quad \frac{1}{2}\left(\ell_{cb}(\phi,\Omega\mu)\right) + \ell_{cb}(\mu,\Delta\phi)\right\}. \end{aligned} \quad (64)$$

All the conditions of Theorem 1, with $\vartheta = 1/9 < 1$, are fulfilled. Therefore, Δ and Ω have a unique common fixed point 1.

Example 3.6. Let $H = P \cup Q$, where $P = [1,2]$ and $Q = \{3,4\}$ be endowed with the partial order $\phi \lesssim \mu$ iff $\mu \leq \phi$. Define $\ell_{cb}: H \times H \longrightarrow C^{+}$ by $\ell_{cb}(\phi,\mu) = (|\phi - \mu|^2 + 2)e^{2ix}$, for all $\phi, \mu \in P$ or $\phi \in P$ and $\mu \in Q$ or $\phi \in Q, \mu \in P$.

It is easy to verify that (H, ℓ_{cb}) is a complete b-CPMS with the coefficient $s \geq 1$ for $x \in [0, \pi/2]$. Define $\Delta, \Omega: H \longrightarrow H$ by

$$\Delta(\phi) = \begin{cases} 1 & \text{if } \phi \in \left[1, \frac{3}{2}\right], \\ 2 & \text{if } \phi \in \left(\frac{3}{2}, 2\right] \cup Q, \end{cases} \quad (65)$$

$$\Omega(\phi) = \begin{cases} 2 & \text{if } \phi \in \left[1, \frac{3}{2}\right], \\ 4 & \text{if } \phi \in \left(\frac{3}{2}, 2\right] \cup Q. \end{cases} \quad (66)$$

Clearly, Δ and Ω are not continuous functions. Now, we consider the following cases:

(A) *If* $\phi, \mu \in [1, 3/2]$, *then* $\Delta(\phi) = 1$ and $\Omega(\mu) = 2$:

$$\ell_{c\mathfrak{b}}(\Delta\phi,\Omega\mu)=e^{2ix}\lesssim\frac{1}{3}3e^{i2x}$$
$$=\vartheta\max\{\ell_{c\mathfrak{b}}(\phi,\mu),\ell_{c\mathfrak{b}}(\phi,\Delta\phi),\ell_{c\mathfrak{b}}(\mu,\Omega\mu)$$
$$\frac{1}{2}(\ell_{c\mathfrak{b}}(\phi,\Omega\mu)+\ell_{c\mathfrak{b}}(\mu,\Delta\phi))\}. \quad (67)$$

(B) If $\phi\in[1,3/2]$ and $\mu\in(3/2,2]\cup Q$, then $\Delta\phi=1$ and $\Omega\mu=4$:

$$\ell_{c\mathfrak{b}}(\Delta\phi,\Omega\mu)=e^{2ix}\lesssim\frac{1}{3}3e^{i2x}$$
$$=\vartheta\max\{\ell_{c\mathfrak{b}}(\phi,\mu),\ell_{c\mathfrak{b}}(\phi,\Delta\phi),\ell_{c\mathfrak{b}}(\mu,\Omega\mu)$$
$$\frac{1}{2}(\ell_{c\mathfrak{b}}(\phi,\Omega\mu)+\ell_{c\mathfrak{b}}(\mu,\Delta\phi))\}. \quad (68)$$

(C) *If* $\mu\in[1,3/2]$ and $\phi\in(3/2]\cup Q$, *then* $\Delta\phi=2$ and $\Omega\mu=2$:

$$\ell_{c\mathfrak{b}}(\Delta\phi,\Omega\mu)=e^{2ix}\lesssim3\frac{1}{3}e^{i2x}$$
$$=\vartheta\max\{\ell_{c\mathfrak{b}}(\phi,\mu),\ell_{c\mathfrak{b}}(\phi,\Delta\phi),\ell_{c\mathfrak{b}}(\mu,\Omega\mu)$$
$$\frac{1}{2}(\ell_{c\mathfrak{b}}(\phi,\Omega\mu)+\ell_{c\mathfrak{b}}(\mu,\Delta\phi))\}. \quad (69)$$

(D) If $\mu,\phi\in(3/2,2]\cup Q$, then $\Delta\phi=2$ and $\Omega\mu=4$:

$$\ell_{c\mathfrak{b}}(\Delta\phi,\Omega\mu)=e^{2ix}\lesssim3\frac{1}{3}e^{i2x}$$
$$=\vartheta\max\{\ell_{c\mathfrak{b}}(\phi,\mu),\ell_{c\mathfrak{b}}(\phi,\Delta\phi),\ell_{c\mathfrak{b}}(\mu,\Omega\mu) \quad (70)$$
$$\frac{1}{2}(\ell_{c\mathfrak{b}}(\phi,\Omega\mu)+\ell_{c\mathfrak{b}}(\mu,\Delta\phi))\}.$$

All the conditions of Theorems 2 and 3, with $\vartheta<1$, are fulfilled except continuous mapping. Therefore, Δ and Ω have no common fixed point.

Remark 1. In view of the fact in Theorems 2 and 3, we cannot drop the continuous mapping.

4. Application

Consider the following systems of integral equations:

$$v(s)=\int_c^d T_1(s,\aleph,v(\aleph))d\aleph, \quad (71)$$

$$\varrho(s)=\int_c^d T_2(s,\aleph,\varrho(\aleph))d\aleph, \quad (72)$$

where

(i) $v(s)$ and $\varrho(s)$ are unknown variables for each $s\in J=[c,d]$, $d>c\geq0$

(ii) $T_1(s,\aleph)$ and $T_2(s,\aleph)$ are deterministic kernels defined for $s,\aleph\in J=[c,d]$

Let $H=(C(J),R^\alpha)$ be the set of continuous functions defined on J. Define $\ell_{c\mathfrak{b}}$: $H\times H\longrightarrow C^+$ by

$$\ell_{c\mathfrak{b}}(v,\varrho)=|v(s)-\varrho(s)|^2+2, \quad (73)$$

$\forall\,v,\varrho\in H$. Then, $\ell_{c\mathfrak{b}}$ is a complete b-CPMS. Define partial order $\lesssim$ given by

$$v,\varrho\in H, v\lesssim\varrho \text{ if and only } v(s)\geq\varrho(s),\ \forall s\in J. \quad (74)$$

Theorem 6. *Assume that*

(A) T_1,T_2: $J\times J\times R^\alpha\longrightarrow R^\alpha$ *are continuous functions satisfying*

$$|T_1(s,\aleph,v(\aleph))-T_2(s,\aleph,\varrho(s))|\lesssim\sqrt{\frac{S(v,\varrho)}{(b-a)e^t}-\frac{2}{b-a}},\quad\forall t>0, \quad (75)$$

where

$$S(v,\varrho)=\max\{\ell_{c\mathfrak{b}}(v,\varrho),\ell_{c\mathfrak{b}}(v,\Delta v),\ell_{c\mathfrak{b}}(\varrho,\Omega\varrho)$$
$$\frac{1}{2}(\ell_{c\mathfrak{b}}(v,\Omega\varrho)+\ell_{c\mathfrak{b}}(\varrho,\Delta v))\}. \quad (76)$$

Then, systems (71) and (72) have a unique common solution.

Proof. For $v,\varrho\in(C(J),R^\alpha)$ and $s\in J$, define the continuous mappings Δ,Ω: $H\longrightarrow H$ by

$$\Delta v(s)=\int_c^d T_1(s,\aleph,v(\aleph))d\aleph, \quad (77)$$

$$\Omega\varrho(s)=\int_c^d T_2(s,\aleph,\varrho(\aleph))d\aleph. \quad (78)$$

Then,

$$\ell_{c\mathfrak{b}}(\Delta v(s),\Omega\varrho(s))=|\Delta v(s)-\Omega\varrho(s)|^2+2$$
$$=\int_c^d|T_1(s,\aleph,v(\aleph))-T_2(s,\aleph,\rangle(s))|^2d\aleph+2$$
$$\lesssim\int_c^d\left(\frac{S(v,\varrho)}{(b-a)e^t}-\frac{2}{b-a}\right)d\aleph+2$$
$$=\frac{S(v,\varrho)}{e^t}$$
$$=\vartheta S(v,\varrho)$$
$$=\vartheta\max\{\ell_{c\mathfrak{b}}(v,\varrho),\ell_{c\mathfrak{b}}(v,\Delta v),\ell_{c\mathfrak{b}}(\varrho,\Omega\varrho)$$
$$\frac{1}{2}(\ell_{c\mathfrak{b}}(v,\Omega\varrho)+\ell_{c\mathfrak{b}}(\varrho,\Delta v))\}. \quad (79)$$

Hence, all the conditions of Theorem 2 are fulfilled for $0 < \vartheta = 1/e^t < 1/s$ with $t > 0$. Therefore, integrals (71) and (72) have a unique common solution. □

5. Conclusion

In this paper, we proved common fixed-point theorems on complex partial b-metric space. An illustrative example and application on complex partial b-metric space is given. Recently, Khalehoghli et al. [24, 25] introduced R-metric spaces and obtained a generalization of Banach fixed-point theorem. It is an interesting open problem to study the relation R instead of complex partial b-metric space and obtain common fixed-point results on R-complete complex partial b-metric spaces.

Data Availability

No data were used to support the findings of the study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

References

[1] I. A. Bakhtin, "The contraction mappings principle in quasimetric spaces," *Functional Analysis*, vol. 30, pp. 26–37, 1989, Russian.

[2] S. Czerwick, "Contraction mappings in b-metric spaces," *Acta Mathematica et Informatica Universitatis Ostraviensis*, vol. 1, pp. 5–11, 1993.

[3] A. Azam, B. Fisher, and M. Khan, "Common fixed point theorems in complex valued metric spaces," *Numerical Functional Analysis and Optimization*, vol. 32, no. 3, pp. 243–253, 2011.

[4] K. P. R. Rao, P. R. Swamy, and J. R. Prasad, "A common fixed point theorem in complex valued b-metric spaces," *Bulletin of Mathematics and Statistics Research*, vol. 1, no. 1, 2013.

[5] P. Dhivya and M. Marudai, "Common fixed point theorems for mappings satisfying a contractive condition of rational expression on a ordered complex partial metric space," *Cogent Mathematics*, vol. 4, no. 1, Article ID 1389622, 2017.

[6] M. Gunaseelan, "Generalized fixed point theorems on complex partial b-metric space," *International Journal of Research and Analytical Reviews*, vol. 6, no. 2, 2019.

[7] T. Gnana Bhaskar and V. Lakshmikantham, "Fixed point theorems in partially ordered metric spaces and applications," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 65, no. 7, pp. 1379–1393, 2006.

[8] M. Gunaseelan and L. N. Mishra, "Coupled fixed point theorems on complex partial metric space using different type of contractive conditions," *Scientific Publications of the State University of Novi Pazar Series A: Applied Mathematics, Informatics and mechanics*, vol. 11, no. 2, pp. 117–123, 2019.

[9] Deepmala and H. K. Pathak, "A study on some problems on existence of solutions for nonlinear functional-integral equations," *Acta Mathematica Scientia*, vol. 33, no. 5, pp. 1305–1313, 2013.

[10] A. Deepmala, "Study on fixed point theorems for nonlinear contractions and its applications," Ph.D. Thesis, Ravishankar Shukla University, Chhatisgarh, India, 2014.

[11] H. K. Pathak and Deepmala, "Common fixed point theorems for PD-operator pairs under relaxed conditions with applications," *Journal of Computational and Applied Mathematics*, vol. 239, pp. 103–113, 2013.

[12] A. Latif, T. Nazir, and M. Abbas, "Stability of fixed points in generalized metric spaces," *Journal Nonlinear Variables Analutic*, vol. 2, pp. 287–294, 2018.

[13] M. Abbas, I. Beg, and B. T. Leyew, "Common fixed points of (R, α)-generalized rational multivalued contractions in R-complete b-metric spaces," *Communications in Optimization Theory*, vol. 2019, Article ID 14, 2019.

[14] M. Gunaseelan, G. Arul Joseph, L. N. Mishra, and V. N. Mishra, "Fixed point theorem for orthogonal F-suzuki contraction mapping on a O-complete metric space with an application," *Malaya Journal of Matematik*, vol. 1, pp. 369–377, 2021.

[15] I. Beg, G. Mani, and A. J. Gnanaprakasam, "Fixed point of orthogonal F-suzuki contraction mapping on O-complete b-metric spaces with applications," *Journal of Function spaces*, vol. 2021, Article ID 6692112, 12 pages, 2021.

[16] M. Gunaseelan, L. N. Mishra, and V. N. Mishra, "Imran abbas baloch, manuel de La sen Application to coupled Fixed point theorems on complex partial b-metric space," *Journal of Mathematics*, vol. 12, pp. 1–11, 2020.

[17] L. M. Prakasam and M. Gunaseelan, "Common fixed point theorems using (CLR) and (EA) properties in complex partial b-metric space," *Advances in Mathematics: Scientific Journal*, vol. 1, pp. 2773–2790, 2020.

[18] F. Gu and W. Shatanawi, "Some new results on common coupled fixed points of two hybrid pairs of mappings in partial metric spaces," *Journal of Nonlinear Functional Analysis*, vol. 2019, Article ID 13, 2019.

[19] D. Baleanu, S. Rezapour, and H. Mohammadi, "Some existence results on nonlinear fractional differential equations," *Philosophical Transactions A*, vol. 371, pp. 1–7, 2013.

[20] W. Sudsutad and J. Tariboon, "Boundary value problems for fractional differential equations with three-point fractional integral boundary conditions," *Advances in Difference Equations*, vol. 2012, no. 1, p. 93, 2012.

[21] M. S. Aslam, M. F. Bota, M. S. R. Chowdhury, L. Guran, and N. Saleem, "Common fixed points technique for existence of a solution of urysohn type integral equations system in complex valued b-metric spaces," *Mathematics*, vol. 9, no. 4, p. 400, 2021.

[22] I. Altun, M. Aslantas, and H. Sahin, "KW-Type nonlinear contractions and their best proximity points," *Numerical Functional Analysis and Optimization*, vol. 42, no. 8, pp. 935–954, 2021.

[23] M. Aslantas, H. Sahin, and I. Altun, "Best proximity point theorems for cyclic p-contractions with some consequences and applications," *Nonlinear Analysis Modelling and Control*, vol. 26, no. 1, pp. 113–129, 2021.

[24] S. Khalehoghli, H. Rahimi, H. Rahimi, and M. Eshaghi Gordji, "Fixed point theorems in R-metric spaces with applications," *AIMS Mathematics*, vol. 5, no. 4, pp. 3125–3137, 2020.

[25] S. Khalehoghli, H. Rahimi, and M. Eshaghi Gordji, "R-topological spaces and SR-topological spaces with their applications," *Mathematical Sciences*, vol. 14, no. 3, pp. 249–255, 2020.