Hindawi
Journal of Function Spaces
Volume 2022, Article ID 1931861, 12 pages
https://doi.org/10.1155/2022/1931861

*Research Article*

# $H^{\beta}$-Hausdorff Functions and Common Fixed Points of Multivalued Operators in a $b$-Metric Space and Their Applications

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Received 14 November 2021; Revised 22 January 2022; Accepted 13 June 2022; Published 8 July 2022

Academic Editor: Alexander Meskhi



$H^{\beta}$-Hausdorff functions for $\beta \in [0, 1]$ are introduced, and common fixed-point theorems for a pair of multivalued operators satisfying generalized contraction conditions are proven in a $b$-metric space. Our results are proper extensions and new variants of many contraction conditions existing in literature. In order to demonstrate applications of our result, we have proven an existence theorem for a unique common multivalued fractal of a pair of iterated multifunction systems and also an existence theorem for a common solution of a pair of Volterra-type integral equations.

## 1. Introduction

In the last few decades, a wide range of extensions, generalizations, and applications of the infamous Banach contraction principle came into existence. In the sequel, Bakhtin [1] initiated the idea of a $b$-metric space followed by Czerwik [2], in which the author by weakening the triangular inequality formally defined a $b$-metric space and proved the Banach contraction principle in a $b$-metric space. Some examples and other details of a $b$-metric space can be found in Kirk and Shahzad [3] whereas a wide range of generalized fixed-point theorems in a $b$-metric space can be found in [4–7]. On the other hand, the study of a metric function on the set of closed and bounded subsets of a metric space was initiated by Pompeiu in [8] and then continued by Hausdorff [9]. Such a metric function is referred to as the Hausdorff-Pompeiu metric. Banach's contraction principle was extended to a multivalued function in a metric space by Nadler [10] and in a $b$-metric space by Czerwik [2] using the Hausdorff-Pompeiu metric $H$. Further generalized results of multivalued contractions can be found in ([11–14]). Czerwik's contraction was also generalized in many directions to name a few: $q$-quasi-contraction [15], Hardy-Rogers contraction [16], weak quasi-contraction [17], Ciric contraction [18], etc. More results on multivalued contraction mappings in a $b$-metric space can be found in [19–23]. Very recently, Debnath [24] proved the set-valued Meir–Keeler-type as well as Geraghty- and Edelstein-type fixed-point theorems in a $b$-metric space whereas Altun et al. [25] and Kumar and Luambano [26] proved fixed-point results for multivalued $F$-contraction mappings in complete metric space and partial metric space, respectively. In [27], the authors introduced the concept of $H^{\beta}$-Hausdorff-Pompeiu $b$-metric for some $0 \leq \beta \leq 1$ and proved fixed-point theorems for multivalued mappings belonging to various classes of multivalued $H^{\beta}$-contractions in a $b$-metric space. Applications of fixed-point results in dealing with solutions of nonlinear problems arising in engineering and science are an important area in present-day research. Fruitful applications of fixed-point problems in solution of various types of integral equations, fractional differential equations, and optimization problems can be found in [28–32]. Barnsley [33] introduced the idea of data interpolation using the fractal methodology of iterated function systems. Nowadays, fractal functions constitute a method of approximation of nondifferentiable mappings, providing suitable tools for the description of irregular signals (see [34–39]). The aim of this work is to prove common fixed-point theorems for a pair of multivalued mappings in a $b$-metric space using $H^\beta$-Hausdorff-Pompeiu $b$-metric and thereby extend and introduce new variants of various fixed-point results for multivalued mappings existing in literature. We have provided two applications of our main results: one to prove the existence of a unique common multivalued fractal of a pair of iterated multifunction system defined on a $b$-metric space and the second to prove the existence of a common solution of a pair of Volterra-type nonlinear integral equations.

## 2. Preliminaries

In this section, we provide some preliminary definitions, lemmas, and propositions required in our main results.

*Definition 1* (see [1]). Let $X$ be a nonempty set and $d_s : X \times X \longrightarrow [0,\infty)$ satisfy the following:

(1) $d_s(\imath, \jmath) = 0$ if and only if $\imath = \jmath$ for all $\imath, \jmath \in X$

(2) $d_s(\imath, \jmath) = d_s(\jmath, \imath)$ for all $\imath, \jmath \in X$

(3) There exists a real number $s \geq 1$ such that $d_s(\imath, \jmath) \leq s[d_s(\imath, \ell) + d_s(\ell, \jmath)]$ for all $\imath, \jmath, \ell \in X$

Then, $d_s$ is a $b$-metric on $X$ and $(X, d_s)$ is a $b$-metric space with coefficient $s$.

Let $CB^{d_s}(X)$ be the collection of all nonempty closed and bounded subsets of a $b$-metric space $(X, d_s)$. For $A, B \in CB^{d_s}(X)$, define $d_s(x, A) = \inf \{d_s(x, a) \colon a \in A\}$, $\delta_{d_s}(A, B) = \sup_{a \in A} d_s(a, B)$, and $H_{d_s}(A, B) = \max \{\delta_{d_s}(A, B), \delta_{d_s}(B, A)\}$. Czerwik [2] has shown that $H_{d_s}$ is a $b$-metric in the set $CB^{d_s}(X)$ and is called the Hausdorff-Pompeiu $b$-metric induced by $d_s$. In [27], the authors introduced the function $H^\beta(A, B) = \max \{\beta\delta_{d_s}(A, B) + (1-\beta)\delta_{d_s}(B, A), \beta\delta_{d_s}(B, A) + (1-\beta)\delta_{d_s}(A, B)\}$ for some $\beta \in [0, 1]$ and showed that $H^\beta$ is a $b$-metric for the set $CB^{d_s}(X)$. They called this function the $H^\beta$-Hausdorff-Pompeiu $b$-metric induced by the $b$-metric $d_s$. Note that for $\beta = 0$ or 1, $H^\beta$ is the Hausdorff-Pompeiu metric $H_{d_s}$.

**Proposition 2** (see [27]). *For any $x, y \in X$, $H^\beta(\{x\}, \{y\}) = d_s(x, y)$.*

*Definition 3* (see [18]). The $b$-metric $d_s$ is $*$-continuous if and only if for any $A \in CB^{d_s}(X)$ and sequence $\{x_n\}$ in $(X, d_s)$ with $\lim_{n\longrightarrow\infty} x_n = x$, we have $\lim_{n\longrightarrow\infty} d_s(x_n, A) = d_s(x, A)$.

**Proposition 4** (see [19]). *For any $A \subseteq X$,*

$$a \in \bar{A} \Leftrightarrow d_s(a, A) = 0. \tag{1}$$

**Lemma 5** (see [18]). *Let $\{x_n\}$ be a sequence in $(X, d_s)$. If there exists $\lambda \in [0, 1)$ such that $d_s(x_n, x_{n+1}) \leq \lambda d_s(x_{n-1}, x_n)$ for all $n \in N$, then $\{x_n\}$ is a Cauchy sequence.*

The following lemma follows immediately from the above lemma.

**Lemma 6.** *If for some $\lambda, \epsilon \in [0, 1)$, with $\lambda < \epsilon$, $d_s(x_n, x_{n+1}) \leq \lambda d_s(x_{n-1}, x_n) + \epsilon^n$ for all $n \in N$, then $\{x_n\}$ is a Cauchy sequence.*

## 3. Main Results

We introduce pairwise $H^\beta$-Hausdorff functions as follows:

*Definition 7.* Let $S, T : X \longrightarrow CB^{d_s}(X)$. For any $\imath \in X$, $\jmath \in T\imath$ (or $S\imath$) and any $\epsilon > 0$, if there exist $\ell \in S\jmath$ (or $T\jmath$) such that

$$d_s(\jmath, \ell) \leq H^\beta(T\imath, S\jmath) + \epsilon \text{ or respectively } d_s(\jmath, \ell) \leq H^\beta(S\imath, T\jmath) + \epsilon, \tag{2}$$

then we say that $T$ and $S$ are pairwise $H^\beta$-Hausdorff functions.

For $S = T$, we get the following.

*Definition 8.* For any $\imath \in X, \jmath \in T\imath$ and any $\epsilon > 0$ if there exist $\ell \in T\jmath$ such that

$$d_s(\jmath, \ell) \leq H^\beta(T\imath, T\jmath) + \epsilon, \tag{3}$$

then we say that $T$ is a $H^\beta$-Hausdorff function.

*Remark 9.*

(i) For $\beta = 1$, $T : X \longrightarrow CB(X)$ is always a $H^\beta$-Hausdorff function

(ii) If for any $0 \leq \beta_1 \leq 1$, the function $T : X \longrightarrow CB(X)$ is a $H^{\beta_1}$-Hausdorff function, then for any $0 \leq \beta_1 \leq \beta_2 \leq 1$, the function $T : X \longrightarrow CB(X)$ is a $H^{\beta_2}$-Hausdorff function

*Example 10.* Let $X = [0, 33/48] \bigcup \{1\}$,

$$d_s(\imath, \jmath) = |\imath - \jmath|^2 \text{ for all } \imath, \jmath \in X. \tag{4}$$

and $S, T : X \longrightarrow CB(X)$ be as follows:

$$S(\imath)=\begin{cases}\left\{\dfrac{\imath}{4}\right\}, & \text{for } \imath\in\left(0,\dfrac{33}{48}\right],\\ \left\{\dfrac{33}{48},1\right\}, & \text{for } \imath\in\{0,1\},\end{cases}$$
$$T(\imath)=\begin{cases}\left\{\dfrac{\imath}{2}\right\}, & \text{for } \imath\in\left(0,\dfrac{33}{48}\right],\\ \left\{\dfrac{1}{3},\dfrac{33}{48},1\right\}, & \text{for } \imath\in\{0,1\}.\end{cases} \tag{5}$$

We will show that the functions $S$ and $T$ satisfy (2). We will consider the values of $\imath$ in $X$ as follows:

(i) $\imath\in(0,33/48]$. In this case, $S\imath$ and $T\jmath$ are singleton sets and so (2) is obviously true

(ii) $\imath=0$. $S\imath=\{33/48,1\}$. If $\jmath=33/48, \jmath=\{33/96\}$, then we have $\ell=33/96$ and $d_s(\jmath,\ell)=1089/9216$, $\delta_s(S\imath,T\jmath)=3969/9216$, $\delta_s(T\jmath,S\imath)=1089/9216$, and $H^{3/4}(S\imath,T\jmath)=3249/9216$. Thus, (2) is true for all $\epsilon>0$. If $\jmath=1$, $T\jmath=\{1/3,33/48,1\}$, then inequality (2) holds with $\ell=1$

(iii) $\imath=1$. $S\imath=\{33/48,1\}$, and the result follows in the same way as in (ii) above.

(iv) $\imath=0$. $T\imath=\{1/3,33/48,1\}$. If $\jmath=1/3, S\jmath=\{1/12\}$, then we have $\ell=1/12$ and $d_s(\jmath,\ell)=9/144$, $\delta_s(T\imath,S\jmath)=121/144$, $\delta_s(S\jmath,T\imath)=9/144$, and $H^{3/4}(S\imath,T\jmath)=93/144$. Thus, (2) is true for all $\epsilon>0$. If $\jmath=33/48$, $S\jmath=\{33/192\}$, then we take $\ell=33/192$ and then $d_s(\jmath,\ell)=1089/4096$, $\delta_s(T\imath,S\jmath)=2809/4096$, $\delta_s(S\jmath,T\imath)=961/36864$, and $H^{3/4}(T\imath,S\jmath)=19201/36864$. Thus, (2) is true for all $\epsilon>0$. If $\jmath=1, S\jmath=\{33/48,1\}$, inequality (2) holds with $\ell=1$

Thus, $S$ and $T$ are pairwise $H^\beta$-Hausdorff functions for $\beta=3/4$. However, $S$ and $T$ are not pairwise $H^\beta$-Hausdorff functions for $\beta=1/2$, as we see that inequality (2) is not satisfied for $i=0$, $T\imath=\{1/3,33/48,1\}$, and $\jmath=33/48$. In fact, $S$ and $T$ are not pairwise $H^\beta$-Hausdorff functions for $34/95<\beta<61/95$.

We now present our main result as follows:

**Theorem 11.** *Let $(X,d_s)$ be a complete b-metric space with constant $s\geq 1$, $d_s$ be $*$-continuous, and $T,S:X\longrightarrow CB^{d_s}(X)$ be multivalued pairwise $H^\beta$-Hausdorff functions for some $1/2\leq\beta\leq 1$ and satisfying the following condition:*

$$H^\beta(T\imath,S\jmath)\leq\alpha_1 d_s(\imath,T\imath)+\alpha_2 d_s(\jmath,S\jmath)+\alpha_3 d_s(\imath,S\jmath)+\alpha_4 d_s(\jmath,T\imath)+\alpha_5\left(\frac{d_s(\imath,S\jmath)+d_s(\jmath,T\imath)}{2}\right)+\alpha_6\frac{d_s(\imath,T\imath)d_s(\jmath,S\jmath)}{1+d_s(\imath,\jmath)}+\alpha_7 d_s(\imath,\jmath), \tag{6}$$

*for all $\imath,\jmath\in X$ and some $\alpha_k\geq 0$, $k=1,2\cdots 7$, with $\alpha_1+\alpha_2+s\alpha_5+\alpha_6+\alpha_7+\max\{2s\alpha_3,2s\alpha_4\}<1$, $s(\alpha_1+\alpha_4+(\alpha_5/2))<\beta$, and $s(\alpha_2+\alpha_3+(\alpha_5/2))<\beta$. Then, $S$ and $T$ have a common fixed point.*

*Proof.* Let $\imath_0\in X$, $\imath_1\in T\imath_0$, and $0<\epsilon<1$. By (2), there exist $\imath_2\in S\imath_1$, such that $d_s(\imath_1,\imath_2)\leq H^\beta(T\imath_0,S\imath_1)+\epsilon$. By (2) again, there exist $\imath_3\in T\imath_2$, such that $d_s(\imath_2,\imath_3)\leq H^\beta(S\imath_1,T\imath_2)+\epsilon^2$.

Continuing these ways, we construct the sequence $<\imath_n>$ such that

$$\begin{gathered}\imath_{2n+1}\in T\imath_{2n},\ \imath_{2n+2}\in S\imath_{2n+1},\\ d_s(\imath_{2n+1},\imath_{2n+2})\leq H^\beta(T\imath_{2n},S\imath_{2n+1})+\epsilon^{2n+1},\\ d_s(\imath_{2n+2},\imath_{2n+3})\leq H^\beta(S\imath_{2n+1},T\imath_{2n+2})+\epsilon^{2n+2}.\end{gathered} \tag{7}$$

Now,

$$\begin{aligned}d_s(\imath_{2n+1},\imath_{2n+2})&\leq H^\beta(T\imath_{2n},S\imath_{2n+1})+\epsilon^{2n+1}\\ &\leq\alpha_1 d_s(\imath_{2n},T\imath_{2n})+\alpha_2 d_s(\imath_{2n+1},S\imath_{2n+1})\\ &\quad+\alpha_3 d_s(\imath_{2n},S\imath_{2n+1})+\alpha_4 d_s(\imath_{2n+1},T\imath_{2n})\\ &\quad+\alpha_5\left[\frac{d_s(\imath_{2n},S\imath_{2n+1})+d_s(\imath_{2n+1},T\imath_{2n})}{2}\right]\\ &\quad+\alpha_6\left[\frac{d_s(\imath_{2n},T\imath_{2n})*d_s(\imath_{2n+1},S\imath_{2n+1})}{1+d_s(\imath_{2n},\imath_{2n+1})}\right]\\ &\quad+\alpha_7 d_s(\imath_{2n},\imath_{2n+1})+\epsilon^{2n+1}\\ &\leq\alpha_1 d_s(\imath_{2n},\imath_{2n+1})+\alpha_2 d_s(\imath_{2n+1},\imath_{2n+2})\\ &\quad+\alpha_3 d_s(\imath_{2n},\imath_{2n+2})+\alpha_4(0)\\ &\quad+\alpha_5\left[\frac{d_s(\imath_{2n},\imath_{2n+2})+0}{2}\right]\\ &\quad+\alpha_6\left[\frac{d_s(\imath_{2n},\imath_{2n+1})*d_s(\imath_{2n+1},\imath_{2n+2})}{1+d_s(\imath_{2n},\imath_{2n+1})}\Big)\right]\\ &\quad+\alpha_7 d_s(\imath_{2n},\imath_{2n+1})+\epsilon^{2n+1}\\ &\leq\alpha_1 d_s(\imath_{2n},\imath_{2n+1})+\alpha_2 d_s(\imath_{2n+1},\imath_{2n+2})\\ &\quad+\alpha_3[s(d_s(\imath_{2n},\imath_{2n+1})+d_s(\imath_{2n+1},\imath_{2n+2}))]\\ &\quad+\alpha_5 s\left[\frac{d_s(\imath_{2n},\imath_{2n+1})+d_s(\imath_{2n+1},\imath_{2n+2})}{2}\right]\\ &\quad+\alpha_6 d_s(\imath_{2n+1},\imath_{2n+2})+\alpha_7 d_s(\imath_{2n},\imath_{2n+1})+\epsilon^{2n+1}.\end{aligned} \tag{8}$$

Therefore,

$$d_s(\imath_{2n+1},\imath_{2n+2})\leq\frac{\alpha_1+s\alpha_3+(s\alpha_5/2)+\alpha_7}{1-\alpha_2-s\alpha_3-(s\alpha_5/2)-\alpha_6}d_s(\imath_{2n},\imath_{2n+1})+\epsilon^{2n+1}. \tag{9}$$

Again,

$$\begin{aligned}d_s(\imath_{2n+2},\imath_{2n+3})&\leq H^\beta(S\imath_{2n+1},T\imath_{2n+2})+\epsilon^{2n+2}\leq\alpha\max\{d_s(\imath_{2n+1},\imath_{2n+2}),\\ &\quad\cdot d_s(\imath_{2n+2},T\imath_{2n+2}),d_s(\imath_{2n+1},S\imath_{2n+1}),\\ &\quad\cdot d_s(\imath_{2n+2},S\imath_{2n+1}),d_s(\imath_{2n+1},T\imath_{2n+2})\}\\ &\quad+L\min\{d_s(\imath_{2n+2},S\imath_{2n+1}),d_s(\imath_{2n+1},T\imath_{2n+2})\}+\epsilon^{2n+2}\\ &\leq\alpha\max\{d_s(\imath_{2n+1},\imath_{2n+2}),d_s(\imath_{2n+2},\imath_{2n+3},\\ &\quad\cdot d_s(\imath_{2n+1},\imath_{2n+2}),d_s(\imath_{2n+2},\imath_{2n+2}),d_s(\imath_{2n+1},\imath_{2n+3})\}\\ &\quad+L\min\{d_s(\imath_{2n+2},\imath_{2n+2}),d_s(\imath_{2n+1},\imath_{2n+3})\}+\epsilon^{2n+2},\end{aligned} \tag{10}$$

$$\text{or } d_s(\imath_{2n+2}, \imath_{2n+3}) \leq \frac{\alpha_2 + s\alpha_4 + (s\alpha_5/2) + \alpha_7}{1 - \alpha_1 - s\alpha_4 - (s\alpha_5/2) - \alpha_6} d_s(\imath_{2n+1}, \imath_{2n+3}) + \epsilon^{2n+2}. \tag{11}$$

Thus, we have

$$d_s(\imath_n, \imath_{n+1}) \leq \lambda d_s(\imath_{n-1}, \imath_n) + \epsilon^n, \tag{12}$$

where $\lambda = \max\{(\alpha_1 + s\alpha_3 + (s\alpha_5/2) + \alpha_7/1 - \alpha_2 - s\alpha_3 - (s\alpha_5/2) - \alpha_6), (\alpha_2 + s\alpha_4 + (s\alpha_5/2) + \alpha_7/1 - \alpha_1 - s\alpha_4 - (s\alpha_5/2) - \alpha_6)\} < 1$.

By Lemma 6, the sequence $<\imath_n>$ is a Cauchy sequence. Since $(X, d_s)$ is complete, there exists $\hbar \in X$ such that the Cauchy sequence $<\imath_n>$ is convergent to $\hbar$. We will show that $\hbar \in T\hbar \bigcap S\hbar$. By the definition of $H^\beta$, we have

$$\begin{aligned}
&\beta\delta_s(S\imath_{2n+1}, T\hbar) + (1-\beta)\delta_s(T\hbar, S\imath_{2n+1}) \\
&\leq H^\beta(S\imath_{2n+1}, T\hbar) \leq \alpha_1 d_s(\hbar, T\hbar) + \alpha_2 d_s(\imath_{2n+1}, S\imath_{2n+1}) \\
&\quad + \alpha_3 d_s(\hbar, S\imath_{2n+1}) + \alpha_4 d_s(\imath_{2n+1}, T\hbar) \\
&\quad + \alpha_5\left[\frac{d_s(\hbar, S\imath_{2n+1} + d_s(\imath_{2n+1}, T\hbar)}{2}\right] \\
&\quad + \alpha_6 \frac{d_s(\hbar, T\hbar) * d_s(\imath_{2n+1}, S\imath_{2n+1})}{1 + d_s(\hbar, \imath_{2n+1})} \\
&\quad + \alpha_7 d_s(\hbar, \imath_{2n+1}) + \epsilon^{2n+1} \\
&\leq \alpha_1 d_s(\hbar, T\hbar) + \alpha_2 d_s(\imath_{2n+1}, \imath_{2n+2}) \\
&\quad + \alpha_3 d_s(\hbar, \imath_{2n+2}) + \alpha_4 d_s(\imath_{2n+1}, T\hbar) \\
&\quad + \alpha_5\left[\frac{d_s(\hbar, \imath_{2n+2}) + d_s(\imath_{2n+1}, T\hbar)}{2}\right] \\
&\quad + \alpha_6 \frac{d_s(\hbar, T\hbar) d_s(\imath_{2n+1}, \imath_{2n+2})}{1 + d_s(\hbar, \imath_{2n+1}} \\
&\quad + \alpha_7 d_s(\hbar, \imath_{2n+1}) + \epsilon^{2n+1}.
\end{aligned} \tag{13}$$

□

It follows that

$$\begin{aligned}
&\lim_{n\longrightarrow\infty} \beta s\delta_s(S\imath_{2n+1}, T\hbar) + (1-\beta)\delta_s(T\hbar, S\imath_{2n+1}) \\
&\leq \lim[\alpha_1 d_s(\hbar, T\hbar) + \alpha_2 d_s(\imath_{2n+1}, \imath_{2n+2}) \\
&\quad + \alpha_3 d_s(\hbar, \imath_{2n+2}) + \alpha_4 d_s(\imath_{2n+1}, T\hbar) \\
&\quad + \alpha_5\left[\frac{d_s(\hbar, \imath_{2n+1}) + d_s(\imath_{2n+1}, T\hbar)}{2}\right] \\
&\quad + \alpha_6 \frac{d_s(\hbar, T\hbar) d_s(\imath_{2n+1}, \imath_{2n+2})}{1 + d_s(\hbar, \imath_{2n+1}} + \alpha_7 d_s(\hbar, \imath_{2n+1})) \\
&\leq \alpha_1 d_s(\hbar, T\hbar) + \alpha_4 d_s(\hbar, T\hbar) + \alpha_5 \frac{d_s(\hbar, T\hbar)}{2} \\
&\leq \left(\alpha_1 + \alpha_4 + \frac{\alpha_5}{2}\right) d_s(\hbar, T\hbar).
\end{aligned} \tag{14}$$

Since

$$\begin{aligned}
&\lim_{n\longrightarrow\infty} \beta\delta_s(S\imath_{2n+1}, T\hbar) + \lim_{n\longrightarrow\infty}(1-\beta)\delta_s(T\hbar, S\imath_{2n+1}) \\
&\leq \lim_{n\longrightarrow\infty} \beta s\delta_s(S\imath_{2n+1}, T\hbar) + (1-\beta)\delta_s(T\hbar, S\imath_{2n+1}),
\end{aligned} \tag{15}$$

we have

$$\begin{aligned}
&\lim_{n\longrightarrow\infty} \beta\delta_s(S\imath_{2n+1}, T\hbar) + \lim_{n\longrightarrow\infty}(1-\beta)\delta_s(T\hbar, S\imath_{2n+1}) \\
&\leq \left(\alpha_1 + \alpha_4 + \frac{\alpha_5}{2}\right) d_s(\hbar, T\hbar).
\end{aligned} \tag{16}$$

This implies

$$\lim_{n\longrightarrow\infty} \beta\delta_s(S\imath_{2n+1}, T\hbar) \leq \left(\alpha_1 + \alpha_4 + \frac{\alpha_5}{2}\right) d_s(\hbar, T\hbar). \tag{17}$$

Again, we have

$$\begin{aligned}
&\beta\delta_s(T\imath_{2n}, S\hbar) + (1-\beta)\delta_s(S\hbar, T\imath_{2n}) \\
&\leq H^\beta(T\imath_{2n}, S\hbar) \leq \alpha_1 d_s(\imath_{2n}, T\imath_{2n}) + \alpha_2 d_s(\hbar, S\hbar) \\
&\quad + \alpha_3 d_s(\imath_{2n}, S\hbar) + \alpha_4 d_s(\hbar, T\imath_{2n}) \\
&\quad + \alpha_5\left[\frac{d_s(\hbar, T\imath_{2n} + d_s(\imath_{2n}, S\hbar))}{2}\right] \\
&\quad + \alpha_6\left[\frac{d_s(\hbar, S\hbar) d_s(\imath_{2n}, T\imath_{2n})}{1 + d_s(\hbar, \imath_{2n})}\right] + \alpha_7 d_s(\hbar, \imath_{2n}) + \epsilon^{2n+1} \\
&\leq \alpha_1 d_s(\imath_{2n}, \imath_{2n+1}) + \alpha_2 d_s(\hbar, S\hbar) + \alpha_3 d_s(\imath_{2n}, S\hbar) \\
&\quad + \alpha_4 d_s(\hbar, \imath_{2n+1}) + \alpha_5\left[\frac{d_s(\hbar, \imath_{2n+1}) + d_s(\imath_{2n}, S\hbar)}{2}\right] \\
&\quad + \alpha_6 \frac{d_s(\hbar, S\hbar) d_s(\imath_{2n}, \imath_{2n+1})}{1 + d_s(\hbar, \imath_{2n})} + \alpha_7 d_s(\hbar, \imath_{2n}) + \epsilon^{2n+1}.
\end{aligned} \tag{18}$$

It follows that

$$\begin{aligned}
&\lim_{n\longrightarrow\infty} \beta s\delta_s(T\imath_{2n}, S\hbar) + (1-\beta)\delta_s(S\hbar, T\imath_{2n}) \\
&\leq \lim\Bigg[\alpha_1 d_s(\imath_{2n}, \imath_{2n+1}) + \alpha_2 d_s(\hbar, S\hbar) \\
&\quad + \alpha_3 d_s(\imath_{2n}, S\hbar) + \alpha_4 d_s(\hbar, i_{2n+1}) \\
&\quad + \alpha_5\left[\frac{d_s(\hbar, \imath_{2n+1}) + d_s(\imath_{2n}, S\hbar)}{2}\right] \\
&\quad + \alpha_6 \frac{d_s(\hbar, S\hbar) d_s(\imath_{2n}, \imath_{2n+1})}{1 + d_s(\hbar, \imath_{2n})} + \alpha_7 d_s(\hbar, \imath_{2n})\Bigg] \\
&\leq \alpha_2 d_s(\hbar, S\hbar) + \alpha_3 d_s(\hbar, S\hbar) + \alpha_5 \frac{d_s(\hbar, S\hbar)}{2} \\
&\leq \left(\alpha_2 + \alpha_3 + \frac{\alpha_5}{2}\right) d_s(\hbar, S\hbar).
\end{aligned} \tag{19}$$

Since

$$\begin{aligned}
&\lim_{n\longrightarrow\infty} \beta\delta_s(T\imath_{2n}, S\hbar) + \lim_{n\longrightarrow\infty}(1-\beta)\delta_s(S\hbar, T\imath_{2n}) \\
&\leq \lim_{n\longrightarrow\infty} \beta s\delta_s(T\imath_{2n}, S\hbar) + (1-\beta)\delta_s(S\hbar, T\imath_{2n}),
\end{aligned} \tag{20}$$

we have

$$\lim_{n\longrightarrow\infty}\beta\delta_s(T\imath_{2n},S\hbar)+\lim_{n\longrightarrow\infty}(1-\beta)\delta_s(S\hbar,T\imath_{2n}) \leq \left(\alpha_2+\alpha_3+\frac{\alpha_5}{2}\right)d_s(\hbar,S\hbar). \tag{21}$$

This implies

$$\lim_{n\longrightarrow\infty}\beta\delta_s(T\imath_{2n},S\hbar)\leq\left(\alpha_2+\alpha_3+\frac{\alpha_5}{2}\right)d_s(\hbar,S\hbar). \tag{22}$$

Now

$$\begin{aligned} d_s(\hbar,T\hbar)&\leq s[d_s(\hbar,\imath_{2n+2})+\delta_s(S\imath_{2n+1},T\hbar),\\ d_s(\hbar,S\hbar)&\leq s[d_s(\hbar,\imath_{2n+1})+\delta_s(T\imath_{2n},S\hbar)). \end{aligned} \tag{23}$$

Using (17) and (22) in the above two inequalities, we get

$$\begin{aligned} d_s(\hbar,T\hbar)&\leq s\lim_{n\longrightarrow\infty}d_s(\hbar,\imath_{2n+2})+s\lim_{n\longrightarrow\infty}\delta_s(S\imath_{2n+1},T\hbar)\\ &\leq\frac{s(\alpha_1+\alpha_4+(\alpha_5/2))}{\beta}d_s(\hbar,T\hbar),\\ d_s(\hbar,S\hbar)&\leq s\lim_{n\longrightarrow\infty}d_s(\hbar,\imath_{2n+1})+s\lim_{n\longrightarrow\infty}\delta_s(T\imath_{2n},S\hbar)\\ &\leq\frac{s(\alpha_2+\alpha_3+(\alpha_5/2))}{\beta}d_s(\hbar,S\hbar). \end{aligned} \tag{24}$$

This gives $d_s(\hbar,T\hbar)=0$ and $d_s(\hbar,S\hbar)=0$. Since $T$ and $S$ are closed, we have $\hbar\in T$ and $\hbar\in S$.

Our next result provides an extension and new variants of Ciric's quasi-contraction [15] and multivalued weak quasi-contraction [17], for a pair of multivalued mappings in a $b$-metric space.

**Theorem 12.** *Let $(X,d_s)$ be a complete $b$-metric space with constant $s\geq 1$, $d_s$ be $*$-continuous, and $T,S:X\longrightarrow CB^{d_s}(X)$ be multivalued pairwise $H^\beta$-Hausdorff functions for some $1/2\leq\beta\leq 1$ and satisfying the following condition:*

$$\begin{aligned} H^\beta(T\imath,S\jmath)\leq{}&\alpha\max\{d_s(\imath,\jmath),d(\imath,T\imath),d_s(\jmath,S\jmath),d_s(\imath,S\jmath),d_s(\jmath,T\imath)\}\\ &+L\min\{d_s(\imath,S\jmath),d_s(\jmath,T\imath)\}, \end{aligned} \tag{25}$$

*for all $\imath,\jmath\in X$, some $\alpha\geq 0$ with $0\leq s\alpha<1/2$ and $L\geq 0$. Then, $S$ and $T$ have a common fixed point.*

*Proof.* Proceeding as in the proof of Theorem 11, for some $\imath_0\in X$, $\imath_1\in T\imath_0$, and $0<\epsilon<1$, we construct the sequence $<\imath_n>$ satisfying (7). Then, we have

$$\begin{aligned} &d_s(\imath_{2n+1},\imath_{2n+2})\\ &\leq H^\beta(T\imath_{2n},S\imath_{2n+1})+\epsilon^{2n+1}\\ &\leq\alpha\max\{d_s(\imath_{2n},\imath_{2n+1}),d_s(\imath_{2n},T\imath_{2n}),d_s(\imath_{2n+1},S\imath_{2n+1}),\\ &\quad\cdot d_s(\imath_{2n},S\imath_{2n+1}),d_s(\imath_{2n+1},T\imath_{2n})\}\\ &\quad+L\min\{d_s(\imath_{2n},S\imath_{2n+1}),d_s(\imath_{2n+1},T\imath_{2n})\}+\epsilon^{2n+1}\\ &\leq\alpha\max\{d_s(\imath_{2n},\imath_{2n+1}),d_s(\imath_{2n},\imath_{2n+1}),d_s(\imath_{2n+1},\imath_{2n+2}),\\ &\quad\cdot d_s(\imath_{2n},\imath_{2n+2}),d_s(\imath_{2n+1},\imath_{2n+1})\}\\ &\quad+L\min\{d_s(\imath_{2n},\imath_{2n+2}),d_s(\imath_{2n+1},\imath_{2n+1})\}+\epsilon^{2n+1}. \end{aligned} \tag{26}$$

Therefore,

$$d_s(\imath_{2n+1},\imath_{2n+2})\leq\frac{s\alpha}{1-s\alpha}d_s(\imath_{2n},\imath_{2n+1})+\epsilon^{2n+1}. \tag{27}$$

Again,

$$\begin{aligned} &d_s(\imath_{2n+2},\imath_{2n+3})\\ &\leq H^\beta(S\imath_{2n+1},T\imath_{2n+2})+\epsilon^{2n+2}\\ &\leq\alpha\max\{d_s(\imath_{2n+1},\imath_{2n+2}),d_s(\imath_{2n+2},T\imath_{2n+2},\\ &\cdot d_s(\imath_{2n+1},S\imath_{2n+1}),d_s(\imath_{2n+2},S\imath_{2n+1}),d_s(\imath_{2n+1},T\imath_{2n+2})\}\\ &\quad+L\min\{d_s(\imath_{2n+2},S\imath_{2n+1}),d_s(\imath_{2n+1},T\imath_{2n+2})\}+\epsilon^{2n+2}\\ &\leq\alpha\max\{d_s(\imath_{2n+1},\imath_{2n+2}),d_s(\imath_{2n+2},\imath_{2n+3},\\ &\cdot d_s(\imath_{2n+1},\imath_{2n+2}),d_s(\imath_{2n+2},\imath_{2n+2}),d_s(\imath_{2n+1},\imath_{2n+3})\}\\ &\quad+L\min\{d_s(\imath_{2n+2},\imath_{2n+2}),d_s(\imath_{2n+1},\imath_{2n+3})\}+\epsilon^{2n+2}, \end{aligned} \tag{28}$$

and we get

$$d_s(\imath_{2n+2},\imath_{2n+3})\leq\frac{s\alpha}{1-s\alpha}d_s({}_s\imath_{2n+1},\imath_{2n+3})+\epsilon^{2n+2}. \tag{29}$$

Thus, we have

$$d_s(\imath_n,\imath_{n+1})\leq\lambda d_s(\imath_{n-1},\imath_n)+\epsilon^n, \tag{30}$$

where $\lambda=(s\alpha/(1-s\alpha))<1$. □

By Lemma 6, the sequence $<\imath_n>$ is a Cauchy sequence. Since $(X,d_s)$ is complete, there exists $\hbar\in X$ such that the Cauchy sequence $<\imath_n>$ is convergent to $\hbar$. We will show that $\hbar\in T\hbar\bigcap S\hbar$. By the definition of $H^\beta$, we have

$$\begin{aligned} &\beta\delta_s(S\imath_{2n+1},T\hbar)+(1-\beta)\delta_s(T\hbar,S\imath_{2n+1})\\ &\leq H^\beta(S\imath_{2n+1},T\hbar)\leq\alpha\max\{d_s(\imath_{2n+1},\hbar),d_s(\hbar,T\hbar),\\ &\quad\cdot d_s(\imath_{2n+1},S\imath_{2n+1}),d_s(\hbar,S\imath_{2n+1}),d_s(\imath_{2n+1},T\hbar)\}\\ &\quad+L\min\{d_s(\hbar,S\imath_{2n+1}),d_s(\imath_{2n+1},T\hbar)\}+\epsilon^{2n+2}\\ &\leq\alpha\max\{d_s(\imath_{2n+1},\hbar),d_s(\hbar,T\hbar,d_s(\imath_{2n+1},\imath_{2n+2}),\\ &\quad\cdot d_s(\hbar,\imath_{2n+2}),d_s(\imath_{2n+1},T\hbar)\}\\ &\quad+L\min\{d_s(\hbar,\imath_{2n+2}),d_s(\imath_{2n+1},T\hbar)\}+\epsilon^{2n+2}. \end{aligned} \tag{31}$$

It follows that

$$\begin{aligned}\lim_{n\longrightarrow\infty}&\beta s\delta_s(S\imath_{2n+1}, T\hbar) + (1-\beta)\delta_s(T\hbar, S\imath_{2n+1})\\ &\le \lim[\alpha \max\{d_s(\imath_{2n+1}, \hbar), d_s(\hbar, T\hbar),\\ &\quad\cdot d_s(\imath_{2n+1}, \imath_{2n+2}), d_s(\hbar, \imath_{2n+2}), d_s(\imath_{2n+1}, T\hbar)\}\\ &\quad + L\min\{d_s(\hbar, \imath_{2n+2}), d_s(\imath_{2n+1}, T\hbar)\}\epsilon^{2n+2}]\\ &\le \alpha\, d_s(\hbar, T\hbar).\end{aligned} \tag{32}$$

Since

$$\begin{aligned}&\lim_{n\longrightarrow\infty}\beta\delta_s(S\imath_{2n+1}, T\hbar) + \lim_{n\longrightarrow\infty}(1-\beta)\delta_s(T\hbar, S\imath_{2n+1})\\ &\quad\le \lim_{n\longrightarrow\infty}\beta s\delta_s(S\imath_{2n+1}, T\hbar) + (1-\beta)\delta_s(T\hbar, S\imath_{2n+1}),\end{aligned} \tag{33}$$

we have

$$\begin{aligned}&\lim_{n\longrightarrow\infty}\beta\delta_s(S\imath_{2n+1}, T\hbar) + \lim_{n\longrightarrow\infty}(1-\beta)\delta_s(T\hbar, S\imath_{2n+1})\\ &\quad\le \alpha\, d_s(\hbar, T\hbar).\end{aligned} \tag{34}$$

This implies

$$\lim_{n\longrightarrow\infty}\beta\delta_s(S\imath_{2n+1}, T\hbar) \le \alpha\, d_s(\hbar, T\hbar). \tag{35}$$

Again, we have

$$\begin{aligned}&\beta\delta_s(T\imath_{2n}, S\hbar) + (1-\beta)\delta_s(S\hbar, T\imath_{2n})\\ &\quad\le H^{\beta}(T\imath_{2n}, S\hbar) \le \alpha\max\{d_s(\imath_{2n}, \hbar), d_s(\imath_{2n}, T\imath_{2n}),\\ &\qquad\cdot d_s(\hbar, S\hbar), d_s(\imath_{2n}, S\hbar), d_s(\hbar, T\imath_{2n})\}\\ &\qquad + L\min\{d_s(\imath_{2n}, S\hbar), d_s(\hbar, T\imath_{2n})\} + \epsilon^{2n+1}\\ &\quad\le \alpha\max\{d_s(\imath_{2n}, \hbar), d_s(\imath_{2n}, \imath_{2n+1}),\\ &\qquad\cdot d_s(\hbar, S\hbar), d_s(\imath_{2n}, S\hbar), d_s(\hbar, \imath_{2n+1})\}\\ &\qquad + L\min\{d_s(\imath_{2n}, S\hbar), d_s(\hbar, \imath_{2n+1})\} + \epsilon^{2n+1}.\end{aligned} \tag{36}$$

It follows that

$$\begin{aligned}\lim_{n\longrightarrow\infty}&\beta s\delta_s(T\imath_{2n}, S\hbar) + (1-\beta)\delta_s(S\hbar, T\imath_{2n})\\ &\le \lim[\alpha\max\{d_s(\imath_{2n}, \hbar), d_s(\imath_{2n}, \imath_{2n+1}),\\ &\quad\cdot d_s(\hbar, S\hbar), d_s(\imath_{2n}, S\hbar), d_s(\hbar, \imath_{2n+1})\\ &\quad + L\min\{d_s(\imath_{2n}, S\hbar), d_s(\hbar, \imath_{2n+1})\} + \epsilon^{2n+1}]\\ &\le \alpha\, d_s(\hbar, S\hbar).\end{aligned} \tag{37}$$

Since

$$\begin{aligned}&\lim_{n\longrightarrow\infty}\beta\delta_s(T\imath_{2n}, S\hbar) + \lim_{n\longrightarrow\infty}(1-\beta)\delta_s(S\hbar, T\imath_{2n})\\ &\quad\le \lim_{n\longrightarrow\infty}\beta s\delta_s(T\imath_{2n}, S\hbar) + (1-\beta)\delta_s(S\hbar, T\imath_{2n}),\end{aligned} \tag{38}$$

we have

$$\lim_{n\longrightarrow\infty}\beta\delta_s(T\imath_{2n}, S\hbar) + \lim_{n\longrightarrow\infty}(1-\beta)\delta_s(S\hbar, T\imath_{2n}) \le \alpha\, d_s(\hbar, S\hbar). \tag{39}$$

This implies

$$\lim_{n\longrightarrow\infty}\beta\delta_s(T\imath_{2n}, S\hbar) \le \alpha\, d_s(\hbar, S\hbar). \tag{40}$$

Now,

$$\begin{aligned}d_s(\hbar, T\hbar) &\le s[d_s(\hbar, \imath_{2n+2}) + \delta_s(S\imath_{2n+1}, T\hbar),\\ d_s(\hbar, S\hbar) &\le s[d_s(\hbar, \imath_{2n+1}) + \delta_s(T\imath_{2n}, S\hbar)).\end{aligned} \tag{41}$$

Using (35) and (40) in the above two inequalities, we get

$$\begin{aligned}d_s(\hbar, T\hbar) &\le s\lim_{n\longrightarrow\infty} d_s(\hbar, \imath_{2n+2}) + s\lim_{n\longrightarrow\infty}\delta_s(S\imath_{2n+1}, T\hbar)\\ &\le \frac{s\alpha}{\beta} d_s(\hbar, T\hbar),\\ d_s(\hbar, S\hbar) &\le s\lim_{n\longrightarrow\infty} d_s(\hbar, \imath_{2n+1}) + s\lim_{n\longrightarrow\infty}\delta_s(T\imath_{2n}, S\hbar)\\ &\le \frac{s\alpha}{\beta} d_s(\hbar, T\hbar).\end{aligned} \tag{42}$$

Since $s\alpha < 1/2$ and $1/2 \le \beta \le 1$, we get $d_s(\hbar, T\hbar) = 0$ and $d_s(\hbar, S\hbar) = 0$. As $T$ and $S$ are closed, we have $\hbar \in T$ and $\hbar \in S$.

Applying the same technique as in the proof of Theorem 12, we can prove the following extension and new variant of Ciric's contraction for a pair of multivalued mappings in a $b$-metric space.

**Theorem 13.** *Let $(X, d_s)$ be a complete b-metric space with constant $s \ge 1$, $d_s$ be $*$-continuous, and $T, S : X \longrightarrow CB^{d_s}(X)(X)$ be multivalued pairwise $H^{\beta}$-Hausdorff functions for some $1/2 \le \beta \le 1$ and satisfying the following condition:*

$$\begin{aligned}H^{\beta}(T\imath, S\jmath) \le \alpha\max\Bigg\{&d_s(\imath, \jmath), d_s(\imath, T\imath), d_s(\jmath, S\jmath),\\ &\cdot\frac{d_s(\imath, S\jmath) + d_s(\jmath, T\imath)}{2}\Bigg\},\end{aligned} \tag{43}$$

*for all $i, j \in X$ and some $\alpha \ge 0$ with $s\alpha < \beta$. Then, $S$ and $T$ have a common fixed point.*

For $S = T$ in Theorem 11, we get the following result:

**Corollary 14.** *Let $(X, d_s)$ be a complete b-metric space with constant $s \ge 1$, $d_s$ be $*$-continuous, and $T : X \longrightarrow CB^{d_s}(X)$ be a multivalued $H^{\beta}$-Hausdorff function for some $1/2 \le \beta \le 1$ and satisfying the following condition:*

$$\begin{aligned}H^{\beta}(T\imath, T\jmath) \le{}& \alpha_1 d_s(\imath, T\imath) + \alpha_2 d_s(\jmath, T\jmath) + \alpha_3 d_s(\imath, T\jmath)\\ &+ \alpha_4 d_s(\jmath, T\imath) + \alpha_5\left(\frac{d_s(\imath, T\jmath) + d_s(\jmath, T\imath)}{2}\right)\\ &+ \alpha_6\frac{d_s(\imath, T\imath) d_s(\jmath, T\jmath)}{1 + d_s(\imath, \jmath)} + \alpha_7 d_s(\imath, \jmath),\end{aligned} \tag{44}$$

*for all $\imath, \jmath \in X$ and some $\alpha_k \geq 0$, $1 \leq k \leq 7$, with $\alpha_1 + \alpha_2 + s\alpha_5 + \alpha_6 + \alpha_7 + \max\{2s\alpha_3, 2s\alpha_4\} < 1$, $s(\alpha_1 + \alpha_4 + (\alpha_5/2)) < \beta$, and $s(\alpha_2 + \alpha_3 + (\alpha_5/2)) < \beta$. Then, $T$ has a fixed point.*

*Example 15.* Let $X = [0, 5/12] \bigcup \{2\}$, $d_s(\imath, \jmath) = |\imath - \jmath|^2$ for all $\imath, \jmath \in X$, and $S, T : X \longrightarrow CB(X)$ be as follows:

$$S(\imath) = \begin{cases} \left\{\frac{\imath}{4}\right\}, & \text{for } \imath \in \left[0, \frac{5}{12}\right], \\ \left\{0, \frac{1}{3}, 2\right\}, & \text{for } \imath = 2, \end{cases}$$
$$T(\imath) = \begin{cases} \left\{\frac{\imath}{4}\right\}, & \text{for } \imath \in \left[0, \frac{5}{12}\right], \\ \left\{0, \frac{5}{12}, 2\right\}, & \text{for } \imath = 2. \end{cases} \quad (45)$$

We will show that the functions $S$ and $T$ satisfy contraction condition (6) for $\beta = 1/2$.

*Case 1.* $\imath, \jmath \in [0, 5/12]$. By Proposition 2, we have

$$H^{1/2}(S\imath, T\jmath) = H^{1/2}\left(\left\{\frac{\imath}{4}\right\}, \left\{\frac{\jmath}{4}\right\}\right) = d_s\left(\frac{\imath}{4}, \frac{\jmath}{4}\right) = \left|\frac{\imath}{4} - \frac{\jmath}{4}\right|^2 \leq \alpha_1 |\imath - \jmath|^2, \quad \text{for any } \alpha_7 \geq \frac{1}{16} = \alpha_7 d_x(\imath, \jmath). \quad (46)$$

*Case 2.* $\imath \in [0, 5/12]$, $\jmath = 2$. We have $d_s(\imath, \jmath) = |2 - \imath|^2$. The minimum value of $d_s(\imath, \jmath)$ for $\imath \in [0, 5/12]$ is 361/144.

$$\delta_s(S\imath, T\jmath) = \delta_s\left(\left\{\frac{\imath}{4}\right\}, \left\{0, \frac{5}{12}, 2\right\}\right) = \frac{\imath^2}{16},$$
$$\delta_s(T\jmath, S\imath) = \delta_s\left(\left\{0, \frac{5}{12}, 1\right\}, \left\{\frac{\imath}{4}\right\}\right) = \left(2 - \frac{\imath}{4}\right)^2, \quad (47)$$
$$H^{1/2}(S\imath, T\jmath) = \frac{1}{2}\left(\frac{\imath^2}{16} + \left(2 - \frac{\imath}{4}\right)^2\right).$$

The maximum value of $H^{1/2}(S\imath, Tj)$ for $\imath \in [0, 5/12]$ is 2 (at $\imath = 0$). Thus, $H^{1/2}(S\imath, T\jmath) \leq \alpha_7 d_s(\imath, \jmath)$ for any $\alpha_7 \geq 288/361$.

*Case 3.* $\imath = 2, \jmath \in [0, 5/12]$. We have $d_s(\imath, \jmath) = |2 - \jmath|^2$. The minimum value of $d_s(\imath, \jmath)$ for $\jmath \in [0, 5/12]$ is 361/144. $\delta_s(S\imath, T\jmath) = \delta_s(\{0, 5/12, 2\}, \{\jmath/4\}) = (2 - (\jmath/4))^2$.

$$\delta_s(T\jmath, S\imath) = \delta_s\left(\left\{\frac{\imath}{4}\right\}, \left\{0, \frac{5}{12}, 1\right\}\right) = \frac{\jmath^2}{16},$$
$$H^{1/2}(S\imath, T\jmath) = \frac{1}{2}\left(\frac{\jmath^2}{16} + \left(2 - \frac{\jmath}{4}\right)^2\right). \quad (48)$$

The maximum value of $H^{1/2}(S\imath, T\jmath)$ for $\jmath \in [0, 5/12]$ is 2 (at $\jmath = 0$). Thus, $H^{1/2}(S\imath, T\jmath) \leq \alpha_7 d_s(\imath, \jmath)$ for any $\alpha_7 \geq 288/361$.

Thus, $S$ and $T$ satisfy contraction condition (6) for $\beta = 1/2$, $288/361 \leq \alpha_7 < 1$ and $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = \alpha_6 = 0$. Simple calculations show that $S$ and $T$ are pairwise $H^\beta$-Hausdorff functions. All conditions of Theorem 11 are satisfied, and 0 is a common fixed point of $S$ and $T$. However, we see that at $\imath = 0, \jmath = 2$, $S$ and $T$ do not satisfy contraction condition (6) for $\beta = 1$ and so do not satisfy Nadler's contraction and Czerwik's contraction.

*Remark 16.* In Example 15, simple calculations show that $S$ and $T$ do not satisfy contraction condition (6) for $62/100 < \beta \leq 1$. However, in view of Remark 9 (i), there may exist functions $S$ and $T$ which satisfy contraction condition (6) for $\beta = 1$ but may not satisfy for $\beta < 1$. Thus, for $\beta = 1$, Theorem 11 is an extension of Nadler's contraction [10], Czerwik's contraction [2], and many of their generalizations. For $\beta < 1$, Theorem 11 provides new variants of Nadler's contraction [10], Czerwik's contraction [2], and many of their generalizations.

*Example 17.* Let $X = \{0, 1/4, 1\}$,

$$d_s(\imath, \jmath) = |\imath - \jmath|^2 \quad \text{for all } \imath, \jmath \in X. \quad (49)$$

and $T : X \longrightarrow CB(X)$ be as follows:

$$T(x) = \begin{cases} \{0\}, & \text{for } \imath \in \left\{0, \frac{1}{4}\right\}, \\ \{0, 1\}, & \text{for } \imath = 1. \end{cases} \quad (50)$$

We will show that $T$ satisfies (44) with $\beta \in (7/16, 9/16)$.

For if $\imath, \jmath \in \{0, 1/4\}$, then the result is clear. Suppose $\imath \in \{0, 1/4\}$ and $\jmath = 1$. Then, $\delta_{d_s}(T\imath, T1) = 0$ and $\delta_{d_s}(T1, T\imath) = 1$ so that $H^\beta(T\imath, T1) = \max\{\beta, 1 - \beta\}$. Also, we have $d_s(\imath, 1) = 1$ or $9/16$.

If $\beta \in (7/16, 1/2]$, then $H^\beta(T\imath, T1) = 1 - \beta$. Now $1 - \beta \in [8/16, 9/16)$. So $1 - \beta = 16/9(1 - \beta)9/16$ and $1 - \beta < (16/9)(1 - \beta)1$, that is, $1 - \beta \leq (16/9)(1 - \beta)d_s(\imath, 1)$. Thus, we have $H^\beta(T\imath, T1) = 1 - \beta \leq kd_s(\imath, 1)$, where $k = 16/9(1 - \beta) < 1$.

Similarly, if $\beta \in [1/2, 9/16)$, we get $H^\beta(T\imath, T1) = \beta \leq kd_s(\imath, 1)$, where $k = 16/9\beta < 1$.

However, for $\imath = 1/4$ and $\jmath = 1$, we have

$$H\left(T\left(\frac{1}{4}\right), T(1)\right\} = \max\left\{\delta_{d_s}\left(T\left(\frac{1}{4}\right), T1\right), \delta_{d_s}\left(T1, T\left(\frac{1}{4}\right)\right)\right\} = 1 \text{ and } d_s\left(\frac{1}{4}, 1\right) = \frac{9}{16}. \quad (51)$$

We see that $T$ does not satisfy condition (2.2) of [24] and condition (2.1) of [26]. Thus, Theorem 2.2 of Debnath [24] and Theorem 2.3 of Kumar and Luambano [26] are not applicable.

*Remark 18* (an open question). Obtain the version of results in fixed points in the sense of Debnath [24], Kumar and Luambano [26], and Altun et al. [25] for two or more mappings using $H^{\beta}$-Hausdorff-Pompeiu $b$-metric, which will give extension and new variants of the respective results and will also generalize Corollary 19.

By taking different values of $\alpha_k$ in Theorem 11, we get the following extension and new variants of well-known contraction principles:

For $\alpha_k = 0$, $k = 1, 2, 3, 4, 5, 6$, we have the following.

**Corollary 19** (Nadler's and Czerwik's contraction). *Let $(X, d_s)$ be a complete b-metric space with constant $s \geq 1$ and $T, S : X \longrightarrow CB^{d_s}(X)$ be multivalued pairwise $H^{\beta}$-Hausdorff functions for some $1/2 \leq \beta \leq 1$ and satisfying the following condition:*

$$H^{\beta}(T\imath, S\jmath) \leq \alpha d_s(\imath, \jmath), \tag{52}$$

*for all $\imath, \jmath \in X$ and $0 \leq \alpha < 1$. Then, S and T have a common fixed point.*

For $\alpha_k = 0$, $k = 3, 4, 5, 6, 7$, we have the following.

**Corollary 20** (Kannan's contraction). *Let $(X, d_s)$ be a complete b-metric space with constant $s \geq 1$ and $T, S : X \longrightarrow CB^{d_s}(X)$ be multivalued pairwise $H^{\beta}$-Hausdorff functions for some $1/2 \leq \beta \leq 1$ and satisfying the following condition:*

$$H^{\beta}(T\imath, S\jmath) \leq \alpha_1 d_s(\imath, T\imath) + \alpha_2 d_s(\jmath, S\jmath), \tag{53}$$

*for all $\imath, \jmath \in X$ and $0 \leq \alpha_1 + \alpha_2 < 1$. Then, S and T have a common fixed point.*

For $\alpha_k = 0$, $k = 1, 2, 5, 6, 7$, we have the following.

**Corollary 21** (Chattarjee contraction). *Let $(X, d_s)$ be a complete b-metric space with constant $s \geq 1$ and $T, S : X \longrightarrow CB^{d_s}(X)$ be multivalued pairwise $H^{\beta}$-Hausdorff functions for some $1/2 \leq \beta \leq 1$ and satisfying the following condition:*

$$H^{\beta}(T\imath, S\jmath) \leq \alpha_3 d_s(\imath, S\imath) + \alpha_4 d_s(\jmath, T\imath), \tag{54}$$

*for all $\imath, \jmath \in X$ and* $\max\{s\alpha_3, s\alpha_4\} < 1/2$. *Then, S and T have a common fixed point.*

For $\alpha_k = 0$, $k = 5, 6$, we have the following.

**Corollary 22** (Hardy-Rogers contraction). *Let $(X, d_s)$ be a complete b-metric space with constant $s \geq 1$ and $T, S : X \longrightarrow CB^{d_s}(X)$ be multivalued pairwise $H^{\beta}$-Hausdorff functions for some $1/2 \leq \beta \leq 1$ and satisfying the following condition:*

$$H^{\beta}(T\imath, S\jmath) \leq \alpha_1 d_s(\imath, T\imath) + \alpha_2 d_s(\jmath, S\jmath) + \alpha_3 d_s(\imath, S\imath) + \alpha_4 d_s(\jmath, T\imath) + \alpha_7 d_s(\imath, \jmath), \tag{55}$$

*for all $\imath, \jmath \in X$ and $\alpha_1 + \alpha_2 + \alpha_7 + \max\{2s\alpha_3, 2s\alpha_4\} < 1$, $s(\alpha_1 + \alpha_4) < \beta$, and $s(\alpha_2 + \alpha_3) < \beta$. Then, S and T have a common fixed point.*

*Remark 23.* Corollary 19 is an extension and new variant of the results of Nadler [10] and Czerwik [2], Corollaries 20 and 21 are extended and new variants of the set-valued versions of the Kannan contraction and Chatterjee contraction, respectively, whereas Corollary 22 is an extended and new variant of the result of Mirmostaffae [16].

If $T, S : X \longrightarrow X$ are single-valued mappings and then by Proposition 2, $H^{\beta}(T\imath, S\jmath) = d_s(T\imath, S\jmath)$ for all $1/2 \leq \beta \leq 1$. So taking $\beta = 1$ in Theorem 11, we get the following results for single-valued mappings.

**Corollary 24.** *Let $(X, d_s)$ be a complete b-metric space with constant $s \geq 1$ and $T, S : X \longrightarrow X$ be single-valued mappings satisfying the following condition:*

$$\begin{aligned} d_s(T\imath, S\jmath) \leq{} & \alpha_1 d_s(\imath, T\imath) + \alpha_2 d_s(\jmath, S\jmath) + \alpha_3 d_s(\imath, S\imath) \\ & + \alpha_4 d_s(\jmath, T\imath) + \alpha_5 \left( \frac{d_s(\imath, S\jmath) + d_s(\jmath, T\imath)}{2} \right) \\ & + \alpha_6 \frac{d_s(\imath, T\imath) d_s(\jmath, S\jmath)}{1 + d_s(\imath, \jmath)} + \alpha_7 d_s(\imath, \jmath), \end{aligned} \tag{56}$$

*for all $\imath, \jmath \in X$ and $\alpha_1 + \alpha_2 + s\alpha_5 + \alpha_6 + \alpha_7 + \max\{2s\alpha_3, 2s\alpha_4\} < 1$, $s(\alpha_1 + \alpha_4 + (s\alpha_5/2)) < 1$, and $s(\alpha_2 + \alpha_3 + (s\alpha_5/2)) < 1$. Then, S and T have a common fixed point.*

*Remark 25.* Corollary 24 is an extension and $b$-metric version of the result of Wong [40].

## 4. Applications

In this section, we provide two applications of our results.

*4.1. Application to Multivalued Fractals.* In this section inspiring from some recent works in [20, 41, 42], we will apply our result to prove the existence of a unique common multivalued fractal for a pair of iterated multifunction systems. Let $P_i, Q_i : X \longrightarrow CB^{d_s}(X)$, $i = 1, 2, \cdots n$, be upper semicontinuous mappings. Then, $P = (P_1, P_2, \cdots P_n)$ and $Q = (Q_1, Q_2, \cdots Q_n)$ form a pair of iterated multifunction systems defined on the $b$-metric space $(X, d_s)$. The extended multifractal operators generated by the iterated multifunction systems $P = (P_1, P_2, \cdots P_n)$ and $Q = (Q_1, Q_2, \cdots Q_n)$ are the operators $T_P, T_Q : CB^{d_s}(X) \longrightarrow CB^{d_s}(X)$ defined by $T_P(Y) = \bigcup_{i=1}^{n} P_i(Y)$ and $T_Q(Y) = \bigcup_{i=1}^{n} Q_i(Y)$, respectively. A common fixed point of $T_P$ and $T_Q$ is called the common multivalued fractal of the iterated multifunction systems $P = (P_1, P_2, \cdots P_n)$ and $Q = (Q_1, Q_2, \cdots Q_n)$.

**Theorem 26.** *Let $P_i, Q_i : X \longrightarrow CB^{d_s}(X)$, $i = 1, 2, \cdots n$, be upper semicontinuous mappings satisfying the following condition:*

*For $i = 1, 2, \cdots n$, there exist $\beta \in [1/2, 1]$ and $a_i, e_i \in (0, 1)$, $a_i + 2se_i < 1$, such that for all $x, y \in X$,*

$$H^{\beta}(P_i x, Q_i y) \le a_i . d_s(x, y) + e_i[d_s(x, Q_i y) + d_s(y, P_i x)]. \quad (57)$$

*Then,*

(i) *For all $U_1, U_2 \in CB(X)$, $H^{\beta}(T_P(U_1), T_Q(U_2)) \le a. H^{\beta}(U_1, U_2) + b.H^{\beta}(U_1, T_P(U_1)) + c.H^{\beta}(U_2, T_Q(U_2)) + e[H^{\beta}(U_1, T_Q(U_2)) + H^{\beta}(U_2, T_P(U_1))]$*

(v) *The pair of systems $P = (P_1, P_2, \cdots P_n)$ and $Q = (Q_1, Q_2, \cdots Q_n)$ has a unique common multivalued fractal*

*Proof.* Suppose condition (57) holds. Then, for $U_1, U_2 \in CB(X)$, we have

$$\begin{aligned} &\beta\delta(P_i(U_1), Q_i(U_2)) + (1-\beta)\delta(Q_i(U_2), P_i(U_1)) \\ &\quad = \beta \sup_{x\in U_1}\left(\inf_{y\in U_2} H^{\beta}(P_i(x), Q_i(y))\right. \\ &\qquad + (1-\beta)\sup_{y\in U_2}\left(\inf_{x\in U_1} H^{\beta}(P_i(x), Q_i(y))\right. \\ &\quad \le \beta \sup_{x\in U_1}\left(\inf_{y\in U_2}\{a_i.d_s(x,y) + e_i[d_s(x, Q_i y)\right. \\ &\qquad + d_s(y, P_i x)]\} + (1-\beta)\sup_{y\in U_2}\left(\inf_{x\in U_1}\{a_i.d_s(x,y)\right. \\ &\qquad + e_i[d_s(x, Q_i y) + d_s(y, P_i x)]\} \\ &\quad = a_i.H^{\beta}(U_1, U_2) + e_i\left[\left(H^{\beta}U_1, Q_i(U_2)\right)\right. \\ &\qquad \left. + H^{\beta}(U_2, P_i(U_1))\right]. \end{aligned} \quad (58)$$

Similarly, we get

$$\begin{aligned} &\beta\delta\delta(Q_i(U_2), P_i(U_1)) + (1-\beta)(P_i(U_1), Q_i(U_2)) \\ &\quad \le a_i.H^{\beta}(U_2, U_1) + e_i\left[H^{\beta}\left(U_2, P_i(U_1) + H^{\beta}(U_1, Q_i(U_2))\right]. \right. \end{aligned} \quad (59)$$

Then, we have

$$\begin{aligned} &H^{\beta}(P_i(U_1), Q_i(U_2)) \\ &\quad \le a_i.H^{\beta}(U_1, U_2) + e_i\left[H^{\beta}\left(U_2, P_i(U_1)\right.\right. \\ &\qquad \left. + H^{\beta}(U_1, Q_i(U_2))\right] \quad i = 1, 2, \cdots n) \\ &\quad \le a.H^{\beta}(U_2, U_1) + e\left[H^{\beta}\left[U_2, P_i(U_1)\right.\right. \\ &\qquad \left. + H^{\beta}(U_1, Q_i(U_2))\right], \end{aligned} \quad (60)$$

where $a = \max\{a_1, a_2, \cdots a_n\}$ and $e = \max\{e_1, e_2, \cdots_n\}$. Note that

$$\begin{aligned} &H^{\beta}\left(\bigcup_{i=1}^{n} P_i(U_1), \bigcup_{i=1}^{n} Q_i(U_2)\right) \\ &\quad \le \max\left\{H^{\beta}(P_1(U_1), Q_1(U_2)),\right. \\ &\qquad \left. \cdot H^{\beta}(P_2(U_1), Q_2(U_2)), \cdots H^{\beta}(P_n(U_1), Q_n(U_2))\right\}, \end{aligned} \quad (61)$$

and so

$$\begin{aligned} H^{\beta}(T_P(U_1), T_Q(U_2)) \le a.H^{\beta}(U_1, U_2) + e\left[H^{\beta}(U_1, T_Q(U_2))\right. \\ \left. + H^{\beta}(U_2, T_P(U_1))\right]. \end{aligned} \quad (62)$$

Thus, $T_P, T_Q : CB(X) \longrightarrow CB(X)$ satisfies the conditions of Corollary 24 in the metric space $\{CB(X), H^{\beta}\}$ and hence has a common fixed point $U^*$ in $CB(X)$, which in turn is the unique common multivalued fractal of the pair of iterated multifunction systems $P = (P_1, P_2, \cdots P_n)$ and $Q = (Q_1, Q_2, \cdots Q_n)$. □

*Remark 27.* Since $H^{\beta}(A, B) \le H(A, B)$, Theorem 26 is a proper improvement and generalization of Theorem 3.4 of [20], Theorem 3.1 of [41], and Theorem 3.8 of [42].

*4.2. Application to the Integral Equation.* In this section, motivated by the applications given in [28–30] and [31], we establish the sufficient conditions for the existence of a common solution of a pair of nonlinear Volterra-type integral equations.

For some real numbers $a, b$ with $0 \le a < b$ and $I = [a, b]$, let $X = C(I, \mathbb{R})$ be the Banach space of real continuous functions defined on $I$ equipped with a norm given by $\|\imath\| = \max_{t\in I}|\imath(t)|$. For some $p \ge 1$, define a $b$-metric $d_s$ on $X$ by

$$d_s(\imath, \jmath) = \max_{t\in I}|\imath(t) - \jmath(t)|^p, \quad \text{for all } \imath, \jmath \in X. \quad (63)$$

Then, $(X, d_s, 2^{p-1})$ is a complete $b$-metric space. Consider the following pair of Volterra-type integral equations:

$$\begin{cases} \imath(t) = q(t) + \int_a^{\mu(t)} \mathscr{P}(t,s)\mathscr{K}_1(t,s,\imath(s))ds + \int_a^{\sigma(t)} \mathscr{Q}(t,s)\mathscr{K}_2(t,s,\imath(s))ds, \\ \jmath(t) = q(t) + \int_a^{\mu(t)} \mathscr{P}(t,s)\mathscr{K}_2(t,s,\jmath(s))ds + \int_a^{\sigma(t)} \mathscr{Q}(t,s)\mathscr{K}_1(t,s,\jmath(s))ds, \end{cases} \quad (64)$$

for all $t, s \in I = [a, b] \subseteq \mathbb{R}$, $|\lambda| > 0$, $\mathscr{K}_{i=1,2} : I \times I \times X \longrightarrow \mathbb{R}$, and $q : I \longrightarrow \mathbb{R}$, and $\mathscr{P}, \mathscr{Q} : I \times I \longrightarrow \mathbb{R}$ are continuous functions and $\mu, \sigma : I \longrightarrow I$.

Suppose $T, S : X \longrightarrow X$ is self-mappings defined by

$$\begin{cases} T\imath(t) = q(t) + \int_a^{\mu(t)} \mathscr{P}(t,s)\mathscr{K}_1(t,s,\imath(s))ds + \int_a^{\sigma(t)} \mathscr{Q}(t,s)\mathscr{K}_2(t,s,\jmath(s))ds, \\ S\jmath(t) = q(t) + \int_a^{\mu(t)} \mathscr{P}(t,s)\mathscr{K}_2(t,s,\jmath(s))ds + \int_a^{\sigma(t)} \mathscr{Q}(t,s)\mathscr{K}_1(t,s,\jmath(s))ds, \end{cases} \tag{65}$$

for all $\imath, \jmath \in X$, where $t \in I$. It is obvious that $\hbar(t)$ is a solution of (64) if and only if it has a common fixed point of $T$ and $S$.

**Theorem 28.** *Suppose that the following hypotheses hold:*

*($H_1$) $T(X)$ and $S(X)$ are closed in $X$*

*($H_2$) There exist nonnegative real numbers $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7$ with $\alpha_1 + \alpha_2 + 2^p \max\{\alpha_3, \alpha_4\} + 2^{p-1}\alpha_5 + \alpha_6 + \alpha_7 < 1$ such that*

$$|\mathscr{K}_1(t,s,\imath(s)) - \mathscr{K}_2(t,s,y(s))|^p \le N(T,S,p,t), \tag{66}$$

*where*

$$\begin{aligned} N(T,S,p,t) = {} & \alpha_1|\imath(t) - T\imath(t)|^p + \alpha_2|\jmath(t) - S\jmath(t)|^p \\ & + \alpha_3|\imath(t) - S\jmath(t)|^p + \alpha_4|\jmath(t) - T\imath(t)|^p \\ & + \alpha_7|\imath(t) - \jmath(t)|^p \\ & + \alpha_5\left(\frac{|\imath(t) - S\jmath(t)|^p + |\jmath(t) - T\imath(t)|^p}{2}\right) \\ & + \alpha_6\frac{|\imath(t) - T\imath(t)|^p|\jmath(t) - S\jmath(t)|^p}{1 + |\imath(t) - \jmath(t)|^p}. \end{aligned} \tag{67}$$

*($H_3$) $\int_a^{\mu(t)}\mathscr{P}(t,s)ds + \int_a^{\sigma(t)}\mathscr{Q}(t,s)ds \le 1/2^{p-1}$*

*Then, the system (64) of integral equations has unique common solutions in $X$.*

*Proof.* Using ($H_2$) and ($H_3$), we have

$$\begin{aligned} d_s(T\imath, S\jmath) = {} & \max_{t\in I}|T\imath(t) - S\jmath(t)|^p \le \max_{t\in I}\left|\int_a^{\mu(t)}\mathscr{P}(t,s)\mathscr{K}_1(t,s,\imath(s))ds\right. \\ & + \int_a^{\sigma(t)}\mathscr{Q}(t,s)\mathscr{K}_2(t,s,\jmath(s))ds - \int_a^{\mu(t)}\mathscr{P}(t,s)\mathscr{K}_2(t,s,\jmath(s))ds \\ & \left. - \int_a^{\sigma(t)}\mathscr{Q}(t,s)\mathscr{K}_1(t,s,\jmath(s))ds\right|^p \\ \le {} & \max_{t\in I}2^{p-1}\left\{\left|\int_a^{\mu(t)}\mathscr{P}(t,s)\mathscr{K}_1(t,s,\imath(s))ds\right.\right. \\ & \left. - \int_a^{\mu(t)}\mathscr{P}(t,s)\mathscr{K}_2(t,s,\jmath(s))ds\right|^p + \left|\int_a^{\sigma(t)}\mathscr{Q}(t,s)\mathscr{K}_2(t,s,\jmath(s))ds\right. \\ & \left.\left. - \int_a^{\sigma(t)}\mathscr{Q}(t,s)\mathscr{K}_1(t,s,\jmath(s))ds\right|^p\right\} \\ \le {} & \max_{t\in I}2^{p-1}\left\{\left|\int_a^{\mu(t)}\mathscr{P}(t,s)(\mathscr{K}_1(t,s,\imath(s)) - \mathscr{K}_2(t,s,\jmath(s)))ds\right|^p\right. \\ & \left. + \left|\int_a^{\sigma(t)}\mathscr{Q}(t,s)(\mathscr{K}_2(t,s,\jmath(s)) - \mathscr{K}_1(t,s,\jmath(s)))ds\right|^p\right\} \\ \le {} & \max_{t\in I}2^{p-1}\left\{\int_a^{\mu(t)}|\mathscr{P}(t,s)|^p|(\mathscr{K}_1(t,s,\imath(s)) - \mathscr{K}_2(t,s,\jmath(s)))|^p ds\right. \\ & \left. + \int_a^{\sigma(t)}|\mathscr{Q}(t,s)|^p|\mathscr{K}_2(t,s,\jmath(s)) - \mathscr{K}_1(t,s,\jmath(s))|^p ds\right\} \\ \le {} & \max_{t\in I}2^{p-1}\left\{\int_a^{\mu(t)}|\mathscr{P}(t,s)|^p N(T,S,p,t)ds\right. \\ & \left. + \int_a^{\sigma(t)}|\mathscr{Q}(t,s)|^p N(T,S,p,t)ds\right. \\ \le {} & \max_{t\in I}2^{p-1}N(T,S,p,t)\left\{\int_a^{\mu(t)}|\mathscr{P}(t,s)|^p ds + \int_a^{\sigma(t)}|\mathscr{Q}(t,s)|^p ds\right. \\ \le {} & \max_{t\in I}N(T,S,p,t) \le \alpha_1 d_s(\imath, T\imath) + \alpha_2 d_s(\jmath, S\jmath) \\ & + \alpha_3 d_s(\imath, S\jmath) + \alpha_4 d_s(\jmath, T\imath) + \alpha_5\left(\frac{d_s(\imath, S\jmath) + d_s(\jmath, T\imath)}{2}\right) \\ & + \alpha_6\frac{d_s(\imath, T\imath)d_s(\jmath, S\jmath)}{1 + d_s(\imath, \jmath)} + \alpha_7 d_s(\imath, \jmath). \end{aligned} \tag{68}$$

Thus, conditions of Theorem 11 are satisfied. Theorem 11 therefore ensures a common fixed point of $T$ and $S$, which in turn is a common solution of the pair of integral equations (64). □

*Remark 29.* Taking $\mathscr{Q}(t,s) = 0, \mathscr{P}(t,s) = 1, q(t) = 0, \mu(t) = t$ and $a = 0$ in (64), we get the Volterra-type integral equations considered in Rasham et al. [31] and Alshoraify et al. [30].

*Remark 30.* Taking $\mathscr{Q}(t,s) = 0, \mu(t) = 1$ and $a = 0$ in (64), we get the Fredholm-type integral equations (III.3) considered in Shoaib et al. [29].

*Remark 31.* Taking $\mathscr{Q}(t,s) = 0, \mathscr{P}(t,s) = 1$ and $\mu(t) = b$ in (64), we get the Fredholm-type integral equations (III.1) considered in Shoaib et al. [29].

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

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