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Research Article

Convergence Analysis of New Construction Explicit Methods for Solving Equilibrium Programming and Fixed Point Problems

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In this paper, we present improved iterative methods for evaluating the numerical solution of an equilibrium problem in a Hilbert space with a pseudomonotone and a Lipschitz-type bifunction. The method is built around two computing phases of a proximal-like mapping with inertial terms. Many such simpler step size rules that do not involve line search are examined, allowing the technique to be enforced more effectively without knowledge of the Lipschitz-type constant of the cost bifunction. When the control parameter conditions are properly defined, the iterative sequences converge weakly on a particular solution to the problem. We provide weak convergence theorems without knowing the Lipschitz-type bifunction constants. A few numerical tests were performed, and the results demonstrated the appropriateness and rapid convergence of the new methods over traditional ones.

1. Introduction

Let Π stand for a certain Hilbert space and Ξ stand for a nonempty closed convex subset of Π. The research is about an iterative technique for solving the equilibrium problem ((1), to make it short). Let $\Gamma : \Pi \times \Pi \longrightarrow \mathbb{R}$ be a bifunction with $\Gamma(y_1, y_1) = 0$, for each $y_1 \in \Xi$. An *equilibrium problem* for granted bifunction Γ on Ξ is interpreted this way: find $\hbar^* \in \Xi$ such that

$$\Gamma(\hbar^*, y_1) \geq 0, \quad \forall y_1 \in \Xi. \tag{1}$$

The numerical evaluation of the equilibrium problem under the following conditions is the focus of this study. We will assume that the following conditions have been satisfied:

For $\Gamma 1$, the solution set of a problem (1) is denoted by $\mathrm{sol}(\Gamma, \Xi)$ and it is nonempty.

For $\Gamma 2$, a bifunction Γ is said to be *pseudomonotone* [1, 2], i.e.,

$$\Gamma(y_1, y_2) \geq 0 \Rightarrow \Gamma(y_2, y_1) \leq 0, \quad \forall y_1, y_2 \in \Xi. \tag{2}$$

For $\Gamma 3$, a bifunction Γ is said to be *Lipschitz-type continuous* [3] on Ξ if there exist two constants $c_1, c_2 > 0$, such that

$$\begin{aligned}\Gamma(y_1, y_3) \leq \Gamma(y_1, y_2) + \Gamma(y_2, y_3) + c_1\|y_1 - y_2\|^2 \\ + c_2\|y_2 - y_3\|^2, \quad \forall y_1, y_2, y_3 \in \Xi.\end{aligned} \tag{3}$$

For $\Gamma 4$, for any sequence $\{y_k\} \subset \Xi$ satisfying $y_k \rightharpoonup y^*$, then, the following inequality holds:

$$\limsup_{k \longrightarrow +\infty} \Gamma(y_k, y_1) \leq \Gamma(y^*, y_1), \quad \forall y_1 \in \Xi. \tag{4}$$

For (Γ5), $\Gamma(y_1, \cdot)$ is convex and subdifferentiable on Π for each fixed $y_1 \in \Pi$.

Let us represent a problem's solution set as $\mathrm{sol}(\Gamma, \Xi)$, and we will assume in the following text that this solution set is not empty. Researchers are interested in the equilibrium problem because it connects many mathematical problems, including fixed point problems, vector and scalar minimization problems, variational inequalities, complementarity problems, saddle point problems, Nash equilibrium problems in noncooperative games, and inverse optimization problems (see for further information [2, 4–9]). It also has a variety of applications in economics [10], the dynamics of offer and demand [11], and it continues to use the theoretical framework of noncooperative games and Nash's equilibrium models [12, 13]. The phrase "equilibrium problem" was first used in the literature in 1992 by Muu and Oettli [9] and was further investigated by Blum [2]. More precisely, we consider two applications for the problem (1). (i) A *variational inequality problem* for an operator $\mathfrak{I}_1 : \Xi \longrightarrow \Pi$ is stated as follows: find $\hbar^* \in \Xi$ such that

$$\langle \mathfrak{I}_1(\hbar^*) y_1 - \hbar^* \rangle \geq 0, \quad \forall y_1, y_2 \in \Xi. \tag{5}$$

Let us define a bifunction Γ as follows:

$$\Gamma(y_1, y_2) \coloneqq \langle \mathfrak{I}_1(y_1), y_2 - y_1 \rangle, \quad \forall y_1, y_2 \in \Xi. \tag{6}$$

Then, the equilibrium problem converts into the problem of variational inequalities defined in (5) and Lipschitz constants of the mapping $\mathfrak{I}_1$ are $L = 2c_1 = 2c_2$. (ii) Letting a mapping $\mathfrak{I}_2 : \Xi \longrightarrow \Xi$ is said to κ-strict pseudocontraction [14] if there exists a constant $\kappa \in (0, 1)$ such that

$$\|\mathfrak{I}_2 y_1 - \mathfrak{I}_2 y_2\|^2 \leq \|y_1 - y_2\|^2 + \kappa \|(y_1 - \mathfrak{I}_2 y_1) - (y_2 - \mathfrak{I}_2 y_2)\|^2, \quad \forall y_1, y_2 \in \Xi. \tag{7}$$

A fixed point problem (FPP) for $\mathfrak{I}_2 : \Xi \longrightarrow \Xi$ is to find $\hbar^* \in \Xi$ such that $\mathfrak{I}_2(\hbar^*) = \hbar^*$. Let us define a bifunction Γ as follows:

$$\Gamma(y_1, y_2) = \langle y_1 - \mathfrak{I}_2 y_1, y_2 - y_1 \rangle, \quad \forall y_1, y_2 \in \Xi. \tag{8}$$

It can be easily seen in [15] that expression (8) satisfies the conditions (Γ1)–(Γ5) as well as the values of Lipschitz constants are $c_1 = c_2 = (3 - 2\kappa)/(2 - 2\kappa)$.

The extragradient method developed by Tran et al. [16] is one useful approach. Take an arbitrary starting point $x_0 \in \Pi$; and the next iteration as follows:

$$\begin{aligned} & x_0 \in \Xi, \\ & y_k = \arg\min_{y \in \Xi} \left\{ \daleth \Gamma(x_k, y) + \frac{1}{2} \|x_k - y\|^2 \right\}, \\ & x_{k+1} = \arg\min_{y \in \Xi} \left\{ \daleth \Gamma(y_k, y) + \frac{1}{2} \|x_k - y\|^2 \right\}, \end{aligned} \tag{9}$$

where $0 < \daleth < \min \{(1/2c_1), (1/2c_2)\}$ and c_1, c_2 are two Lipschitz-type constants.

The main goal is to create an inertial-type technique in the case of [16] that will be designed to increase the convergence rate of the iterative sequence. Such techniques have already been established as a result of the oscillator equation with damping and conservative force restoration. This second-order dynamical system is known as a "heavy friction ball," and it was first proposed by Polyak in [17]. The important feature of this method is that the next iteration is built on the previous two iterations. Numerical results show that inertial terms improve the performance of the approaches in terms of the number of iterations and elapsed time in this context. Inertial-type approaches have been extensively studied in recent years for certain classes of equilibrium problems [18–26] and others in [27–33].

As a result, the following natural question arises: Is it possible to develop new inertial-type weakly convergent extragradient-type methods with monotone and nonmonotone step size rules to solve equilibrium problems?

In our study, we provide a positive answer to this question, namely, that the gradient approach still generates a weak convergence sequence when solving equilibrium problems involving pseudomonotone bifunctions using a novel monotone and nonmonotone variable step size rule. Motivated by the work of Censor et al. [34] and Tran et al. [16], we will describe new inertial extragradient-type approaches to solving problem (1) in the context of an infinite-dimensional real Hilbert space. Our primary contributions to this work are as follows:

(i) We build an inertial subgradient extragradient technique with a novel monotone variable step size rule to solve equilibrium problems in a real Hilbert space and show that the resulting sequence is weakly convergent

(ii) To solve equilibrium problems, we devise another inertial subgradient extragradient technique that leverages a novel variable nonmonotone step size rule that is independent of the Lipschitz constants

(iii) Some results are investigated in order to address different kinds of equilibrium problems in a real Hilbert space

(iv) We offer numerical demonstrations of the suggested methodologies for the verification of theoretical conclusions and compare them to earlier results [22, 35, 36]. Our numerical results indicate that the new approaches are useful and outperform the current ones

The paper is structured as follows: in Section 2, preliminary results were presented. Section 3 gives all new approaches and their convergence analysis. Finally, Section 5 gives some numerical results to explain the practical efficiency of the proposed methods.

2. Preliminaries

In this part, we will go over several fundamental identities as well as crucial lemmas and definitions. A *metric projection* $P_{\Xi}(y_1)$ of $y_1 \in \Pi$ is defined by

$$P_{\Xi}(y_1) = \operatorname{argmin}\{\|y_1 - y_2\| : y_2 \in \Xi\}. \tag{10}$$

The following sections outline the key characteristics of projection mapping.

Lemma 1 (see [37]). *Let $P_{\Xi} : \Pi \longrightarrow \Xi$ be a metric projection. Then, there are the following features:*

$$\|y_1 - P_{\Xi}(y_2)\|^2 + \|P_{\Xi}(y_2) - y_2\|^2 \le \|y_1 - y_2\|^2, \quad y_1 \in \Xi, y_2 \in \Pi,$$

$$y_3 = P_{\Xi}(y_1), \tag{11}$$

if and only if

$$\begin{aligned} &\langle y_1 - y_3, y_2 - y_3 \rangle \le 0, \quad \forall y_2 \in \Xi, \\ &\|y_1 - P_{\Xi}(y_1)\| \le \|y_1 - y_2\|, \quad y_2 \in \Xi, y_1 \in \Pi. \end{aligned} \tag{12}$$

Lemma 2 (see [37]). *For any $y_1, y_2 \in \Pi$ and $\ell \in \mathbb{R}$. Then, the following conditions were met:*

$$\|\ell y_1 + (1-\ell) y_2\|^2 = \ell \|y_1\|^2 + (1-\ell)\|y_2\|^2 - \ell(1-\ell)\|y_1 - y_2\|^2,$$

$$\|y_1 + y_2\|^2 \le \|y_1\|^2 + 2\langle y_2, y_1 + y_2 \rangle. \tag{13}$$

A normal cone of Ξ at $y_1 \in \Xi$ is defined by

$$N_{\Xi}(y_1) = \{y_3 \in \Pi : \langle y_3, y_2 - y_1 \rangle \le 0, \forall y_2 \in \Xi\}. \tag{14}$$

Assume that $\mho : \Xi \longrightarrow \mathbb{R}$ is a convex function and subdifferential of $\mho$ at $y_1 \in \Xi$ is defined by

$$\partial \mho(y_1) = \{y_3 \in \Pi : \mho(y_2) - \mho(y_1) \ge \langle y_3, y_2 - y_1 \rangle, \forall y_2 \in \Xi\}. \tag{15}$$

Lemma 3 (see [38]). *Let $\mho : \Xi \longrightarrow \mathbb{R}$ be a subdifferentiable, convex, and lower semicontinuous function on Ξ. An element $x \in \Xi$ is a minimizer of a function $\mho$ if and only if*

$$0 \in \partial \mho(x) + N_{\Xi}(x), \tag{16}$$

where $\partial \mho(x)$ stands for the subdifferential of $\mho$ at $x \in \Xi$ and $N_{\Xi}(x)$ the normal cone of Ξ at x.

Lemma 4 (see [39]). *Let Ξ be a nonempty subset of Π and $\{x_k\}$ be a sequence in Π satisfying two conditions:*

(i) For each $x \in \Xi$, $\lim_{k \longrightarrow +\infty} \|x_k - x\|$ exists

(ii) Each sequentially weak cluster point of $\{x_k\}$ belongs to Ξ

Then, sequence $\{x_k\}$ weakly converges to an element in Ξ.

Lemma 5 (see [40]). *Suppose that $\{a_k\}$ and $\{t_k\}$ are two sequences of nonnegative real numbers satisfying the inequality*

$$a_{k+1} \le a_k + t_k, \quad \text{for all } k \in \mathbb{N}. \tag{17}$$

If $\sum t_k < +\infty$, then, $\lim_{k \longrightarrow +\infty} a_k$ exists.

3. Main Results

In this section, we present a numerical iterative method for accelerating the rate of convergence of an iterative sequence by combining two strong convex optimization problems with an inertial term. We propose the techniques listed below for solving equilibrium problems.

Remark 6. (i) If $\zeta = 0$ is used in the abovementioned method, then, it is equivalent to the default extragradient method [16] with the updated step size rule. (ii) From the expressions in Algorithm 1, we have

$$\sum_{k=1}^{+\infty} \zeta_k \|x_k - x_{k-1}\| \le \sum_{k=1}^{+\infty} \beta_k \|x_k - x_{k-1}\| < +\infty. \tag{18}$$

It further implies that

$$\lim_{k \longrightarrow +\infty} \beta_k \|x_k - x_{k-1}\| = 0. \tag{19}$$

Lemma 7. *A sequence $\{\sqsupset_k\}$ is converged to $\sqsupset$ and*

$$\min\left\{\frac{\varkappa\left(2 - \sqrt{2} - \phi\right)}{\max\{2c_1, 2c_2\}}, \sqsupset_0\right\} \le \sqsupset \le \sqsupset_0. \tag{20}$$

Proof. Assume that $\Gamma(v_k, x_{k+1}) - \Gamma(v_k, y_k) - \Gamma(y_k, x_{k+1}) > 0$, such that

$$\begin{aligned} &\frac{\varkappa\left(2 - \sqrt{2} - \phi\right)\left(\|v_k - y_k\|^2 + \|x_{k+1} - y_k\|^2\right)}{2[\Gamma(v_k, x_{k+1}) - \Gamma(v_k, y_k) - \Gamma(y_k, x_{k+1})]} \\ &\quad \ge \frac{\varkappa\left(2 - \sqrt{2} - \phi\right)\left(\|v_k - y_k\|^2 + \|x_{k+1} - y_k\|^2\right)}{2\left[c_1\|v_k - y_k\|^2 + c_2\|x_{k+1} - y_k\|^2\right]} \\ &\quad \ge \frac{\varkappa\left(2 - \sqrt{2} - \phi\right)}{2\max\{c_1, c_2\}}. \end{aligned} \tag{21}$$

STEP 0: Choose $_0 > 0, x_{-1}, x_0 \in \Pi, \zeta \in (0,1), \varkappa \in (0,1)$, $\phi \in (0, 2-\sqrt{2})$ with a sequence $\{\psi_k\} \subset [0, +\infty)$ such that $\sum_{k=0}^{+\infty} \psi_k < +\infty$.

Moreover, choose ζ_k such that $0 \le \zeta_k \le \beta_k$ such that

$$\beta_k = \begin{cases} \min\{\zeta, (\psi_k / \|x_k - x_{k-1}\|)\} & \text{if } x_k \ne x_{k-1}, \\ \zeta & \text{otherwise.} \end{cases}$$

STEP 1: Compute

$$y_k = \arg\min_{y \in \Xi} \{\beth_k \Gamma(\nu_k, y) + 1/2 \|\nu_k - y\|^2\} \text{wwhere } \nu_k = x_k + \zeta_k (x_k - x_{k-1}).$$

STEP 2: Given the current iterates x_{k-1}, x_k, y_k. Firstly choose $\omega_k \in \partial_2 \Gamma(\nu_k, y_k)$ satisfying $\nu_k - \beth_k \omega_k - y_k \in N_\Xi(y_k)$ and generate a half-space

$$\Pi_k = \{z \in \Pi : \langle \nu_k - \beth_k \omega_k - y_k, z - y_k \rangle \le 0\}.$$

Compute

$$x_{k+1} = \arg\min_{y \in \Pi_k} \{\beth_k \Gamma(y_k, y) + 1/2 \|\nu_k - y\|^2\}.$$

STEP 3: Compute

$$\beth_{k+1} = \begin{cases} \min\{\beth_k, ((2-\sqrt{2}-\phi)\varkappa\|\nu_k - y_k\|^2 + (2-\sqrt{2}-\phi)\varkappa\|x_{k+1} - y_k\|^2 / 2[\Gamma(\nu_k, x_{k+1}) - \Gamma(\nu_k, y_k) - \Gamma(y_k, x_{k+1})])\} \\ \text{if } \Gamma(\nu_k, x_{k+1}) - \Gamma(\nu_k, y_k) - \Gamma(y_k, x_{k+1}) > 0, \\ \beth_k, \text{otherwise.} \end{cases}$$

STEP 4: If $y_k = \nu_k$, then complete the computation. Otherwise, set $k \coloneqq k+1$ and go back **STEP 1**.

Algorithm 1

Thus, we obtain $\lim_{k \longrightarrow +\infty} \beth_k = \beth$ This completes the proof of the lemma. □

Lemma 8. *A sequence $\{\beth_k\}$ is converged to $\beth$ and*

$$\min\left\{\frac{\varkappa\left(2-\sqrt{2}-\phi\right)}{\max\{2c_1, 2c_2\}}, \beth_0\right\} \le \beth \le \beth_0, \tag{22}$$

where $P = \sum_{k=1}^{+\infty} p_k$.

Proof. Assume that $\Gamma(\nu_k, x_{k+1}) - \Gamma(\nu_k, y_k) - \Gamma(y_k, x_{k+1}) > 0$ such that

$$\begin{aligned} &\frac{\varkappa\left(2-\sqrt{2}-\phi\right)\left(\|\nu_k - y_k\|^2 + \|x_{k+1} - y_k\|^2\right)}{2[\Gamma(\nu_k, x_{k+1}) - \Gamma(\nu_k, y_k) - \Gamma(y_k, x_{k+1})]} \\ &\quad \ge \frac{\varkappa\left(2-\sqrt{2}-\phi\right)\left(\|\nu_k - y_k\|^2 + \|x_{k+1} - y_k\|^2\right)}{2\left[c_1\|\nu_k - y_k\|^2 + c_2\|x_{k+1} - y_k\|^2\right]} \\ &\quad \ge \frac{\varkappa\left(2-\sqrt{2}-\phi\right)}{2\max\{c_1, c_2\}}. \end{aligned} \tag{23}$$

Applying mathematical induction on the concept of $\beth_{k+1}$, we have

$$\min\left\{\frac{\varkappa\left(2-\sqrt{2}-\phi\right)}{\max\{2c_1, 2c_2\}}, \le \beth_k \le \beth_0 + P\right\}. \tag{24}$$

Suppose that $[\beth_{k+1} - \beth_k]^+ = \max\{0, \beth_{k+1} - \beth_k\}$ and $[\beth_{k+1} - \beth_k]^- = \max\{0, -(\beth_{k+1} - \beth_k)\}$. Due to the definition of $\{\beth_k\}$, we get

$$\sum_{k=1}^{+\infty} (\beth_{k+1} - \beth_k)^+ = \sum_{k=1}^{+\infty} \max\{0, \beth_{k+1} - \beth_k\} \le P < +\infty. \tag{25}$$

That is, the series $\sum_{k=1}^{+\infty} (\beth_{k+1} - \beth_k)^+$ is convergent. The convergence must now be proven of $\sum_{k=1}^{+\infty} (\beth_{k+1} - \beth_k)^-$. Let $\sum_{k=1}^{+\infty} (\beth_{k+1} - \beth_k)^- = +\infty$. Due to the fact that $\beth_{k+1} - \beth_k = (\beth_{k+1} - \beth_k)^+ - (\beth_{k+1} - \beth_k)^-$, we could get

$$\beth_{k+1} - \beth_0 = \sum_{k=0}^{k} (\beth_{k+1} - \beth_k) = \sum_{k=0}^{k} (\beth_{k+1} - \beth_k)^+ - \sum_{k=0}^{k} (\beth_{k+1} - \beth_k)^-. \tag{26}$$

Letting $k \longrightarrow +\infty$ in (26), we have $\beth_k \longrightarrow -\infty$ as $k \longrightarrow +\infty$. This is an absurdity. As a result of the series convergence $\sum_{k=0}^{k} (\beth_{k+1} - \beth_k)^+$ and $\sum_{k=0}^{k} (\beth_{k+1} - \beth_k)^-$ taking $k \longrightarrow +\infty$ in expression (26), we obtain $\lim_{k \longrightarrow +\infty} \beth_k = \beth$. This brings the proof to a conclusion. □

Lemma 9. *The following useful inequality is derived in Algorithm 3.*

$$\beth_k \Gamma(y_k, y) - \beth_k \Gamma(y_k, x_{k+1}) \ge \langle \nu_k - x_{k+1}, y - x_{k+1} \rangle, \quad \forall y \in \Pi_k. \tag{27}$$

STEP 0: Choose $_0 > 0, x_{-1}, x_0 \in \Pi, \zeta \in (0,1), \varkappa \in (0,1)$, $\phi \in (0, 2-\sqrt{2})$ with a sequence $\{\psi_k\} \subset [0,+\infty)$ such that $\sum_{k=0}^{+\infty}\psi_k < +\infty$.
Moreover, choose a non-negative real sequence $\{p_k\}$ such that $\sum_{k=1}^{+\infty}p_k < +\infty$ and ζ_k such that $0 \le \zeta_k \le \beta_k$ such that

$$\beta_k = \begin{cases} \min\{\zeta, (\psi_k/\|x_k - x_{k-1}\|)\} & \text{if } x_k \neq x_{k-1}, \\ \zeta & \text{otherwise.} \end{cases}$$

STEP 1: Compute

$$y_k = \arg\min_{y\in\Xi}\{\beth_k \Gamma(\nu_k, y) + 1/2\|\nu_k - y\|^2\} \text{where } \nu_k = x_k + \zeta_k(x_k - x_{k-1}).$$

STEP 2: Given the current iterates x_{k-1}, x_k, y_k. Firstly choose $\omega_k \in \partial_2\Gamma(\nu_k, y_k)$ satisfying $\nu_k - \beth_k\omega_k - y_k \in N_\Xi(y_k)$ and generate a half-space

$$\Pi_k = \{z \in \Pi : \langle \nu_k - \beth_k\omega_k - y_k, z - y_k\rangle \le 0\}.$$

Compute

$$x_{k+1} = \arg\min_{y\in\Pi_k}\{\beth_k \Gamma(y_k, y) + 1/2\|\nu_k - y\|^2\}.$$

STEP 3: Compute

$$\beth_{k+1} = \begin{cases} \min\{\beth_k + p_k, ((2-\sqrt{2}-\phi)\varkappa\|\nu_k - y_k\|^2 + (2-\sqrt{2}-\phi)\varkappa\|x_{k+1} - y_k\|^2/2[\Gamma(\nu_k, x_{k+1}) - \Gamma(\nu_k, y_k) - \Gamma(y_k, x_{k+1})])\} \\ \text{if } \Gamma(\nu_k, x_{k+1}) - \Gamma(\nu_k, y_k) - \Gamma(y_k, x_{k+1}) > 0, \\ \beth_k + p_k, \text{otherwise.} \end{cases}$$

STEP 4: If $y_k = \nu_k$, then complete the computation. Otherwise, set $k := k+1$ and go back **STEP 1**.

Algorithm 2

STEP 0: Choose $_0 > 0, x_{-1}, x_0 \in \Pi, \zeta \in (0,1), \varkappa \in (0,1)$, $\phi \in (0, 2-\sqrt{2})$ with a sequence $\{\psi_k\} \subset [0,+\infty)$ such that $\sum_{k=0}^{+\infty}\psi_k < +\infty$.
Moreover, choose ζ_k such that $0 \le \zeta_k \le \beta_k$ such that

$$\beta_k = \begin{cases} \min\{\zeta, (\psi_k/\|x_k - x_{k-1}\|)\} & \text{if } x_k \neq x_{k-1}, \\ \zeta & \text{otherwise.} \end{cases}$$

STEP 1: Compute

$$y_k = \arg\min_{y\in\Xi}\{\beth_k \Gamma(\nu_k, y) + 1/2\|\nu_k - y\|^2\}, \text{wwhere } \nu_k = x_k + \zeta_k(x_k - x_{k-1}).$$

STEP 2: Compute

$$x_{k+1} = \arg\min_{y\in\Xi}\{\beth_k \Gamma(y_k, y) + 1/2\|\nu_k - y\|^2\}.$$

STEP 3: Compute

$$\beth_{k+1} = \begin{cases} \min\{\beth_k, ((2-\sqrt{2}-\phi)\varkappa\|\nu_k - y_k\|^2 + (2-\sqrt{2}-\phi)\varkappa\|x_{k+1} - y_k\|^2/2[\Gamma(\nu_k, x_{k+1}) - \Gamma(\nu_k, y_k) - \Gamma(y_k, x_{k+1})])\} \\ \text{if } \Gamma(\nu_k, x_{k+1}) - \Gamma(\nu_k, y_k) - \Gamma(y_k, x_{k+1}) > 0, \\ \beth_k, \text{otherwise.} \end{cases}$$

STEP 4: If $y_k = \nu_k$, then complete the computation. Otherwise, set $k := k+1$ and go back **STEP 1**.

Algorithm 3

Proof. By use of Lemma 3, we have

$$0 \in \partial_2\left\{\beth_k \Gamma(y_k, \cdot) + \frac{1}{2}\|\nu_k - \cdot\|^2\right\}(x_{k+1}) + N_{\Pi_k}(x_{k+1}). \quad (28)$$

Thus, for $\upsilon \in \partial\Gamma(y_k, x_{k+1})$, there exists a vector $\bar{\upsilon} \in N_{\Pi_k}(x_{k+1})$ such that

$$\beth_k\upsilon + x_{k+1} - \nu_k + \bar{\upsilon} = 0. \quad (29)$$

Thus, we have

$$\langle \nu_k - x_{k+1}, y - x_{k+1}\rangle = \beth_k\langle \upsilon, y - x_{k+1}\rangle + \langle \bar{\upsilon}, y - x_{k+1}\rangle, \quad \forall y \in \Pi_k. \quad (30)$$

Since $\bar{\upsilon} \in N_{\Pi_k}(x_{k+1})$ implies that $\langle \bar{\upsilon}, y - x_{k+1}\rangle \le 0$ for all $y \in \Pi_k$, thus, we have

$$\langle \nu_k - x_{k+1}, y - x_{k+1}\rangle \le \beth_k\langle \upsilon, y - x_{k+1}\rangle, \quad \forall y \in \Pi_k. \quad (31)$$

Since $\upsilon \in \partial \Gamma(y_k, x_{k+1})$, we have

$$\Gamma(y_k, y) - \Gamma(y_k, x_{k+1}) \geq \langle \upsilon, y - x_{k+1} \rangle, \quad \forall y \in \Pi. \tag{32}$$

Combining expressions (31) and (32), we have

$$\beth_k \Gamma(y_k, y) - \beth_k \Gamma(y_k, x_{k+1}) \geq \langle \nu_k - x_{k+1}, y - x_{k+1} \rangle, \quad \forall y \in \Pi_k. \tag{33}$$

□

Lemma 10. *In Algorithm 3, we also have the following useful inequality:*

$$\beth_k \Gamma(\nu_k, y) - \beth_k \Gamma(\nu_k, y_k) \geq \langle \nu_k - y_k, y - y_k \rangle, \quad \forall y \in \Xi. \tag{34}$$

Proof. The proof is analogous to the proof of Lemma 9. Next, substituting $y = x_{k+1}$, we have

$$\beth_k \{\Gamma(\nu_k, x_{k+1}) - \Gamma(\nu_k, y_k)\} \geq \langle \nu_k - y_k, x_{k+1} - y_k \rangle. \tag{35}$$

□

Theorem 11. *Let $\{x_k\}$ be a sequence generated by Algorithm 3, and the conditions (Γ1)–(Γ5) are satisfied. Then, the sequence $\{x_k\}$ converges weakly to $\hbar^*$.*

Proof. By substituting $y = \hbar^*$ into Lemma 9, we have

$$\beth_k \Gamma(y_k, \hbar^*) - \beth_k \Gamma(y_k, x_{k+1}) \geq \langle \nu_k - x_{k+1}, \hbar^* - x_{k+1} \rangle. \tag{36}$$

By the use of condition Γ2, we obtain

$$\langle \nu_k - x_{k+1}, x_{k+1} - \hbar^* \rangle \geq \beth_k \Gamma(y_k, x_{k+1}). \tag{37}$$

From the expression in Algorithm 1, we obtain

$$\begin{aligned} &\Gamma(\nu_k, x_{k+1}) - \Gamma(\nu_k, y_k) - \Gamma(y_k, x_{k+1}) \\ &\quad \leq \frac{\left(2 - \sqrt{2} - \phi\right) \varkappa \left(\|\nu_k - y_k\|^2 + \|x_{k+1} - y_k\|^2\right)}{2\beth_{k+1}}, \end{aligned} \tag{38}$$

which after multiplying both sides by $\beth_k > 0$ implies that

$$\begin{aligned} \beth_k \Gamma(y_k, x_{k+1}) &\geq \beth_k \Gamma(\nu_k, x_{k+1}) - \beth_k \Gamma(\nu_k, y_k) \\ &\quad - \frac{\left(2 - \sqrt{2} - \phi\right) \beth_k \varkappa \left(\|\nu_k - y_k\|^2 + \|x_{k+1} - y_k\|^2\right)}{2\beth_{k+1}}. \end{aligned} \tag{39}$$

Combining expressions (37) and (39), we obtain

$$\begin{aligned} \langle \nu_k - x_{k+1}, x_{k+1} - \hbar^* \rangle &\geq \beth_k \{\Gamma(\nu_k, x_{k+1}) - \Gamma(\nu_k, y_k)\} \\ &\quad - \frac{\left(2 - \sqrt{2} - \phi\right) \beth_k \varkappa \left(\|\nu_k - y_k\|^2 + \|x_{k+1} - y_k\|^2\right)}{2\beth_{k+1}}. \end{aligned} \tag{40}$$

By using expression (35), we have

$$\beth_k \{\Gamma(\nu_k, x_{k+1}) - \Gamma(\nu_k, y_k)\} \geq \langle \nu_k - y_k, x_{k+1} - y_k \rangle. \tag{41}$$

Combining expressions (40) and (41), we have

$$\begin{aligned} \langle \nu_k - x_{k+1}, x_{k+1} - \hbar^* \rangle &\geq \langle \nu_k - y_k, x_{k+1} - y_k \rangle \\ &\quad - \frac{\left(2 - \sqrt{2} - \phi\right) \beth_k \varkappa \left(\|\nu_k - y_k\|^2 + \|x_{k+1} - y_k\|^2\right)}{2\beth_{k+1}}. \end{aligned} \tag{42}$$

The following facts are available to us:

$$\begin{aligned} 2\langle \nu_k - x_{k+1}, x_{k+1} - \hbar^* \rangle &= \|\nu_k - \hbar^*\|^2 - \|x_{k+1} - \nu_k\|^2 - \|x_{k+1} - \hbar^*\|^2, \\ 2\langle y_k - \nu_k, y_k - x_{k+1} \rangle &= \|\nu_k - y_k\|^2 + \|x_{k+1} - y_k\|^2 - \|\nu_k - x_{k+1}\|^2. \end{aligned} \tag{43}$$

Thus, we have

$$\begin{aligned} \|x_{k+1} - \hbar^*\|^2 &\leq \|\nu_k - \hbar^*\|^2 - \|\nu_k - y_k\|^2 - \|x_{k+1} - y_k\|^2 \\ &\quad + \frac{\left(2 - \sqrt{2} - \phi\right) \beth_k \varkappa \left(\|\nu_k - y_k\|^2 + \|x_{k+1} - y_k\|^2\right)}{\beth_{k+1}}. \end{aligned} \tag{44}$$

Since $\beth_k \longrightarrow \beth$, thus, there exists a fixed natural number $k_1 \in \mathbb{N}$ such that

$$\lim_{k \longrightarrow +\infty} \frac{\varkappa \beth_k}{\beth_{k+1}} \leq 1. \tag{45}$$

Thus, we have

$$\begin{aligned} \|x_{k+1} - \hbar^*\|^2 &\leq \|\nu_k - \hbar^*\|^2 - \|\nu_k - y_k\|^2 - \|x_{k+1} - y_k\|^2 \\ &\quad + \left(2 - \sqrt{2} - \phi\right) \left(\|\nu_k - y_k\|^2 + \|x_{k+1} - y_k\|^2\right). \end{aligned} \tag{46}$$

Furthermore, it implies that

$$\begin{aligned} \|x_{k+1} - \hbar^*\|^2 &\leq \|\nu_k - \hbar^*\|^2 - \left(\sqrt{2} - 1\right) \|\nu_k - y_k\|^2 \\ &\quad - \left(\sqrt{2} - 1\right) \|x_{k+1} - y_k\|^2 - \phi \left(\|\nu_k - y_k\|^2 + \|x_{k+1} - y_k\|^2\right). \end{aligned} \tag{47}$$

From expression (47), we obtain

$$\|x_{k+1} - \hbar^*\|^2 \leq \|\nu_k - \hbar^*\|^2, \quad \forall k \geq k_1. \tag{48}$$

It is possible to write as an expression for every $k \geq k_1$ such that

$$\|x_{k+1} - \hbar^*\| \leq \|x_k + \zeta_k (x_k - x_{k-1}) - \hbar^*\| \leq \|x_k - \hbar^*\| + \zeta_k \|x_k - x_{k-1}\|. \tag{49}$$

Combining expressions (18) and (49) and Lemma 5 implies that

$$\lim_{k\longrightarrow+\infty}\|x_k-\hbar^*\|=l,\quad \text{for some finite } l\ge 0. \tag{50}$$

By using the definition of ν_k, we have

$$\begin{aligned}\|\nu_k-\hbar^*\|^2 &= \|x_k+\zeta_k(x_k-x_{k-1})-\hbar^*\|^2=\|(1+\zeta_k)(x_k-\hbar^*)\\ &\quad -\zeta_k(x_{k-1}-\hbar^*)\|^2=(1+\zeta_k)\|x_k-\hbar^*\|^2-\zeta_k\|x_{k-1}\\ &\quad -\hbar^*\|^2+\zeta_k(1+\zeta_k)\|x_k-x_{k-1}\|^2\le(1+\zeta_k)\|x_k\\ &\quad -\hbar^*\|^2-\zeta_k\|x_{k-1}-\hbar^*\|^2+2\zeta_k\|x_k-x_{k-1}\|^2.\end{aligned} \tag{51}$$

By using expressions (50) and (19) in the abovementioned formula, we may deduce that

$$\lim_{k\longrightarrow+\infty}\|\nu_k-\hbar^*\|=l. \tag{52}$$

Thus, we have

$$\begin{aligned}\|x_{k+1}-\hbar^*\|^2 &\le \|\nu_k-\hbar^*\|^2-\|\nu_k-y_k\|^2-\|x_{k+1}-y_k\|^2\\ &\quad +\frac{\left(2-\sqrt{2}-\phi\right)\beth_k\varkappa\left(\|\nu_k-y_k\|^2+\|x_{k+1}-y_k\|^2\right)}{\beth_{k+1}}.\end{aligned} \tag{53}$$

By using expressions (51) and (53), we obtain

$$\begin{aligned}\|x_{k+1}-\hbar^*\|^2 &\le (1+\zeta_k)\|x_k-\hbar^*\|^2-\zeta_k\|x_{k-1}-\hbar^*\|^2+2\zeta_k\|x_k\\ &\quad -x_{k-1}\|^2-\left(1-\frac{\left(2-\sqrt{2}-\phi\right)\varkappa\beth_k}{\beth_{k+1}}\right)\|\nu_k-y_k\|^2\\ &\quad -\left(1-\frac{\left(2-\sqrt{2}-\phi\right)\varkappa\beth_k}{\beth_{k+1}}\right)\|y_k-x_{k+1}\|^2.\end{aligned} \tag{54}$$

Consequently, this implies that

$$\begin{aligned}&\left(1-\frac{\left(2-\sqrt{2}-\phi\right)\varkappa\beth_k}{\beth_{k+1}}\right)\|\nu_k-y_k\|^2\\ &\quad +\left(1-\frac{\left(2-\sqrt{2}-\phi\right)\varkappa\beth_k}{\beth_{k+1}}\right)\|y_k-x_{k+1}\|^2\le\|x_k-\hbar^*\|^2\\ &\quad -\|x_{k+1}-\hbar^*\|^2+\zeta_k\left(\|x_k-\hbar^*\|^2-\|x_{k-1}-\hbar^*\|^2\right)\\ &\quad +2\zeta_k\|x_k-x_{k-1}\|^2.\end{aligned} \tag{55}$$

By taking the limit as $k\longrightarrow+\infty$ in expression (55), we obtain

$$\lim_{k\longrightarrow+\infty}\|\nu_k-y_k\|=\lim_{k\longrightarrow+\infty}\|y_k-x_{k+1}\|=0. \tag{56}$$

Thus, expressions (52) and (56) give that

$$\lim_{k\longrightarrow+\infty}\|y_k-\hbar^*\|=l. \tag{57}$$

By using expressions (50), (52), and (57), so that the sequences $\{x_k\}$, $\{\nu_k\}$, and $\{y_k\}$ are bounded, therefore $\{x_k\}$, $\{\nu_k\}$, and $\{y_k\}$ exist. Thus, $\lim_{k\longrightarrow+\infty}\|x_k-\hbar^*\|^2$, $\lim_{k\longrightarrow+\infty}\|y_k-\hbar^*\|^2$, $\lim_{k\longrightarrow+\infty}\|\nu_k-\hbar^*\|^2$. Following that, we will show that the sequence $\{x_k\}$ weakly converges to $\hbar^*$. As a result, all sequences $\{x_k\}$, $\{\nu_k\}$, and $\{y_k\}$ are bounded. We now demonstrate that each sequential weak cluster point in the sequence $\{x_k\}$ is in $\mathrm{sol}(\Gamma,\Xi)$. Consider that z is a weak cluster point of $\{x_k\}$, which means that there is a subsequence of $\{x_k\}$ that is weakly convergent to z. Then, $z\in\Xi$,$\{y_{k_m}\}$ is also weakly convergent to z.Now let demonstrate that$z\in\mathrm{sol}(\Gamma,\Xi)$. We have obtained the following by combining Lemma 9 with expressions (39) and (35):

$$\begin{aligned}\beth_{k_m}\Gamma\left(y_{k_m},y\right) &\ge \beth_{k_m}\Gamma\left(y_{k_m},x_{k_m+1}\right)+\left\langle \nu_{k_m}-x_{k_m+1},y-x_{k_m+1}\right\rangle\\ &\ge \beth_{k_m}\Gamma\left(\nu_{k_m},x_{k_{m+1}}\right)-\beth_{k_m}\Gamma\left(\nu_{k_m},y_{k_m}\right)\\ &\quad -\frac{\left(2-\sqrt{2}-\phi\right)\varkappa\beth_{k_m}}{2\beth_{k_m+1}}\left\|y_{k_m}-\nu_{k_m}\right\|^2\\ &\quad -\frac{\left(2-\sqrt{2}-\phi\right)\varkappa\beth_{k_m}}{2\beth_{k_m+1}}\left\|y_{k_m}-x_{k_m+1}\right\|^2\\ &\quad +\left\langle \nu_{k_m}-x_{k_m+1},y-x_{k_m+1}\right\rangle\ge\left\langle \nu_{k_m}-y_{k_m},x_{k_m+1}-y_{k_m}\right\rangle\\ &\quad -\frac{\left(2-\sqrt{2}-\phi\right)\varkappa\beth_{k_m}}{2\beth_{k_m+1}}\left\|y_{k_m}-\nu_{k_m}\right\|^2\\ &\quad -\frac{\left(2-\sqrt{2}-\phi\right)\varkappa\beth_{k_m}}{2\beth_{k_m+1}}\left\|y_{k_m}-x_{k_m+1}\right\|^2\\ &\quad +\left\langle \nu_{k_m}-x_{k_m+1},y-x_{k_m+1}\right\rangle,\end{aligned} \tag{58}$$

where y is any member of Π_k. The use of expression (56) and the boundedness of the sequence $\{x_k\}$ implies that the right-hand side of the last inequality is convergent to zero. By using the condition $\Gamma 4$ and $y_{k_m}\rightharpoonup z$, we have $\beth_{k_m}\ge\beth>0$ such as

$$0\le\limsup_{m\longrightarrow+\infty}\left(y_{k_m},y\right)\le\Gamma(z,y),\quad \forall y\in\Pi_k. \tag{59}$$

Since Ξ is a subset of half-space Π_k, it follows that $\Gamma(z,y)\ge 0,\forall y\in\Xi$. This proves that $z\in\mathrm{sol}(\Gamma,\Xi)$. Thus, Lemma

STEP 0: Choose $_0 > 0, x_{-1}, x_0 \in \Pi, \zeta \in (0,1), \varkappa \in (0,1)$, $\phi \in (0, 2-\sqrt{2})$ with a sequence $\{\psi_k\} \subset [0,+\infty)$ such that $\sum_{k=0}^{+\infty} \psi_k < +\infty$.

Moreover, choose a non-negative real sequence $\{p_k\}$ such that $\sum_{k=1}^{+\infty} p_k < +\infty$ and ζ_k such that $0 \le \zeta_k \le \beta_k$ such that

$$\beta_k = \begin{cases} \min\{\zeta, (\psi_k/\|x_k - x_{k-1}\|)\} & \text{if } x_k \neq x_{k-1}, \\ \zeta & \text{otherwise.} \end{cases}$$

STEP 1: Compute

$$y_k = \arg\min_{y\in\Xi}\{\beth_k \Gamma(\nu_k, y) + 1/2\|\nu_k - y\|^2\}, \text{wwhere } \nu_k = x_k + \zeta_k(x_k - x_{k-1}).$$

STEP 2: Compute

$$x_{k+1} = \arg\min_{y\in\Xi}\{\beth_k \Gamma(y_k, y) + 1/2\|\nu_k - y\|^2\}.$$

STEP 3: Compute

$$\beth_{k+1} = \begin{cases} \min\{\beth_k + p_k, ((2-\sqrt{2}-\phi)\varkappa\|\nu_k - y_k\|^2 + (2-\sqrt{2}-\phi)\varkappa\|x_{k+1} - y_k\|^2/2[\Gamma(\nu_k, x_{k+1}) - \Gamma(\nu_k, y_k) - \Gamma(y_k, x_{k+1})])\} \\ \text{if } \Gamma(\nu_k, x_{k+1}) - \Gamma(\nu_k, y_k) - \Gamma(y_k, x_{k+1}) > 0, \\ \beth_k + p_k, \text{otherwise.} \end{cases}$$

STEP 4: If $y_k = \nu_k$, then complete the computation. Otherwise, set $k := k+1$ and go back **STEP 1**.

Algorithm 4

4 assures that $\{\nu_k\}, \{x_k\}$, and $\{y_k\}$ converge weakly to $\hbar^*$ as $k \longrightarrow +\infty$. □

We now present two iterative methods based on a monotone and nonmonotone variable step size rule and two strongly convex minimization problems without the need for subgradient methods. The following is a description of the second major result.

4. Results to Solve the Fixed Point Problem and Variational Inequalities

In this section, we solve fixed point problems and variational inequalities using the results from our main results. Expressions (6) and (8) are employed to obtain the following conclusions. All the methods are based on our main findings, which are interpreted as follows.

Corollary 12. *Assume that $\mathfrak{I}_1 : \Xi \longrightarrow \Pi$ is a pseudomonotone, weakly continuous, and L -Lipschitz continuous operator and the solution set $sol(\mathfrak{I}_1, \Xi) \neq \varnothing$. Choose $\beth_0 > 0$, $x_{-1}, x_0 \in \Pi, \zeta \in (0,1), \varkappa \in (0,1), \phi \in (0, 2-\sqrt{2})$ with a sequence $\{\psi_k\} \subset [0,+\infty)$ such that*

$$\sum_{k=0}^{+\infty} \psi_k < +\infty. \tag{60}$$

Moreover, choose ζ_k such that $0 \le \zeta_k \le \beta_k$ such that

$$\beta_k = \begin{cases} \min\left\{\zeta, \dfrac{\psi_k}{\|x_k - x_{k-1}\|}\right\}, & \text{if } x_k \neq x_{k-1}, \\ \zeta, & \text{otherwise.} \end{cases} \tag{61}$$

First, we have to compute

$$\begin{aligned} \nu_k &= x_k + \zeta_k(x_k - x_{k-1}), \\ y_k &= P_\Xi(\nu_k - \hat{\mathrm{u}}_k \mathfrak{I}_1(\nu_k)). \end{aligned} \tag{62}$$

Having x_{k-1}, x_k, y_k with

$$\Pi_k = \{z \in \Pi : \langle \nu_k - \beth_k \mathfrak{I}_1(\nu_k) - y_k, z - y_k \rangle \le 0\}, \quad \text{for each } k \ge 0. \tag{63}$$

Compute

$$x_{k+1} = P_{\Pi_k}(\nu_k - \beth_k \mathfrak{I}_1(y_k)). \tag{64}$$

Update the step size in the following way:

$$_{k+1} = \begin{cases} \min\left\{\beth_k, \dfrac{\left(2-\sqrt{2}-\phi\right)\varkappa\|\nu_k - y_k\|^2 + \left(2-\sqrt{2}-\phi\right)\varkappa\|x_{k+1} - y_k\|^2}{2\langle \mathfrak{I}_1(\nu_k) - \mathfrak{I}_1(y_k), x_{k+1} - y_k\rangle}\right\}, & \text{if } \langle \mathfrak{I}_1(\nu_k) - \mathfrak{I}_1(y_k), x_{k+1} - y_k \rangle > 0, \\ \beth_k, \quad \text{otherwise.} \end{cases} \tag{65}$$

Then, the sequences $\{x_k\}$ converge weakly to $\hbar^* \in \mathrm{sol}(\mathfrak{I}_1, \Xi)$.

Corollary 13. *Assume that $\mathfrak{I}_1 : \Xi \longrightarrow \Pi$ is a pseudomonotone, weakly continuous, and L -Lipschitz continuous operator and the solution set $sol(\mathfrak{I}_1, \Xi) \neq \varnothing$. Choose $\beth_0 > 0$, $x_{-1}, x_0 \in \Pi$,$\zeta \in (0, 1)$,$\varkappa \in (0, 1)$,$\phi \in (0, 2 - \sqrt{2})$ with a sequence $\{\psi_k\} \subset [0,+\infty)$ such that*

$$\sum_{k=0}^{+\infty} \psi_k < +\infty. \tag{66}$$

Moreover, choose a non-negative real sequence $\{p_k\}$ such that $\sum_{k=1}^{+\infty} p_k < +\infty$ and ζ_k such that $0 \leq \zeta_k \leq \beta_k$ such that

$$\beta_k = \begin{cases} \min\left\{\zeta, \dfrac{\psi_k}{\|x_k - x_{k-1}\|}\right\}, & \text{if } x_k \neq x_{k-1}, \\ \zeta, & \text{otherwise.} \end{cases} \tag{67}$$

First, we have to compute

$$\begin{aligned} \nu_k &= x_k + \zeta_k(x_k - x_{k-1}), \\ y_k &= P_\Xi(\nu_k - \hat{\mathrm{u}}_k \mathfrak{I}_1(\nu_k)). \end{aligned} \tag{68}$$

Having x_{k-1},x_k,y_k with

$$\Pi_k = \{z \in \Pi : \langle \nu_k - \beth_k \mathfrak{I}_1(\nu_k) - y_k, z - y_k \rangle \leq 0\}, \quad \text{for each } k \geq 0. \tag{69}$$

Compute

$$x_{k+1} = P_{\Pi_k}(\nu_k - \beth_k \mathfrak{I}_1(y_k)). \tag{70}$$

Update the step size in the following way:

$$_{k+1} = \begin{cases} \min\left\{\beth_k + p_k, \dfrac{\left(2 - \sqrt{2} - \phi\right)\varkappa\|\nu_k - y_k\|^2 + \left(2 - \sqrt{2} - \phi\right)\varkappa\|x_{k+1} - y_k\|^2}{2\langle \mathfrak{I}_1(\nu_k) - \mathfrak{I}_1(y_k), x_{k+1} - y_k\rangle}\right\}, & \text{if } \langle \mathfrak{I}_1(\nu_k) - \mathfrak{I}_1(y_k), x_{k+1} - y_k\rangle > 0, \\ \beth_k + p_k, \quad \text{otherwise.} \end{cases} \tag{71}$$

Then, the sequences $\{x_k\}$ converge weakly to $\hbar^* \in \mathrm{sol}(\mathfrak{I}_1, \Xi)$.

Corollary 14. *Assume that $\mathfrak{I}_1 : \Xi \longrightarrow \Pi$ is a pseudomonotone, weakly continuous, and L -Lipschitz continuous operator and the solution set $sol(\mathfrak{I}_1, \Xi) \neq \varnothing$. Choose $\beth_0 > 0$, $x_{-1}, x_0 \in \Pi$,$\zeta \in (0, 1)$,$\varkappa \in (0, 1)$,$\phi \in (0, 2 - \sqrt{2})$ with a sequence $\{\psi_k\} \subset [0,+\infty)$ such that*

$$\sum_{k=0}^{+\infty} \psi_k < +\infty. \tag{72}$$

Moreover, choose ζ_k such that $0 \leq \zeta_k \leq \beta_k$ such that

$$\beta_k = \begin{cases} \min\left\{\zeta, \dfrac{\psi_k}{\|x_k - x_{k-1}\|}\right\}, & \text{if } x_k \neq x_{k-1}, \\ \zeta, & \text{otherwise.} \end{cases} \tag{73}$$

First, we have to compute

$$\begin{aligned} \nu_k &= x_k + \zeta_k(x_k - x_{k-1}), \\ y_k &= P_\Xi(\nu_k - \beth_k \mathfrak{I}_1(\nu_k)), \\ x_{k+1} &= P_\Xi(\nu_k - \beth_k \mathfrak{I}_1(y_k)). \end{aligned} \tag{74}$$

Update the step size in the following way:

$$_{k+1} = \begin{cases} \min\left\{\beth_k, \dfrac{\left(2 - \sqrt{2} - \phi\right)\varkappa\|\nu_k - y_k\|^2 + \left(2 - \sqrt{2} - \phi\right)\varkappa\|x_{k+1} - y_k\|^2}{2\langle \mathfrak{I}_1(\nu_k) - \mathfrak{I}_1(y_k), x_{k+1} - y_k\rangle}\right\}, & \text{if } \langle \mathfrak{I}_1(\nu_k) - \mathfrak{I}_1(y_k), x_{k+1} - y_k\rangle > 0, \\ \beth_k, \quad \text{otherwise.} \end{cases} \tag{75}$$

Then, the sequences $\{x_k\}$ converge weakly to $\hbar^* \in \mathrm{sol}(\mathfrak{I}_1, \Xi)$.

Corollary 15. *Assume that $\mathfrak{I}_1 : \Xi \longrightarrow \Pi$ is a pseudomonotone, weakly continuous, and L -Lipschitz continuous operator and the solution set $sol(\mathfrak{I}_1, \Xi) \neq \emptyset$. Choose $\beth_0 > 0$, $x_{-1}, x_0 \in \Pi, \zeta \in (0,1), \varkappa \in (0,1), \phi \in (0, 2-\sqrt{2})$ with a sequence $\{\psi_k\} \subset [0,+\infty)$ such that*

$$\sum_{k=0}^{+\infty} \psi_k < +\infty. \quad (76)$$

Moreover, choose a non-negative real sequence $\{p_k\}$ such that $\sum_{k=1}^{+\infty} p_k < +\infty$ and ζ_k such that $0 \le \zeta_k \le \beta_k$ such that

$$\beta_k = \begin{cases} \min\left\{\zeta, \dfrac{\psi_k}{\|x_k - x_{k-1}\|}\right\}, & \text{if } x_k \neq x_{k-1}, \\ \zeta, & \text{otherwise.} \end{cases} \quad (77)$$

First, we have to compute

$$\begin{aligned} \nu_k &= x_k + \zeta_k (x_k - x_{k-1}), \\ y_k &= P_\Xi(\nu_k - \beth_k \mathfrak{I}_1(\nu_k)), \\ x_{k+1} &= P_\Xi(\nu_k - \beth_k \mathfrak{I}_1(y_k)). \end{aligned} \quad (78)$$

Update the step size in the following way:

$${}_{k+1} = \begin{cases} \min\left\{\beth_k + p_k, \dfrac{\left(2-\sqrt{2}-\phi\right)\varkappa\|\nu_k - y_k\|^2 + \left(2-\sqrt{2}-\phi\right)\varkappa\|x_{k+1} - y_k\|^2}{2\langle \mathfrak{I}_1(\nu_k) - \mathfrak{I}_1(y_k), x_{k+1} - y_k\rangle}\right\}, & \text{if } \langle \mathfrak{I}_1(\nu_k) - \mathfrak{I}_1(y_k), x_{k+1} - y_k\rangle > 0, \\ \beth_k + p_k, & \text{otherwise.} \end{cases} \quad (79)$$

Then, the sequences $\{x_k\}$ converge weakly to $\hbar^* \in \mathrm{sol}(\mathfrak{I}_1, \Xi)$.

Corollary 16. *Assume that $\mathfrak{I}_2 : \Xi \longrightarrow \Pi$ is a κ-strict pseudocontraction, weakly continuous, and L-Lipschitz continuous operator and the solution set $sol(\mathfrak{I}_2, \Xi) \neq \emptyset$. Choose $\beth_0 > 0$, $x_{-1}, x_0 \in \Pi, \zeta \in (0,1), \varkappa \in (0,1), \phi \in (0, 2-\sqrt{2})$ with a sequence $\{\psi_k\} \subset [0,+\infty)$ such that*

$$\sum_{k=0}^{+\infty} \psi_k < +\infty. \quad (80)$$

Moreover, choose ζ_k such that $0 \le \zeta_k \le \beta_k$ such that

$$\beta_k = \begin{cases} \min\left\{\zeta, \dfrac{\psi_k}{\|x_k - x_{k-1}\|}\right\}, & \text{if } x_k \neq x_{k-1}, \\ \zeta, & \text{otherwise.} \end{cases} \quad (81)$$

Compute

$$\begin{aligned} \nu_k &= x_k + \zeta_k (x_k - x_{k-1}), \\ y_k &= P_\Xi[\nu_k - \beth_k(\nu_k - \mathfrak{I}_2(\nu_k))]. \end{aligned} \quad (82)$$

Having x_{k-1}, x_k, y_k, with

$$\Pi_k = \{z \in \mathscr{E} : \langle (1-\beth_k)\nu_k + \beth_k \mathfrak{I}_2(\nu_k) - y_k, z - y_k\rangle \le 0\}. \quad (83)$$

Compute

$$x_{k+1} = P_{\Pi_k}[\nu_k - \beth_k(y_k - \mathfrak{I}_2(y_k))]. \quad (84)$$

The step size rule for the next iteration is evaluated as follows:

$${}_{k+1} = \begin{cases} \min\left\{\beth_k, \dfrac{\left(2-\sqrt{2}-\phi\right)\varkappa\|\nu_k - y_k\|^2 + \left(2-\sqrt{2}-\phi\right)\varkappa\|x_{k+1} - y_k\|^2}{2\langle (\nu_k - y_k) - [\mathfrak{I}_2(\nu_k) - \mathfrak{I}_2(y_k)], x_{k+1} - y_k\rangle}\right\}, & \text{if } \langle (\nu_k - y_k) - [\mathfrak{I}_2(\nu_k) - \mathfrak{I}_2(y_k)], x_{k+1} - y_k\rangle > 0, \\ \beth_k, & \text{otherwise.} \end{cases} \quad (85)$$

Then, the sequence $\{x_k\}$ converges weakly to $\hbar^* \in \mathrm{sol}(\mathfrak{I}_1, \Xi)$.

Corollary 17. *Assume that $\mathfrak{I}_2 : \Xi \longrightarrow \Pi$ is a κ -strict pseudocontraction, weakly continuous, and L-Lipschitz continuous operator and the solution set $sol(\mathfrak{I}_2, \Xi) \neq \varnothing$. Choose $\daleth_0 > 0$, $x_{-1}, x_0 \in \Pi, \zeta \in (0, 1), \varkappa \in (0, 1)$, $\phi \in (0, 2 - \sqrt{2})$ with a sequence $\{\psi_k\} \subset [0,+\infty)$ such that*

$$\sum_{k=0}^{+\infty} \psi_k < +\infty. \tag{86}$$

Moreover, choose a non-negative real sequence $\{p_k\}$ such that $\sum_{k=1}^{+\infty} p_k < +\infty$ and ζ_k such that $0 \leq \zeta_k \leq \beta_k$ such that

$$\beta_k = \begin{cases} \min\left\{\zeta, \dfrac{\psi_k}{\|x_k - x_{k-1}\|}\right\}, & \text{if } x_k \neq x_{k-1}, \\ \zeta, & \text{otherwise.} \end{cases} \tag{87}$$

Compute

$$\begin{aligned} v_k &= x_k + \zeta_k (x_k - x_{k-1}), \\ y_k &= P_\Xi[v_k - \daleth_k (v_k - \mathfrak{I}_2(v_k))]. \end{aligned} \tag{88}$$

Having x_{k-1}, x_k, y_k, with

$$\Pi_k = \{z \in \mathscr{E} : \langle (1 - \daleth_k) v_k + \daleth_k \mathfrak{I}_2(v_k) - y_k, z - y_k \rangle \leq 0\}. \tag{89}$$

Compute

$$x_{k+1} = P_{\Pi_k}[v_k - \daleth_k (y_k - \mathfrak{I}_2(y_k))]. \tag{90}$$

The step size rule for the next iteration is evaluated as follows:

$$_{k+1} = \begin{cases} \min\left\{\daleth_k + p_k, \dfrac{\left(2 - \sqrt{2} - \phi\right)\varkappa\|v_k - y_k\|^2 + \left(2 - \sqrt{2} - \phi\right)\varkappa\|x_{k+1} - y_k\|^2}{2\langle (v_k - y_k) - [\mathfrak{I}_2(v_k) - \mathfrak{I}_2(y_k)], x_{k+1} - y_k\rangle}\right\}, & \text{if } \langle (v_k - y_k) - [\mathfrak{I}_2(v_k) - \mathfrak{I}_2(y_k)], x_{k+1} - y_k\rangle > 0, \\ \daleth_k + p_k, \quad \text{otherwise.} \end{cases} \tag{91}$$

Then, the sequence $\{x_k\}$ converges weakly to $\hbar^* \in \mathrm{sol}(\mathfrak{I}_2, \Xi)$.

Corollary 18. *Assume that $\mathfrak{I}_2 : \Xi \longrightarrow \Pi$ is a κ-strict pseudocontraction, weakly continuous, and L-Lipschitz continuous operator and the solution set $sol(\mathfrak{I}_2, \Xi) \neq \varnothing$. Choose $\daleth_0 > 0$, $x_{-1}, x_0 \in \Pi, \zeta \in (0, 1)$, $\varkappa \in (0, 1)$, $\phi \in (0, 2 - \sqrt{2})$ with a sequence $\{\psi_k\} \subset [0,+\infty)$ such that*

$$\sum_{k=0}^{+\infty} \psi_k < +\infty. \tag{92}$$

Moreover, choose ζ_k such that $0 \leq \zeta_k \leq \beta_k$ such that

$$\beta_k = \begin{cases} \min\left\{\zeta, \dfrac{\psi_k}{\|x_k - x_{k-1}\|}\right\}, & \text{if } x_k \neq x_{k-1}, \\ \zeta, & \text{otherwise.} \end{cases} \tag{93}$$

Compute

$$\begin{aligned} v_k &= x_k + \zeta_k (x_k - x_{k-1}), \\ y_k &= P_\Xi[v_k - \daleth_k (v_k - \mathfrak{I}_2(v_k))], \\ x_{k+1} &= P_\Xi[v_k - \daleth_k (y_k - \mathfrak{I}_2(y_k))]. \end{aligned} \tag{94}$$

$$_{k+1} = \begin{cases} \min\left\{\daleth_k, \dfrac{\left(2 - \sqrt{2} - \phi\right)\varkappa\|v_k - y_k\|^2 + \left(2 - \sqrt{2} - \phi\right)\varkappa\|x_{k+1} - y_k\|^2}{2\langle (v_k - y_k) - [\mathfrak{I}_2(v_k) - \mathfrak{I}_2(y_k)], x_{k+1} - y_k\rangle}\right\}, & \text{if } \langle (v_k - y_k) - [\mathfrak{I}_2(v_k) - \mathfrak{I}_2(y_k)], x_{k+1} - y_k\rangle > 0, \\ \daleth_k, \quad \text{otherwise.} \end{cases} \tag{95}$$

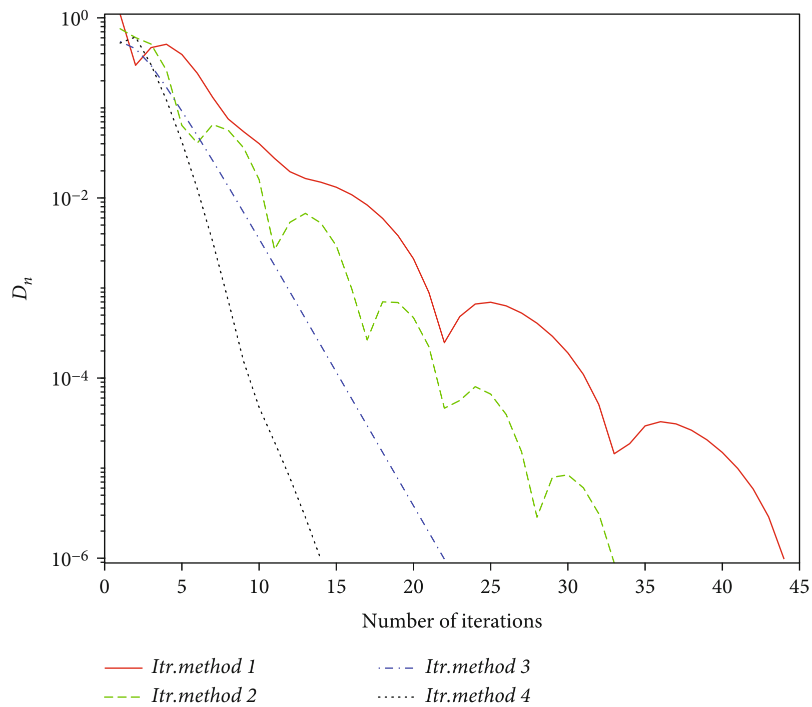

FIGURE 1: All methods are compared computationally while $x_0 = (0, 0, 0, 0, 0)^T$.

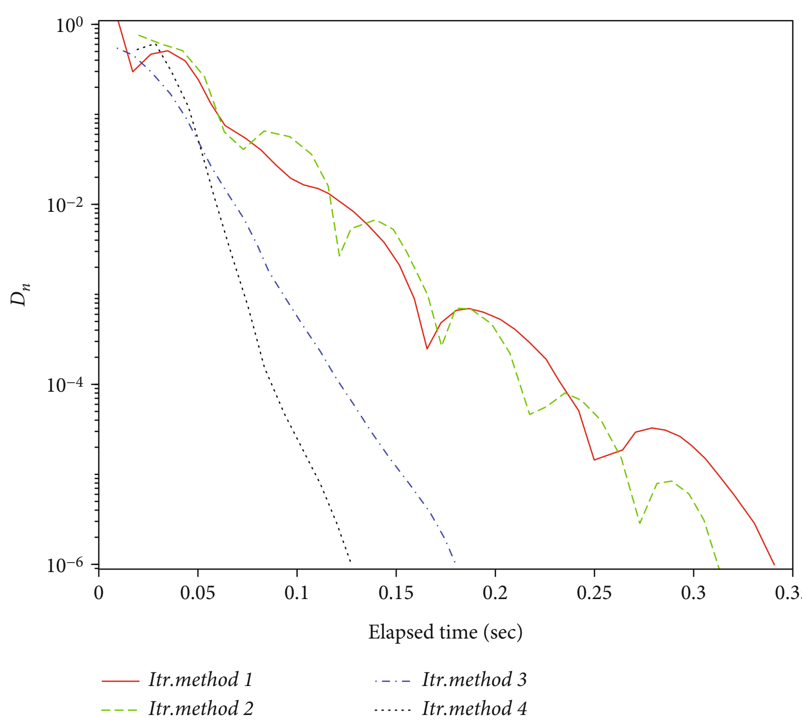

FIGURE 2: All methods are compared computationally while $x_0 = (0, 0, 0, 0, 0)^T$.

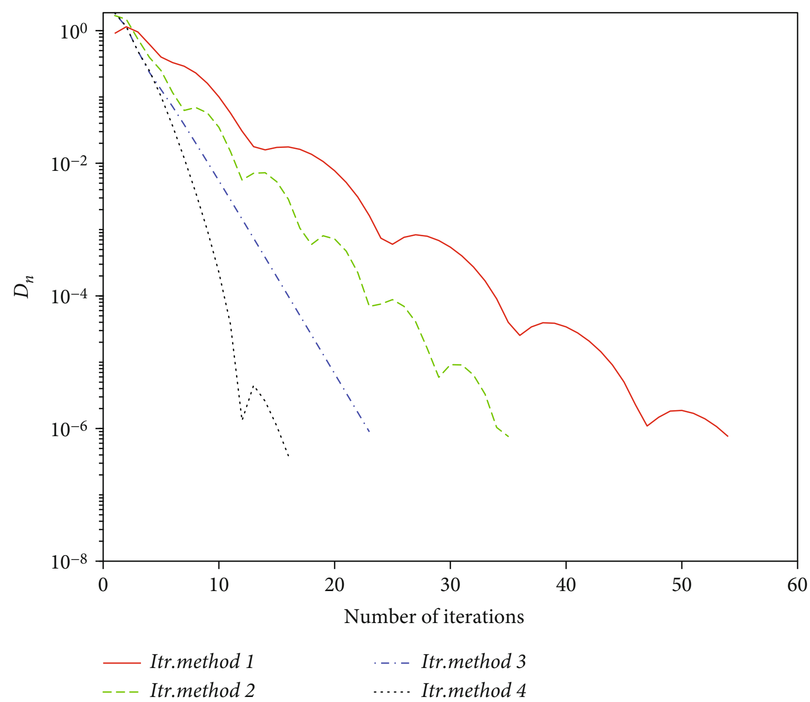

FIGURE 3: All methods are compared computationally while $x_0 = (1, 2, 1, 2, 1)^T$.

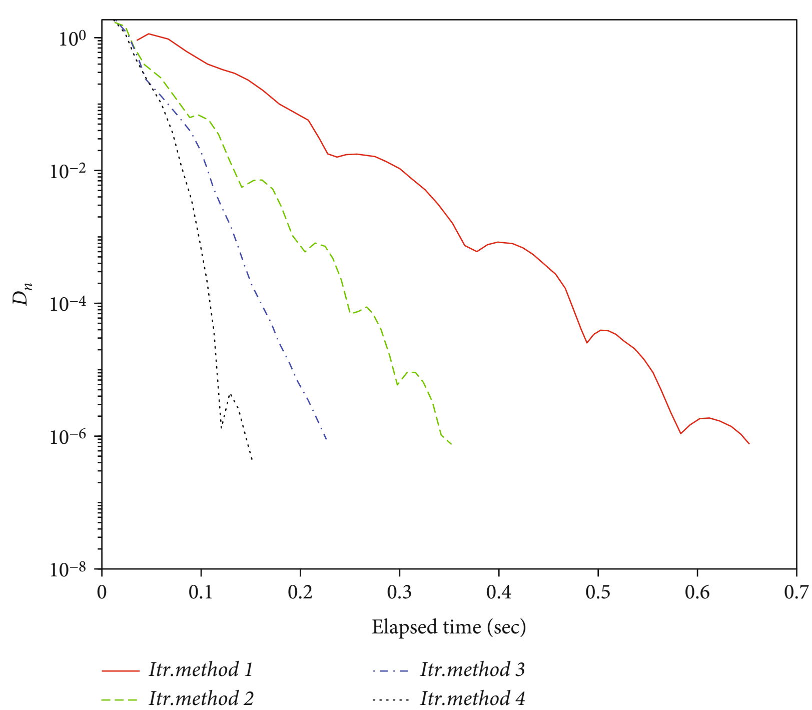

FIGURE 4: All methods are compared computationally while $x_0 = (1, 2, 1, 2, 1)^T$.

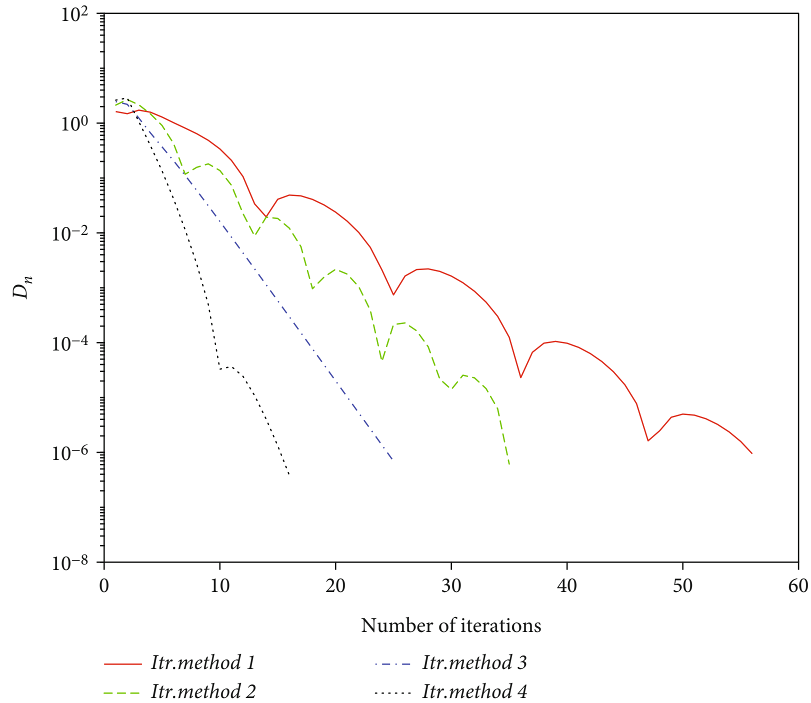

FIGURE 5: All methods are compared computationally while $x_0 = (1, 2, 3, -4, 5)^T$.

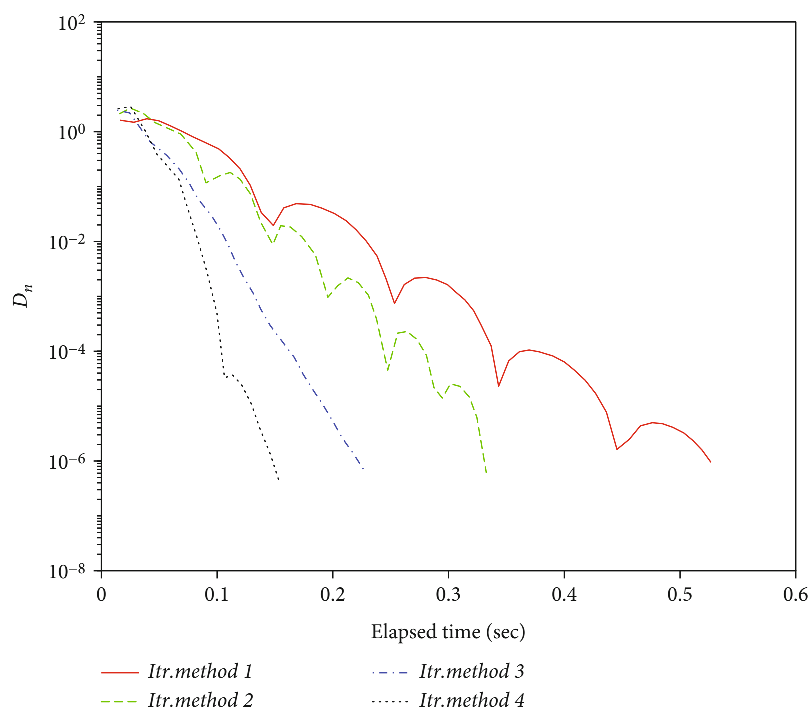

FIGURE 6: All methods are compared computationally while $x_0 = (1, 2, 3, -4, 5)^T$.

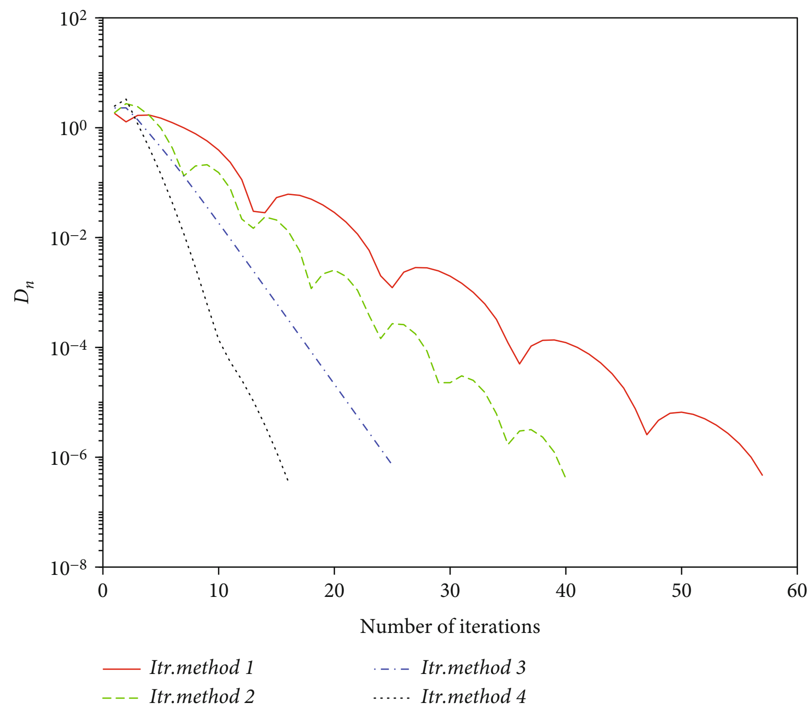

FIGURE 7: All methods are compared computationally while $x_0 = (2,-1,3,-4,5)^T$.

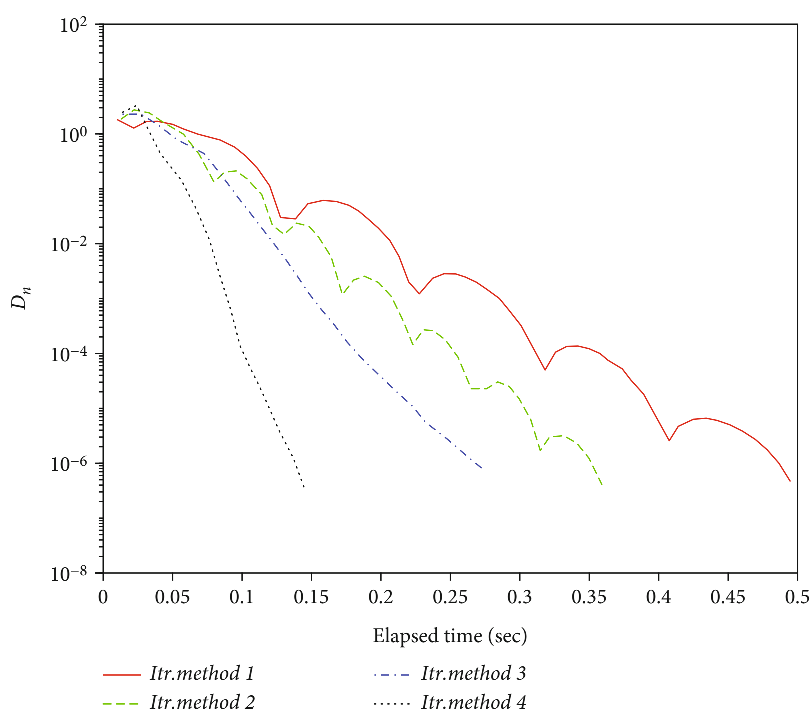

FIGURE 8: All methods are compared computationally while $x_0 = (2,-1,3,-4,5)^T$.

Table 1: All methods' numerical values for Figures 1–8.

x_0	Number of iterations		Execution time in seconds	
	Algorithm 1	Algorithm 2	Algorithm 1	Algorithm 2
$(0, 0, 0, 0, 0)^T$	22	14	0.180260200000000	0.127609500000000
$(1, 2, 1, 2, 1)^T$	23	16	0.226162200000000	0.152221400000000
$(1, 2, 3, -4, 5)^T$	25	16	0.226667900000000	0.154296300000000
$(2, -1, 3, -4, 5)^T$	25	16	0.275009100000000	0.144512100000000

Table 2: All methods' numerical values for Figures 1–8.

x_0	Number of iterations		Execution time in seconds	
	Algorithm 1 n [22]	Algorithm 2 in [35]	Algorithm 1 in [22]	Algorithm 2 in [35]
$(0, 0, 0, 0, 0)^T$	44	33	0.340814700000000	0.312906600000000
$(1, 2, 1, 2, 1)^T$	54	35	0.652377900000000	0.351818000000000
$(1, 2, 3, -4, 5)^T$	56	35	0.526694900000000	0.332574400000000
$(2, -1, 3, -4, 5)^T$	57	40	0.494837300000000	0.359039600000000

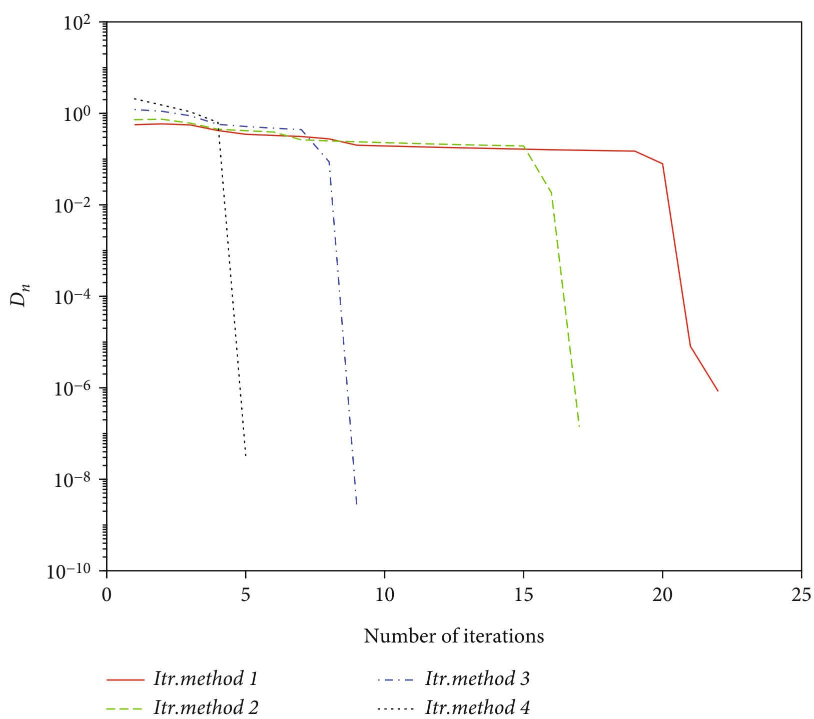

Figure 9: All methods are compared computationally while $x_0 = (2, 3, 2, 5, 2)^T$.

The step size rule for the next iteration is evaluated as follows:

Then, the sequence $\{x_k\}$ converges weakly to $\hbar^* \in \mathrm{sol}(\mathfrak{I}_2, \Xi)$.

Corollary 19. *Assume that $\mathfrak{I}_2 : \Xi \longrightarrow \Pi$ is a κ-strict pseudo-contraction, weakly continuous, and L-Lipschitz continuous operator and the solution set $sol(\mathfrak{I}_2, \Xi) \neq \varnothing$. Choose $\beth_0 > 0$, $x_{-1}, x_0 \in \Pi, \zeta \in (0, 1), \varkappa \in (0, 1)$, $\phi \in (0, 2 - \sqrt{2})$ with a sequence $\{\psi_k\} \subset [0, +\infty)$ such that*

$$\sum_{k=0}^{+\infty} \psi_k < +\infty. \tag{96}$$

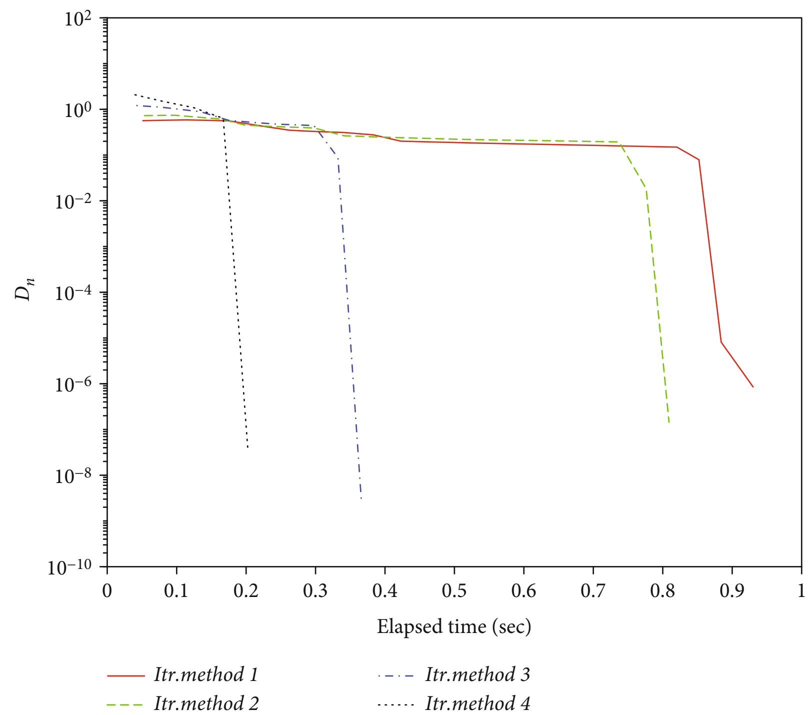

FIGURE 10: All methods are compared computationally while $x_0 = (2, 3, 2, 5, 2)^T$.

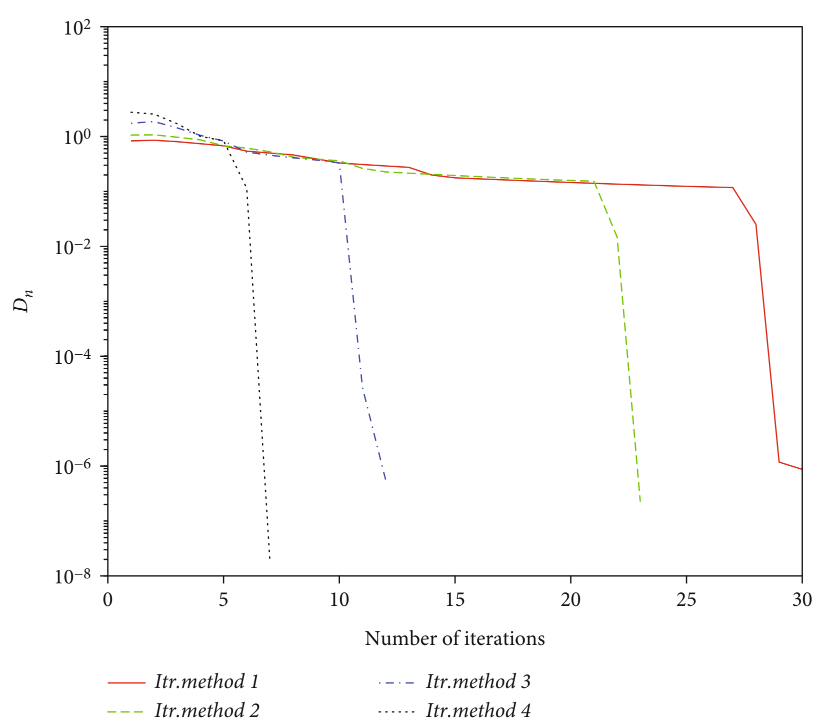

FIGURE 11: All methods are compared computationally while $x_0 = (1, 3, 5, 4, 7)^T$.

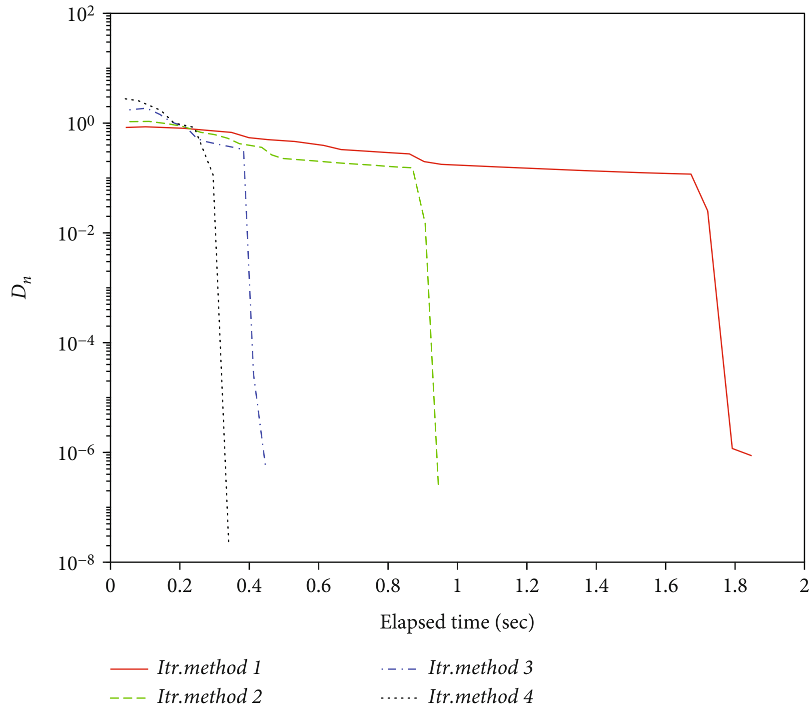

FIGURE 12: All methods are compared computationally while $x_0 = (1, 3, 5, 4, 7)^T$.

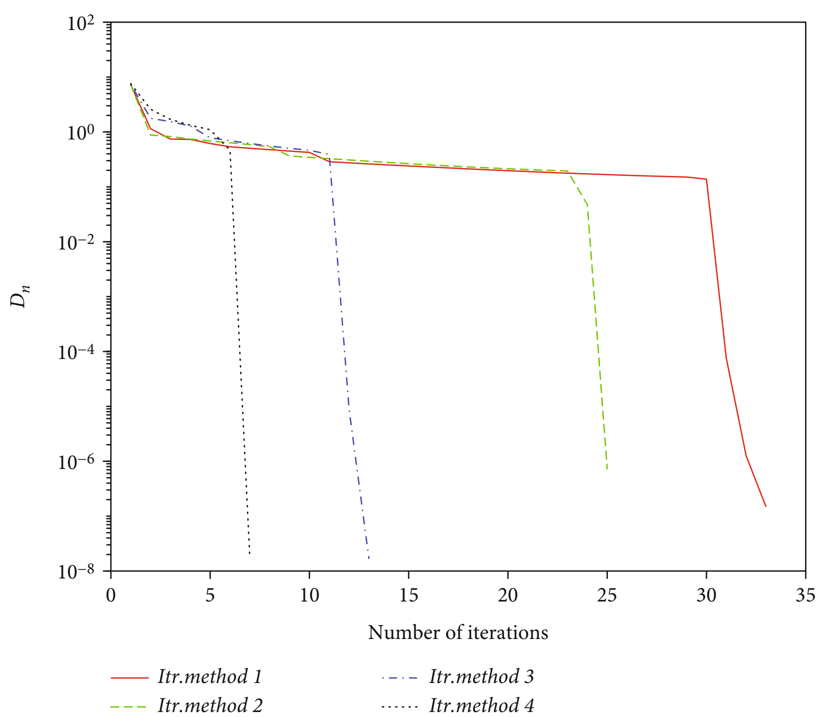

FIGURE 13: All methods are compared computationally while $x_0 = (2, -3, 5, 9, -5)^T$.

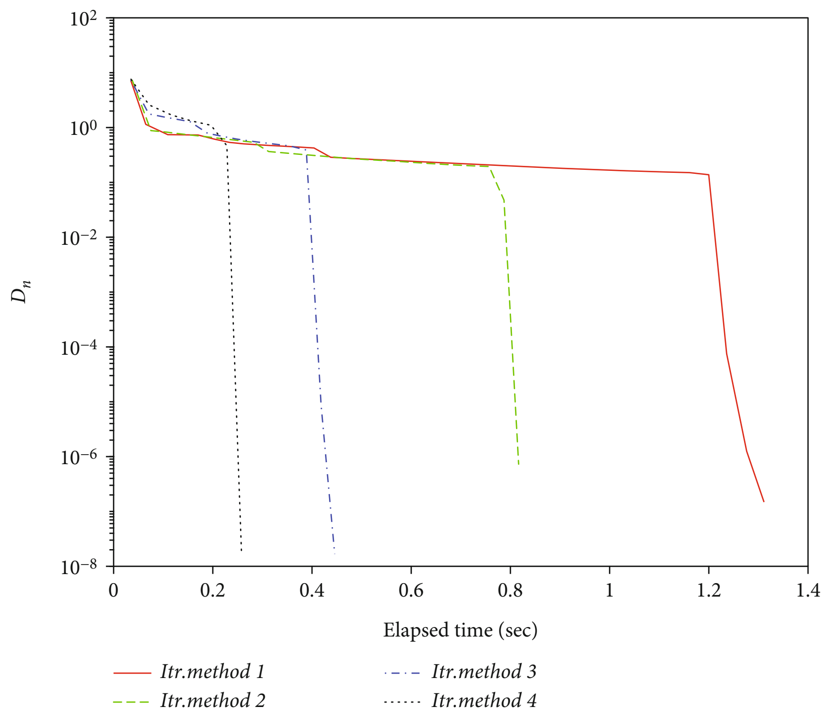

FIGURE 14: All methods are compared computationally while $x_0 = (2,-3,5,9,-5)^T$.

TABLE 3: All methods' numerical values for Figures 9–14.

x_0	Number of iterations		Execution time in seconds	
	Algorithm 1 in [22]	Algorithm 2 in [35]	Algorithm 1 in [22]	Algorithm 2 in [35]
$(2,3,2,5,2)^T$	22	17	0.9305202000	0.808993700
$(1,3,5,4,7)^T$	30	23	1.8477304000	0.945203900
$(2,-3,5,9,-5)^T$	33	25	1.3113005000	0.816565900

TABLE 4: All methods' numerical values for Figures 9–14.

x_0	Number of iterations		Execution time in seconds	
	Algorithm 1	Algorithm 2	Algorithm 1	Algorithm 2
$(2,3,2,5,2)^T$	09	05	0.366167800000000	0.202759300000000
$(1,3,5,4,7)^T$	12	07	0.446752600000000	0.341142700000000
$(2,-3,5,9,-5)^T$	13	07	0.445763600000000	0.257909300000000

Moreover, choose a non-negative real sequence $\{p_k\}$ such that $\sum_{k=1}^{+\infty} p_k < +\infty$ and ζ_k such that $0 \le \zeta_k \le \beta_k$ such that

$$\beta_k = \begin{cases} \min\left\{\zeta, \dfrac{\psi_k}{\|x_k - x_{k-1}\|}\right\}, & \text{if } x_k \neq x_{k-1}, \\ \zeta, & \text{otherwise.} \end{cases} \tag{97}$$

Compute

$$\begin{aligned} v_k &= x_k + \zeta_k (x_k - x_{k-1}), \\ y_k &= P_\Xi[v_k - \beth_k(v_k - \mathfrak{I}_2(v_k))], \\ x_{k+1} &= P_\Xi[v_k - \beth_k(y_k - \mathfrak{I}_2(y_k))]. \end{aligned} \tag{98}$$

The step size rule for the next iteration is evaluated as follows:

$$
{}_{k+1} = \begin{cases} \min\left\{ \daleth_k + p_k, \dfrac{\left(2-\sqrt{2}-\phi\right)\varkappa\|\nu_k - y_k\|^2 + \left(2-\sqrt{2}-\phi\right)\varkappa\|x_{k+1}-y_k\|^2}{2\langle(\nu_k - y_k) - [\mathfrak{I}_2(\nu_k) - \mathfrak{I}_2(y_k)], x_{k+1} - y_k\rangle} \right\}, & \text{if } \langle(\nu_k - y_k) - [\mathfrak{I}_2(\nu_k) - \mathfrak{I}_2(y_k)], x_{k+1} - y_k\rangle > 0, \\ \daleth_k + p_k, \quad \text{otherwise.} \end{cases} \tag{99}
$$

Then, the sequence $\{x_k\}$ converges weakly to $\hbar^* \in \mathrm{sol}(\mathfrak{I}_2, \varXi)$.

5. Numerical Illustrations

This section describes a number of numerical experiments conducted to demonstrate the validity of the proposed methods. Some of these numerical experiments provide a thorough understanding of how to select effective control parameters. Some of them demonstrate the advantages of the proposed methods over existing ones in the literature. All MATLAB codes were run in MATLAB 9.5 (R2018b) on an Intel(R) Core(TM) i5-6200 Processor CPU @ 2.30 GHz 2.40 GHz, with 8.00 GB RAM.

Example 20. The first sample problem here is drawn from the Nash-Cournot oligopolistic equilibrium model in [16]. In this example, the bifunction $\varGamma$ can be formulated as having

$$
\varGamma(x, y) = \langle Px + Qy + c, y - x\rangle, \tag{100}
$$

where P, Q, and vector c are defined by

$$
P = \begin{pmatrix} 3.1 & 2 & 0 & 0 & 0 \\ 2 & 3.6 & 0 & 0 & 0 \\ 0 & 0 & 3.5 & 2 & 0 \\ 0 & 0 & 2 & 3.3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}, Q = \begin{pmatrix} 1.6 & 1 & 0 & 0 & 0 \\ 1 & 1.6 & 0 & 0 & 0 \\ 0 & 0 & 1.5 & 1 & 0 \\ 0 & 0 & 1 & 1.5 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}, c = \begin{pmatrix} 1 \\ -2 \\ -1 \\ 2 \\ -1 \end{pmatrix}. \tag{101}
$$

The eigenvalues of the matrix $Q - P$ are as follows: $-2.9050, -2.7808, -1.0000, -0.8950, -0.7192$. As a result, the matrices $Q - P$ and Q are symmetrically negative semidefinite and symmetrically positive semidefinite, respectively. Furthermore, the values for Lipschitz-like parameters are $c_1 = c_2 = 1/2\|P - Q\| = 1.4525$. The constraint set $\varXi \subset \mathbb{R}^M$ is regarded as

$$
\varXi \coloneqq \left\{ x \in \mathbb{R}^M : -2 \le x_i \le 5 \right\}. \tag{102}
$$

The beginning points for these numerical investigations vary, as does the error term $D_k = \|x_{k+1} - x_k\|$. Figures 1–8 and Tables 1 and 2 show several results for the error term 10^{-5}. Consider the following information regarding control settings:

(1) For Algorithm 1 in [22] (in short, *Itr.Method1*), we use

$$
\begin{aligned} \phi &= 0.45, \\ \daleth &= \frac{1}{2c_2 + 8c_1} \end{aligned} \tag{103}
$$

(2) For Algorithm 2 in [41] (in short, *Itr.Method2*), we use

$$
\begin{aligned} \zeta_k &= 0.12, \\ \varkappa &= 0.11, \\ \daleth_0 &= 1 \end{aligned} \tag{104}
$$

(3) For Algorithm 1 (in short, *Itr.Method3*), we use

$$
\begin{aligned} \daleth_0 &= 0.50, \\ \zeta &= 0.50, \\ \varkappa &= 0.55, \\ \phi &= 0.05, \\ \psi_k &= \frac{1}{k^2} \end{aligned} \tag{105}
$$

(4) For Algorithm 2 (in short, *Itr.Method4*), we use

$$\begin{aligned} \beth_0 &= 0.50, \\ \zeta &= 0.50, \\ \varkappa &= 0.55, \\ \phi &= 0.05, \\ \psi_k &= \frac{1}{k^2}, \\ p_k &= \frac{100}{(1+k)^2}. \end{aligned} \tag{106}$$

Example 21. Consider that the possible set $\Xi \subset \mathbb{R}^N$ is defined as follows:

$$\Xi = \left\{ u \in \mathbb{R}^N : Au \le b \right\}, \tag{107}$$

where matrix A has an order $100 \times N$. Consider that $\Gamma : \Xi \times \Xi \longrightarrow \mathbb{R}$ is expressed by

$$\Gamma(u, y) = \langle \mathscr{L}(u), y - u \rangle, \quad \forall u, y \in \Xi, \tag{108}$$

where $\mathscr{L} : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ is an operator evaluated as $\mathscr{L}(u) = Pu + r$ with $r \in \mathbb{R}^N$ and $P = QQ^T + R + S$, where Q is an $N \times N$ matrix, R is an $N \times N$ skew-symmetric matrix, and S is an $N \times N$ positive definite diagonal matrix. It is simple to demonstrate that Γ is monotone and that the Lipschitz constants are $2c_1 = 2c_2 = \|M\|$ (for more information, see [42, 43]). The beginning points for these numerical investigations vary, as does the error term $D_k = \|x_{k+1} - x_k\|$. Figures 9–14 and Tables 3 and 4 show several results for the error term 10^{-5}. Consider the following information regarding control settings:

(1) For Algorithm 1 in [22] (in short, *Itr.Method1*), we use

$$\begin{aligned} \phi &= 0.45, \\ \beth &= \frac{1}{2c_2 + 8c_1} \end{aligned} \tag{109}$$

(2) For Algorithm 2 in [41] (in short, *Itr.Method2*), we use

$$\begin{aligned} \zeta_k &= 0.12, \\ \varkappa &= 0.11, \\ \beth_0 &= 1 \end{aligned} \tag{110}$$

(3) For Algorithm 1 (in short, *Itr.Method3*), we use

$$\begin{aligned} \beth_0 &= 0.50, \\ \zeta &= 0.50, \\ \varkappa &= 0.55, \\ \phi &= 0.05, \\ \psi_k &= \frac{1}{k^2} \end{aligned} \tag{111}$$

(4) For Algorithm 2 (in short, *Itr.Method4*), we use

$$\begin{aligned} \beth_0 &= 0.50, \\ \zeta &= 0.50, \\ \varkappa &= 0.55, \\ \phi &= 0.05, \\ \psi_k &= \frac{1}{k^2}, \\ p_k &= \frac{100}{(1+k)^2} \end{aligned} \tag{112}$$

6. Conclusion

The research proposed four explicit extragradient-like strategies for dealing with an equilibrium problem in a real Hilbert space involving a pseudomonotone and a Lipschitz-type bifunction. A novel step size rule that does not rely on Lipschitz-type constant information has been proposed. The convergence theorems and applications of the main results have been demonstrated. Several experiments are given to show the numerical behavior of our two algorithms and to compare them to other well-known algorithms in the literature.

Data Availability

The numerical data used to support the findings of this study are included within the article.

Conflicts of Interest

No potential conflict of interest was reported by the authors.

Acknowledgments

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