Hindawi
Journal of Function Spaces
Volume 2022, Article ID 1938290, 8 pages
https://doi.org/10.1155/2022/1938290

Research Article

Remarks on the Initial and Terminal Value Problem for Time and Space Fractional Diffusion Equation

Hoang Luc Nguyen

Division of Applied Mathematics, Thu Dau Mot University, Binh Duong, Vietnam

Correspondence should be addressed to Hoang Luc Nguyen; nguyenhoangluc@tdmu.edu.vn

Received 6 August 2022; Revised 3 September 2022; Accepted 12 September 2022; Published 4 October 2022

Academic Editor: Yusuf Gurefe

The fractional problem for partial differential equation has many applications in science and technology. The main objective of the paper is to investigate the convergence of the mild solution of the diffusion equation with time and space fractional. We consider the problem in two cases which are forward problem and inverse problem. We use new techniques to overcome some of the complex assessments.

1. Introduction

Fractional calculation has been shown to provide many important applications in natural sciences, such as in biological systems, signal processing, fluid mechanics, electrical networks, optical, and viscosity [1–8]. With the development of mathematics, there are now many different definitions of fractional derivatives, for example, Riemann-Liouville, Caputo, Hadamard, and Riesz. Let us refer many various papers on fractional differential equation, for example, Manimaran et al., Tuan et al., Long et al., Long L.D. et al., and Ngoc et al. [9–14]; Adiguzel et al., Li et al., Afshari et al., Alqahtani et al., Karapınar et al., Salim et al., Karapinar et al., and Abdeljawad et al. [15–22]; and Bachir et al., Salim et al., and Baitichea et al. [23–25]. Although most of them have been extensively studied, most mathematicians are interested and studied two derivatives which are Caputo and Riemann-Liouville derivatives.

In this paper, for $\alpha, \beta \in (0, 1)$, we are interested to study the following problem:

$$\begin{cases} \partial_t^\alpha u(x,t) + (-\Delta)^\beta u(x,t) = H(x,t), & x,t) \in (0,\pi)\times(0,T), \\ (u(0,t) = u(\pi,t) = 0, & t \in (0,T), \end{cases} \tag{1}$$

with the initial condition

$$u(x,0) = u_0(x), 0 < x < \pi, \tag{2}$$

or the terminal condition

$$u(x,T) = f(x), 0 < x < \pi. \tag{3}$$

There are many results related to the Problem (1) in both aspects: theoretical analysis and numerical analysis. The existence and well-posedness of Problems (1)–(2) and (1)–(3) has been studied in [26]. Jin et al. [27] applied two semidiscrete schemes of Galerkin FEM method in order to approximate the solution of Problems (1) and (2). In [28], the authors investigated a reaction-diffusion equation with a Caputo fractional derivative in time. In [29], the authors established the existence and uniqueness of the weak solution and the regularity of the solution for coupled fractional diffusion system. Mu et al. [30] investigated some initial-boundary value problems for time-fractional diffusion equations. Let us now mention some previous works on terminal value problem Problems (1)–(3). The main current applications of the terminal value problem are hydrodynamic inversion and spoil the image. In [31], the authors used variable total variation to approximate the backward problem for a time-space fractional diffusion equation. Under the interesting paper [32], Ngoc et al. considered the terminal value problem for nonlinear model.

$$\mathbf{D}_{0^+}^{\alpha} u - u_{xx} = F(u). \tag{4}$$

Our main purpose of this paper is to study the convergence of Problem (1) when $\beta \longrightarrow 1^-$. This result gives us the relationship between the solutions of the two Problem (1) with the case $0 < \beta < 1$ and $\beta = 1$. To the best of our knowledge, the research direction on this convergence topic is still limited. The main techniques to solve the our problem is to use Mittag-Leffler evaluations with the combination of the Wright function.

This paper is organized as follows. In Section 2, we focus premilinaries with some background on the definition and evaluations of Mittag-Leffler functions.

2. Premilinaries

Let us consider the Mittag-Leffler function, which is defined by

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha + \beta)}. \tag{5}$$

($z \in \mathbb{C}$), for $\alpha > 0$ and $\beta \in \mathbb{R}$. When $\beta = 1$, it is abbreviated as $E_\alpha(z) = E_{\alpha,1}(z)$.

Lemma 2.1. *The following equality holds (See [33]):*

$$E_{\alpha,1}(-z) = \int_0^\infty \mathbf{\Phi}_\alpha(\theta) e^{-z\theta} d\theta, \quad \text{for } z \in \mathbb{C}, \tag{6}$$

where the Wright function $\mathbf{\Phi}_\alpha(\theta)$ is defined by

$$\mathbf{\Phi}_\alpha(\theta) \coloneqq \sum_{j=0}^{\infty} \frac{\theta^j}{j!\Gamma(-\alpha j + 1 - \alpha)}, \quad 0 < \alpha < 1. \tag{7}$$

In addition, $\mathbf{\Phi}_\alpha(\theta)$ is a probability density function, that is,

$$\mathbf{\Phi}_\alpha(\theta) \geq 0, \quad \text{for } \theta > \text{and} \int_0^\infty \mathbf{\Phi}_\alpha(\theta) = 1. \tag{8}$$

Lemma 2.2. *For $\alpha \in (0, 1)$ and $b > -1$, the following properties hold (See [33]):*

$$\int_0^\infty \theta^b \mathbf{\Phi}_\alpha(\theta) d\theta = \frac{\Gamma(b+1)}{\Gamma(b\alpha + 1)}. \tag{9}$$

Let a given positive number $\sigma \geq 0$. Let us also define the Hilbert scale space as follows:

$$\mathbb{H}^\sigma(\Omega) = \left\{ \psi \in L^2(\Omega) \colon \sum_{j=1}^{\infty} j^{2\sigma} \left\langle \psi, \varphi_j \right\rangle^2 < +\infty \right\}, \tag{10}$$

with the following norm $\|\psi\|_{\mathbb{H}^\sigma(\Omega)} = (\sum_{j=1}^\infty j^{2\sigma} \langle \psi, \varphi_j \rangle^2)^{(1/2)}$.

Here we give the following lemma, which will help our proofs later:

Lemma 2.3. *Let $\varepsilon, \varepsilon' > 0$. Then we get the following:*

$$E_{\alpha,1}\left(-j^{2\beta} t^\alpha\right) - E_{\alpha,1}\left(-j^2 t^\alpha\right) \leq C_1(\alpha, \varepsilon) t^{\alpha\varepsilon} (1-\beta)^{\varepsilon\varepsilon'} j^{2\varepsilon + \varepsilon\varepsilon'}. \tag{11}$$

$$E_{\alpha,\alpha}\left(-j^{2\beta} t^\alpha\right) - E_{\alpha,\alpha}\left(-j^2 t^\alpha\right) \leq C_2(\alpha, \varepsilon) t^{\alpha\varepsilon} (1-\beta)^{\varepsilon\varepsilon'} j^{2\varepsilon + \varepsilon\varepsilon'}. \tag{12}$$

Proof. Let us now to study the difference $|E_{\alpha,1}(-j^{2\beta} t^\alpha) - E_{\alpha,1}(-j^2 t^\alpha)|$ for $0 < \beta < 1$. Since the definition of Wright function as in Lemma 2.1, we get that

$$\begin{aligned} E_{\alpha,1}\left(-j^{2\beta} t^\alpha\right) - E_{\alpha,1}\left(-j^2 t^\alpha\right) &= \int_0^\infty \Phi_\alpha(\theta) \exp\left(-j^{2\beta} t^\alpha \theta\right) d\theta \\ &\quad - \int_0^\infty \Phi_\alpha(\theta) \exp\left(-j^2 t^\alpha \theta\right) d\theta. \end{aligned} \tag{13}$$

Since $j \geq 1$ and $0 < \beta \leq 1$, we know easily that $\exp(-j^{2\beta} t^\alpha \theta) > \exp(-j^2 t^\alpha \theta)$. Hence, we find that

$$\begin{aligned} &\exp\left(-j^{2\beta} t^\alpha \theta\right) - \exp\left(-j^2 t^\alpha \theta\right) \\ &\quad = \exp\left(-j^{2\beta} t^\alpha \theta\right)\left(1 - \exp\left(-\left(j^2 - j^{2\beta}\right) t^\alpha \theta\right)\right). \end{aligned} \tag{14}$$

Using the inequality $1 - e^{-z} \leq C_\varepsilon z^\varepsilon$ for any $\varepsilon > 0$, we find that

$$\exp\left(-j^{2\beta} t^\alpha \theta\right) - \exp\left(-j^2 t^\alpha \theta\right) \leq C_\varepsilon \left(j^2 - j^{2\beta}\right)^\varepsilon t^{\alpha\varepsilon} \theta^\varepsilon. \tag{15}$$

Combining Problems (13) and (15), we derive that

$$\begin{aligned} E_{\alpha,1}\left(-j^{2\beta} t^\alpha\right) - E_{\alpha,1}\left(-j^2 t^\alpha\right) &\leq C_\varepsilon \left(j^2 - j^{2\beta}\right)^\varepsilon t^{\alpha\varepsilon} \left(\int_0^\infty \theta^\varepsilon \Phi_\alpha(\theta) d\theta\right) \\ &= C_\varepsilon \frac{\Gamma(\varepsilon + 1)}{\alpha\varepsilon + 1} \left(j^2 - j^{2\beta}\right)^\varepsilon t^{\alpha\varepsilon}. \end{aligned} \tag{16}$$

For any $\varepsilon' > 0$ and noting that $\log(j) \leq j$ for any $j \geq 1$, it is obvious to see that

$$\begin{aligned} j^2 - j^{2\beta} &= j^2(1 - \exp(-(2 - 2\beta)\log(j))) \\ &\leq j^2(2 - 2\beta)^{\varepsilon'} |\log(j)|^{\varepsilon'} \\ &\lesssim (1 - \beta)^{\varepsilon'} j^{2+\varepsilon'}. \end{aligned} \tag{17}$$

This implies that

$$\left(j^2 - j^{2\beta}\right)^\varepsilon \lesssim (1 - \beta)^{\varepsilon\varepsilon'} j^{2\varepsilon + \varepsilon\varepsilon'}. \tag{18}$$

From some above observations, we get that

$$E_{\alpha,1}\left(-j^{2\beta}t^{\alpha}\right) - E_{\alpha,1}\left(-j^{2}t^{\alpha}\right) \le C_1(\alpha,\varepsilon)t^{\alpha\varepsilon}(1-\beta)^{\varepsilon\varepsilon'} j^{2\varepsilon+\varepsilon\varepsilon'}. \tag{19}$$

By a similar argument as above, we also obtain the desired result, Problem (12). □

3. Initial Value Problem

In this section, we focus the following initial value problem under the linear case:

$$\begin{cases} \partial_t^{\alpha} v = -(-\Delta)^{\beta} v(x,t) + H(x,t), & (x,t) \in (0,\pi)\times(0,T), \\ v(0,t) = v(\pi,t) = 0, & t \in (0,T), \\ v(x,0) = v_0(x), & x \in (0,\pi), \end{cases} \tag{20}$$

where v_0 and source function H are defined later.

Theorem 3.1. *Let $v_0 \in \mathbb{H}^p(\Omega)$ and $H \in L^{\infty}(0,T;\mathbb{H}^p(\Omega))$ for any $p>0$. Then we get*

$$\begin{aligned} &\left\| v_{\beta}(.,t) - v^*(.,t) \right\|_{\mathbb{H}^s(\Omega)} \\ &\quad \lesssim (1-\beta)^{p-s/2}\left[\|v_0\|_{\mathbb{H}^p(\Omega)} + \|H\|_{L^{\infty}(0,T;\mathbb{H}^p(\Omega))} \right] \end{aligned} \tag{21}$$

for any $0<s<p$.

Proof. The mild solution to Problem (20) with $0<\beta<1$ is defined by

$$\begin{aligned} v_{\beta}(x,t) &= \sum_{j=1}^{\infty} E_{\alpha,1}\left(-j^{2\beta}t^{\alpha}\right)\left(\int_0^{\pi} v_0(x)\varphi_j(x)dx\right)\varphi_j(x) \\ &\quad + \sum_{j=1}^{\infty}\left[\int_0^t (t-r)^{\alpha-1}E_{\alpha,\alpha}\left(-j^{2\beta}(t-r)^{\alpha}\right)H_j(r)dr\right]\varphi_j(x), \end{aligned} \tag{22}$$

and the mild solution to Problem (20) with $\beta=1$ is defined by

$$\begin{aligned} v^*(x,t) &= \sum_{j=1}^{\infty} E_{\alpha,1}\left(-j^{2}t^{\alpha}\right)\left(\int_0^{\pi} v_0(x)\varphi_j(x)dx\right)\varphi_j(x) \\ &\quad + \sum_{j=1}^{\infty}\left[\int_0^t (t-r)^{\alpha-1}E_{\alpha,\alpha}\left(-j^{2}(t-r)^{\alpha}\right)H_j(r)dr\right]\varphi_j(x). \end{aligned} \tag{23}$$

By subtracting both sides of the two expressions above, we get the following difference:

$$\begin{aligned} &v_{\beta}(x,t) - v^*(x,t) \\ &\quad = \sum_{j=1}^{\infty}\left[E_{\alpha,1}\left(-j^{2\beta}t^{\alpha}\right) - E_{\alpha,1}\left(-j^{2}t^{\alpha}\right)\right]\left(\int_0^{\pi} v_0(x)\varphi_j(x)dx\right)\varphi_j(x) \\ &\qquad + \sum_{j=1}^{\infty}\left[\int_0^t (t-r)^{\alpha-1}\left(E_{\alpha,\alpha}\left(-j^{2\beta}(t-r)^{\alpha}\right)\right.\right. \\ &\qquad \left.\left. - E_{\alpha,\alpha}\left(-j^{2}(t-r)^{\alpha}\right)\right)H_j(r)dr\right]\varphi_j(x) \\ &\quad = \mathcal{M}_1(x,t) + \mathcal{M}_2(x,t). \end{aligned} \tag{24}$$

Let us first consider the term $\mathcal{M}_1$. By applying Parseval's equality and Lemma 2.3, we find that

$$\begin{aligned} \|\mathcal{M}_1(.,t)\|^2_{\mathbb{H}^s(\Omega)} &= \sum_{j=1}^{\infty} j^{2s}\left[E_{\alpha,1}\left(-j^{2\beta}t^{\alpha}\right) - E_{\alpha,1}\left(-j^{2}t^{\alpha}\right)\right]^2 \\ &\quad \cdot \left(\int_0^{\pi} v_0(x)\varphi_j(x)dx\right)^2 \\ &\le |C_1(\alpha,\varepsilon,\delta)|^2 t^{2\alpha\varepsilon}(1-\beta)^{2\varepsilon\delta}\sum_{j=1}^{\infty} j^{2s+4\varepsilon+2\varepsilon\delta} \\ &\quad \cdot \left(\int_0^{\pi} v_0(x)\varphi_j(x)dx\right)^2, \end{aligned} \tag{25}$$

where any *delta* >0. Hence, we know that the upper bound

$$\|\mathcal{M}_1(.,t)\|_{\mathbb{H}^s(\Omega)} \lesssim (1-\beta)^{\varepsilon\delta}\|v_0\|_{\mathbb{H}^{s+2\varepsilon+\varepsilon\delta}(\Omega)}. \tag{26}$$

Let us now treat the second term $\mathcal{M}_2$. By using Parseval's equality, we get that

$$\begin{aligned} \|\mathcal{M}_2(.,t)\|^2_{\mathbb{H}^s(\Omega)} &= \sum_{j=1}^{\infty} j^{2s}\left[\int_0^t (t-r)^{\alpha-1}\left(E_{\alpha,\alpha}\left(-j^{2\beta}(t-r)^{\alpha}\right)\right.\right. \\ &\quad \left.\left. - E_{\alpha,\alpha}\left(-j^{2}r(t-)^{\alpha}\right)\right)H_j(r)dr\right]^2 \\ &\quad \cdot \sum_{j=1}^{\infty} j^{2s}\left[\int_0^t (t-r)^{\alpha-1}\left(E_{\alpha,\alpha}\left(-j^{2\beta}(t-r)^{\alpha}\right)\right.\right. \\ &\quad \left.\left. - E_{\alpha,\alpha}\left(-j^{2}(t-r)^{\alpha}\right)\right)^2 \left|H_j(r)\right|^2 dr\right]. \end{aligned} \tag{27}$$

In view of the second estimate of Lemma 2.3, we derive that

$$\begin{aligned} &\left(E_{\alpha,\alpha}\left(-j^{2\beta}(t-r)^{\alpha}\right) - E_{\alpha,\alpha}\left(-j^{2}(t-r)^{\alpha}\right)\right)^2 \\ &\quad \le \left|C_2\left(\alpha,\varepsilon,\varepsilon'\right)\right|^2 (t-r)^{2\alpha\varepsilon}(1-\beta)^{2\varepsilon\varepsilon'} j^{\left(4\varepsilon+2\varepsilon\varepsilon'\right)} \end{aligned} \tag{28}$$

Combining Problems (27) and (28), we derive that

$$\begin{aligned}
\|\mathscr{M}_2(.,t)\|^2_{\mathbb{H}^s(\Omega)} &\le \left|C_2\left(\alpha,\varepsilon,\varepsilon'\right)\right|^2(1-\beta)^{2\varepsilon\varepsilon'}\int_0^t (t-r)^{\alpha-1+2\alpha\varepsilon} \\
&\quad\cdot\left(\sum_{j=1}^{\infty} j^{2s+4\varepsilon+2\varepsilon\varepsilon'}\left|H_j(r)\right|^2\right)dr \\
&= \left|C_2\left(\alpha,\varepsilon,\varepsilon'\right)\right|^2(1-\beta)^{2\varepsilon\varepsilon'} \\
&\quad\cdot\int_0^t (t-r)^{\alpha-1+2\alpha\varepsilon}\|H(r)\|_{\mathbb{H}^{s+2\varepsilon+\varepsilon\varepsilon'}(\Omega)}dr \\
&\le \left|C_2\left(\alpha,\varepsilon,\varepsilon'\right)\right|^2(1-\beta)^{2\varepsilon\varepsilon'} \\
&\quad\cdot\left(\int_0^t (t-r)^{\alpha-1+2\alpha\varepsilon}dr\right)\|H\|^2_{L^\infty\left(0,T;\mathbb{H}^{s+2\varepsilon+\varepsilon\varepsilon'}(\Omega)\right)}.
\end{aligned} \tag{29}$$

It is obvious to see that the integral term $\int_0^t (t-r)^{(\alpha-1+2\alpha\varepsilon)}dr$ is convergent. Hence, we obtain that the following estimate:

$$\|\mathscr{M}_2(.,t)\|_{\mathbb{H}^s(\Omega)} \le \left|C_2\left(\alpha,\varepsilon,\varepsilon'\right)\right|(1-\beta)^{\varepsilon\varepsilon'}\|H\|_{L^\infty\left(0,T;\mathbb{H}^{s+2\varepsilon+\varepsilon\varepsilon'}(\Omega)\right)}. \tag{30}$$

Combining Problems (24), (25), and (30), we find that

$$\begin{aligned}
&\left\|v_\beta(.,t)-v^*(.,t)\right\|_{\mathbb{H}^s(\Omega)} \\
&\quad\le \|\mathscr{M}_1(.,t)\|_{\mathbb{H}^s(\Omega)}+\|\mathscr{M}_2(.,t)\|_{\mathbb{H}^s(\Omega)} \\
&\quad\lesssim (1-\beta)^{\varepsilon\delta}\|v_0\|_{\mathbb{H}^{s+2\varepsilon+\varepsilon\delta}(\Omega)} + (1-\beta)^{\varepsilon\varepsilon'}\|H\|_{L^\infty\left(0,T;\mathbb{H}^{s+2\varepsilon+\varepsilon\varepsilon'}(\Omega)\right)}.
\end{aligned} \tag{31}$$

Since $p > s$, we can choose

$$\varepsilon = \frac{p-s}{4}, \delta = \varepsilon' = 2. \tag{32}$$

This implies that

$$\left\|v_\beta(.,t)-v^*(.,t)\right\|_{\mathbb{H}^s(\Omega)} \lesssim (1-\beta)^{p-s/2}\left[\|v_0\|_{\mathbb{H}^p(\Omega)} + \|H\|_{L^\infty(0,T;\mathbb{H}^p(\Omega))}\right]. \tag{33}$$

□

4. Terminal Value Problem

Theorem 4.1. *Let $f \in \mathbb{H}^b(\Omega)$ and $H \in L^\infty(0,T;\mathbb{H}^b(\Omega))$. Then we get*

$$\begin{aligned}
&\left\|u_\beta(.,t)-u_*(.,t)\right\|_{L^m(0,T;\mathbb{H}^s(\Omega))} \\
&\quad\lesssim (1-\beta)^{b-s-2\beta-2/2}\left(\|f\|_{\mathbb{H}^b(\Omega)} + \|H\|_{L^\infty\left(0,T;\mathbb{H}^b(\Omega)\right)}\right) \\
&\qquad + (1-\beta)^{b-s+2\beta+2/2}\|H\|_{L^\infty\left(0,T;\mathbb{H}^b(\Omega)\right)},
\end{aligned} \tag{34}$$

for $1 < m < 1/\alpha$ and $b > s + 2\beta + 2$.

Proof. The mild solution to terminal value Problem (1) for $0 < \beta < 1$ is given by

$$\begin{aligned}
u_\beta(x,t) &= \sum_{j=1}^{\infty}\frac{E_{\alpha,1}\left(-j^{2\beta}t^\alpha\right)}{E_{\alpha,1}\left(-j^{2\beta}T^\alpha\right)}\left(\int_0^\pi f(x)\varphi_j(x)dx\right)\varphi_j(x) \\
&\quad - \sum_{j=1}^{\infty}\frac{E_{\alpha,1}\left(-j^{2\beta}t^\alpha\right)}{E_{\alpha,1}\left(-j^{2\beta}T^\alpha\right)}\left(\int_0^T (T-r)^{\alpha-1}E_{\alpha,\alpha}\right. \\
&\quad \left.\cdot\left(-j^{2\beta}(T-r)^\alpha\right)H_j(r)dr\right)\varphi_j(x) \\
&\quad + \sum_{j=1}^{\infty}\left[\int_0^t (t-r)^{\alpha-1}E_{\alpha,\alpha}\left(-j^{2\beta}(t-r)^\alpha\right)H_j(r)dr\right]\varphi_j(x),
\end{aligned} \tag{35}$$

where

$$H_j(r) = \int_0^\pi H(x,r)\varphi_j(x)dx. \tag{36}$$

The mild solution to terminal value Problem (1) for $\beta = 1$ is given by

$$\begin{aligned}
u_*(x,t) &= \sum_{j=1}^{\infty}\frac{E_{\alpha,1}\left(-j^2t^\alpha\right)}{E_{\alpha,1}\left(-j^2T^\alpha\right)}\left(\int_0^\pi f(x)\varphi_j(x)dx\right)\varphi_j(x) \\
&\quad - \sum_{j=1}^{\infty}\frac{E_{\alpha,1}\left(-j^2t^\alpha\right)}{E_{\alpha,1}\left(-j^2T^\alpha\right)}\left(\int_0^T (T-r)^{\alpha-1}E_{\alpha,\alpha}\right. \\
&\quad \left.\cdot\left(-j^{2\beta}(T-r)^\alpha\right)H_j(r)dr\right)\varphi_j(x) \\
&\quad + \sum_{j=1}^{\infty}\left[\int_0^t (t-r)^{\alpha-1}E_{\alpha,\alpha}(-j^2(t-r)^\alpha)H_j(r)dr\right]\varphi_j(x).
\end{aligned} \tag{37}$$

Taking the difference of Problems (35) and (37) on both sides, we get the following bound:

$$\begin{aligned}
&u_\beta(x,t)-u_*(x,t) \\
&\quad = \sum_{j=1}^{\infty}\left(\frac{E_{\alpha,1}\left(-j^{2\beta}t^\alpha\right)}{E_{\alpha,1}\left(-j^{2\beta}T^\alpha\right)} - \frac{E_{\alpha,1}\left(-j^2t^\alpha\right)}{E_{\alpha,1}\left(-j^2T^\alpha\right)}\right)\left(\int_0^\pi f(x)\varphi_j(x)dx\right)\varphi_j(x) \\
&\quad - \sum_{j=1}^{\infty}\frac{E_{\alpha,1}\left(-j^{2\beta}t^\alpha\right)}{E_{\alpha,1}\left(-j^{2\beta}T^\alpha\right)}\left(\int_0^T (T-r)^{\alpha-1}\left(E_{\alpha,\alpha}\left(-j^{2\beta}(T-r)^\alpha\right)\right.\right. \\
&\quad \left.\left. - E_{\alpha,\alpha}\left(-j^2(T-r)^\alpha\right)\right)H_j(r)dr\right)\varphi_j(x) \\
&\quad + \sum_{j=1}^{\infty}\left(\frac{E_{\alpha,1}\left(-j^{2\beta}t^\alpha\right)}{E_{\alpha,1}\left(-j^{2\beta}T^\alpha\right)} - \frac{E_{\alpha,1}\left(-j^2t^\alpha\right)}{E_{\alpha,1}\left(-j^2T^\alpha\right)}\right) \\
&\quad \cdot\left(\int_0^T (T-r)^{\alpha-1}E_{\alpha,\alpha}\left(-j^2(T-r)^\alpha\right)H_j(r)dr\right)\varphi_j(x)
\end{aligned}$$

$$+\sum_{j=1}^{\infty}\left[\int_0^t (t-r)^{\alpha-1}\left(E_{\alpha,\alpha}\left(-j^{2\beta}(t-r)^{\alpha}\right)\right.\right.$$
$$\left.\left.-E_{\alpha,\alpha}\left(-j^2(t-r)^{\alpha}\right)\right)H_j(r)dr\right]\varphi_j(x) \quad (38)$$
$$=J_1(x,t)+J_2(x,t)+J_3(x,t)+J_4(x,t).$$

Step 1. Estimation of the Term J_1.

In order to evaluate J_1, we need to control the component

$$M_1=\frac{E_{\alpha,1}\left(-j^{2\beta}t^{\alpha}\right)}{E_{\alpha,1}\left(-j^{2\beta}T^{\alpha}\right)}-\frac{E_{\alpha,1}\left(-j^{2}t^{\alpha}\right)}{E_{\alpha,1}\left(-j^{2}T^{\alpha}\right)}. \quad (39)$$

It is obvious to compute the above term as follows:

$$M_1=\frac{E_{\alpha,1}\left(-j^{2\beta}t^{\alpha}\right)-E_{\alpha,1}\left(-j^{2}t^{\alpha}\right)}{E_{\alpha,1}\left(-j^{2\beta}T^{\alpha}\right)}-\frac{E_{\alpha,1}\left(-j^{2\beta}T^{\alpha}\right)-E_{\alpha,1}\left(-j^{2}T^{\alpha}\right)}{E_{\alpha,1}\left(-j^{2\beta}T^{\alpha}\right)E_{\alpha,1}\left(-j^{2}T^{\alpha}\right)}. \quad (40)$$

Since the fact that

$$E_{\alpha,1}\left(-j^{2\beta}T^{\alpha}\right)\geq\frac{C_{\alpha}^{-}}{1+j^{2\beta}T^{\alpha}}\leq\frac{C_{\alpha}}{j^{2\beta}(T^{\alpha}+1)}, \quad (41)$$

we know that

$$\frac{E_{\alpha,1}\left(-j^{2\beta}t^{\alpha}\right)-E_{\alpha,1}\left(-j^{2}t^{\alpha}\right)}{E_{\alpha,1}\left(-j^{2\beta}T^{\alpha}\right)}\leq\frac{C_1(\alpha,\varepsilon)(T^{\alpha}+1)}{C_{\alpha}}t^{\alpha\varepsilon}(1-\beta)^{\varepsilon\varepsilon'}j^{2\varepsilon+\varepsilon\varepsilon'+2\beta}. \quad (42)$$

By a similar explanation as above, we find that

$$\frac{E_{\alpha,1}\left(-j^{2\beta}T^{\alpha}\right)-E_{\alpha,1}\left(-j^{2}T^{\alpha}\right)}{E_{\alpha,1}\left(-j^{2\beta}T^{\alpha}\right)E_{\alpha,1}\left(-j^{2}T^{\alpha}\right)}\lesssim T^{\alpha\varepsilon}(1-\beta)^{\varepsilon\varepsilon'}j^{2\varepsilon+\varepsilon\varepsilon'+2\beta+2}, \quad (43)$$

where the hidden constant depends on $\alpha, T, \varepsilon, \varepsilon'$. From two above observation, we find that

$$M_1=\frac{E_{\alpha,1}\left(-j^{2\beta}t^{\alpha}\right)}{E_{\alpha,1}\left(-j^{2\beta}T^{\alpha}\right)}-\frac{E_{\alpha,1}\left(-j^{2}t^{\alpha}\right)}{E_{\alpha,1}\left(-j^{2}T^{\alpha}\right)}\lesssim(1-\beta)^{\varepsilon\varepsilon'}j^{2\varepsilon+\varepsilon\varepsilon'+2\beta+2}, \quad (44)$$

where the hidden constant depends on α, T, ε. Hence, we obtain that

$$\|J_1\|^2_{\mathbb{H}^s(\Omega)}=\sum_{j=1}^{\infty}j^{2s}\left(\frac{E_{\alpha,1}\left(-j^{2\beta}t^{\alpha}\right)}{E_{\alpha,1}\left(-j^{2\beta}T^{\alpha}\right)}-\frac{E_{\alpha,1}\left(-j^{2}t^{\alpha}\right)}{E_{\alpha,1}\left(-j^{2}T^{\alpha}\right)}\right)^2\cdot\left(\int_0^{\pi}f(x)\varphi_j(x)dx\right)^2\lesssim(1-\beta)^{2\varepsilon\varepsilon'}\sum_{j=1}^{\infty}j^{2s+4\varepsilon+2\varepsilon\varepsilon'+4\beta+4}\left(\int_0^{\pi}f(x)\varphi_j(x)dx\right)^2. \quad (45)$$

It implies that the following bound

$$\|J_1\|_{\mathbb{H}^s(\Omega)}\lesssim(1-\beta)^{\varepsilon\varepsilon'}\|f\|_{\mathbb{H}^{s+2\varepsilon+\varepsilon\varepsilon'+2\beta+2}(\Omega)}. \quad (46)$$

Step 2. Estimation of the Term J_3.

By using Parseval's equality and noting that Problem (44), we find that

$$\|J_3\|^2_{\mathbb{H}^s(\Omega)}=\sum_{j=1}^{\infty}j^{2s}\left(\frac{E_{\alpha,1}\left(-j^{2\beta}t^{\alpha}\right)}{E_{\alpha,1}\left(-j^{2\beta}T^{\alpha}\right)}-\frac{E_{\alpha,1}\left(-j^{2}t^{\alpha}\right)}{E_{\alpha,1}\left(-j^{2}T^{\alpha}\right)}\right)^2\cdot\left(\int_0^T(T-r)^{\alpha-1}E_{\alpha,\alpha}\left(-j^2(T-r)^{\alpha}\right)H_j(r)dr\right)^2\lesssim(1-\beta)^{2\varepsilon\varepsilon'}\sum_{j=1}^{\infty}j^{2s+4\varepsilon+2\varepsilon\varepsilon'+4\beta+4}\left(\int_0^T(T-r)^{\alpha-1}dr\right)\cdot\left(\int_0^T(T-r)^{\alpha-1}\left|H_j(r)\right|^2dr\right)^2, \quad (47)$$

where we have used the fact that $E_{\alpha,\alpha}(-j^2(T-r)^{\alpha})\leq C_{\alpha}$. Hence, we find that

$$\|J_3\|^2_{\mathbb{H}^s(\Omega)}\lesssim\frac{T^{\alpha}}{\alpha}(1-\beta)^{2\varepsilon\varepsilon'}\left(\int_0^T(T-r)^{\alpha-1}\|H(r)\|_{\mathbb{H}^{s+2\varepsilon+\varepsilon\varepsilon'+2\beta+2}(\Omega)}dr\right)\lesssim(1-\beta)^{2\varepsilon\varepsilon'}\|H\|^2_{L^{\infty}\left(0,T;\mathbb{H}^{s+2\varepsilon+\varepsilon\varepsilon'+2\beta+2}(\Omega)\right)}. \quad (48)$$

Step 3. Estimation of the Term J_2.

By using Parseval's equality, we derive that

$$\|J_2\|^2_{\mathbb{H}^s(\Omega)}=\sum_{j=1}^{\infty}j^{2s}\left(\frac{E_{\alpha,1}\left(-j^{2\beta}t^{\alpha}\right)}{E_{\alpha,1}\left(-j^{2\beta}T^{\alpha}\right)}\right)^2\left(\int_0^T(T-r)^{\alpha-1}\cdot\left(E_{\alpha,\alpha}\left(-j^{2\beta}(T-r)^{\alpha}\right)-E_{\alpha,\alpha}\left(-j^2(T-r)^{\alpha}\right)\right)H_j(r)dr\right)^2. \quad (49)$$

It is easy to verify that

$$\frac{E_{\alpha,1}\left(-j^{2\beta}t^{\alpha}\right)}{E_{\alpha,1}\left(-j^{2\beta}T^{\alpha}\right)} \lesssim \frac{1+T^{\alpha}j^{2\beta}}{1+t^{\alpha}j^{2\beta}} \le T^{\alpha}t^{-\alpha}. \tag{50}$$

Using Hölder's inequality, we derive that

$$\begin{aligned}&\left(\int_0^T (T-r)^{\alpha-1}\left(E_{\alpha,\alpha}\left(-j^{2\beta}(T-r)^{\alpha}\right)-E_{\alpha,\alpha}\left(-j^{2}(T-r)^{\alpha}\right)\right)H_j(r)dr\right)^2\\ &\le \left(\int_0^T (T-r)^{\alpha-1}dr\right)\left(\int_0^T (T-r)^{\alpha-1}\left(E_{\alpha,\alpha}\left(-j^{2\beta}(T-r)^{\alpha}\right)\right.\right.\\ &\quad \left.\left.-E_{\alpha,\alpha}\left(-j^{2}(T-r)^{\alpha}\right)\right)^2\left|H_j(r)\right|^2dr\right)\\ &\lesssim \int_0^T (T-r)^{\alpha-1}\left(E_{\alpha,\alpha}\left(-j^{2\beta}(T-r)^{\alpha}\right)\right.\\ &\quad \left.-E_{\alpha,\alpha}\left(-j^{2}(T-r)^{\alpha}\right)\right)^2\left|H_j(r)\right|^2dr.\end{aligned} \tag{51}$$

By a similar explanation, we can get that the following bound:

$$E_{\alpha,\alpha}\left(-j^{2\beta}t^{\alpha}\right)-E_{\alpha,\alpha}\left(-j^{2}t^{\alpha}\right) \le C_2(\alpha,\varepsilon,\gamma)t^{\alpha\varepsilon}(1-\beta)^{\varepsilon\gamma}j^{2\varepsilon+\varepsilon\gamma}, \tag{52}$$

for any $\gamma > 0$. This implies that

$$\begin{aligned}&\left(E_{\alpha,\alpha}\left(-j^{2\beta}(T-r)^{\alpha}\right)-E_{\alpha,\alpha}\left(-j^{2}(T-r)^{\alpha}\right)\right)^2\\ &\quad\lesssim (T-r)^{2\alpha\varepsilon}(1-\beta)^{2\varepsilon\gamma}j^{4\varepsilon+2\varepsilon\gamma}.\end{aligned} \tag{53}$$

Hence, we get that the following bound:

$$\begin{aligned}&\int_0^T (T-r)^{\alpha-1}\left(E_{\alpha,\alpha}\left(-j^{2\beta}(T-r)^{\alpha}\right)-E_{\alpha,\alpha}\left(-j^{2}(T-r)^{\alpha}\right)\right)^2\left|H_j(r)\right|^2dr\\ &\lesssim (1-\beta)^{2\varepsilon\gamma}\int_0^T (T-r)^{\alpha+2\alpha\varepsilon-1}j^{4\varepsilon+2\varepsilon\gamma}\left|H_j(r)\right|^2dr.\end{aligned} \tag{54}$$

Combining Problems (49), (50), and (54), we derive that

$$\begin{aligned}\|J_2\|^2_{\mathbb{H}^s(\Omega)} &\lesssim t^{-2\alpha}(1-\beta)^{2\varepsilon\gamma}\int_0^T (T-r)^{\alpha+2\alpha\varepsilon-1}\\ &\quad\cdot\left(\sum_{j=1}^{\infty} j^{2s+4\varepsilon+2\varepsilon\gamma}\left|H_j(r)\right|^2\right)dr\\ &= t^{-2\alpha}(1-\beta)^{2\varepsilon\gamma}\int_0^T (T-r)^{\alpha+2\alpha\varepsilon-1}\|H(r)\|^2_{\mathbb{H}^{s+2\varepsilon+\varepsilon\gamma}(\Omega)}dr\\ &\le t^{-2\alpha}(1-\beta)^{2\varepsilon\gamma}\left(\int_0^T (T-r)^{\alpha+2\alpha\varepsilon-1}dr\right)\\ &\quad\cdot\|H\|^2_{L^{\infty}\left(0,T;\mathbb{H}^{s+2\varepsilon+\varepsilon\gamma}(\Omega)\right)}.\end{aligned} \tag{55}$$

It is obvious to see that

$$\int_0^T (T-r)^{\alpha+2\alpha\varepsilon-1}dr = \frac{T^{\alpha+2\alpha\varepsilon}}{\alpha+2\alpha\varepsilon}. \tag{56}$$

So, we obtain that the following confirmation

$$\|J_2\|_{\mathbb{H}^s(\Omega)} \lesssim (1-\beta)^{\varepsilon\gamma}t^{-\alpha}\|H\|_{L^{\infty}\left(0,T;\mathbb{H}^{s+2\varepsilon+\varepsilon\gamma}(\Omega)\right)}. \tag{57}$$

Step 4. Estimation of the Term J_4.

By using Parseval's equality and Hölder's inequality, we get that

$$\begin{aligned}\|J_4\|^2_{\mathbb{H}^s(\Omega)} &= \sum_{j=1}^{\infty} j^{2s}\left(\int_0^t (t-r)^{\alpha-1}\left(E_{\alpha,\alpha}\left(-j^{2\beta}(t-r)^{\alpha}\right)\right.\right.\\ &\quad\left.\left.-E_{\alpha,\alpha}\left(-j^{2}(t-r)^{\alpha}\right)\right)H_j(r)dr\right)^2\\ &\le \left(\int_0^t (t-r)^{\alpha-1}dr\right)\sum_{j=1}^{\infty} j^{2s}\\ &\quad\cdot\left(\int_0^t (t-r)^{\alpha-1}\left(E_{\alpha,\alpha}\left(-j^{2\beta}(T-r)^{\alpha}\right)\right.\right.\\ &\quad\left.\left.-E_{\alpha,\alpha}\left(-j^{2}(t-r)^{\alpha}\right)\right)^2\left|H_j(r)\right|^2dr\right).\end{aligned} \tag{58}$$

By a similar techniques as in Probelem (54), we derive that

$$\begin{aligned}&\int_0^t (t-r)^{\alpha-1}\left(E_{\alpha,\alpha}\left(-j^{2\beta}(t-r)^{\alpha}\right)-E_{\alpha,\alpha}\left(-j^{2}(t-r)^{\alpha}\right)\right)^2\left|H_j(r)\right|^2dr\\ &\lesssim (1-\beta)^{2\varepsilon\gamma}\int_0^t (t-r)^{\alpha+2\alpha\varepsilon-1}j^{4\varepsilon+2\varepsilon\gamma}\left|H_j(r)\right|^2dr.\end{aligned} \tag{59}$$

By review two latter observations, we can deduce that

$$\begin{aligned}\|J_4\|^2_{\mathbb{H}^s(\Omega)} &\lesssim (1-\beta)^{2\varepsilon\gamma}\int_0^t (t-r)^{\alpha+2\alpha\varepsilon-1}\left(\sum_{j=1}^{\infty} j^{2s+4\varepsilon+2\varepsilon\gamma}\left|H_j(r)\right|^2\right)dr\\ &= (1-\beta)^{2\varepsilon\gamma}\int_0^t (t-r)^{\alpha+2\alpha\varepsilon-1}\|H(r)\|^2_{\mathbb{H}^{s+2\varepsilon+\varepsilon\gamma}(\Omega)}dr\\ &\le (1-\beta)^{2\varepsilon\gamma}\left(\int_0^t (t-r)^{\alpha+2\alpha\varepsilon-1}dr\right)\|H\|^2_{L^{\infty}\left(0,T;\mathbb{H}^{s+2\varepsilon+\varepsilon\gamma}(\Omega)\right)}.\end{aligned} \tag{60}$$

The above inequality implies that the following estimate:

$$\|J_4\|_{\mathbb{H}^s(\Omega)} \lesssim (1-\beta)^{\varepsilon\gamma}\left(\int_0^t (t-r)^{\alpha+2\alpha\varepsilon-1}dr\right)\|H\|_{L^{\infty}\left(0,T;\mathbb{H}^{s+2\varepsilon+\varepsilon\gamma}(\Omega)\right)} \tag{61}$$

By similar computation as above, we deduce that

$$\|J_4\|_{\mathbb{H}^s(\Omega)} \lesssim (1-\beta)^{\varepsilon\gamma}\|H\|_{L^{\infty}\left(0,T;\mathbb{H}^{s+2\varepsilon+\varepsilon\gamma}(\Omega)\right)}. \tag{62}$$

Combining four steps as above, we deduce that

$$\begin{aligned}\left\|u_\beta(.,t)-u_*(.,t)\right\|_{\mathbb{H}^s(\Omega)} &\le \|J_1\|_{\mathbb{H}^s(\Omega)}+\|J_2\|_{\mathbb{H}^s(\Omega)}+\|J_3\|_{\mathbb{H}^s(\Omega)}+\|J_4\|_{\mathbb{H}^s(\Omega)}\\ &\lesssim (1-\beta)^{\varepsilon\varepsilon'}\left(\|f\|_{\mathbb{H}^{s+2\varepsilon+\varepsilon\varepsilon'+2\beta+2}(\Omega)}+\|H\|_{L^\infty\left(0,T;\mathbb{H}^{s+2\varepsilon+\varepsilon\varepsilon'+2\beta+2}(\Omega)\right)}\right)\\ &\quad+(1-\beta)^{\varepsilon\gamma}\left(t^{-\alpha}+1\right)\|H\|_{L^\infty\left(0,T;\mathbb{H}^{s+2\varepsilon+\varepsilon\gamma}(\Omega)\right)}.\end{aligned} \tag{63}$$

Let us choose

$$\varepsilon=\frac{b-s-2\beta-2}{4},\ \varepsilon'=2,\ \gamma=2\frac{b-s+2\beta+2}{b-s-2\beta-2}. \tag{64}$$

Then from some above observations, we deduce that the following estimate:

$$\begin{aligned}\left\|u_\beta(.,t)-u_*(.,t)\right\|_{\mathbb{H}^s(\Omega)} &\lesssim (1-\beta)^{b-s-2\beta-2/2}\left(\|f\|_{\mathbb{H}^b(\Omega)}+\|H\|_{L^\infty\left(0,T;\mathbb{H}^b(\Omega)\right)}\right)\\ &\quad+(1-\beta)^{b-s+2\beta+2/2}\left(t^{-\alpha}+1\right)\|H\|_{L^\infty\left(0,T;\mathbb{H}^b(\Omega)\right)}.\end{aligned} \tag{65}$$

This estimate implies that the desired result, Problem (34). □

5. Conclusion

In this work, we consider the fractional problem for partial differential equation. We investigate the convergence of the mild solution of the diffusion equation with time and space fractional. Moreover, we consider the problem in two cases which are forward problem and inverse problem by using new techniques to overcome some of the complex assessments.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that they have no conflicts of interest.

Acknowledgments

This work is supported by Thu Dau Mot University.

References

[1] B. D. Coleman and W. Noll, “Foundations of linear viscoelasticity,” *Reviews of Modern Physics*, vol. 33, no. 2, pp. 239–249, 1961.

[2] P. Clement and J. A. Nohel, “Asymptotic behavior of solutions of nonlinear Volterra equations with completely positive kernels,” *SIAM Journal on Mathematical Analysis*, vol. 12, no. 4, pp. 514–535, 1981.

[3] B. de Andrade and A. Viana, “Abstract Volterra integrodifferential equations with applications to parabolic models with memory,” *Mathematische Annalen*, vol. 369, no. 3-4, pp. 1131–1175, 2017.

[4] O. Khan, S. Araci, and M. Saif, “Fractional calculus formulas for Mathieu-type series and generalized Mittag-Leffler function,” *Journal of Mathematics and Computer Science*, vol. 20, no. 2, pp. 122–130, 2020.

[5] R. S. Ali, S. Mubeen, and M. M. Ahmad, “A class of fractional integral operators with multi-index Mittag-Leffler k-function and Bessel k-function of first kind,” *Journal of Mathematics and Computer Science*, vol. 22, no. 3, pp. 266–281, 2021.

[6] R. Agarwal, U. P. Sharma, and R. P. Agarwal, “Bicomplex Mittag-Leffler function and associated properties,” *Journal of Nonlinear Sciences and Applications (JNSA)*, vol. 15, no. 1, pp. 48–60, 2022.

[7] P. Long, G. Murugusundaramoorthy, H. Tang, and W. Wang, “Subclasses of analytic and bi-univalent functions involving a generalized Mittag-Leffler function based on quasi-subordination,” *Journal of Mathematics and Computer Science*, vol. 26, no. 4, pp. 379–394, 2022.

[8] N. H. Tuan, Y. Zhou, T. N. Thach, and N. H. Can, “Initial inverse problem for the nonlinear fractional Rayleigh-Stokes equation with random discrete data,” *Communications in Nonlinear Science and Numerical Simulation*, vol. 78, article 104873, 2019.

[9] J. Manimaran, L. Shangerganesh, and A. Debbouche, “Finite element error analysis of a time-fractional nonlocal diffusion equation with the Dirichlet energy,” *Computational and Applied Mathematics*, vol. 382, article 113066, 2021.

[10] J. Manimaran, L. Shangerganesh, and A. Debbouche, “A time-fractional competition ecological model with cross-diffusion,” *Mathematical Methods in the Applied Sciences*, vol. 43, no. 8, pp. 5197–5211, 2020.

[11] N. H. Tuan, A. Debbouche, and T. B. Ngoc, “Existence and regularity of final value problems for time fractional wave equations,” *Computers & Mathematics with Applications*, vol. 78, no. 5, pp. 1396–1414, 2019.

[12] L. D. Long, H. D. Binh, D. Kumar, N. H. Luc, and N. H. Can, “Stability of fractional order of time nonlinear fractional diffusion equation with Riemann–Liouville derivative,” *Mathematical Methods in the Applied Sciences*, vol. 45, no. 10, pp. 6194–6216, 2022.

[13] L. D. Long, N. H. Luc, S. Tatar, D. Baleanu, and N. H. Can, “An inverse source problem for pseudo-parabolic equation with Caputo derivative,” *Journal of Applied Mathematics and Computing*, vol. 68, no. 2, pp. 739–765, 2022.

[14] T. B. Ngoc, V. V. Tri, Z. Hammouch, and N. H. Can, “Stability of a class of problems for time-space fractional pseudo-parabolic equation with datum measured at terminal time,” *Applied Numerical Mathematics*, vol. 167, pp. 308–329, 2021.

[15] R. S. Adiguzel, U. Aksoy, and E. Karapinar, “New anisotropic models from isotropic solutions,” *Mathematical Methods in the Applied Sciences*, vol. 29, no. 1, pp. 67–83, 2006.

[16] C. F. Li, X. N. Luo, and Y. Zhou, “Existence of positive solutions of the boundary value problem for nonlinear fractional differential equations,” *Computers & Mathematics with Applications*, vol. 59, no. 3, pp. 1363–1375, 2010.

[17] H. Afshari, S. Kalantari, and E. Karapinar, “Solution of fractional differential equations via coupled fixed point,” *Electronic*

Journal of Differential Equations, vol. 2015, no. 286, pp. 1–12, 2015.

[18] B. Alqahtani, H. Aydi, E. Karapınar, and V. Rakočević, "A solution for Volterra fractional integral equations by hybrid contractions," *Mathematics*, vol. 7, no. 8, p. 694, 2019.

[19] E. Karapınar, A. Fulga, M. Rashid, L. Shahid, and H. Aydi, "Large contractions on quasi-metric spaces with an application to nonlinear fractional differential equations," *Mathematics*, vol. 7, no. 5, p. 444, 2019.

[20] A. Salim, M. Benchohra, E. Karapınar, and J. E. Lazreg, "Existence and Ulam stability for impulsive generalized Hilfer-type fractional differential equations," *Advances in Difference Equations*, vol. 2020, no. 1, Article ID 601, 21 pages, 2020.

[21] E. Karapinar, T. Abdeljawad, and F. Jarad, "Applying new fixed point theorems on fractional and ordinary differential equations," *Advances in Difference Equations*, vol. 2019, no. 1, Article ID 421, 2019.

[22] A. Abdeljawad, R. P. Agarwal, E. Karapinar, and P. S. Kumari, "Solutions of the nonlinear integral equation and fractional differential equation using the technique of a fixed point with a numerical experiment in extended b-metric space," *Symmetry*, vol. 11, no. 5, p. 686, 2019.

[23] F. S. Bachir, S. Abbas, M. Benbachir, and M. Benchohra, "Hilfer-Hadamard fractional differential equations; existence and attractivity," *Advances in the Theory of Nonlinear Analysis and Its Application*, vol. 5, no. 1, pp. 49–57, 2021.

[24] A. Salim, M. Benchohra, J. Lazreg, and J. Henderson, "Nonlinear implicit generalized Hilfer-type fractional differential equations with non-instantaneous impulses in Banach spaces," *Advances in the Theory of Nonlinear Analysis and Its Application*, vol. 4, no. 4, pp. 332–348, 2020.

[25] Z. Baitichea, C. Derbazia, and M. Benchohrab, "ψ-Caputo fractional differential equations with multi-point boundary conditions by topological degree theory," *Results in Nonlinear Analysis*, vol. 3, no. 4, pp. 167–178, 2020.

[26] K. Sakamoto and M. Yamamoto, "Initial value/boundary value problems for fractional diffusion-wave equations and applications to some inverse problems," *Journal of Mathematical Analysis and Applications*, vol. 382, no. 1, pp. 426–447, 2011.

[27] B. Jin, R. Lazarov, and Z. Zhou, "Error estimates for a semidiscrete finite element method for fractional order parabolic equations," *SIAM Journal on Numerical Analysis*, vol. 51, no. 1, pp. 445–466, 2013.

[28] R. Tapdigoglu and B. Torebek, "Global existence and blow-up of solutions of the time-fractional space-involution reaction-diffusion equation," *Turkish Journal of Mathematics*, vol. 44, no. 3, pp. 960–969, 2020.

[29] L. Li, L. Jin, and S. Fang, "Existence and uniqueness of the solution to a coupled fractional diffusion system," *Advances in Difference Equations*, vol. 2015, no. 1, Article ID 370, 2015.

[30] J. Mu, B. Ahmad, and S. Huang, "Existence and regularity of solutions to time-fractional diffusion equations," *Computers & Mathematics with Applications*, vol. 73, no. 6, pp. 985–996, 2017.

[31] J. Jia, J. Peng, J. Gao, and Y. Li, "Backward problem for a time-space fractional diffusion equation," *Inverse Problems & Imaging*, vol. 12, no. 3, pp. 773–799, 2018.

[32] T. B. Ngoc, Y. Zhou, D. O'Regan, and N. H. Tuan, "On a terminal value problem for pseudoparabolic equations involving Riemann- Liouville fractional derivatives," *Applied Mathematics Letters*, vol. 106, article 106373, 2020.

[33] R. Gorenflo, Y. Luchko, and F. Mainardi, "Analytical properties and applications of the Wright function," *Fractional Calculus and Applied Analysis*, vol. 2, pp. 383–414, 1999.