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Research Article

Kannan Nonexpansive Operators on Variable Exponent Cesàro Sequence Space of Fuzzy Functions

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In general, we have constructed the operator ideal generated by extended s-fuzzy numbers and a certain space of sequences of fuzzy numbers. An investigation into the conditions sufficient for variable exponent Cesàro sequence space of fuzzy functions furnished with the definite function to create pre-quasi-Banach and closed is carried out. The (R) and the normal structural properties of this space are shown. Fixed points for Kannan contraction and nonexpansive mapping have been introduced. Lastly, we explore whether the Kannan contraction mapping has a fixed point in its associated pre-quasioperator ideal. The existence of solutions to nonlinear difference equations is illustrated with a few real-world examples and applications.

1. Introduction

Probability theory, fuzzy set theory, soft sets, and rough sets have contributed substantially to the study of uncertainty. But there are drawbacks to these theories that must be considered. After Zadeh [1] established the concept of fuzzy sets and fuzzy set operations, many researchers adopted the concept of fuzziness in cybernetics and artificial intelligence as well as in expert systems and fuzzy control. For more information and real-world examples, some comparable fixed point results were discussed by Javed et al. [2] to ensure that a fixed point exists and is unique in R-fuzzy b-metric spaces. The viability of the proposed methodologies was demonstrated through a challenging case study. There was no doubt about the superiority of the findings delivered. For the first type of Fredholm-type integral equation, an application was described. In [3], Al-Masarwah and Ahmad defined and investigated the m-Polar (α, β)-Fuzzy Ideals in BCK/BCI-Algebras and explored some pertinent properties. There are many other orthogonal fuzzy metric spaces; however, Javed et al. [4] expanded the orthogonal image fuzzy metric space concept. In the context of the newly specified structure, they displayed some fixed point outcomes. Fuzzy sequence spaces were introduced, and their various features were studied by many workers on sequence spaces and summability theory. Nuray and Savas [5] defined and studied the Nakano sequences of fuzzy numbers, $\ell^F(\tau)$ equipped with the function h. The operator ideal is very important in fixed point theory, Banach space geometry, normal series theory, approximation theory, and ideal transformations. See [6–8] for further proof. Pre-quasioperator ideals are more extensive than quasioperator ideals, according to Faried and Bakery [9]. The learning about the variable exponent Lebesgue spaces obtained impetus from the mathematical description of the hydrodynamics of non-Newtonian fluids (see [10, 11]). There are numerous uses for electrorheological fluids, which include military science, civil engineering, and orthopedic. There have been many developments in mathematics since the Banach fixed point theorem [12] was first published. While contractions have fixed point actions, Kannan [13] cited an example of a type of mapping that is not continuous. In Reference [14], the only attempt was made to explain Kannan operators in modular vector spaces. For more details on Kannan's fixed point theorems, see

[15–20]. Given that the proof of many fixed point theorems in a given space requires either growing the space itself or expanding the self-mapping that acts on it, both options are viable. Hence, we have constructed the Cesàro sequence spaces of fuzzy functions and have presented the solutions of a fuzzy nonlinear dynamical system in this newly created space. This work is aimed at introducing the certain space of sequences of fuzzy numbers, in short (cssf), under a certain function to be pre-quasi (cssf). This space and s-numbers have been used to describe the structure of the ideal operators. We explain the sufficient conditions of variable exponent Cesàro sequence space of fuzzy functions, which is denoted by $C^F_{\tau(.)}$, equipped with the definite function h to be pre-quasi-Banach and closed (cssf). The (R) and the normal structure property of this space are shown. Fixed points for Kannan contraction and nonexpansive mapping have been introduced. Lastly, we explore whether the Kannan contraction mapping has a fixed point in its associated pre-quasioperator ideal. The existence of solutions to nonlinear difference equations is illustrated with a few real-world examples and applications.

2. Definitions and Preliminaries

As a reminder, Matloka [21] presented the notion of ordinary convergence of sequences of fuzzy numbers, where he introduced bounded and convergent fuzzy numbers, explored some of their features, and proved that any convergent fuzzy number sequence is bounded. Nanda [22] studied the sequences of fuzzy numbers and showed the set of all convergent sequences of fuzzy numbers from a complete metric space. Kumar et al. [23] investigated the notion of limit points and cluster points of sequences of fuzzy numbers. Assume Ω is the set of all closed and bounded intervals on the real line $\mathfrak{R}$. For $f = [f_1, f_2]$ and $g = [g_1, g_2]$ in Ω, suppose

$$f \leq g, \text{ if and only if } f_1 \leq g_1 \text{ and } f_2 \leq g_2. \tag{1}$$

Define a metric ρ on Ω by

$$\rho(f, g) = \max \{|f_1 - g_1|, |f_2 - g_2|\}. \tag{2}$$

Matloka [21] showed that ρ is a metric on Ω and (Ω, ρ) is a complete metric space. Also, the relation $\leq$ is a partial order on Ω.

Definition 1. A fuzzy number g is a fuzzy subset of $\mathfrak{R}$, i.e., a mapping $g : \mathfrak{R} \longrightarrow [0, 1]$ which verifies the following four settings:

(a) g is fuzzy convex, i.e., for $x, y \in \mathfrak{R}$ and $\alpha \in [0, 1]$, $g(\alpha x + (1-\alpha)y) \geq \min \{g(x), g(y)\}$

(b) g is normal, i.e., there is $y_0 \in \mathfrak{R}$ such that $g(y_0) = 1$

(c) g is an upper semicontinuous, i.e., for all $\alpha > 0$, $g^{-1}([0, x+\alpha))$ for all $x \in [0, 1]$ is open in the usual topology of $\mathfrak{R}$

(d) the closure of $g^0 := \{y \in \mathfrak{R} : g(y) > 0\}$ is compact

The β-level set of a fuzzy real number g, $0 < \beta < 1$, indicated by g^β is defined as

$$g^\beta = \{y \in \mathfrak{R} : g(y) \geq \beta\}. \tag{3}$$

The set of every upper semicontinuous, normal, convex fuzzy number, and g^β is compact is denoted by $\mathfrak{R}([0, 1])$. The set $\mathfrak{R}$ can be embedded in $\mathfrak{R}([0, 1])$, if we define $r \in \mathfrak{R}([0, 1])$ by

$$\bar{r}(t) = \begin{cases} 1, & t = r, \\ 0, & t \neq r. \end{cases} \tag{4}$$

The additive identity and multiplicative identity in $\mathfrak{R}[0, 1]$ are denoted by $\bar{0}$ and $\bar{1}$, respectively.

The arithmetic operations on $\mathfrak{R}[0, 1]$ are defined as follows:

$$\begin{aligned}
(f \oplus g)(y) &= \sup_{y \in \mathfrak{R}} \min \{f(x), g(y-x)\}, \\
(f!g)(y) &= \sup_{y \in \mathfrak{R}} \min \{f(x), g(x-y)\}, \\
(f \otimes g)(y) &= \sup_{y \in \mathfrak{R}} \min \left\{f(x), g\left(\frac{y}{x}\right)\right\}, \\
\left(\frac{f}{g}\right)(y) &= \sup_{y \in \mathfrak{R}} \min \{f(xy), g(x)\}, \\
xf(y) &= \begin{cases} f\left(x^{-1}y\right), & x \neq 0, \\ 0, & x = 0. \end{cases}
\end{aligned} \tag{5}$$

The absolute value $|f|$ of $f \in \mathfrak{R}[0, 1]$ is defined by

$$|f|(y) = \begin{cases} \max \{f(y), f(-y)\}, & \text{if } y \geq 0, \\ 0, & \text{if } y < 0. \end{cases} \tag{6}$$

Suppose $f, g \in \mathfrak{R}[0, 1]$ and the β-level sets are $[f]^\beta = [f^\beta_1, f^\beta_2]$, $[g]^\beta = [g^\beta_1, g^\beta_2]$, and $\beta \in [0, 1]$. A partial ordering for any $f, g \in \mathfrak{R}[0, 1]$ as follows: $f^\circ g$, if and only if $f^\beta \leq g^\beta$, for all $\beta \in [0, 1]$. Then, the above operations can be defined in terms of β-level sets as follows:

$$\begin{aligned}
[f \oplus g]^\beta &= \left[f^\beta_1 + g^\beta_1, f^\beta_2 + g^\beta_2\right], \\
[f!g]^\beta &= \left[f^\beta_1 - g^\beta_2, f^\beta_2 - g^\beta_1\right], \\
[f \otimes g]^\beta &= \left[\min_{j \in \{1,2\}} f^\beta_j g^\beta_j, \max_{j \in \{1,2\}} f^\beta_j g^\beta_j\right], \\
\left[f^{-1}\right]^\beta &= \left[\left(f^\beta_2\right)^{-1}, \left(f^\beta_1\right)^{-1}\right], f^\beta_j > 0, \text{for every } \beta \in (0, 1], \\
[xf]^\beta &= \begin{cases} \left[xf^\beta_1, xf^\beta_2\right], & x \geq 0, \\ \left[xf^\beta_2, xf^\beta_1\right], & x < 0. \end{cases}
\end{aligned} \tag{7}$$

Assume $\bar{\rho} : \mathfrak{R}[0,1] \times \mathfrak{R}[0,1] \longrightarrow \mathfrak{R}^+ \cup \{0\}$ is defined by $\bar{\rho}(f,g) = \sup_{0\le\beta\le1}\rho(f^\beta, g^\beta)$.

Recall that

(1) $(\mathfrak{R}[0,1], \bar{\rho})$ is a complete metric space

(2) $\bar{\rho}(f+k, g+k) = \bar{\rho}(f,g)$ for all $f, g, k \in \mathfrak{R}[0,1]$

(3) $\bar{\rho}(f+k, g+l) \le \bar{\rho}(f,g) + \bar{\rho}(k,l)$.

(4) $\bar{\rho}(\xi f, \xi g) = |\xi|\bar{\rho}(f,g)$, for all $\xi \in \mathfrak{R}$.

Definition 2. A sequence $f = (f_j)$ of fuzzy numbers is said to be

(a) bounded if the set $\{f_j : j \in \mathcal{N}\}$ of fuzzy numbers is bounded, i.e., if a sequence (f_j) is bounded, then there are two fuzzy numbers g, l such that $g \le f_j \le l$

(b) convergent to a fuzzy real number f_0 if for every $\varepsilon > 0$, there exists $n_0 \in \mathcal{N}$ such that $\bar{\rho}(f_j, f_0) < \varepsilon$, for all $j \ge j_0$

Lemma 3 (see [24]). *Suppose $\tau_a \ge 1$ and $v_a, t_a \in \mathfrak{R}$, for every $a \in \mathcal{N}$, then $|v_a + t_a|^{\tau_a} \le 2^{K-1}(|v_a|^{\tau_a} + |t_a|^{\tau_a})$, where $K = \max\{1, \sup_a \tau_a\}$.*

3. Main Results

3.1. Some Properties of $C^F_{\tau(.)}$. In this section, we have introduced the certain space of sequences of fuzzy numbers or in short (cssf), under the definite function to form pre-quasi (cssf). We explain the sufficient setting of $C^F_{\tau(.)}$ equipped with the definite function h to construct pre-quasi-Banach and closed (cssf). The Fatou property of various pre-quasinorms h on $C^F_{\tau(.)}$ has been investigated. We have presented this space's k-nearly uniformly convex, the property (R), and the h-normal structure-property, which are connected with the fixed point theorem.

By ℓ_∞ and ℓ_r, we denote the spaces of bounded and r-absolutely summable sequences of real numbers, respectively. Let $\omega(F)$ denote the classes of all sequence spaces of fuzzy real numbers. Suppose $\tau = (\tau_a) \in \mathfrak{R}^{+\mathcal{N}}$, where $\mathfrak{R}^{+\mathcal{N}}$ is the space of positive real sequences. The variable exponent Cesàro sequence space of fuzzy functions is denoted by the following: $C^F_{\tau(.)} = \{v = (v_a) \in \omega(F)\colon h(\mu v)<\infty, \text{for some } \mu > 0\}$, when $h(v) = \sum_{a=0}^\infty \left(\sum_{k=0}^a \bar{\rho}(v_k, \bar{0})/a+1\right)^{\tau_a}$. If $(\tau_a) \in \ell_\infty$, then

$$\begin{aligned} C^F_{\tau(.)} &= \{v = (v_a) \in \omega(F)\colon h(\mu v)<\infty, \text{for some } \mu > 0\} \\ &= \left\{ v = (v_a) \in \omega(F)\colon \inf_a |\mu|^{\tau_a} \sum_{a=0}^{\infty} \left(\frac{\sum_{k=0}^a \bar{\rho}(v_k, \bar{0})}{a+1}\right)^{\tau_a} \right. \\ &\quad \left. \le \sum_{a=0}^{\infty} \left(\frac{\sum_{k=0}^a \bar{\rho}(\mu v_k, \bar{0})}{a+1}\right)^{\tau_a} <\infty, \text{for some } \mu > 0 \right\} \\ &= \left\{ v = (v_a) \in \omega(F)\colon \sum_{a=0}^{\infty} \left(\frac{\sum_{k=0}^a \bar{\rho}(v_k, \bar{0})}{a+1}\right)^{\tau_a} <\infty \right\} \\ &= \{v = (v_a) \in \omega(F)\colon h(\mu v)<\infty, \text{for any } \mu > 0\}. \end{aligned} \tag{8}$$

Definition 4 (see [25]). The linear space U is said to be a certain space of sequences of fuzzy numbers (cssf), if

(1) $\{\bar{\mathfrak{b}}_q\}_{q\in\mathcal{N}} \subseteq \mathbf{U}$, where $\bar{\mathfrak{b}}_q = \{\bar{0}, \bar{0}, \cdots, \bar{1}, \bar{0}, \bar{0}, \cdots\}$, while $\bar{1}$ displays at the q^{th} place

(2) suppose $Y = (Y_q) \in \omega(F)$, $Z = (Z_q) \in \mathbf{U}$ and $|Y_q| \le |Z_q|$, for all $q \in \mathcal{N}$, then $Y \in \mathbf{U}$

(3) $(Y_{[q/2]})_{q=0}^\infty \in \mathbf{U}$, where $[q/2]$ marks the integral part of $q/2$, if $(Y_q)_{q=0}^\infty \in \mathbf{U}$

Definition 5 (see [25]). A subclass U_h of U is called a pre-modular (cssf), if there is $h \in [0,\infty)^U$ satisfies the next settings:

(i) If $Y \in \mathbf{U}$, $Y = \bar{\vartheta} \Leftrightarrow h(Y) = 0$ with $h(Y) \ge 0$, where $\bar{\vartheta} = (\bar{0}, \bar{0}, \bar{0},)$

(ii) There is $Q \ge 1$, and the inequality $h(\alpha Y) \le Q|\alpha|h(Y)$ holds, for every $Y \in \mathbf{U}$ and $\alpha \in \mathfrak{R}$

(iii) There is $P \ge 1$, and the inequality $h(Y+Z) \le P(h(Y) + h(Z))$ holds, for every $Y, Z \in \mathbf{U}$

(iv) If $|Y_q| \le |Z_q|$, for every $q \in \mathcal{N}$, one has $h((Y_q)) \le h((Z_q))$

(v) The inequality $h((Y_q)) \le h((Y_{[q/2]})) \le P_0 h((Y_q))$ holds, for some $P_0 \ge 1$

(vi) Let E be the space of finite sequences of fuzzy numbers; then, the closure of $E = \mathbf{U}_h$

(vii) There is $\sigma > 0$ with $h(\bar{\alpha}, \bar{0}, \bar{0}, \bar{0}, \cdots) \ge \sigma|\alpha|h(\bar{1}, \bar{0}, \bar{0}, \bar{0}, \cdots)$, where

$$\bar{\alpha}(y) = \begin{cases} 1, & y = \alpha, \\ 0, & y \ne \alpha. \end{cases} \tag{9}$$

Definition 6 (see [25]). Suppose U is a (cssf). The function $h \in [0,\infty)^U$ is called a pre-quasinorm on U, if it satisfies the following conditions:

(i) If $Y \in \mathbf{U}$, $Y = \bar{\vartheta} \Leftrightarrow h(Y) = 0$ with $h(Y) \ge 0$, where $\bar{\vartheta} = (\bar{0}, \bar{0}, \bar{0},)$

(ii) There is $Q \ge 1$, and the inequality $h(\alpha Y) \le Q|\alpha|h(Y)$ satisfies, for every $Y \in \mathbf{U}$ and $\alpha \in \mathfrak{R}$

(iii) There is $P \ge 1$, and the inequality $h(Y+Z) \le P(h(Y) + h(Z))$ holds, for each $Y, Z \in \mathbf{U}$

Clearly, from the last two definitions, we conclude the following two theorems:

Theorem 7 (see [25]). *If U is a premodular (cssf), then it is pre-quasinormed (cssf).*

Theorem 8 (see [25]). *U is a pre-quasinormed (cssf) if it is quasinormed (cssf).*

Definition 9.

(a) The function h on $C^F_{\tau(.)}$ is named h-convex, if

$$h(\alpha Y + (1-\alpha)Z) \leq \alpha h(Y) + (1-\alpha)h(Z), \quad (10)$$

for every $\alpha \in [0,1]$ and $Y, Z \in C^F_{\tau(.)}$.

(b) $\{Y_q\}_{q\in\mathcal{N}} \subseteq (C^F_{\tau(.)})_h$ is h-convergent to $Y \in (C^F_{\tau(.)})_h$, if and only if $\lim_{q\longrightarrow\infty} h(Y_q - Y) = 0$. When the h-limit exists, then it is unique

(c) $\{Y_q\}_{q\in\mathcal{N}} \subseteq (C^F_{\tau(.)})_h$ is h-Cauchy, if $\lim_{q,r\longrightarrow\infty} h(Y_q - Y_r) = 0$

(d) $\Gamma \subset (C^F_{\tau(.)})_h$ is h-closed, when for all h-converges $\{Y_q\}_{a\in\mathcal{N}} \subset \Gamma$ to Y, then $Y \in \Gamma$

(e) $\Gamma \subset (C^F_{\tau(.)})_h$ is h-bounded, if $\delta_h(\Gamma) = \sup\{h(Y-Z) : Y, Z \in \Gamma\} < \infty$

(f) The h-ball of radius $\varepsilon \geq 0$ and center Y, for every $Y \in (C^F_{\tau(.)})_h$, is described as follows:

$$\mathbf{B}_h(Y,\varepsilon) = \left\{Z \in \left(C^F_{\tau(.)}\right)_h : h(Y-Z) \leq \varepsilon\right\}. \quad (11)$$

(g) A pre-quasinorm h on $C^F_{\tau(.)}$ satisfies the Fatou property, if for every sequence $\{Z^q\} \subseteq (C^F_{\tau(.)})_h$ under $\lim_{q\longrightarrow\infty} h(Z^q - Z) = 0$ and all $Y \in (C^F_{\tau(.)})_h$, one has $h(Y-Z) \leq \sup_r \inf_{q\geq r} h(Y - Z^q)$

Note that the Fatou property implies the h-closed of the h-balls. We will denote the space of all increasing sequences of real numbers by **I**.

Theorem 10. *$(C^F_{\tau(.)})_h$, where $h(Y) = [\sum_{q=0}^{\infty}(\sum_{p=0}^{q}\bar{\rho}(Y_p,\bar{0})/q+1)^{\tau_q}]^{1/K}$, for all $Y \in C^F_{\tau(.)}$, is a premodular (cssf), when $(\tau_q)_{q\in\mathcal{N}} \in \ell_\infty \cap I$ with $\tau_0 > 1$.*

Proof. (i) Evidently, $h(Y) \geq 0$ and $h(Y) = 0 \Leftrightarrow Y = \vartheta$

(1-i) Let $Y, Z \in C^F_{\tau(.)}$. One has

$$h(Y+Z) = \left[\sum_{q=0}^{\infty}\left(\frac{\sum_{p=0}^{q}\bar{\rho}(Y_p+Z_p,\bar{0})}{q+1}\right)^{\tau_q}\right]^{1/K} \leq \left[\sum_{q=0}^{\infty}\left(\frac{\sum_{p=0}^{q}\bar{\rho}(Y_p,\bar{0})}{q+1}\right)^{\tau_q}\right]^{1/K} + \left[\sum_{q=0}^{\infty}\left(\frac{\sum_{p=0}^{q}\bar{\rho}(Z_p,\bar{0})}{q+1}\right)^{\tau_q}\right]^{1/K} = h(Y) + h(Z) < \infty, \quad (12)$$

and then, $Y + Z \in C^F_{\tau(.)}$.

(iii) One gets $P \geq 1$ with $h(Y+Z) \leq P(h(Y) + h(Z))$, for all $Y, Z \in C^F_{\tau(.)}$

(1-ii) Assume $\alpha \in \mathfrak{R}$ and $Y \in C^F_{\tau(.)}$, and we obtain

$$h(\alpha Y) = \left[\sum_{q=0}^{\infty}\left(\frac{\sum_{p=0}^{q}\bar{\rho}(\alpha Y_p,\bar{0})}{q+1}\right)^{\tau_q}\right]^{1/K} \leq \sup_q |\alpha|^{\tau_q/K} \cdot \left[\sum_{q=0}^{\infty}\left(\frac{\sum_{p=0}^{q}\bar{\rho}(Y_p,\bar{0})}{q+1}\right)^{\tau_q}\right]^{1/K} \leq Q|\alpha|h(Y) < \infty. \quad (13)$$

As $\alpha Y \in C^F_{\tau(.)}$. Hence, from conditions (1-i) and (1-ii), one has $C^F_{\tau(.)}$ is linear. Also, $\bar{\mathfrak{b}}_r \in C^F_{\tau(.)}$, for all $r \in \mathcal{N}$, since $h(\bar{\mathfrak{b}}_r) = [\sum_{q=0}^{\infty}(\sum_{p=0}^{q}\bar{\rho}(\bar{\mathfrak{b}}_r,\bar{0})/q+1)^{\tau_q}]^{1/K} \leq [\sum_{q=0}^{\infty}(1/q+1)^{\tau_0}]^{1/K} < \infty$.

(ii) There is $Q = \max\{1, \sup_q |\alpha|^{\tau_q/K-1}\} \geq 1$ with $h(\alpha Y) \leq Q|\alpha|h(Y)$, for all $Y \in C^F_{\tau(.)}$ and $\alpha \in \mathfrak{R}$

(2) Assume $|Y_q| \leq |Z_q|$, for all $q \in \mathcal{N}$ and $Z \in C^F_{\tau(.)}$. One finds

$$h(Y) = \left[\sum_{q=0}^{\infty}\left(\frac{\sum_{p=0}^{q}\bar{\rho}(Y_p,\bar{0})}{q+1}\right)^{\tau_q}\right]^{1/K} \leq \left[\sum_{q=0}^{\infty}\left(\frac{\sum_{p=0}^{q}\bar{\rho}(Z_p,\bar{0})}{q+1}\right)^{\tau_q}\right]^{1/K} = h(Z) < \infty, \quad (14)$$

and then, $Y \in C^F_{\tau(.)}$.

(iv) Obviously, from (2)

(3) Let $(Y_q) \in C^F_{\tau(.)}$, and we get

$$\begin{aligned} h\left(\left(Y_{[q/2]}\right)\right) &= \left[\sum_{q=0}^{\infty}\left(\frac{\sum_{p=0}^{q}\bar{\rho}\left(Y_{[p/2]},\bar{0}\right)}{q+1}\right)^{\tau_q}\right]^{1/K} = \left[\sum_{q=0}^{\infty}\left(\frac{\sum_{p=0}^{2q}\bar{\rho}\left(Y_{[p/2]},\bar{0}\right)}{2q+1}\right)^{\tau_{2q}} + \sum_{q=0}^{\infty}\left(\frac{\sum_{p=0}^{2q+1}\bar{\rho}\left(Y_{[p/2]},\bar{0}\right)}{2q+2}\right)^{\tau_{2q+1}}\right]^{1/K} \leq 2^{1/K}\left[\sum_{q=0}^{\infty}\left(\frac{\sum_{p=0}^{q}\bar{\rho}(Y_p,\bar{0})}{q+1}\right)^{\tau_q}\right]^{1/K} \\ &\leq \left[\sum_{q=0}^{\infty}\left(\frac{\bar{\rho}(Y_q,\bar{0}) + 2\sum_{p=0}^{q}\bar{\rho}(Y_p,\bar{0})}{q+1}\right)^{\tau_q} + \sum_{q=0}^{\infty}\left(\frac{2\sum_{p=0}^{q}\bar{\rho}(Y_p,\bar{0})}{q+1}\right)^{\tau_q}\right]^{1/K} \\ &\leq 2^{1/K}\left[\sum_{q=0}^{\infty}\left(\frac{\sum_{p=0}^{q}\bar{\rho}(Y_p,\bar{0})}{q+1}\right)^{\tau_q}\right]^{1/K} \leq \left[\sum_{q=0}^{\infty}\left(\frac{3\sum_{p=0}^{q}\bar{\rho}(Y_p,\bar{0})}{q+1}\right)^{\tau_q} + \sum_{q=0}^{\infty}\left(\frac{2\sum_{p=0}^{q}\bar{\rho}(Y_p,\bar{0})}{q+1}\right)^{\tau_q}\right]^{1/K} \leq \left(3^K + 2^K\right)^{1/K}\left[\sum_{q=0}^{\infty}\left(\frac{\sum_{p=0}^{q}\bar{\rho}(Y_p,\bar{0})}{q+1}\right)^{\tau_q}\right]^{1/K} \\ &= \left(3^K + 2^K\right)^{1/K} h\left(\left(Y_q\right)\right), \end{aligned} \quad (15)$$

and then, $(Y_{[p/2]}) \in C^F_{\tau(.)}$.

(v) From (4), we obtain $P_0 = (3^K + 2^K)^{1/K} \geq 1$

(vi) Evidently the closure of $E = C^F_{\tau(.)}$

(vii) There is $0 < \sigma \leq \sup_q |\alpha|^{\tau_q/K-1}$, for $\alpha \neq 0$ or $\sigma > 0$, for $\alpha = 0$ with $h(\bar{\alpha},\bar{0},\bar{0},\bar{0},\cdots) \geq \sigma|\alpha|h(\bar{1},\bar{0},\bar{0},\bar{0},\cdots)$ □

Theorem 11. *If $(\tau_q)_{q\in\mathcal{N}} \in \ell_\infty \cap I$ with $\tau_0 > 1$, then $(C^F_{\tau(.)})_h$ is a pre-quasi-Banach (cssf), where $h(Y) = [\sum_{q=0}^{\infty}(\sum_{p=0}^{q}\bar{\rho}(Y_p, \bar{0})/q+1)^{\tau_q}]^{1/K}$, for every $Y \in C^F_{\tau(.)}$.*

Proof. In view of Theorem 10 and Theorem 7, the space $(C^F_{\tau(.)})_h$ is a pre-quasinormed (cssf). Assume $Y^l = (Y^l_q)_{q=0}^{\infty}$ is a Cauchy sequence in $(C^F_{\tau(.)})_h$. Hence, for every $\varepsilon \in (0,1)$, one has $l_0 \in \mathcal{N}$ such that for all $l, m \geq l_0$, one gets

$$h\left(Y^l - Y^m\right) = \left[\sum_{q=0}^{\infty}\left(\frac{\sum_{p=0}^{q}\bar{\rho}\left(Y^l_p - Y^m_p, \bar{0}\right)}{q+1}\right)^{\tau_q}\right]^{1/K} < \varepsilon. \tag{16}$$

That implies $\bar{\rho}(Y^l_q - Y^m_q, \bar{0}) < \varepsilon$. As $(\mathfrak{R}[0,1], \bar{\rho})$ is a complete metric space. Then, (Y^m_q) is a Cauchy sequence in $\mathfrak{R}[0,1]$, for fixed $q \in \mathcal{N}$, which implies $\lim_{m\longrightarrow\infty} Y^m_q = Y^0_q$, for constant $q \in \mathcal{N}$. Hence, $h(Y^l - Y^0) < \varepsilon$, for every $l \geq l_0$, since $h(Y^0) = h(Y^0 - Y^l + Y^l) \leq h(Y^l - Y^0) + h(Y^l) < \infty$. So $Y^0 \in C^F_{\tau(.)}$. □

Theorem 12. *Suppose $(\tau_q)_{q\in\mathcal{N}} \in \ell_\infty \cap I$ with $\tau_0 > 1$, then $(C^F_{\tau(.)})_h$ is a pre-quasiclosed (cssf), where $h(Y) = [\sum_{q=0}^{\infty}(\sum_{p=0}^{q}\bar{\rho}(Y_p, \bar{0})/q+1)^{\tau_q}]^{1/K}$, for every $Y \in C^F_{\tau(.)}$.*

Proof. In view of Theorem 10 and Theorem 7, the space $(C^F_{\tau(.)})_h$ is a pre-quasinormed (cssf). Assume $Y^l = (Y^l_q)_{q=0}^{\infty} \in (C^F_{\tau(.)})_h$ and $\lim_{l\longrightarrow\infty} h(Y^l - Y^0) = 0$; then, for all $\varepsilon \in (0,1)$, there is $l_0 \in \mathcal{N}$ such that for all $l \geq l_0$, we obtain

$$\varepsilon > h\left(Y^l - Y^0\right) = \left[\sum_{q=0}^{\infty}\left(\frac{\sum_{p=0}^{q}\bar{\rho}\left(Y^l_p - Y^0_p, \bar{0}\right)}{q+1}\right)^{\tau_q}\right]^{1/K}, \tag{17}$$

which implies $\bar{\rho}(Y^l_q - Y^0_q, \bar{0}) < \varepsilon$. As $(\mathfrak{R}[0,1], \bar{\rho})$ is a complete metric space, therefore, (Y^l_q) is a convergent sequence in $\mathfrak{R}[0,1]$, for fixed $q \in \mathcal{N}$. So, $\lim_{l\longrightarrow\infty} Y^l_q = Y^0_q$, for fixed $q \in \mathcal{N}$. Since $h(Y^0) = h(Y^0 - Y^l + Y^l) \leq h(Y^l - Y^0) + h(Y^l) < \infty$, one has $Y^0 \in C^F_{\tau(.)}$. □

Theorem 13. *The function $h(Y) = [\sum_{q=0}^{\infty}(\sum_{p=0}^{q}\bar{\rho}(Y_p, \bar{0})/q+1)^{\tau_q}]^{1/K}$ verifies the Fatou property, when $(\tau_q)_{q\in\mathcal{N}} \in \ell_\infty \cap I$ with $\tau_0 > 1$, for all $Y \in C^F_{\tau(.)}$.*

Proof. Let $\{Z^r\} \subseteq (C^F_{\tau(.)})_h$ such that $\lim_{r\longrightarrow\infty} h(Z^r - Z) = 0$. Since $(C^F_{\tau(.)})_h$ is a pre-quasiclosed space, one has $Z \in (C^F_{\tau(.)})_h$. For all $Y \in (C^F_{\tau(.)})_h$, one gets

$$\begin{aligned} h(Y-Z) &= \left[\sum_{q=0}^{\infty}\left(\frac{\sum_{p=0}^{q}\bar{\rho}\left(Y_p - Z_p, \bar{0}\right)}{q+1}\right)^{\tau_q}\right]^{1/K} \\ &\leq \left[\sum_{q=0}^{\infty}\left(\frac{\sum_{p=0}^{q}\bar{\rho}\left(Y_p - Z^r_p, \bar{0}\right)}{q+1}\right)^{\tau_q}\right]^{1/K} \\ &\quad + \left[\sum_{q=0}^{\infty}\left(\frac{\sum_{p=0}^{q}\bar{\rho}\left(Z^r_p - Z_p, \bar{0}\right)}{q+1}\right)^{\tau_q}\right]^{1/K} \\ &\leq \sup_m \inf_{r\geq m} h(Y - Z^r). \end{aligned} \tag{18}$$

□

Theorem 14. *The function $h(Y) = \sum_{q=0}^{\infty}(\sum_{p=0}^{q}\bar{\rho}(Y_p, \bar{0})/q+1)^{\tau_q}$ does not satisfy the Fatou property, for all $Y \in C^F_{\tau(.)}$, when $(\tau_q) \in \ell_\infty$ and $\tau_q > 1$, for all $q \in \mathcal{N}$.*

Proof. Let $\{Z^r\} \subseteq (C^F_{\tau(.)})_h$ so that $\lim_{r\longrightarrow\infty} h(Z^r - Z) = 0$. Since $(C^F_{\tau(.)})_h$ is a pre-quasiclosed space, one gets $Z \in (C^F_{\tau(.)})_h$. For every $Z \in (C^F_{\tau(.)})_h$, we obtain

$$\begin{aligned} h(Y-Z) &= \sum_{q=0}^{\infty}\left(\frac{\sum_{p=0}^{q}\bar{\rho}\left(Y_p - Z_p, \bar{0}\right)}{q+1}\right)^{\tau_q} \\ &\leq 2^{K-1}\left(\sum_{q=0}^{\infty}\left(\frac{\sum_{p=0}^{q}\bar{\rho}\left(Y_p - Z^r_p, \bar{0}\right)}{q+1}\right)^{\tau_q} + \sum_{q=0}^{\infty}\left(\frac{\sum_{p=0}^{q}\bar{\rho}\left(Z^r_p - Z_p, \bar{0}\right)}{q+1}\right)^{\tau_q}\right) \\ &\leq 2^{K-1}\sup_m \inf_{r\geq m} h(Y - Z^r). \end{aligned} \tag{19}$$

□

Example 1. For $(\tau_q) \in [1,\infty)^{\mathcal{N}}$, the function $h(Y) = \inf\{\alpha > 0 : \sum_{q\in\mathcal{N}}(\sum_{p=0}^{q}\bar{\rho}(Y_p/\alpha, \bar{0})/q+1)^{\tau_q} \leq 1\}$ is a norm on $C^F_{\tau(.)}$.

Example 2. The function $h(Y) = \sqrt[3]{\sum_{q\in\mathcal{N}}(\sum_{p=0}^{q}\bar{\rho}(Y_p, \bar{0})/q+1)^{3q+2/q+1}}$ is a pre-quasinorm (not a norm) on $C^F((3q+2/q+1)_{q=0}^{\infty})$.

Example 3. The function $h(Y) = \sum_{q\in\mathcal{N}}(\sum_{p=0}^{q}\bar{\rho}(Y_p, \bar{0})/q+1)^{3q+2/q+1}$ is a pre-quasinorm (not a quasinorm) on $C^F((3q+2/q+1)_{q=0}^{\infty})$.

Example 4. The function $h(Y) = \sqrt[d]{\sum_{q\in\mathcal{N}}(\sum_{p=0}^{q}\bar{\rho}(Y_p, \bar{0})/q+1)^{d}}$ is a pre-quasinorm, quasinorm, and not a norm on C^F_d, for $0 < d < 1$.

In the next part of this section, we will use the function h as $h(Y) = [\sum_{q=0}^{\infty}(\sum_{p=0}^{q}\bar{\rho}(Y_p, \bar{0})/q+1)^{\tau_q}]^{1/K}$, for every $Y \in C_{\tau(.)}^{F}$.

Definition 15 [26]. The function h is said to be strictly convex, (SC), if for all $Y, Z \in U_h$ such that $h(Y) = h(Z)$ and $h(Y + Z/2) = h(Y) + h(Z)/2$, we get $Y = Z$.

Definition 16 [27]. A sequence $\{Y_p\} \subseteq U$ is said to be ε -separated sequence for some $\varepsilon > 0$, if

$$\operatorname{sep}(Y_p) = \inf\{h(Y_p - Y_q): p \neq q\} > \varepsilon. \tag{20}$$

Definition 17 (see [27]). Let $k \geq 2$ be an integer, and a Banach space U is called k-nearly uniformly convex (k-NUC), if for any $\varepsilon > 0$, there exists $\delta \in (0,1)$ such that for any sequence $\{Y_p\} \subseteq B_h(0,1)$, with $sep(Y_p) \geq \varepsilon$, there are $p_1, p_2, p_3, \cdots, p_k \in \mathcal{N}$, such that

$$h\left(\frac{Y_{p_1} + Y_{p_2} + Y_{p_3} + \cdots + Y_{p_k}}{k}\right) < 1 - \delta. \tag{21}$$

Definition 18 (see [28]). A function h is said to satisfy the δ_2 -condition ($h \in \delta_2$), if for any $\varepsilon > 0$, there exists a constant $k \geq 2$ and $a > 0$ such that $h(2u) \leq kh(u) + \varepsilon$, for each $u \in X_h$, with $h(u) \leq a$.

If h satisfies the δ_2-condition for any $a > 0$ with $k \geq 2$ depending on a, we say that h satisfies the strong δ_2-condition ($\rho \in \delta_2^s$).

The following known results are very important for our consideration.

Theorem 19 (see [28], Lemma 2.1). *If $h \in \delta_2^s$, then for any $L > 0$ and $\varepsilon > 0$, there exists $\delta > 0$ such that $|h(x+y) - h(x)| < \varepsilon$, where $x, y \in X_h$, with $h(x) \leq L$ and $h(y) \leq \delta$.*

Theorem 20. *Pick an $(\tau_q)_{q\in\mathcal{N}} \in \ell_\infty \cap I$ with $\tau_0 > 1$; then, for any $L > 0$ and $\varepsilon > 0$, there exists $\delta > 0$ such that $|h(x+y) - h(x)| < \varepsilon$, for all $x, y \in (C_{\tau(.)}^{F})_h$, with $h(x) \leq L$ and $h(y) \leq \delta$.*

Proof. Since (τ_q) is bounded, it is easy to see that $h \in \delta_2^s$. Hence, the proposition is obtained directly from Theorem 19. □

Theorem 21. *Suppose $(\tau_q)_{q\in\mathcal{N}} \in \ell_\infty \cap I$ with $\tau_0 > 1$; then, $(C_{\tau(.)}^{F})_h$ is k-NUC, for any integer $k \geq 2$.*

Proof. Let $\varepsilon \in (0,1)$ and $\{x_n\} \subseteq \mathbf{B}_h(0,1)$ with $sep(x_n) \geq \varepsilon$, for each $m \in \mathcal{N}$, and let $x_n^m = (0, 0, 0, \cdots, x_n(m), x_n(m+1), \cdots)$. Since for each $i \in \mathcal{N}$, $(x_n(i))_{n=0}^{\infty}$ is bounded, and by using the diagonal method, we can find a subsequence (x_{n_j}) of (x_n) such that $(x_{n_j}(i))$ converges for each $i \in \mathcal{N}$, $0 \leq i \leq m$. Therefore, there exists an increasing sequence of positive integers (t_m) such that $sep((x_{n_j}^m)_{j>t_m}) \geq \varepsilon$. Hence, there is a sequence of positive integers $(r_m)_{m=0}^{\infty}$ with $r_0 < r_1 < r_2 < \cdots$, such that

$$h^K\left(x_{r_m}^m\right) \geq \frac{\varepsilon}{2}, \tag{22}$$

for each $m \in \mathcal{N}$. For fixed integer $k \geq 2$, let $\varepsilon_1 = (k^{p_0-1} - 1/(k-1)k^{p_0})(\varepsilon/4)$; then, by Theorem 20, there exists $\delta > 0$ such that

$$\left|h^K(x+y) - h^K(x)\right| < \varepsilon_1, \tag{23}$$

whenever $h^K(x) \leq 1$ and $h^K(y) \leq \delta$. Since $h^K(x_n) \leq 1$, for any $n \in \mathcal{N}$, then there exist positive integers $m_i (i = 0, 1, 2, \cdots, k-2)$ with $m_0 < m_1 < m_2 < \cdots < m_{k-2}$ such that $h^K(x_i^{m_i}) \leq \delta$. Define $m_{k-1} = m_{k-2} + 1$. By inequality (1), we have $h(x_{r_{m_k}}^{m_k}) \geq \varepsilon/2$. Let $s_i = i$ for $0 \leq i \leq k-2$ and $s_{k-1} = r_{m_{k-1}}$. Then, in virtue of inequality (1), inequality (2), and convexity of the function $f_n(u) = |u|^{\tau_n}$ for any $n \in \mathcal{N}$, we have

$$\begin{aligned}
&h^K\left(\frac{x_{s_0} + x_{s_1} + x_{s_2} + . \cdots + x_{s_{k-1}}}{k}\right) \\
&= \sum_{n=0}^{\infty}\left(\frac{\sum_{i=0}^{n}\bar{\rho}\left(x_{s_0}(i) + x_{s_1}(i) + + x_{s_{k-1}}(i)/k, \bar{0}\right)}{n+1}\right)^{\tau_n} \\
&= \sum_{n=0}^{m_1-1}\left(\frac{\sum_{i=0}^{n}\bar{\rho}\left(x_{s_0}(i) + x_{s_1}(i) + + x_{s_{k-1}}(i)/n+1, \bar{0}\right)}{n+1}\right)^{\tau_n} \\
&\quad + \sum_{n=m_1}^{\infty}\left(\frac{\sum_{i=0}^{n}\bar{\rho}\left(x_{s_0}(i) + x_{s_1}(i) + + x_{s_{k-1}}(i)/k, \bar{0}\right)}{n+1}\right)^{\tau_n} \\
&\leq \sum_{n=0}^{m_1-1}\left(\frac{\sum_{i=0}^{n}\bar{\rho}\left(x_{s_0}(i) + x_{s_1}(i) + + x_{s_{k-1}}(i)/k, \bar{0}\right)}{n+1}\right)^{\tau_n} \\
&\quad + \sum_{n=m_1}^{\infty}\left(\frac{\sum_{i=0}^{n}\bar{\rho}\left(x_{s_1}(i) + x_{s_2}(i) + + x_{s_{k-1}}(i)/k, \bar{0}\right)}{n+1}\right)^{\tau_n} \\
&\quad + \varepsilon_1 \leq \sum_{n=0}^{m_1-1}\frac{1}{k}\sum_{j=0}^{k-1}\left(\frac{\sum_{i=0}^{n}\bar{\rho}\left(x_{s_j}(i), \bar{0}\right)}{n+1}\right)^{\tau_n} \\
&\quad + \sum_{n=m_1}^{m_2-1}\left(\frac{\sum_{i=0}^{n}\bar{\rho}\left(x_{s_1}(i) + x_{s_2}(i) + + x_{s_{k-1}}(i)/k, \bar{0}\right)}{n+1}\right)^{\tau_n} \\
&\quad + \sum_{n=m_2}^{\infty}\left(\frac{\sum_{i=0}^{n}\bar{\rho}\left(x_{s_1}(i) + x_{s_2}(i) + + x_{s_{k-1}}(i)/k, \bar{0}\right)}{n+1}\right)^{\tau_n} \\
&\quad + \varepsilon_1 \leq \sum_{n=0}^{m_1-1}\frac{1}{k}\sum_{j=0}^{k-1}\left(\frac{\sum_{i=0}^{n}\bar{\rho}\left(x_{s_j}(i), \bar{0}\right)}{n+1}\right)^{\tau_n} \\
&\quad + \sum_{n=m_1}^{m_2-1}\left(\frac{\sum_{i=0}^{n}\bar{\rho}\left(x_{s_1}(i) + x_{s_2}(i) + + x_{s_{k-1}}(i)/k, \bar{0}\right)}{n+1}\right)^{\tau_n}
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{n=m_2}^{\infty}\left(\frac{\sum_{i=0}^{n}\bar{\rho}\left(x_{s_2}(i)+x_{s_3}(i)++x_{s_{k-1}}(i)/k,\bar{0}\right)}{n+1}\right)^{\tau_n}+2\varepsilon_1\\
&\le\sum_{n=0}^{m_1-1}\frac{1}{k}\sum_{j=0}^{k-1}\left(\frac{\sum_{i=0}^{n}\bar{\rho}\left(x_{s_j}(i),\bar{0}\right)}{n+1}\right)^{\tau_n}\\
&+\sum_{n=m_1}^{m_2-1}\frac{1}{k}\sum_{j=1}^{k-1}\left(\frac{\sum_{i=0}^{n}\bar{\rho}\left(x_{s_j}(i),\bar{0}\right)}{n+1}\right)^{\tau_n}\\
&+\sum_{n=m_2}^{m_3-1}\frac{1}{k}\sum_{j=2}^{k-1}\left(\frac{\sum_{i=0}^{n}\bar{\rho}\left(x_{s_j}(i),\bar{0}\right)}{n+1}\right)^{\tau_n}\\
&++\sum_{n=m_{k-1}}^{m_k-1}\frac{1}{k}\sum_{j=k-2}^{k-1}\left(\frac{\sum_{i=0}^{n}\bar{\rho}\left(x_{s_j}(i),\bar{0}\right)}{n+1}\right)^{\tau_n}\\
&+\sum_{n=m_k}^{\infty}\left(\frac{\sum_{i=0}^{n}\bar{\rho}\left(x_{s_k}(i)/k,\bar{0}\right)}{n+1}\right)^{\tau_n}+(k-1)\varepsilon_1\\
&\le\frac{h^K\left(x_{s_0}+x_{s_1}+x_{s_2}+.\cdots+x_{s_{k-2}}\right)}{k}\\
&+\frac{1}{k}\sum_{n=0}^{m_k-1}\left(\frac{\sum_{i=0}^{n}\bar{\rho}\left(x_{s_k}(i),\bar{0}\right)}{n+1}\right)^{\tau_n}\\
&+\sum_{n=m_k}^{\infty}\left(\frac{\sum_{i=0}^{n}\bar{\rho}\left(x_{s_k}(i)/k,\bar{0}\right)}{n+1}\right)^{\tau_n}+(k-1)\varepsilon_1\le\frac{k-1}{k}\\
&+\frac{1}{k}\sum_{n=0}^{m_k-1}\left(\frac{\sum_{i=0}^{n}\bar{\rho}\left(x_{s_k}(i),\bar{0}\right)}{n+1}\right)^{\tau_n}\\
&+\frac{1}{k^{p_0}}\sum_{n=m_k}^{\infty}\left(\frac{\sum_{i=0}^{n}\bar{\rho}\left(x_{s_k}(i),\bar{0}\right)}{n+1}\right)^{\tau_n}+(k-1)\varepsilon_1\le 1-\frac{1}{k}\\
&+\frac{1}{k}\left(1-\sum_{n=m_k}^{\infty}\left(\frac{\sum_{i=0}^{n}\bar{\rho}\left(x_{s_k}(i),\bar{0}\right)}{n+1}\right)^{\tau_n}\right)+\frac{1}{k^{p_0}}\sum_{n=m_k}^{\infty}\\
&\cdot\left(\frac{\sum_{i=0}^{n}\bar{\rho}\left(x_{s_k}(i),\bar{0}\right)}{n+1}\right)^{\tau_n}+(k-1)\varepsilon_1=1+(k-1)\varepsilon_1\\
&-\left(\frac{k^{p_0-1}-1}{k^{p_0}}\right)\sum_{n=m_k}^{\infty}\left(\frac{\sum_{i=0}^{n}\bar{\rho}\left(x_{s_k}(i),\bar{0}\right)}{n+1}\right)^{\tau_n}\\
&\le 1+(k-1)\varepsilon_1-\left(\frac{k^{p_0-1}-1}{k^{p_0}}\right)\frac{\varepsilon}{2}=1-\left(\frac{k^{p_0-1}-1}{k^{p_0}}\right)\frac{\varepsilon}{4}.
\end{aligned}
\tag{24}
$$

Therefore, $(C^F_{\tau(.)})_h$ is k-NUC. □

Recall that k-NUC implies reflexivity.

Definition 22. The space U_h satisfies the property (R), if and only if, for all decreasing sequence $\{\Gamma_j\}_{j\in\mathcal{N}}$ of h-closed and h-convex nonempty subsets of U_h with $\sup_{j\in\mathcal{N}}\mathfrak{K}_h(Y,\Gamma_j)<\infty$, for some $Y\in U_h$, one has $\bigcap_{j\in\mathcal{N}}\Gamma_j\neq\varnothing$.

By fixing Γ a nonempty h-closed and h-convex subset of $(C^F_{\tau(.)})_h$.

Theorem 23. *If $(\tau_q)_{q\in\mathcal{N}}\in\ell_\infty\cap\mathbf{I}$ with $\tau_0>1$, one has the following:*

(i) Suppose $Y\in(C^F_{\tau(.)})_h$ with $\mathfrak{K}_h(Y,\Gamma)=\inf\{h(Y-Z):Z\in\Gamma\}<\infty$. There is a unique $\lambda\in\Gamma$ so that $\mathfrak{K}_h(Y,\Gamma)=h(Y-\lambda)$

(ii) $(C^F_{\tau(.)})_h$ verifies the property (R).

Proof. To prove (i), assume $Y\notin\Gamma$ as Γ is h-closed. One has $C:=\mathfrak{K}_h(Y,\Gamma)>0$. Hence, for all $r\in\mathcal{N}$, one has $Z_r\in\Gamma$ with $h(Y-Z_r)<C(1+1/r)$. If $\{Z_r/2\}$ is not h-Cauchy, one gets a subsequence $\{Z_{g(r)}/2\}$ and $l_0>0$ with $h(Z_{g(r)}-Z_{g(j)}/2)\ge l_0$, for every $r>j\ge 0$, since

$$
\begin{aligned}
&\max\left(h\left(Y-Z_{g(r)}\right),h\left(Y-Z_{g(j)}\right)\right)\le C\left(1+\frac{1}{g(j)}\right),\\
&h\left(\frac{Z_{g(r)}-Z_{g(j)}}{2}\right)\ge l_0\ge C\left(1+\frac{1}{g(j)}\right)\frac{l_0}{2C},
\end{aligned}
\tag{25}
$$

for every $r>j\ge 0$. Since $(\tau_q)_{q\in\mathcal{N}}\in\ell_\infty\cap\mathbf{I}$ with $\tau_0>1$, then the function $f_n(u)=|u|^{\tau_n}$ is strictly convex, for any $n\in\mathcal{N}$. Therefore, the space $(C^F_{\tau(.)})_h$ is strictly convex; hence,

$$
h\left(Y-\frac{Z_{g(r)}+Z_{g(j)}}{2}\right)<C\left(1+\frac{1}{g(j)}\right). \tag{26}
$$

Then,

$$
C=\mathfrak{K}_h(Y,\Gamma)<C\left(1+\frac{1}{g(j)}\right), \tag{27}
$$

for all $j\in\mathcal{N}$. By putting $j\longrightarrow\infty$, one has a contradiction. So $\{Z_r/2\}$ is h-Cauchy. As $(C^F_{\tau(.)})_h$ is h-complete, then $\{Z_r/2\}h$-converges to some Z. For all $j\in\mathcal{N}$, one gets $\{Z_r+Z_j/2\}h$-converges to $Z+Z_j/2$. Since Γ is h-closed and h-convex, then $Z+Z_j/2\in\Gamma$. Since $Z+Z_j/2h$-converges to $2Z$, then $2Z\in\Gamma$. Let $\lambda=2z$, and from Theorem 13, since h satisfies the Fatou property, one has

$$
\begin{aligned}
\mathfrak{K}_h(Y,\Gamma)&\le h(Y-\lambda)\le\sup_i\inf_{j\ge i}h\left(Y-\left(Z+\frac{Z_j}{2}\right)\right)\\
&\le\sup_i\inf_{j\ge i}\sup_i\inf_{r\ge i}h\left(Y-\frac{Z_r+Z_j}{2}\right)\\
&\le\frac{1}{2}\sup_i\inf_{r\ge i}\sup_i\inf_{r\ge i}\left[h(Y-Z_r)+h(Y-Z_j)\right]\\
&=\mathfrak{K}_h(Y,\Gamma).
\end{aligned}
\tag{28}
$$

Then $h(Y-\lambda)=\mathfrak{K}_h(Y,\Gamma)$. Since h is (SC), this implies the uniqueness of λ. To prove (ii), assume $Y \notin \Gamma_{r_0}$, for some $r_0 \in \mathcal{N}$. Since $(\mathfrak{K}_h(Y,\Gamma_r))_{r\in\mathcal{N}} \in \ell_\infty$ is increasing, put $\lim_{r\longrightarrow\infty}\mathfrak{K}_h(Y,\Gamma_r)=C$, when $C>0$. Otherwise, $Y\in\Gamma_r$, for all $r\in\mathcal{N}$. According to (i), there is one point $Z_r\in\Gamma_r$ with $\mathfrak{K}_h(Y,\Gamma_r)=h(Y-Z_r)$, for every $r\in\mathcal{N}$. A similar proof will prove that $\{Z_r/2\}h$-converges to some $Z\in(C^F_{\tau(.)})_h$. As $\{\Gamma_r\}$ is h-convex, decreasing, and h-closed, one has $2Z\in\cap_{r\in\mathcal{N}}\Gamma_r$. □

Definition 24. The space U_h verifies the h-normal structure-property, if and only if, for all nonempty h-bounded, h-convex and h-closed subset Γ of U_h not decreased to one point, and one has $Y\in\Gamma$ with

$$\sup_{Z\in\Gamma} h(Y-Z) < \delta_h(\Gamma) \coloneqq \sup\{h(Y-Z): Y,Z\in\Gamma\} < \infty. \tag{29}$$

Definition 25 (see [29]). U_h is a real Banach space, and $S(U_h)$ is the unit sphere of U_h. The weakly convergent sequence coefficient of U_h, denoted by $WCS(U_h)$, is defined as follows:

$$\begin{aligned} WCS(\mathbf{U}_h) = \inf\{&A(\{x_n\}): \{x_n\}_{n=1}^{\infty}\subset S(\mathbf{U}_h), A(\{x_n\}) \\ &= A_1(\{x_n\}), x_n \overset{w}{\longrightarrow} 0\}, \end{aligned} \tag{30}$$

where

$$\begin{aligned} A(\{x_n\}) &= \limsup_{n\longrightarrow\infty}\{\|x_i-x_j\|: i,j\ge n, i\ne j\}, \\ A_1(\{x_n\}) &= \liminf_{n\longrightarrow\infty}\{\|x_i-x_j\|: i,j\ge n, i\ne j\}. \end{aligned} \tag{31}$$

Theorem 26 (see [30]). *A reflexive Banach space U_h with $WCS(U_h)>1$ has normal structure-property.*

Theorem 27. *If $(\tau_q)_{q\in\mathcal{N}}\in\ell_\infty\cap I$ with $\tau_0>1$, then $(C^F_{\tau(.)})_h$ holds the h-normal structure-property.*

Proof. Take any $\varepsilon>0$ and an asymptotic equidistant sequence $\{x_n\}\subset S((C^F_{\tau(.)})_h)$ with $x_n \overset{w}{\longrightarrow} 0$ and put $v_1=x_1$. There exists $i_1\in\mathcal{N}$ such that $h(\sum_{i=i_1+1}^{\infty}v_1(i)\bar{\mathfrak{b}}_i)<\varepsilon$. Since $x_n \longrightarrow 0$ coordinate-wise, there exists $n_2\in\mathcal{N}$ such that $h(\sum_{i=1}^{i_1}x_n(i)\bar{\mathfrak{b}}_i)<\varepsilon$, whenever $n\ge n_2$. Take $v_2=x_{n_2}$; then, there is $i_2>i_1$ such that $h(\sum_{i=i_2+1}^{\infty}v_1(i)\bar{\mathfrak{b}}_i)<\varepsilon$. Since $x_n(i)\longrightarrow 0$ coordinate-wise, there exists $n_3\in\mathcal{N}$ such that $h(\sum_{i=1}^{i_2}x_n(i)\bar{\mathfrak{b}}_i)<\varepsilon$, whenever $n\ge n_3$. Continuing this process in such a way by induction, we get a subsequence $\{v_n\}$ of $\{x_n\}$ such that

$$\begin{aligned} h\left(\sum_{i=i_n+1}^{\infty} v_n(i)\bar{\mathfrak{b}}_i\right) &< \varepsilon, \\ h\left(\sum_{i=1}^{i_n} v_{n+1}(i)\bar{\mathfrak{b}}_i\right) &< \varepsilon. \end{aligned} \tag{32}$$

Put $z_n=\sum_{i=i_{n-1}+1}^{i_n}v_n(i)\bar{\mathfrak{b}}_i$, for $n=2,3,\cdots$ Then,

$$\begin{aligned} 1\ge h(z_n) &= h\left(\sum_{i=1}^{\infty}v_n(i)\bar{\mathfrak{b}}_i - \sum_{i=1}^{i_n-1}v_n(i)\bar{\mathfrak{b}}_i - \sum_{i=i_n+1}^{\infty}v_n(i)\bar{\mathfrak{b}}_i\right) \\ &\ge h\left(\sum_{i=1}^{\infty}v_n(i)\bar{\mathfrak{b}}_i\right) - h\left(\sum_{i=1}^{i_n-1}v_n(i)\bar{\mathfrak{b}}_i\right) \\ &\quad - h\left(\sum_{i=i_n+1}^{\infty}v_n(i)\bar{\mathfrak{b}}_i\right) > 1-2\varepsilon. \end{aligned} \tag{33}$$

Moreover, for any $n,m\in\mathcal{N}$ with $n\ne m$, we have

$$\begin{aligned} h(v_n-v_m) &= h\left(\sum_{i=1}^{\infty}v_n(i)\bar{\mathfrak{b}}_i - \sum_{i=1}^{\infty}v_m(i)\bar{\mathfrak{b}}_i\right) \\ &\ge h\left(\sum_{i=i_{n-1}+1}^{i_n}v_n(i)\bar{\mathfrak{b}}_i - \sum_{i=i_{m-1}+1}^{i_m}v_m(i)\bar{\mathfrak{b}}_i\right) \\ &\quad - h\left(\sum_{i=1}^{i_{n-1}}v_n(i)\bar{\mathfrak{b}}_i\right) - h\left(\sum_{i=i_n+1}^{\infty}v_n(i)\bar{\mathfrak{b}}_i\right) \\ &\quad - h\left(\sum_{i=1}^{i_{m-1}}v_m(i)\bar{\mathfrak{b}}_i\right) - h\left(\sum_{i=i_m+1}^{\infty}v_m(i)\bar{\mathfrak{b}}_i\right) \\ &\ge h(z_n-z_m)-4\varepsilon. \end{aligned} \tag{34}$$

This means that $A(\{x_n\})=A(\{v_n\})\ge A(\{z_n\})-4\varepsilon$. Put $u_n=z_n/\|z_n\|$, for $n=2,3,\cdots$ Then,

$$u_n\in S\left(\left(C^F_{\tau(.)}\right)_h\right), \tag{35}$$

$$A(\{x_n\})\ge 1-\varepsilon A(\{u_n\})-4\varepsilon. \tag{36}$$

On the other hand,

$$h(v_n-v_m)\le h(z_n-z_m)+4\varepsilon\le h(u_n-u_m)+4\varepsilon, \tag{37}$$

for any $n,m\in\mathcal{N}$ with $n\ne m$. Therefore,

$$A(\{u_n\})\ge A(\{x_n\})-4\varepsilon. \tag{38}$$

By the arbitrariness of $\varepsilon>0$, we have from the relations (35), (36), and (38) that

$$\mathrm{WCS}\left(\left(C^F_{\tau(.)}\right)_h\right)=\inf\{A(\{u_n\})\}, \tag{39}$$

such that

$$u_n = \sum_{i=i_{n-1}+1}^{i_n} u_n(i)\bar{\mathfrak{b}}_i \in S\left(\left(C^F_{\tau(.)}\right)_h\right), 0 = i_0 < i_1 < \cdots, u_n \overset{w}{\longrightarrow} 0 \text{ and } \{u_n\} \text{ is asymptotic equidistant.} \tag{40}$$

Take $m \in \mathcal{N}$ large enough such that $\sum_{k=i_{m-1}+1}^{\infty}(b/k)^{\tau_k} < \varepsilon$, where $b := \sum_{i=i_{n-1}+1}^{i_n}|u_n(i)|$. We have for $n < m$ that

$$\begin{aligned} h^K(u_n - u_m) &= \sum_{k=i_{n-1}+1}^{i_{m-1}} \left(\frac{1}{k}\sum_{i=1}^{k}\bar{\rho}(u_n(i),\bar{0})\right)^{\tau_k} + \sum_{k=i_{m-1}+1}^{\infty}\left(\frac{1}{k}\left(b + \sum_{i=1}^{k}\bar{\rho}(u_m(i),\bar{0})\right)\right)^{\tau_k} \\ &\geq \sum_{k=i_{n-1}+1}^{i_{m-1}} \left(\frac{1}{k}\sum_{i=1}^{k}\bar{\rho}(u_n(i),\bar{0})\right)^{\tau_k} + \sum_{k=i_{m-1}+1}^{\infty}\left(\frac{1}{k}\sum_{i=1}^{k}\bar{\rho}(u_m(i),\bar{0})\right)^{\tau_k} \\ &= \sum_{k=i_{n-1}+1}^{\infty} \left(\frac{1}{k}\sum_{i=1}^{k}\bar{\rho}(u_n(i),\bar{0})\right)^{\tau_k} - \sum_{k=i_{m-1}+1}^{\infty}\left(\frac{b}{k}\right)^{\tau_k} + \sum_{k=i_{m-1}+1}^{\infty}\left(\frac{1}{k}\sum_{i=1}^{k}\bar{\rho}(u_m(i),\bar{0})\right)^{\tau_k} \\ &> 1 - \varepsilon + 1 = 2 - \varepsilon, \end{aligned} \tag{41}$$

that is, $A_n(\{u_n\}) \geq (2-\varepsilon)^{1/K}$. Note that

$$\left[\sum_{k=i_{m-1}+1}^{\infty}\left(\frac{1}{k}\left(b + \sum_{i=1}^{k}\bar{\rho}(u_m(i),\bar{0})\right)\right)^{\tau_k}\right]^{1/K} \leq \left[\sum_{k=i_{m-1}+1}^{\infty}\left(\frac{b}{k}\right)^{\tau_k}\right]^{1/K} + \left[\sum_{k=i_{m-1}+1}^{\infty}\left(\frac{1}{k}\sum_{i=1}^{k}\bar{\rho}(u_m(i),\bar{0})\right)^{\tau_k}\right]^{1/K} < \varepsilon^{1/K} + 1. \tag{42}$$

Therefore,

$$\begin{aligned} h^K(u_n - u_m) &= \sum_{k=i_{n-1}+1}^{i_{m-1}} \left(\frac{1}{k}\sum_{i=1}^{k}\bar{\rho}(u_m(i),\bar{0})\right)^{\tau_k} + \sum_{k=i_{m-1}+1}^{\infty}\left(\frac{1}{k}\left(b + \sum_{i=1}^{k}\bar{\rho}(u_m(i),\bar{0})\right)\right)^{\tau_k} \\ &\leq \sum_{k=i_{n-1}+1}^{\infty} \left(\frac{1}{k}\sum_{i=1}^{k}\bar{\rho}(u_m(i),\bar{0})\right)^{\tau_k} + \sum_{k=i_{m-1}+1}^{\infty}\left(\frac{1}{k}\left(b + \sum_{i=1}^{k}\bar{\rho}(u_m(i),\bar{0})\right)\right)^{\tau_k} \\ &\leq 1 + \left(1 + \varepsilon^{1/K}\right)^K, \end{aligned} \tag{43}$$

for any $n, m \in \mathcal{N}$ with $n \neq m$. Therefore, $A_n(\{u_n\}) \leq \left(1 + \left(1 + \varepsilon^{1/K}\right)^K\right)^{1/K}$, and by the arbitrariness of $\varepsilon > 0$, we obtain $WCS((C^F_{\tau(.)})_h) = 2^{1/K}$. From Theorem 21 and Theorem 26, the sequence space $(C^F_{\tau(.)})_h$ has the h-normal structure-property. □

4. Kannan Contraction Mapping on $C^F_{\tau(.)}$

In this section, we look at how to configure $(C^F_{\tau(.)})_h$ with different h so that there is only one fixed point of Kannan contraction mapping.

Definition 28. An operator $V : U_h \longrightarrow U_h$ is said to be a Kannan h-contraction, if one gets $\alpha \in [0, 1/2)$ with $h(VY - VZ) \leq \alpha(h(VY - Y) + h(VZ - Z))$, for all $Y, Z \in U_h$. The operator V is called Kannan h-nonexpansive, when $\alpha = 1/2$.

An element $Y \in \mathbf{U}_h$ is called a fixed point of V when $V(Y) = Y$.

Theorem 29. *If $(\tau_q)_{q\in\mathcal{N}} \in \ell_\infty \cap I$ with $\tau_0 > 1$, and $V : (C^F_{\tau(.)})_h \longrightarrow (C^F_{\tau(.)})_h$ is Kannan h-contraction mapping, where $h(Y) = [\sum_{q=0}^{\infty}(\sum_{p=0}^{q}\bar{\rho}(Y_p, \bar{0})/q + 1)^{\tau_q}]^{1/K}$, for all $Y \in C^F_{\tau(.)}$, then V has a unique fixed point.*

Proof. If $Y \in C^F_{\tau(.)}$, one has $V^pY \in C^F_{\tau(.)}$. As V is a Kannan h-contraction mapping, one gets

$$\begin{aligned} h\left(V^{l+1}Y - V^lY\right) &\leq \alpha\left(h\left(V^{l+1}Y - V^lY\right) + h\left(V^lY - V^{l-1}Y\right)\right) \\ &\Rightarrow h\left(V^{l+1}Y - V^lY\right) \leq \frac{\alpha}{1-\alpha}h\left(V^lY - V^{l-1}Y\right) \\ &\leq \left(\frac{\alpha}{1-\alpha}\right)^2 h\left(V^{l-1}Y - V^{l-2}Y\right) \\ &\leq \leq \left(\frac{\alpha}{1-\alpha}\right)^l h(VY - Y). \end{aligned} \tag{44}$$

So for all $l, m \in \mathcal{N}$ with $m > l$, one gets

$$\begin{aligned} h\left(V^lY - V^mY\right) &\leq \alpha\left(h\left(V^lY - V^{l-1}Y\right) + h\left(V^mY - V^{m-1}Y\right)\right) \\ &\leq \alpha\left(\left(\frac{\alpha}{1-\alpha}\right)^{l-1} + \left(\frac{\alpha}{1-\alpha}\right)^{m-1}\right)h(VY - Y). \end{aligned} \tag{45}$$

Then, $\{V^lY\}$ is a Cauchy sequence in $(C^F_{\tau(.)})_h$. As the space $(C^F_{\tau(.)})_h$ is pre-quasi-Banach space, one has $Z \in (C^F_{\tau(.)})_h$ with $\lim_{l\longrightarrow\infty}V^lY = Z$. To prove that $VZ = Z$, since h has the Fatou property, one obtains

$$\begin{aligned} h(VZ - Z) &\leq \sup_i \inf_{l\geq i} h\left(V^{l+1}Y - V^lY\right) \\ &\leq \sup_i \inf_{l\geq i}\left(\frac{\alpha}{1-\alpha}\right)^l h(VY - Y) = 0, \end{aligned} \tag{46}$$

and then, $VZ = Z$. So Z is a fixed point of V. To show the uniqueness. Let $Y, Z \in (C^F_{\tau(.)})_h$ be two not equal fixed points of V. One has

$$h(Y - Z) \leq h(VY - VZ) \leq \alpha(h(VY - Y) + h(VZ - Z)) = 0. \tag{47}$$

So, $Y = Z$. □

Corollary 30. *If $(\tau_q)_{q\in\mathcal{N}} \in \ell_\infty \cap I$ with $\tau_0 > 1$, and $V : (C^F_{\tau(.)})_h \longrightarrow (C^F_{\tau(.)})_h$ is Kannan h-contraction mapping, where $h(Y) = [\sum_{q=0}^{\infty}(\sum_{p=0}^{q}\bar{\rho}(Y_p, \bar{0})/q + 1)^{\tau_q}]^{1/K}$, for all $Y \in C^F_{\tau(.)}$, one has V has unique fixed point Z so that $h(V^l Y - Z) \leq \alpha(a/1 - \alpha)^{l-1} h(VY - Y)$.*

Proof. In view of Theorem 29, one has a unique fixed point Z of V. So

$$\begin{aligned} h\left(V^l Y - Z\right) &= h\left(V^l Y - VZ\right) \\ &\leq \alpha\left(h\left(V^l Y - V^{l-1} Y\right) + h(VZ - Z)\right) \\ &= \alpha\left(\frac{\alpha}{1-\alpha}\right)^{l-1} h(VY - Y). \end{aligned} \tag{48}$$

□

Example 5. Assume $V : (C^F((2q + 3/q + 2)_{q=0}^{\infty}))_h \longrightarrow (C^F((2q + 3/q + 2)_{q=0}^{\infty}))_h$, where $h(g) = \sqrt{\sum_{q=0}^{\infty}(\sum_{p=0}^{q}\bar{\rho}(g_p, \bar{0})/q + 1)^{2q+3/q+2}}$, for every $g \in C^F((2q + 3/q + 2)_{q=0}^{\infty})$ and

$$V(g) = \begin{cases} \dfrac{g}{4}, & h(g) \in [0, 1), \\ \dfrac{g}{5}, & h(g) \in [1, \infty). \end{cases} \tag{49}$$

As for each $g_1, g_2 \in (C^F((2q + 3/q + 2)_{q=0}^{\infty}))_h$ with $h(g_1), h(g_2) \in [0, 1)$, one has

$$\begin{aligned} h(Vg_1 - Vg_2) &= h\left(\frac{g_1}{4} - \frac{g_2}{4}\right) \leq \frac{1}{\sqrt[4]{27}}\left(h\left(\frac{3g_1}{4}\right) + h\left(\frac{3g_2}{4}\right)\right) \\ &= \frac{1}{\sqrt[4]{27}}(h(Vg_1 - g_1) + h(Vg_2 - g_2)). \end{aligned} \tag{50}$$

For all $g_1, g_2 \in (C^F((2q + 3/q + 2)_{q=0}^{\infty}))_h$ with $h(g_1), h(g_2) \in [1, \infty)$, one has

$$\begin{aligned} h(Vg_1 - Vg_2) &= h\left(\frac{g_1}{5} - \frac{g_2}{5}\right) \leq \frac{1}{\sqrt[4]{64}}\left(h\left(\frac{4g_1}{5}\right) + h\left(\frac{4g_2}{5}\right)\right) \\ &= \frac{1}{\sqrt[4]{64}}(h(Vg_1 - g_1) + h(Vg_2 - g_2)). \end{aligned} \tag{51}$$

For all $g_1, g_2 \in (C^F((2q + 3/q + 2)_{q=0}^{\infty}))_h$ with $h(g_1) \in [0, 1)$ and $h(g_2) \in [1, \infty)$, we get

$$\begin{aligned} h(Vg_1 - Vg_2) &= h\left(\frac{g_1}{4} - \frac{g_2}{5}\right) \leq \frac{1}{\sqrt[4]{27}} h\left(\frac{3g_1}{4}\right) + \frac{1}{\sqrt[4]{64}} h\left(\frac{4g_2}{5}\right) \\ &\leq \frac{1}{\sqrt[4]{27}}\left(h\left(\frac{3g_1}{4}\right) + h\left(\frac{4g_2}{5}\right)\right) \\ &= \frac{1}{\sqrt[4]{27}}(h(Vg_1 - g_1) + h(Vg_2 - g_2)). \end{aligned} \tag{52}$$

Hence, V is Kannan h-contraction. As h satisfies the Fatou property, from Theorem 29, one has V holds one fixed point $\bar{\vartheta} \in (C^F((2q + 3/q + 2)_{q=0}^{\infty}))_h$.

Definition 31. Pick up U_h be a pre-quasinormed (cssf), $V : U_h \longrightarrow U_h$, and $Z \in U_h$. The operator V is called h-sequentially continuous at Z, if and only if when $\lim_{q\longrightarrow\infty} h(Y_q - Z) = 0$, then $\lim_{q\longrightarrow\infty} h(VY_q - VZ) = 0$.

Example 6. Suppose $V : (C^F((q + 1/2q + 4)_{q=0}^{\infty}))_h \longrightarrow (C^F((q + 1/2q + 4)_{q=0}^{\infty}))_h$, where $h(Z) = [\sum_{q=0}^{\infty}(\sum_{p=0}^{q}\bar{\rho}(Z_p, \bar{0})/q + 1)^{q+1/2q+4}]^4$, for every $Z \in C^F((q + 1/2q + 4)_{q=0}^{\infty})$ and

$$V(Z) = \begin{cases} \dfrac{1}{18}\left(\bar{\mathfrak{b}}_0 + Z\right), & Z_0(y) \in \left[0, \dfrac{1}{17}\right), \\ \dfrac{1}{17}\bar{\mathfrak{b}}_0, & Z_0(y) = \dfrac{1}{17}, \\ \dfrac{1}{18}\bar{\mathfrak{b}}_0, & Z_0(y) \in \left(\dfrac{1}{17}, 1\right]. \end{cases} \tag{53}$$

V is clearly both h-sequentially continuous and discontinuous at $1/17\bar{\mathfrak{b}}_0 \in (C^F((q + 1/2q + 4)_{q=0}^{\infty}))_h$.

Example 7. Assume V is defined as in Example 5. Suppose $\{Z^{(n)}\} \subseteq (C^F((2q + 3/q + 2)_{q=0}^{\infty}))_h$ such that $\lim_{n\longrightarrow\infty} h(Z^{(n)} - Z^{(0)}) = 0$, where $Z^{(0)} \in (C^F((2q + 3/q + 2)_{q=0}^{\infty}))_h$ with $h(Z^{(0)}) = 1$.

As the pre-quasinorm h is continuous, we have

$$\lim_{n\longrightarrow\infty} h\left(VZ^{(n)} - VZ^{(0)}\right) = \lim_{n\longrightarrow\infty} h\left(\frac{Z^{(n)}}{4} - \frac{Z^{(0)}}{5}\right) = h\left(\frac{Z^{(0)}}{20}\right) > 0. \tag{54}$$

Therefore, V is not h-sequentially continuous at $Z^{(0)}$.

Theorem 32. *If $(\tau_q)_{q\in\mathcal{N}} \in \ell_\infty \cap I$ with $\tau_0 > 1$, $V : (C^F_{\tau(.)})_h \longrightarrow (C^F_{\tau(.)})_h$, where $h(Y) = \sum_{q=0}^{\infty} (\sum_{p=0}^{q} \bar{\rho}(Y_p, \bar{0})/q+1)^{\tau_q}$, for all $Y \in C^F_{\tau(.)}$. Suppose*

(1) V is Kannan h-contraction mapping

(2) V is h-sequentially continuous at $Z \in (C^F_{\tau(.)})_h$

(3) there is $Y \in (C^F_{\tau(.)})_h$ with $\{V^l Y\}$ has $\{V^{l_j} Y\}$ converging to Z

Then, $Z \in (C^F_{\tau(.)})_h$ is the only fixed point of V.

Proof. Assume Z is not a fixed point of V, and one has $VZ \neq Z$. From parts (2) and (4), we get

$$\lim_{l_j \longrightarrow \infty} h\left(V^{l_j} Y - Z\right) = 0,$$
$$\lim_{l_j \longrightarrow \infty} h\left(V^{l_j+1} Y - VZ\right) = 0. \tag{55}$$

As V is Kannan h-contraction, one obtains

$$\begin{aligned} 0 < h(VZ - Z) &= h\left(\left(VZ - V^{l_j+1}Y\right) + \left(V^{l_j}Y - Z\right) + \left(V^{l_j+1}Y - V^{l_j}Y\right)\right) \leq 2^{2\sup_i \tau_i - 2} \\ &\cdot h\left(V^{l_j+1}Y - VZ\right) + 2^{2\sup_i \tau_i - 2} h\left(V^{l_j}Y - Z\right) \\ &+ 2^{\sup_i \tau_i - 1} \alpha \left(\frac{\alpha}{1-\alpha}\right)^{l_j - 1} h(VY - Y). \end{aligned} \tag{56}$$

As $l_j \longrightarrow \infty$, one has a contradiction. Then, Z is a fixed point of V. To show the uniqueness, let $Z, Y \in (C^F_{\tau(.)})_h$ be two not equal fixed points of V. One obtains

$$h(Z - Y) \leq h(VZ - VY) \leq \alpha(h(VZ - Z) + h(VY - Y)) = 0. \tag{57}$$

Hence, $Z = Y$. □

Example 8. Assume V is defined as in Example 5. Let $h(Y) = \sum_{q\in\mathcal{N}} (\sum_{p=0}^{q} \bar{\rho}(Y_p, \bar{0})/q+1)^{2q+3/q+2}$, for all $v \in C^F((2q+3/q+2)_{q=0}^{\infty})$. Since for all $Y_1, Y_2 \in (C^F((2q+3/q+2)_{q=0}^{\infty}))_h$ with $h(Y_1), h(Y_2) \in [0,1)$, one gets $h(VY_1 - VY_2) = h(Y_1/4 - Y_2/4) \leq 2/\sqrt{27}(h(3Y_1/4) + h(3Y_2/4)) = 2/\sqrt{27}(h(VY_1 - Y_1) + h(VY_2 - Y_2))$. For all $Y_1, Y_2 \in (C^F((2q+3/q \;+2)_{q=0}^{\infty}))_h$ with $h(Y_1), h(Y_2) \in [1,\infty)$, one gets

$$\begin{aligned} h(VY_1 - VY_2) &= h\left(\frac{Y_1}{5} - \frac{Y_2}{5}\right) \leq \frac{1}{4}\left(h\left(\frac{4Y_1}{5}\right) + h\left(\frac{4Y_2}{5}\right)\right) \\ &= \frac{1}{4}(h(VY_1 - Y_1) + h(VY_2 - Y_2)). \end{aligned} \tag{58}$$

For all $Y_1, Y_2 \in (C^F((2q+3/q+2)_{q=0}^{\infty}))_h$ with $h(Y_1) \in [0,1)$ and $h(Y_2) \in [1,\infty)$, one gets

$$\begin{aligned} h(VY_1 - VY_2) &= h\left(\frac{Y_1}{4} - \frac{Y_2}{5}\right) \leq \frac{2}{\sqrt{27}} h\left(\frac{3Y_1}{4}\right) + \frac{1}{4} h\left(\frac{4Y_2}{5}\right) \\ &\leq \frac{2}{\sqrt{27}}\left(h\left(\frac{3Y_1}{4}\right) + h\left(\frac{4Y_2}{5}\right)\right) \\ &= \frac{2}{\sqrt{27}}(h(VY_1 - Y_1) + h(VY_2 - Y_2)). \end{aligned} \tag{59}$$

So V is Kannan h-contraction and $V^p(Y) = \begin{cases} Y/4^p, & h(Y) \in [0,1), \\ Y/5^p, & h(Y) \in [1,\infty). \end{cases}$

Obviously, V is h-sequentially continuous at $\bar{\vartheta} \in (C^F((2q+3/q+2)_{q=0}^{\infty}))_h$, and $\{V^p Y\}$ holds $\{V^{l_j} Y\}$ converges to $\bar{\vartheta}$. By Theorem 32, the point $\bar{\vartheta} \in (C^F((2q+3/q+2)_{q=0}^{\infty}))_h$ is the only fixed point of V.

5. Kannan Nonexpansive Mapping on $(C^F_{\tau(.)})_h$

We introduce the sufficient conditions of $(C^F_{\tau(.)})_h$, where $h(g) = [\sum_{m=0}^{\infty} \bar{\rho}(g_m, \bar{0})^{\tau_m}]^{1/K}$, for every $g \in C^F_{\tau(.)}$, such that the Kannan nonexpansive mapping on it has a fixed point, by fixing Γ a nonempty h-bounded, h-convex, and h-closed subset of $(C^F_{\tau(.)})_h$.

Lemma 33. *If $(C^F_{\tau(.)})_h$ verifies the (R) property and the h-quasinormal property. Assume $V : \Gamma \longrightarrow \Gamma$ is a Kannan h-nonexpansive mapping. For $t > 0$, let $G_t = \{Y \in \Gamma : h(Y - V(Y)) \leq t\} \neq \varnothing$. Put*

$$\Gamma_t = \bigcap \{\mathbf{B}_h(r, j) \colon V(G_t) \subset \mathbf{B}_h(r, j)\} \cap \Gamma. \tag{60}$$

Then, $\Gamma_t \neq \varnothing$, h-convex, h-closed subset of Γ, and $V(\Gamma_t) \subset \Gamma_t \subset G_t$ and $\delta_h(\Gamma_t) \leq t$.

Proof. Since $V(G_t) \subset \Gamma_t$, then $\Gamma_t \neq \varnothing$. As the h-balls are h-convex and h-closed, then Γ_t is a h-closed and h-convex subset of Γ. To show that $\Gamma_t \subset G_t$, assume $Y \in \Gamma_t$. When $h(Y - V(Y)) = 0$, one has $Y \in G_t$. Else, assume $h(Y - V(Y)) > 0$. Put

$$r = \sup \{h(V(Z) - V(Y)) \colon Z \in G_t\}. \tag{61}$$

From the definition of r, one gets $V(G_t) \subset \mathbf{B}_h(V(Y), r)$.

Therefore, $\Gamma_t \subset \mathbf{B}_h(V(Y), r)$, then $h(Y - V(Y)) \le r$. Let $l > 0$. One has $Z \in G_t$ with $r - l \le h(V(Z) - V(Y))$. So

$$\begin{aligned} h(Y - V(Y)) - l \le r - l \le h(V(Z) - V(Y)) \\ \le \frac{1}{2}(h(Y - V(Y)) + h(Z - V(Z))) \\ \le \frac{1}{2}(h(Y - V(Y)) + t). \end{aligned} \tag{62}$$

As l is an arbitrary positive, one obtains $h(Y - V(Y)) \le t$; then, $Y \in G_t$. Since $V(G_t) \subset \Gamma_t$, one gets $V(\Gamma_t) \subset V(G_t) \subset \Gamma_t$, so Γ_t is V-invariant, to show that $\delta_h(\Gamma_t) \le t$, since

$$h(V(Y) - V(Z)) \le \frac{1}{2}(h(Y - V(Y))) + h(Z - V(Z))), \tag{63}$$

for all $Y, Z \in G_t$. Let $Y \in G_t$. Then, $V(G_t) \subset \mathbf{B}_h(V(Y), t)$. The definition of Γ_t gives $\Gamma_t \subset \mathbf{B}_h(V(Y), t)$. Therefore, $V(Y) \in \bigcap_{t\in\Gamma_t} \mathbf{B}_h(Z, t)$. One has $h(Z - Y) \le t$, for all $Z, Y \in \Gamma_t$, so $\delta_h(\Gamma_t) \le t$. □

Theorem 34. *If $(C^F_{\tau(.)})_h$ satisfies the h-quasinormal property and the (R) property, let $V : \Gamma \longrightarrow \Gamma$ be a Kannan h-nonexpansive mapping. Then, V has a fixed point.*

Proof. Let $t_0 = \inf\{h(Y - V(Y)) : Y \in \Gamma\}$ and $t_r = t_0 + 1/r$, for every $r \ge 1$. By the definition of t_0, one gets $G_{t_r} = \{Y \in \Gamma : h(Y - V(Y)) \le t_r\} \ne \varnothing$, for every $r \ge 1$. Assume Γ_{t_r} is defined as in Lemma 33. Clearly, $\{\Gamma_{t_r}\}$ is a decreasing sequence of nonempty h-bounded, h-closed, and h-convex subsets of Γ. The property (R) investigates that $\Gamma_\infty = \bigcap_{r\ge1} \Gamma_{t_r} \ne \varnothing$. Let $Y \in \Gamma_\infty$, and one has $h(Y - V(Y)) \le t_r$, for all $r \ge 1$. Suppose $r \longrightarrow \infty$; then, $h(Y - V(Y)) \le t_0$, so $h(Y - V(Y)) = t_0$. Therefore, $G_{t_0} \ne \varnothing$. Then, $t_0 = 0$. Else, $t_0 > 0$; then, V fails to have a fixed point. Let Γ_{t_0} be defined in Lemma 33. As V fails to have a fixed point and Γ_{t_0} is V-invariant, then Γ_{t_0} has more than one point, so $\delta_h(\Gamma_{t_0}) > 0$. By the h-quasinormal property, one has $Y \in \Gamma_{t_0}$ with

$$h(Y - Z) < \delta_h(\Gamma_{t_0}) \le t_0, \tag{64}$$

for all $Z \in \Gamma_{t_0}$. From Lemma 33, we get $\Gamma_{t_0} \subset G_{t_0}$. From definition of Γ_{t_0}, $V(Y) \in G_{t_0} \subset \Gamma_{t_0}$. Then,

$$h(Y - V(Y)) < \delta_h(\Gamma_{t_0}) \le t_0, \tag{65}$$

which contradicts the definition of t_0. Then, $t_0 = 0$ which gives that any point in G_{t_0} is a fixed point of V. □

According to Theorems 23, 27, and 34, we conclude the following:

Corollary 35. *Assume $(\tau_q)_{q\in\mathcal{N}} \in \ell_\infty \cap I$ with $\tau_0 > 1$, and $V : \Gamma \longrightarrow \Gamma$ is a Kannan h-nonexpansive mapping. Then, V has a fixed point.*

Example 9. Assume $V : \Gamma \longrightarrow \Gamma$ with $V(Y) = \begin{cases} Y/4, & h(Y) \in [0, 1), \\ Y/5, & h(Y) \in [1, \infty), \end{cases}$ where $\Gamma = \{Y \in (C^F((2q + 3/q + 2)_{q=0}^\infty))_h : Y_0 = Y_1 = \bar{0}\}$ and $h(Y) = \sqrt{\sum_{q\in\mathcal{N}} \bar{\rho}(Y_q, \bar{0})^{2q+3/q+2}}$, for every $Y \in (C^F((2q + 3/q + 2)_{q=0}^\infty))_h$. By using Example 8, V is Kannan h-contraction. So it is Kannan h-nonexpansive. By Corollary 35, V has a fixed point $\bar{\vartheta}$ in Γ.

6. Kannan Contraction and Structure of Operator Ideal

The structure of the operator ideal by $(C^F_{\tau(.)})_h$ equipped with the definite function h, where $h(g) = [\sum_{q=0}^\infty (\sum_{p=0}^q \bar{\rho}(g_p, \bar{0})/q + 1)^{\tau_q}]^{1/K}$, for every $g \in C^F_{\tau(.)}$, and s-numbers has been explained. Finally, we examine the idea of Kannan contraction mapping in its associated pre-quasioperator ideal. As well, the existence of a fixed point of Kannan contraction mapping has been introduced. We indicate the space of all bounded, finite rank linear operators from a Banach space Δ into a Banach space Λ by $\mathscr{L}(\Delta, \Lambda)$, and $\mathfrak{F}(\Delta, \Lambda)$, and if $\Delta = \Lambda$, we inscribe $\mathscr{L}(\Delta)$ and $\mathfrak{F}(\Delta)$.

Definition 36 (see [31]). An s-number function is $s : \mathscr{L}(\Delta, \Lambda) \longrightarrow \mathfrak{R}^{+\mathcal{N}}$ which sorts every $V \in \mathscr{L}(\Delta, \Lambda)$ a $(s_d(V))_{d=0}^\infty$ verifies the following settings:

(a) $\|V\| = s_0(V) \ge s_1(V) \ge s_2(V) \ge \cdots \ge 0$, for all $V \in \mathscr{L}(\Delta, \Lambda)$

(b) $s_{l+d-1}(V_1 + V_2) \le s_l(V_1) + s_d(V_2)$, for all $V_1, V_2 \in \mathscr{L}(\Delta, \Lambda)$ and $l, d \in \mathcal{N}$

(c) $s_d(VYW) \le \|V\| s_d(Y) \|W\|$, for all $W \in \mathscr{L}(\Delta_0, \Delta)$, $Y \in \mathscr{L}(\Delta, \Lambda)$, and $V \in \mathscr{L}(\Lambda, \Lambda_0)$, where Δ_0 and Λ_0 are arbitrary Banach spaces

(d) If $V \in \mathscr{L}(\Delta, \Lambda)$ and $\gamma \in \mathfrak{R}$, then $s_d(\gamma V) = |\gamma| s_d(V)$

(e) Suppose rank $(V) \le d$, and then, $s_d(V) = 0$, for each $V \in \mathscr{L}(\Delta, \Lambda)$

(f) $s_{l\ge a}(I_a) = 0$ or $s_{l<a}(I_a) = 1$, where I_a denotes the unit map on the a-dimensional Hilbert space ℓ_2^a

Definition 37 (see [8]).

(i) $\mathscr{L}$ is the class of all bounded linear operators within any two arbitrary Banach spaces. A subclass $\mathscr{U}$ of $\mathscr{L}$ is said to be an operator ideal, if all $\mathscr{U}(\Delta, \Lambda) = \mathscr{U} \cap \mathscr{L}(\Delta, \Lambda)$ verifies the following conditions: $I_\Gamma \in \mathscr{U}$, where Γ denotes Banach space of one dimension

(ii) The space $\mathscr{U}(\Delta, \Lambda)$ is linear over $\mathfrak{R}$

(iii) Assume $W \in \mathscr{L}(\Delta_0, \Delta)$, $X \in \mathscr{U}(\Delta, \Lambda)$, and $Y \in \mathscr{L}(\Lambda, \Lambda_0)$, then $YXW \in \mathscr{U}(\Delta_0, \Lambda_0)$

Notation 38.

$$\Psi_{\mathbf{U}} := \{\Psi_{\mathbf{U}}(\Delta, \Lambda)\} \quad (66)$$

,where

$$\Psi_{\mathbf{U}}(\Delta, \Lambda) := \left\{V \in \mathscr{L}(\Delta, \Lambda): \left(\left(s_d(\bar{V})\right)_{d=0}^{\infty} \in \mathbf{U}\right\}, \quad (67)$$

where

$$s_d(\bar{V})(x) = \begin{cases} 1, & x = s_d(V), \\ 0, & x \neq s_d(V). \end{cases} \quad (68)$$

Theorem 39. *Suppose U is a (cssf); then, Ψ_U is an operator ideal.*

Proof.

(i) Assume $V \in \mathfrak{F}(\Delta, \Lambda)$ and rank $(V) = n$ for all $n \in \mathscr{N}$; as $\bar{\mathfrak{b}}_i \in \mathbf{U}$ for all $i \in \mathscr{N}$ and $\mathbf{U}$ is a linear space, one has

$(s_i(V)\bar{\ })_{i=0}^{\infty} = (s_0(\bar{V}), s_1(\bar{V}), \cdots, s_{n1}(\bar{V}), \bar{0}, \bar{0}, \bar{0}, \cdots) = \sum_{i=0}^{n-1} s_i(\bar{V})\bar{\mathfrak{b}}_i \in \mathbf{U}$; for that $V \in \Psi_{\mathbf{U}}(\Delta, \Lambda)$ then $\mathfrak{F}(\Delta, \Lambda) \subseteq \Psi_E(\Delta, \Lambda)$.

(ii) Suppose $V_1, V_2 \in \Psi_{\mathbf{U}}(\Delta, \Lambda)$ and $\beta_1, \beta_2 \in \mathfrak{R}$, then by Definition 4 condition (33), one has $(s_{[i/2]}(\bar{V}_1))_{i=0}^{\infty} \in \mathbf{U}$ and $(s_{[i/2]}(\bar{V}_1))_{i=0}^{\infty} \in \mathbf{U}$, as $i \geq 2[i/2]$; by the definition of s-numbers and $s_i(P)$ is a decreasing sequence, one gets $s_i(\beta_1 V_1 \bar{+} \beta_2 V_2) \leq s_{2[i/2]}(\bar{\beta}_1 V_1 + \beta_2 V_2) \leq s_{[i/2]}(\beta_1 V_1) \bar{+} s_{[i/2]}(\beta_2 V_2) = |\beta_1| s_{[i/2]}(\bar{V}_1) + |\beta_2| s_{[i/2]}(\bar{V}_2)$, for each $i \in \mathscr{N}$. In view of Definition 4 condition (23) and $\mathbf{U}$ is a linear space, one obtains $(s_i(\beta_1 V_1 \bar{+} \beta_2 V_2))_{i=0}^{\infty} \in \mathbf{U}$; hence, $\beta_1 V_1 + \beta_2 V_2 \in \Psi_{\mathbf{U}}(\Delta, \Lambda)$.

(iii) Suppose $P \in \mathscr{L}(\Delta_0, \Delta)$, $T \in \Psi_{\mathbf{U}}(\Delta, \Lambda)$, and $R \in \mathscr{L}(\Lambda, \Lambda_0)$, one has $(s_i(\bar{T}))_{i=0}^{\infty} \in \mathbf{U}$, and as $s_i(R\bar{T}P) \leq \|R\| s_i(\bar{T}) \|P\|$, by Definition 4 conditions (22) and (23), one gets

$(s_i(R\bar{T}P))_{i=0}^{\infty} \in \mathbf{U}$, and then, $RTP \in \Psi_{\mathbf{U}}(\Delta_0, \Lambda_0)$. □

According to Theorems 10 and 39, one concludes the following theorem.

Theorem 40. *Let $(\tau_q)_{q\in\mathscr{N}} \in \ell_\infty \cap I$ with $\tau_0 > 1$, and one has $\Psi_{(C^F_{\tau(.)})_h}$ is an operator ideal.*

Definition 41 (see [9]). A function $H \in [0,\infty)^{\mathscr{U}}$ is called a pre-quasinorm on the ideal $\mathscr{U}$ if the next conditions hold:

(1) Let $V \in \mathscr{U}(\Delta, \Lambda)$, $H(V) \geq 0$, and $H(V) = 0$, if and only if $V = 0$

(2) We have $Q \geq 1$ so as to $H(\alpha V) \leq D|\alpha| H(V)$, for every $V \in \mathscr{U}(\Delta, \Lambda)$ and $\alpha \in \mathfrak{R}$

(3) We have $P \geq 1$ so that $H(V_1 + V_2) \leq P[H(V_1) + H(V_2)]$, for each $V_1, V_2 \in \mathscr{U}(\Delta, \Lambda)$

(4) We have $\sigma \geq 1$ for $V \in \mathscr{L}(\Delta_0, \Delta)$, $X \in \mathscr{U}(\Delta, \Lambda)$, and $Y \in \mathscr{L}(\Lambda, \Lambda_0)$; then, $H(YXV) \leq \sigma \|Y\| H(X) \|V\|$.

Theorem 42 (see [9]). *H is a pre-quasinorm on the ideal $\mathscr{U}$ if H is a quasinorm on the ideal $\mathscr{U}$.*

Theorem 43. *If $(\tau_q)_{q\in\mathscr{N}} \in \ell_\infty \cap I$ with $\tau_0 > 1$, then the function H is a pre-quasinorm on $\Psi_{(C^F_{\tau(.)})_h}$, with $H(Z) = h(s_q(\bar{Z}))_{q=0}^{\infty}$, for all $Z \in \Psi_{(C^F_{\tau(.)})_h}(\Delta, \Lambda)$.*

Proof.

(1) When $X \in \Psi_{(C^F_{\tau(.)})_h}(\Delta, \Lambda)$, $H(X) = h(s_q(\bar{X}))_{q=0}^{\infty} \geq 0$ and $H(X) = h(s_q(\bar{X}))_{q=0}^{\infty} = 0$, if and only if $s_q(\bar{X}) = \bar{0}$, for all $q \in \mathscr{N}$, if and only if $X = 0$

(2) There is $Q \geq 1$ with $H(\alpha X) = h(s_q(\bar{\alpha} X))_{q=0}^{\infty} \leq Q|\alpha| H(X)$, for all $X \in \Psi_{(C^F_{\tau(.)})_h}(\Delta, \Lambda)$ and $\alpha \in \mathfrak{R}$

(3) One has $PP_0 \geq 1$ so that for $X_1, X_2 \in \Psi_{(C^F_{\tau(.)})_h}(\Delta, \Lambda)$, one can see

$$\begin{aligned} H(X_1 + X_2) &= h\left(s_q(X_1 \bar{+} X_2)\right)_{q=0}^{\infty} \\ &\leq P\left(h\left(s_{[q/2]}(\bar{X}_1)\right)_{q=0}^{\infty} + h\left(s_{[q/2]}(\bar{X}_2)\right)_{q=0}^{\infty}\right) \\ &\leq PP_0\left(h\left(s_q(\bar{X}_1)\right)_{q=0}^{\infty} + h\left(s_q(\bar{X}_2)\right)_{q=0}^{\infty}\right) \end{aligned} \quad (69)$$

(4) We have $\rho \geq 1$, if $X \in \mathscr{L}(\Delta_0, \Delta)$, $Y \in \Psi_{(C^F_{\tau(.)})_h}(\Delta, \Lambda)$, and $Z \in \mathscr{L}(\Lambda, \Lambda_0)$, and then, $H(ZYX) = h(s_q(\bar{Z}YX))_{q=0}^{\infty} \leq h(\|X\|\|Z\| s_q(\bar{Y}))_{q=0}^{\infty} \leq \rho \|X\| H(Y) \|Z\|$.

□

In the next theorems, we will use the notation $(\Psi_{(C^F_{\tau(.)})_h}, H)$, where $H(V) = h((s_q(\bar{V}))_{q=0}^{\infty})$, for all $V \in \Psi_{(C^F_{\tau(.)})_h}$.

Theorem 44. *Suppose $(\tau_q)_{q\in\mathscr{N}} \in \ell_\infty \cap I$ with $\tau_0 > 1$, and one has $(\Psi_{(C^F_{\tau(.)})_h}, H)$ is a pre-quasi-Banach operator ideal.*

Proof. Suppose $(V_a)_{a\in\mathcal{N}}$ is a Cauchy sequence in $\bar{\Psi}_{(C^F_{\tau(.)})_h}(\Delta,\Lambda)$. As $\mathscr{L}(\Delta,\Lambda)\supseteq S_{(C^F_{\tau(.)})_h}(\Delta,\Lambda)$, one has

$$H(V_r-V_a)=h\Big(\big(s_q(\bar{V_r V_a})\big)_{q=0}^{\infty}\Big)\geq h\big(s_0(\bar{V_r V_a}),\bar{0},\bar{0},\bar{0},\cdots\big)\geq\inf_q\|V_r-V_a\|^{\tau_q/K}\left[\sum_{q=0}^{\infty}\left(\frac{1}{q+1}\right)^{\tau_q}\right]^{1/K}. \tag{70}$$

Hence, $(V_a)_{a\in\mathcal{N}}$ is a Cauchy sequence in $\mathscr{L}(\Delta,\Lambda)$. $\mathscr{L}(\Delta,\Lambda)$ is a Banach space, so there exists $V\in\mathscr{L}(\Delta,\Lambda)$ so that $\lim_{a\longrightarrow\infty}\|V_a-V\|=0$ and since $(s_q(\bar{V}_a))_{q=0}^{\infty}\in(C^F_{\tau(.)})_h$, for all $a\in\mathcal{N}$, and $(C^F_{\tau(.)})_h$ is a premodular (cssf). Hence, one can see

$$H(V)=h\Big(\big(s_q(\bar{V})\big)_{q=0}^{\infty}\Big)\leq h\Big(\big(s_{[q/2]}(\bar{V}V_a)\big)_{q=0}^{\infty}\Big)+h\Big(\big(s_{[q/2]}(\bar{V_a})_{q=0}^{\infty}\big)\Big)\leq h\Big(\big(\|V_a-V\|\bar{1}\big)_{q=0}^{\infty}\Big)+\left(3^K+2^K\right)^{1/K}h\Big(\big(s_q(\bar{V}_a)\big)_{q=0}^{\infty}\Big)<\varepsilon. \tag{71}$$

We obtain $(s_q(\bar{V}))_{q=0}^{\infty}\in(C^F_{\tau(.)})_h$, and hence, $V\in\bar{\Psi}_{(C^F_{\tau(.)})_h}(\Delta,\Lambda)$. □

Theorem 45. *If $(\tau_q)_{q\in\mathcal{N}}\in\ell_\infty\cap I$ with $\tau_0>1$, one has $(\bar{\Psi}_{(C^F_{\tau(.)})_h}, H)$ is a pre-quasiclosed operator ideal.*

Proof. Suppose $V_a\in\bar{\Psi}_{(C^F_{\tau(.)})_h}(\Delta,\Lambda)$, for all $a\in\mathcal{N}$ and $\lim_{a\longrightarrow\infty}H(V_a-V)=0$. As $\mathscr{L}(\Delta,\Lambda)\supseteq S_{(C^F_{\tau(.)})_h}(\Delta,\Lambda)$, one has

$$H(V_a-V)=h\Big(\big(s_q(\bar{V}_a V)\big)_{q=0}^{\infty}\Big)\geq h\big(s_0(\bar{V}_a V),\bar{0},\bar{0},\bar{0},\cdots\big)\geq\inf_q\|V_a-V\|^{\tau_q/K}\left[\sum_{q=0}^{\infty}\left(\frac{1}{q+1}\right)^{\tau_q}\right]^{1/K}. \tag{72}$$

So $(V_a)_{a\in\mathcal{N}}$ is convergent in $\mathscr{L}(\Delta,\Lambda)$. i.e., $\lim_{a\longrightarrow\infty}\|V_a-V\|=0$, and since $(s_q(\bar{V}_a))_{q=0}^{\infty}\in(C^F_{\tau(.)})_h$, for all $q\in\mathcal{N}$ and $(C^F_{\tau(.)})_h$ is a premodular (cssf). Hence, one can see

$$H(V)=h\Big(\big(s_q(\bar{V})\big)_{q=0}^{\infty}\Big)\leq h\Big(\big(s_{[q/2]}(\bar{V}V_a)\big)_{q=0}^{\infty}\Big)+h\Big(\big(s_{[q/2]}(\bar{V_a})_{q=0}^{\infty}\big)\Big)\leq h\Big(\big(\|V_a-V\|\bar{1}\big)_{q=0}^{\infty}\Big)+\left(3^K+2^K\right)^{1/K}h\Big(\big(s_q(\bar{V}_a)\big)_{q=0}^{\infty}\Big)<\varepsilon. \tag{73}$$

We obtain $(s_q(\bar{V}))_{q=0}^{\infty}\in(C^F_{\tau(.)})_h$, and hence, $V\in\bar{\Psi}_{(C^F_{\tau(.)})_h}(\Delta,\Lambda)$. □

Definition 46. A pre-quasinorm H on the ideal $\bar{\Psi}_{U_h}$ verifies the Fatou property if for every $\{T_q\}_{q\in\mathcal{N}}\subseteq\bar{\Psi}_{U_h}(\Delta,\Lambda)$ so that $\lim_{q\longrightarrow\infty}H(T_q-T)=0$ and $M\in\bar{\Psi}_{U_h}(\Delta,\Lambda)$, one gets

$$H(M-T)\leq\sup_q\inf_{j\geq q}H\big(M-T_j\big). \tag{74}$$

Theorem 47. *Suppose $(\tau_q)_{q\in\mathcal{N}}\in\ell_\infty\cap I$ with $\tau_0>1$, then $(\bar{\Psi}_{(C^F_{\tau(.)})_h}, H)$ does not satisfy the Fatou property.*

Proof. Assume $\{T_q\}_{q\in\mathcal{N}}\subseteq\bar{\Psi}_{(C^F_{\tau(.)})_h}(\Delta,\Lambda)$ with $\lim_{q\longrightarrow\infty}H(T_q-T)=0$. Since $\bar{\Psi}_{(C^F_{\tau(.)})_h}$ is a pre-quasiclosed ideal, then $T\in\bar{\Psi}_{(C^F_{\tau(.)})_h}(\Delta,\Lambda)$. So for every $M\in\bar{\Psi}_{(C^F_{\tau(.)})_h}(\Delta,\Lambda)$, one has

$$\begin{aligned}H(M-T)&=\left[\sum_{q=0}^{\infty}\left(\frac{\sum_{p=0}^{q}\bar{\rho}\big(s_p(\bar{M}T),\bar{0}\big)}{q+1}\right)^{\tau_q}\right]^{1/K}\\&\leq\left[\sum_{q=0}^{\infty}\left(\frac{\sum_{p=0}^{q}\bar{\rho}\big(s_{[p/2]}(\bar{M}T_j),\bar{0}\big)}{q+1}\right)^{\tau_q}\right]^{1/K}+\left[\sum_{q=0}^{\infty}\left(\frac{\sum_{p=0}^{q}\bar{\rho}\big(s_{[p/2]}(\bar{T}_jT),\bar{0}\big)}{q+1}\right)^{\tau_q}\right]^{1/K}\\&\leq\left(3^K+2^K\right)^{1/K}\sup_r\inf_{j\geq r}\left[\sum_{q=0}^{\infty}\left(\frac{\sum_{p=0}^{q}\bar{\rho}\big(s_p(\bar{M}T_j),\bar{0}\big)}{q+1}\right)^{\tau_q}\right]^{1/K}.\end{aligned} \tag{75}$$

□

Definition 48. An operator $V:\bar{\Psi}_{U_h}(\Delta,\Lambda)\longrightarrow\bar{\Psi}_{U_h}(\Delta,\Lambda)$ is said to be a Kannan H-contraction, if one has $\alpha\in[0,1/2)$ with $H(VT-VM)\leq\alpha(H(VT-T)+H(VM-M))$, for all $T,M\in\bar{\Psi}_{U_h}(\Delta,\Lambda)$.

Definition 49. An operator $V:\bar{\Psi}_{U_h}(\Delta,\Lambda)\longrightarrow\bar{\Psi}_{U_h}(\Delta,\Lambda)$ is said to be H-sequentially continuous at M, where $M\in\bar{\Psi}_{U_h}(\Delta,\Lambda)$, if and only if $\lim_{r\longrightarrow\infty}H(T_r-M)=0\Rightarrow\lim_{r\longrightarrow\infty}H(VT_r-VM)=0$.

Example 10. $V:\bar{\Psi}_{(C^F((2q+3/q+2)_{q=0}^{\infty}))_h}(\Delta,\Lambda)\longrightarrow\bar{\Psi}_{(C^F((2q+3/q+2)_{q=0}^{\infty}))_h}(\Delta,\Lambda)$,

where $H(T)=\sqrt{\sum_{q=0}^{\infty}(\sum_{p=0}^{q}\bar{\rho}(s_p(\bar{T}),\bar{0})/q+1)^{2q+3/q+2}}$, for every $T\in\bar{\Psi}_{(C^F((2q+3/q+2)_{q=0}^{\infty}))_h}(\Delta,\Lambda)$ and

$$V(T)=\begin{cases}\dfrac{T}{6}, & H(T)\in[0,1),\\[2mm]\dfrac{T}{7}, & H(T)\in[1,\infty).\end{cases} \tag{76}$$

Evidently, V is H-sequentially continuous at the zero operator $\Theta\in\bar{\Psi}_{(C^F((2q+3/q+2)_{q=0}^{\infty}))_h}$. Let $\{T^{(j)}\}\subseteq\bar{\Psi}_{(C^F((2q+3/q+2)_{q=0}^{\infty}))_h}$ be such that $\lim_{j\longrightarrow\infty}H(T^{(j)}-T^{(0)})=0$, where $T^{(0)} \in \bar{\mp}_{(C^F((2q+3/q+2)_{q=0}^{\infty}))_h}$ with $H(T^{(0)}) = 1$. Since the pre-quasinorm H is continuous, one gets

$$\lim_{j\longrightarrow\infty} H\left(VT^{(j)} - VT^{(0)}\right) = \lim_{j\longrightarrow\infty} H\left(\frac{T^{(0)}}{6} - \frac{T^{(0)}}{7}\right) = H\left(\frac{T^{(0)}}{42}\right) > 0. \quad (77)$$

Therefore, V is not H-sequentially continuous at $T^{(0)}$.

Theorem 50. *Pick up* $(\tau_q)_{q\in\mathcal{N}} \in \ell_\infty \cap I$ *with* $\tau_0 > 1$ *and* $V : \bar{\mp}_{(C^F_{\tau(.)})_h}(\Delta, \Lambda) \longrightarrow \bar{\mp}_{(C^F_{\tau(.)})_h}(\Delta, \Lambda)$. *Assume*

(i) V is Kannan H-contraction mapping

(ii) V is H-sequentially continuous at an element $A \in \bar{\mp}_{(C^F_{\tau(.)})_h}(\Delta, \Lambda)$

(iii) there are $G \in \bar{\mp}_{(C^F_{\tau(.)})_h}(\Delta, \Lambda)$ such that the sequence of iterates $\{V^r G\}$ has a $\{V^{r_m} G\}$ converging to M

Then, $M \in \bar{\mp}_{(C^F_{\tau(.)})_h}(\Delta, \Lambda)$ is the unique fixed point of V.

Proof. Let M be not a fixed point of V; hence, $VM \neq M$. By using parts (ii) and (iii), we get

$$\begin{aligned}
&\lim_{r_m\longrightarrow\infty} H(V^{r_m}G - M) = 0,\\
&\lim_{r_m\longrightarrow\infty} H\left(V^{r_m+1}G - VM\right) = 0.
\end{aligned} \quad (78)$$

Since V is Kannan H-contraction, one obtains

$$\begin{aligned}
0 < H(VM - M) &= H\left(\left(VM - V^{r_m+1}G\right) + \left(V^{r_m}G - M\right) + \left(V^{r_m+1}G - V^{r_m}G\right)\right)\\
&\leq \left(3^K + 2^K\right)^{1/K} H\left(V^{r_m+1}G - VM\right)\\
&\quad + \left(3^K + 2^K\right)^{2/K} H\left(V^{r_m}G - M\right)\\
&\quad + \left(3^K + 2^K\right)^{2/K} \alpha\left(\frac{\alpha}{1-\alpha}\right)^{r_m-1} H(VG - G).
\end{aligned} \quad (79)$$

As $r_m \longrightarrow \infty$, there is a contradiction. Hence, M is a fixed point of V. To prove the uniqueness of the fixed point M, suppose one has two not equal fixed points $M, J \in \bar{\mp}_{(C^F_{\tau(.)})_h}(\Delta, \Lambda)$ of V. So, one gets $H(M - J) \leq H(VM - VJ) \leq \alpha(H(VM - M) + H(VJ - J)) = 0$. Then, $M = J$. □

Example 11. Given Example 10, since for all $T_1, T_2 \in \bar{\mp}_{(C^F((2q+3/q+2)_{q=0}^{\infty}))_h}$ with $H(T_1), H(T_2) \in [0, 1)$, we have

$$\begin{aligned}
H(VT_1 - VT_2) &= H\left(\frac{T_1}{6} - \frac{T_2}{6}\right)\\
&\leq \frac{\sqrt{2}}{\sqrt[4]{125}}\left(H\left(\frac{5T_1}{6}\right) + H\left(\frac{5T_2}{6}\right)\right)\\
&= \frac{\sqrt{2}}{\sqrt[4]{125}}(H(VT_1 - T_1) + H(VT_2 - T_2)).
\end{aligned} \quad (80)$$

For all $T_1, T_2 \in \bar{\mp}_{(C^F((2q+3/q+2)_{q=0}^{\infty}))_h}$ with $H(T_1), H(T_2) \in [1,\infty)$, we have

$$\begin{aligned}
H(VT_1 - VT_2) &= H\left(\frac{T_1}{7} - \frac{T_2}{7}\right)\\
&\leq \frac{\sqrt{2}}{\sqrt[4]{216}}\left(H\left(\frac{6T_1}{7}\right) + H\left(\frac{6T_2}{7}\right)\right)\\
&= \frac{\sqrt{2}}{\sqrt[4]{216}}(H(VT_1 - T_1) + H(VT_2 - T_2)).
\end{aligned} \quad (81)$$

For all $T_1, T_2 \in \bar{\mp}_{(C^F((2q+3/q+2)_{q=0}^{\infty}))_h}$ with $H(T_1) \in [0, 1)$ and $H(T_2) \in [1,\infty)$, we have

$$\begin{aligned}
H(VT_1 - VT_2) &= H\left(\frac{T_1}{6} - \frac{T_2}{7}\right) \leq \frac{\sqrt{2}}{\sqrt[4]{125}} H\left(\frac{5T_1}{6}\right)\\
&\quad + \frac{\sqrt{2}}{\sqrt[4]{216}} H\left(\frac{6T_2}{7}\right) \leq \frac{\sqrt{2}}{\sqrt[4]{125}}(H(VT_1 - T_1)\\
&\quad + H(VT_2 - T_2)).
\end{aligned} \quad (82)$$

Hence, V is Kannan H-contraction and $V^r(T) = \begin{cases} T/6^r, & H(T) \in [0, 1), \\ T/7^r, & H(T) \in [1,\infty). \end{cases}$

Obviously, V is H-sequentially continuous at $\Theta \in \bar{\mp}_{(C^F((2q+3/q+2)_{q=0}^{\infty}))_h}$, and $\{V^r T\}$ has a subsequence $\{V^{r_m} T\}$ converges to Θ. By Theorem 50, Θ is the only fixed point of G.

7. Applications

Theorem 51. *Consider the summable equation*

$$Y_p = R_p + \sum_{r=0}^{\infty} D(p, r) m(r, Y_r), \quad (83)$$

which presented by many authors [32, 33, 34], and assume $V : (C^F_{\tau(.)})_h \longrightarrow (C^F_{\tau(.)})_h$, where $(\tau_q)_{q\in\mathcal{N}} \in \ell_\infty \cap \mathbf{I}$ with

$\tau_0 > 1$ and $h(Y) = [\sum_{q=0}^{\infty}(\sum_{p=0}^{q}\bar{\rho}(Y_p, \bar{0})/q+1)^{\tau_q}]^{1/K}$, for all $Y \in C^F_{\tau(.)}$, is defined by

$$V(Y_p)_{p\in\mathcal{N}} = \left(R_p + \sum_{r=0}^{\infty} D(p,r)m(r,Y_r)\right)_{p\in\mathcal{N}}. \quad (84)$$

The summable equation (83) has a unique solution in $(C^F_{\tau(.)})_h$, if $D : \mathcal{N}^2 \longrightarrow \mathfrak{R}$, $m : \mathcal{N} \times \mathfrak{R}[0,1] \longrightarrow \mathfrak{R}[0,1]$, $R : \mathcal{N} \longrightarrow \mathfrak{R}[0,1]$, and $Z : \mathcal{N} \longrightarrow \mathfrak{R}[0,1]$; assume there is $\delta \in \mathfrak{R}$ such that $\sup_q|\delta|^{\tau_q/K} \in [0, 0.5)$, and for all $q \in \mathcal{N}$, let

$$\begin{aligned}\sum_{p=0}^{q}\left[\sum_{r=0}^{\infty} D(p,r)(m(r,Y_r) - m(r,Z_r))\right] &\le |\delta|\left[\sum_{p=0}^{q}\left(R_p - Y_p + \sum_{r=0}^{\infty} D(p,r)m(r,Y_r)\right)\right.\\ &\quad \left.+ \sum_{p=0}^{q}\left(R_p - Z_p + \sum_{r=0}^{\infty} D(p,r)m(r,Z_r)\right)\right].\end{aligned} \quad (85)$$

Proof. One has

$$\begin{aligned}h(VY - VZ) &= \left[\sum_{q=0}^{\infty}\left(\frac{\sum_{p=0}^{q}\bar{\rho}(VY_p - VZ_p, \bar{0})}{q+1}\right)^{\tau_q}\right]^{1/K}\\ &= \left[\sum_{q=0}^{\infty}\left(\frac{\sum_{p=0}^{q}\bar{\rho}(\sum_{r=0}^{\infty}D(p,r)[m(r,Y_r) - m(r,Z_r)], \bar{0})}{q+1}\right)^{\tau_q}\right]^{1/K}\\ &\le \sup_q|\delta|^{\tau_q/K}\left[\sum_{q=0}^{\infty}\left(\frac{\sum_{p=0}^{q}\bar{\rho}(R_p - Y_p + \sum_{r=0}^{\infty}D(p,r)m(r,Y_r), \bar{0})}{q+1}\right)^{\tau_q}\right]^{1/K}\\ &\quad + \sup_q|\delta|^{\tau_q/K}\left[\sum_{q=0}^{\infty}\left(\frac{\sum_{p=0}^{q}\bar{\rho}(R_p - Z_p + \sum_{r=0}^{\infty}D(p,r)m(r,Z_r), \bar{0})}{q+1}\right)^{\tau_q}\right]^{1/K}\\ &= \sup_q|\delta|^{\tau_q/K}(h(VY - Y) + h(VZ - Z)).\end{aligned} \quad (86)$$

By Theorem 29, one gets a unique solution of equation (83) in $(C^F_{\tau(.)})_h$. □

Example 12. Suppose $(C^F((2q+3/q+2)_{q=0}^{\infty}))_h$, where $h(Y) = \sqrt{\sum_{q=0}^{\infty}(\sum_{p=0}^{q}\bar{\rho}(Y_p, \bar{0})/q+1)^{2q+3/q+2}}$, for all $Y \in C^F((2q+3/q+2)_{q=0}^{\infty})$. Consider the summable equation

$$Y_p = R_p + \sum_{r=0}^{\infty}(-1)^{p+r}\left(\frac{Y_p}{p^2 + r^2 + 1}\right)^t, \quad (87)$$

with $p \ge 2$ and $t > 0$. Suppose $\Gamma = \{Y \in (C^F((2q+3/q+2)_{q=0}^{\infty}))_h : Y_0 = Y_1 = \bar{0}\}$. Indeed, Γ is a nonempty, h-convex, h-closed, and h-bounded subset of $(C^F((2q+3/q+2)_{q=0}^{\infty}))_h$. Let $V : \Gamma \longrightarrow \Gamma$ be defined by

$$V(Y_p)_{p\ge2} = \left(R_p + \sum_{r=0}^{\infty}(-1)^{p+r}\left(\frac{Y_p}{p^2 + r^2 + 1}\right)^t\right)_{p\ge2}. \quad (88)$$

Obviously,

$$\begin{aligned}\sum_{p=0}^{q}\sum_{r=0}^{\infty}(-1)^p\left(\frac{Y_p}{p^2 + r^2 + 1}\right)^t((-1)^r - (-1)^r) &\le \frac{1}{\sqrt{2}}\left[\sum_{p=0}^{q}\left(R_p - Y_p + \sum_{r=0}^{\infty}(-1)^{p+r}\left(\frac{Y_p}{p^2 + r^2 + 1}\right)^t\right)\right.\\ &\quad \left.+ \sum_{p=0}^{q}\left(R_p - Z_p + \sum_{r=0}^{\infty}(-1)^{p+r}\left(\frac{Z_p}{p^2 + r^2 + 1}\right)^t\right)\right].\end{aligned} \quad (89)$$

By Corollary 35 and Theorem 51, the summable equation (87) has a solution in Γ.

Example 13. Suppose $(C^F((2q+3/q+2)_{q=0}^{\infty}))_h$, where $h(Y) = \sqrt{\sum_{q=0}^{\infty}(\sum_{p=0}^{q}\bar{\rho}(Y_p, \bar{0})/q+1)^{2q+3/q+2}}$, for every $Y \in C^F((2q+3/q+2)_{q=0}^{\infty})$. Consider the following nonlinear difference equation:

$$Y_p = R_p + \sum_{l=0}^{\infty}(-1)^{p+l}\frac{Y_{p-2}^r}{Y_{p-1}^p + l^2 + 1}, \quad (90)$$

with $r, p > 0$, $Y_{-2}(x), Y_{-1}(x) > 0$, for all $x \in \mathfrak{R}$, and assume $V : C^F((2q+3/q+2)_{q=0}^{\infty}) \longrightarrow C^F((2q+3/q+2)_{q=0}^{\infty})$ is defined by

$$V(Y_p)_{p=0}^{\infty} = \left(R_p + \sum_{l=0}^{\infty}(-1)^{p+l}\frac{Y_{p-2}^r}{Y_{p-1}^p + l^2 + 1}\right)_{p=0}^{\infty}. \quad (91)$$

Evidently,

$$\begin{aligned}\sum_{p=0}^{q}\sum_{l=0}^{\infty}(-1)^p\frac{Y_{p-2}^r}{Y_{p-1}^p + l^2 + 1}\left((-1)^l - (-1)^l\right) &\le \frac{1}{\sqrt{2}}\left[\sum_{p=0}^{q}\left(R_p - Y_p + \sum_{l=0}^{\infty}(-1)^{p+l}\frac{Y_{p-2}^r}{Y_{p-1}^p + l^2 + 1}\right)\right.\\ &\quad \left.+ \sum_{p=0}^{q}\left(R_p - Z_p + \sum_{l=0}^{\infty}(-1)^{p+l}\frac{Z_{p-2}^r}{Z_{p-1}^p + l^2 + 1}\right)\right].\end{aligned} \quad (92)$$

By Theorem 51, the nonlinear difference equation (90) has a unique solution in $C^F((2q+3/q+2)_{q=0}^{\infty})$.

8. Conclusion

Rather than simply referring to a "quasi-normed" place, we used the term "prequasi-normed." It is the concept of a fixed point of the Kannan pre-quasinorm contraction mapping in the pre-quasi-Banach variable exponent Cesàro sequence spaces of fuzzy functions (cssf). Pre-quasinormal structure and (R) are supported. The Kannan nonexpansive mapping's presence of a fixed point was investigated. The presence of a fixed point of Kannan contraction mapping in the pre-quasi-Banach operator ideal produced by variable exponent Cesàro sequence spaces of fuzzy functions (cssf) and s-fuzzy numbers has also been examined. To put our findings to the test, we introduce several numerical experiments. In addition, various effective implementations of the stochastic nonlinear dynamical system are discussed. The fixed points of any Kannan contraction and nonexpansive mappings on this new fuzzy functions space, its associated pre-quasi-ideal, and a new general space of solutions for many stochastic nonlinear dynamical systems are investigated.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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