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Retraction

Retracted: Decompositions of Circulant-Balanced Complete Multipartite Graphs Based on a Novel Labelling Approach

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This article has been retracted by Hindawi, as publisher, following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of systematic manipulation of the publication and peer-review process. We cannot, therefore, vouch for the reliability or integrity of this article.

Please note that this notice is intended solely to alert readers that the peer-review process of this article has been compromised.

Wiley and Hindawi regret that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

References

[1] A. El-Mesady and O. Bazighifan, "Decompositions of Circulant-Balanced Complete Multipartite Graphs Based on a Novel Labelling Approach," *Journal of Function Spaces*, vol. 2022, Article ID 2017936, 17 pages, 2022.

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Research Article

Decompositions of Circulant-Balanced Complete Multipartite Graphs Based on a Novel Labelling Approach

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For applied scientists and engineers, graph theory is a strong and vital tool for evaluating and inventing solutions for a variety of issues. Graph theory is extremely important in complex systems, particularly in computer science. Many scientific areas use graph theory, including biological sciences, engineering, coding, and operational research. A strategy for the orthogonal labelling of a bipartite graph G with n edges has been proposed in the literature, yielding cyclic decompositions of balanced complete bipartite graphs $K_{n,n}$ by the graph G. A generalization to circulant-balanced complete multipartite graphs $K_{\underbrace{n,n,\cdots,n}_{m}}$; $m, n \geq 2$, is our objective here. In this paper, we expand the orthogonal labelling approach used to generate cyclic decompositions for $K_{n,n}$ to a generalized orthogonal labelling approach that may be used for decomposing $K_{\underbrace{n,n,\cdots,n}_{m}}$. We can decompose $K_{\underbrace{n,n,\cdots,n}_{m}}$ into distinct graph classes based on the proposed generalized orthogonal labelling approach.

1. Introduction

As is well known, discrete mathematics is a field of mathematics that deals with countable processes and components. One of the most significant and intriguing disciplines in discrete mathematics is graph theory [1–3]. Graph theory is the study of structural models called graphs, which are made up of a collection of vertices and edges. Graph theory is extremely important in complex systems, particularly in computer science. Many scientific areas use graph theory, including engineering, coding [4, 5], operational research, biological sciences, and management sciences. For applied scientists and engineers, graph theory is a strong and vital science for evaluating and inventing solutions for a variety of issues. Graphs have recently been utilized as structural models for characterizing World Wide Web connections and the number of links necessary to move between web pages [6].

Circulant graphs are a significant category of graphs [7–10]. Circulant graphs have gained a lot of attention in recent decades. The circulant graphs class includes complete graphs and classic rings topologies. The algebraic properties of circulant graphs have been studied in thousands of publications. Circulant graphs have been handled in a variety of graph applications, including wide area communication graphs, local area computer graphs, parallel processing architectures, very large-scale integrated circuit design, and distributed computing [11–13].

Several traditional parallel and distributed systems were built on the foundation of circulant graphs [14–16]. Circulant graphs have a wide range of practical uses, such as a structure in chemical reaction models [17], multiprocessor cluster systems [18], small-world graph models [19], discrete cellular neural graphs [20], and as a basic structure for optical graphs [21], and so on.

The study of circulant graphs, including their characterization, analysis, and applications, is currently a popular issue in research. Several papers have been published that deal with graph decompositions by simpler graphs [22–24]. Decompositions of circulant graphs have several excellent contributions. For Cayley graphs labelled with Abelian groups, the Hamilton decomposition was investigated in [25]. The circulant graph is a particular case of the Cayley graph. It has been demonstrated that two Hamilton cycles may be used to decompose four-regular connected Cayley graphs [26].

For a certain recursive circulant graph, the Hamilton decompositions have been proven [27]. Every circulant graph has a corresponding circulant matrix [28]. Excellent descriptions of circulant matrices have been published in [28].

Definition 1. A circulant-balanced complete multipartite graph $K_{\underbrace{n,n,\cdots,n}_{m}}$ is a simple graph having $mn = \sum_{l=1}^{m} n$ vertices. The vertices of $K_{\underbrace{n,n,\cdots,n}_{m}}$ are divided into m partitions of cardinality n; two vertices are said to be adjacent if they are found in two different partitions. The graph $K_{\underbrace{n,n,\cdots,n}_{m}}$ has a degree equal to $(m-1)n$. The circulant graph $K_{\underbrace{n,n,\cdots,n}_{m}}$ can be divided into $\delta K_{n,n}$, $\delta = \begin{pmatrix} m \\ 2 \end{pmatrix}$.

Definition 2. A caterpillar graph $C_a(b_1, b_2, \cdots, b_a)$ is a tree formed by the path $P_a = y_1 y_2 \cdots y_a$ by linking a vertex y_i to b_i new vertices where $a \geq 1, b_1, b_2, \cdots, b_a$ are integers greater than zero, $b_1, b_a \geq 1$ and $b_i \geq 0$ for $i \in \{2, 3, \cdots, a-1\}$.

El-Mesady et al. have proposed an orthogonal labelling approach to decompose a certain circulant graph class with $2n$ vertices and n degree [29]. Circulant-balanced complete bipartite graphs are the name for this type of graph which is denoted by $K_{n,n}$. In cognitive radio graphs and cloud computing, bipartite circulant graphs can address a variety of challenges. For a good survey on several decompositions of circulant graphs, see [30–34].

In this study, we generalize the orthogonal labelling approach proposed in [29] to create edge decompositions of the graphs $K_{\underbrace{n,n,\cdots,n}_{m}}$; $m, n \geq 2$ which are considered a generalization to the graphs $K_{n,n}$. The following sections make up the current paper: The second section deals with the proposed novel orthogonal labelling approach. In the third section, the graph $K_{\underbrace{n,n,\cdots,n}_{m}}$ is decomposed by infinite classes of graphs. We generate many decompositions of $K_{\underbrace{n,n,\cdots,n}_{m}}$ by connected caterpillars in the fourth section. The fifth section introduces concluding remarks and future work.

2. A Novel Labelling Approach

Consider now the circulant-balanced complete multipartite graph with vertex set $V = \bigcup_{l=0}^{m-1} V_l$, where $V, l \in \{0, 1, \cdots, m-1\}$ are m independent sets of vertices. There are bijective mappings $\varphi_l : V_l \longrightarrow \mathbb{Z}_n \times \{l\}, l \in \{0, 1, \cdots, m-1\}$ where the vertices in V_l are labelled by $\mathbb{Z}_n \times \{l\}$, see Figure 1.

The distance between two vertices $x_i \in \{0_i, 1_i, \cdots, (n-1)_i\}$ and $y_j \in \{0_j, 1_j, \cdots, (n-1)_j\}, 0 \leq i < j \leq m-1$ is the usual circular distance defined by $d\{x_i, y_j\} = \min\{|x_i - y_j|, n - |x_i - y_j|\}$. The edge $\{x_i, y_j\}$ is said to have length $d\{x_i, y_j\}$. Suppose $G = (V, E)$ is a subgraph with mn vertices and $\begin{pmatrix} m \\ 2 \end{pmatrix} n$ edges, a labelling

$$\psi_k : V\left(G_k^{i,j}\right) \longrightarrow \mathbb{Z}_n \times \{i, j\}, 0 \leq i < j \leq m-1, k = \begin{cases} j & \text{if } i = 0, \\ mi + j (\operatorname{mod}\ (i+1)) & \text{if } i > 0. \end{cases} \tag{1}$$

is considered an orthogonal labelling of $G \cong \bigcup_{k=1}^{w} G_k^{i,j}$, $w = \begin{pmatrix} m \\ 2 \end{pmatrix}$, $0 \leq i < j \leq m-1$ if,

(i) Each graph $G_k^{i,j}$ has precisely two edges of length $\lambda \in \{1, 2, \cdots, \lfloor (n-1)/2 \rfloor\}$, the length 0 is found once in $G_k^{i,j}$, and the length $n/2$ is found once in $G_k^{i,j}$ if n is even

(ii) For every $\lambda \in \{1, 2, \cdots, \lfloor (n-1)/2 \rfloor\}$,$G$ has precisely $2.\begin{pmatrix} m \\ 2 \end{pmatrix} = m(m-1)$ edges of length λ,

(iii) The length 0 is found $\begin{pmatrix} m \\ 2 \end{pmatrix}$ times in G,

(iv) The length $n/2$ is found $\begin{pmatrix} m \\ 2 \end{pmatrix}$ times in G if n is even

Example 1. An orthogonal labelling of $K_{1,3}^{0,1} \cup P_4^{0,2} \cup K_{1,3}^{1,2}$ is shown in Figure 2.

Definition 3. Suppose G is a subgraph of $K_{\underbrace{n,n,\cdots,n}_{m}}$, $x \in \mathbb{Z}_n$. Then $G + x$ with $E(G + x) = \{\{a + x, b + x\}: \{a, b\} \in E(G)\}$ is called the x-translate of G.

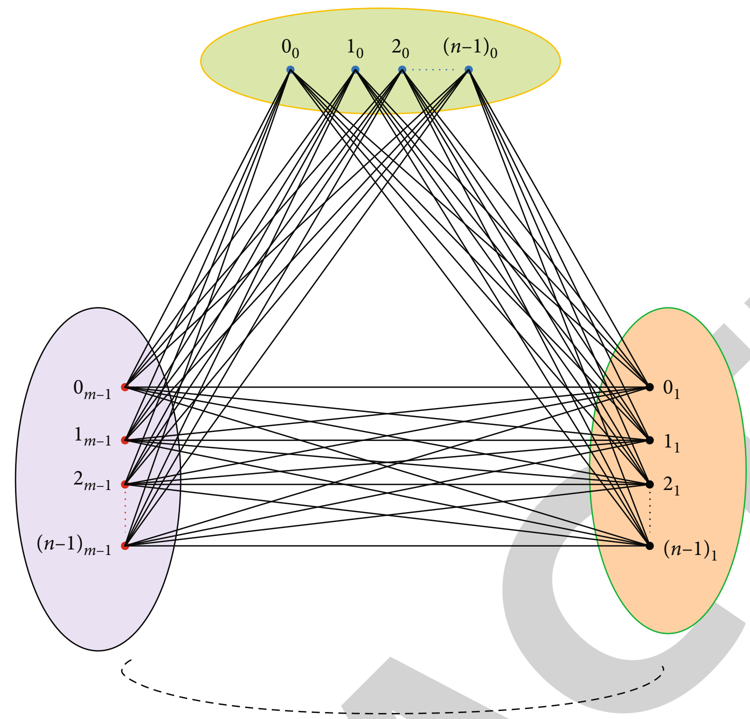

Figure 1: The labelling for $K_{\underbrace{n,n,\cdots,n}_{m}}$.

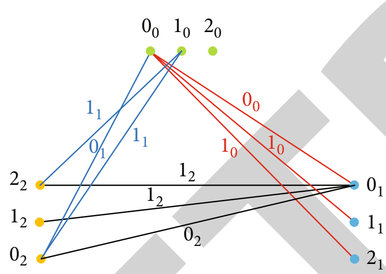

Figure 2: An orthogonal labelling of $K_{1,3}^{0,1} \cup P_4^{0,2} \cup K_{1,3}^{1,2}$.

The edge decomposition of circulant-balanced complete multipartite graphs and orthogonal labelling are linked in the next proposition.

Proposition 4. *If and only if there is an orthogonal labelling of* $G \cong \bigcup_{k=1}^{w} G_k^{i,j}, 0 \le i < j \le m-1$, *an edge decomposition of* $K_{\underbrace{n,n,\cdots,n}_{m}}$ *can be constructed by G.*

Proof. Our goal is to show that $E(G^{i,j}+\omega) \cap E(G^{i,j}+\sigma) = \phi$ for all $\omega, \sigma \in \mathbb{Z}_n$. We assume, by way of contradiction, that $|E(G^{i,j}+\omega) \cap E(G^{i,j}+\sigma)| \ge 1$ for $\omega, \sigma \in \mathbb{Z}_n$ with $\omega \ne \sigma$. For the lengths $\lambda \in \{1, 2, \cdots, \lfloor (n-1)/2 \rfloor\}$, which are repeated twice in $G^{i,j}$, let $\{a, b\}$ and $\{c, d\}$ be two edges of $E(G^{i,j}+\omega) \cap E(G^{i,j}+\sigma)$ with length λ, then $\{a-\omega, b-\omega\}, \{c-\omega, d-\omega\}$ and $\{a-\sigma, b-\sigma\}, \{c-\sigma, d-\sigma\}$ are various edges with length λ in $E(G^{i,j})$. However, this is a contradiction because $G^{i,j}$ verifies the orthogonal labelling requirement (i). Let $\{a, b\}$ belong to $E(G^{i,j}+\omega) \cap E(G^{i,j}+\sigma)$ with length $l \in \{0, n/2\}$,n is even, then $\{a-\omega, b-\omega\}$ and $\{a-\sigma, b-\sigma\}$ are distinct edges in $E(G^{i,j})$, both with length l. However, this is a contradiction because $G^{i,j}$ verifies the orthogonal labelling requirement (i). Hence, $\cap_{x \in \mathbb{Z}_n} E(G+x) = \varphi$. Also, for every $\lambda \in \{1, 2, \cdots, \lfloor (n-1)/2 \rfloor\}$,$G$ has precisely $2.\binom{m}{2} = m(m-1)$ edges with length λ, the length 0 is only found $\binom{m}{2}$ times in G, the length $n/2$ is only found $\binom{m}{2}$ times in G if n is even. Consequently,

$$\cup_{x \in \mathbb{Z}_n} E(G+x) = E\left(K_{\underbrace{n,n,\cdots,n}_{m}}\right). \tag{2}$$

□

Example 2. An example of edge decomposition of $K_{3,3,3}$ by $K_{1,3}^{0,1} \cup P_4^{0,2} \cup K_{1,3}^{1,2}$ is shown in Figure 3.

In what follows, based on the aforementioned orthogonal labelling approach, we will decompose the circulant-balanced complete multipartite graph $K_{\underbrace{n,n,\cdots,n}_{m}}$ by the $G \cong \bigcup_{k=1}^{w} G_k^{i,j}$, where the graphs $G_k^{i,j}, k \in \{1, 2, \cdots, w\}, w = \binom{m}{2}, i \ne j \in \{0, 1, \cdots, m-1\}$ are isomorphic. Also, we will consider

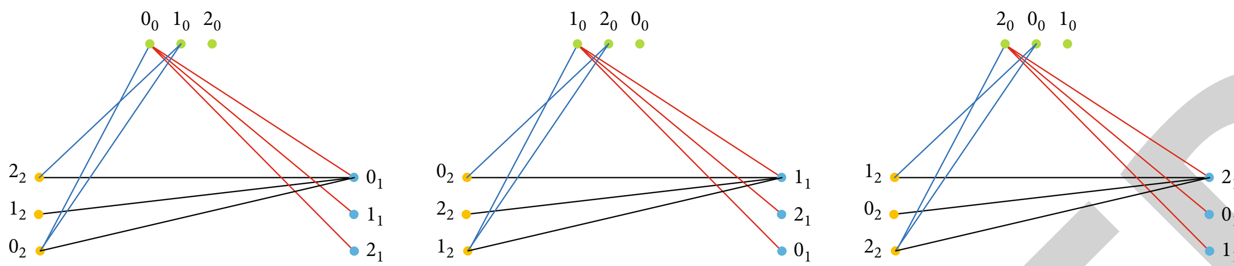

FIGURE 3: An edge decomposition of $K_{3,3,3}$ by $K_{1,3}^{0,1} \cup P_4^{0,2} \cup K_{1,3}^{1,2}$.

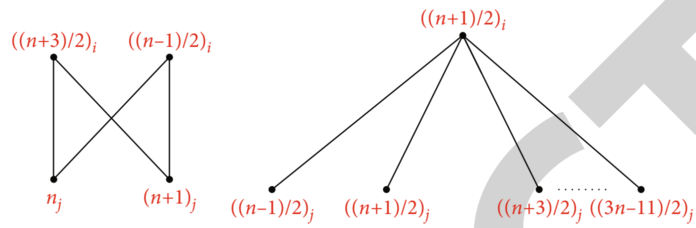

FIGURE 4: The labelling for $(K_{2,2} \cup K_{1,n-4})^{i,j}$.

$$k = \begin{cases} j & \text{if } i = 0, \\ im + j (\operatorname{mod}\ (i+1)) & \text{if } i > 0. \end{cases} \tag{3}$$

3. Decompositions of $K_{\underbrace{n,n,\cdots,n}_{m}}$ by Several Classes of Graphs

Theorem 5. *Let $n \geq 5, m \geq 2$ be integers. Then, there is an orthogonal labelling for $G_1 \cong \bigcup_{0 \leq i < j \leq m-1} (K_{2,2} \cup K_{1,n-4})^{i,j}$.*

Proof. Suppose $V((K_{2,2} \cup K_{1,n-4})^{i,j}) = \{v_s : s \in \{0, 1, 2, \cdots, n\}\}$. The mapping ψ_k can be used to define an orthogonal labelling for the subgraph G_1, which can be defined by $\psi_k : V((K_{2,2} \cup K_{1,n-4})^{i,j}) \longrightarrow \mathbb{Z}_n \times \{i, j\}$ which is defined by $\psi_k(v_0) = ((n+3)/2)_i, \psi_k(v_1) = ((n-1)/2)_i$,

$\psi_k(v_2) = ((n+1)/2)_i, \psi_k(v_{s+3}) = (((n-1)/2) + s)_j, s \in \{0, \cdots, n-5\}$, and the edge set of $(K_{2,2} \cup K_{1,n-4})^{i,j}$ is

$$E((K_{2,2} \cup K_{1,n-4})^{i,j}) = \left\{ \left\{ \left(\frac{n+3}{2}\right)_i, n_j \right\}, \left\{ \left(\frac{n+3}{2}\right)_i, (n+1)_j \right\}, \left\{ \left(\frac{n-1}{2}\right)_i, n_j \right\}, \left\{ \left(\frac{n-1}{2}\right)_i, (n+1)_j \right\}, \left\{ \left(\frac{n+1}{2}\right)_i, \left(\frac{n-1}{2}\right)_j \right\} \right\} \cup \left\{ \left\{ \left(\frac{n+1}{2}\right)_i, \left(\frac{n+1}{2} + s\right)_j \right\} : s \in \{0, 1, \cdots, n-6\} \right\} \tag{4}$$

see Figure 4. From the edge set of G_1, the following conditions are verified: Each graph $(K_{2,2} \cup K_{1,n-4})^{i,j}$ has precisely two edges of length $\lambda \in \{1, 2, \cdots, \lfloor (n-1)/2 \rfloor\}$, the length 0 is found once in $(K_{2,2} \cup K_{1,n-4})^{i,j}$, the length $n/2$ is found once in $(K_{2,2} \cup K_{1,n-4})^{i,j}$ if n is even, for every $\lambda \in \{1, 2, \cdots, \lfloor (n-1)/2 \rfloor\}, G_1$ has precisely $2.\binom{m}{2} = m(m-1)$ edges of length λ, the length 0 is found $\binom{m}{2}$ times in G_1, and the length $n/2$ is found $\binom{m}{2}$ times in G_1 if n is even. Hence, $K_{\underbrace{n,n,\cdots,n}_{m}}$ can be decomposed by G_1. □

Theorem 6. *Let $n > 1, m \geq 2$ be integers. Then, there is an orthogonal labelling for $G_2 \cong \bigcup_{0 \leq i < j \leq m-1} (K_{2,n})^{i,j}$.*

Proof. Suppose $V((K_{2,n})^{i,j})$ is $V((K_{2,n})^{i,j}) = \{v_s : s \in \{0, 1, 2, \cdots, n+1\}\}$. The mapping ψ_k can be used to define an orthogonal labelling for the subgraph G_2, which can be defined by $\psi_k : V((K_{2,n})^{i,j}) \longrightarrow \mathbb{Z}_{2n} \times \{i, j\}$ which is defined by

$$\psi_k(v_s) = s_i, s \in \{0, 1\}, \psi_k(v_{s+2}) = ((2(s-1)) (\operatorname{mod}\ 2n))_j, s \in \{1, \cdots, n\}, \tag{5}$$

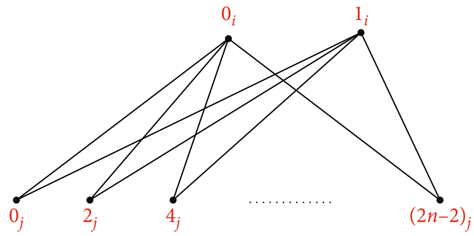

Figure 5: The labelling for $(K_{2,n})^{i,j}$.

and the edge set of $(K_{2,n})^{i,j}$ is

$$E\left((K_{2,n})^{i,j}\right) = \left\{\left\{0_i, (2s)_j\right\} : s \in \{0, 1, \cdots, n-1\}\right\} \cup \left\{\left\{1_i, ((2s)(\bmod 2n))_j\right\} : s \in \{1, \cdots, n\}\right\}, \tag{6}$$

see Figure 5. From the edge set of G_2, the following conditions are verified: Each graph $(K_{2,n})^{i,j}$ has precisely two edges of length $\lambda \in \{1, 2, \cdots, \lfloor (2n-1)/2 \rfloor\}$, the length 0 is found once in $(K_{2,n})^{i,j}$, the length n is found once in $(K_{2,n})^{i,j}$, for every $\lambda \in \{1, 2, \cdots, \lfloor (2n-1)/2 \rfloor\}, G_2$ has precisely $2.\binom{m}{2} = m(m-1)$ edges of length λ, the length 0 is found $\binom{m}{2}$ times in G_2, and the length n is found $\binom{m}{2}$ times in G_2. Hence, $K_{\underbrace{2n,2n,\cdots,2n}_{m}}$ can be decomposed by G_2. □

Theorem 7. *Let $n \equiv 2 \bmod 6$ or $n \equiv 4 \bmod 6, m \geq 2$. Then, there is an orthogonal labelling for*

$$G_3 \cong \bigcup_{0 \leq i < j \leq m-1} \left(\frac{n}{2} K_{1,2}\right)^{i,j}. \tag{7}$$

Proof. Suppose $V(((n/2)K_{1,2})^{i,j}) = \{v_s : s \in \{0, 1, 2, \cdots, 2(n-1)\}\}$. The mapping ψ_k can be used to define an orthogonal labelling for the subgraph G_3, which can be defined by $\psi_k : V(((n/2)K_{1,2})^{i,j}) \longrightarrow \mathbb{Z}_n \times \{i, j\}$ which is defined by $\psi_k(v_s) = s_i, s \in \{0, 1, \cdots, n-1\}, \psi_k(v_{n+s}) = ((2s)(\bmod n))_j, s \in \{0, 1, \cdots, n-1\}$, and the edge set of $((n/2)K_{1,2})^{i,j}$ is $E(((n/2)K_{1,2})^{i,j}) = \{\{s_i, ((2s)(\bmod n))_j\} : s \in \{0, 1, 2, \cdots, n-1\}\}$, see Figure 6. From the edge set of G_3, the following conditions are verified: Each graph $((n/2)K_{1,2})^{i,j}$ has precisely two edges of length $\lambda \in \{1, 2, \cdots, \lfloor (n-1)/2 \rfloor\}$, the length 0 is found once in $((n/2)K_{1,2})^{i,j}$, the length $n/2$ is found once in $((n/2)K_{1,2})^{i,j}$, for every $\lambda \in \{1, 2, \cdots, \lfloor (n-1)/2 \rfloor\}, G_3$ has precisely $2.\binom{m}{2} = m(m-1)$ edges of length λ, the length 0 is found $\binom{m}{2}$ times in G_3, and the length $n/2$ is found $\binom{m}{2}$ times in G_3. Hence, $K_{\underbrace{n,n,\cdots,n}_{m}}$ can be decomposed by G_3. □

Theorem 8. *Let $n \geq 9, m \geq 2$ be integers. Then, there is an orthogonal labelling for*

$$G_4 \cong \bigcup_{0 \leq i < j \leq m-1} (C_8 \cup K_{1,n-8})^{i,j}. \tag{8}$$

Proof. Suppose $V((C_8 \cup K_{1,n-8})^{i,j}) = \{v_s : s \in \{0, 1, 2, \cdots, n\}\}$. The mapping ψ_k can be used to define an orthogonal labelling for the subgraph G_4, which can be defined by $\psi_k : V((C_8 \cup K_{1,n-8})^{i,j}) \longrightarrow \mathbb{Z}_n \times \{i, j\}$ which is defined by

$$\psi_k(v_0) = 0_0, \psi_k(v_1) = 1_0, \psi_k(v_2) = 2_0, \psi_k(v_3) = 4_0, \psi_k(v_4) = 8_0, \psi_k(v_s) = (s-4)_1, s \in \{5, \cdots, n\}, \tag{9}$$

and the edge set of $(C_8 \cup K_{1,n-8})^{i,j}$ is

$$E((C_8 \cup K_{1,n-8})^{i,j}) = \{\{0_i, 2_j\}, \{0_i, 4_j\}, \{4_i, 2_j\}, \{4_i, 3_j\}, \{2_i, 3_j\}, \{2_i, 5_j\}, \{8_i, 4_j\}, \{8_i, 5_j\}, \{1_i, 1_j\}\} \tag{10}$$

$\cup \{\{1_i, s_j\} : s \in \{6, 7, \cdots, n-4\}\}$, see Figure 7. From the edge set of G_4, the following conditions are verified: Each graph $(C_8 \cup K_{1,n-8})^{i,j}$ has precisely two edges of length $\lambda \in \{1, 2, \cdots, \lfloor (n-1)/2 \rfloor\}$, the length 0 is found once in $(C_8 \cup K_{1,n-8})^{i,j}$, the length $n/2$ is found once in $(C_8 \cup K_{1,n-8})^{i,j}$ if n is even, for every $\lambda \in \{1, 2, \cdots, \lfloor (n-1)/2 \rfloor\}, G_4$ has precisely $2.\binom{m}{2} = m(m-1)$ edges of length λ, the length 0 is found $\binom{m}{2}$ times in G_4, and the length $n/2$ is found $\binom{m}{2}$ times in G_4 if n is even. Hence, $K_{\underbrace{n,n,\cdots,n}_{m}}$ can be decomposed by G_4. □

Theorem 9. *Let $n \geq 7, m \geq 2$ be integers. Then, there is an orthogonal labelling for $G_5 \cong \bigcup_{0 \leq i < j \leq m-1} (C_6 \cup K_{1,1} \cup K_{1,n-7})^{i,j}$..*

Proof. Suppose $V((K_{1,1} \cup C_6 \cup K_{1,n-7})^{i,j}) = \{v_s : s \in \{0, 1, 2, \cdots, n\}\}$. The mapping ψ_k can be used to define an

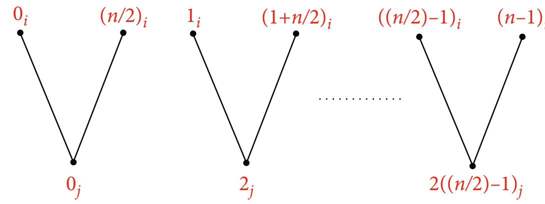

FIGURE 6: The labelling for $((n/2)K_{1,2})^{i,j}$.

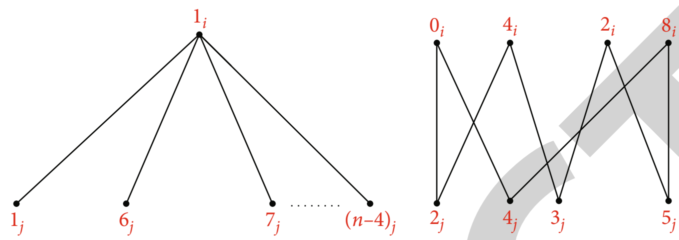

FIGURE 7: The labelling for $(C_8 \cup K_{1,n-8})^{i,j}$.

orthogonal labelling for the subgraph G_5, which can be defined by $\psi_k : V(K_{1,1} \cup C_6 \cup K_{1,n-7}) \longrightarrow \mathbb{Z}_n \times \{i, j\}$ which is defined by

$$\psi_k(v_0) = 0_i, \psi_k(v_1) = 1_i, \psi_k(v_2) = 3_i, \psi_k(v_3) = 4_i, \psi_k(v_4) = 6_i, \psi_k(v_5) = 1_j, \psi_k(v_6) = 2_j, \psi_k(v_7) = 3_j, \quad (11)$$

$\psi_k(v_8) = 5_j, \psi_k(v_s) = (s-2)_j, s \in \{9, \cdots, n+1\}$, and the edge set of $(K_{1,1} \cup C_6 \cup K_{1,n-7})^{i,j}$ is

$$E((K_{1,1} \cup C_6 \cup K_{1,n-7})^{i,j}) = \{\{1_i, 1_j\}, \{0_i, 2_j\}, \{0_i, 3_j\}, \{4_i, 2_j\}, \{4_i, 5_j\}, \{6_i, 3_j\}, \{6_i, 5_j\}\} \cup \{\{3_i, s_j\}: s \in \{7, \cdots, n-1\}\}, \quad (12)$$

see Figure 8. From the edge set of G_5, the following conditions are verified: Each graph $(K_{1,1} \cup C_6 \cup K_{1,n-7})^{i,j}$ has precisely two edges of length $\lambda \in \{1, 2, \cdots, \lfloor (n-1)/2 \rfloor\}$, the length 0 is found once in $(K_{1,1} \cup C_6 \cup K_{1,n-7})^{i,j}$, the length $n/2$ is found once in $(K_{1,1} \cup C_6 \cup K_{1,n-7})^{i,j}$ if n is even, for every $\lambda \in \{1, 2, \cdots, \lfloor (n-1)/2 \rfloor\}$,$G_5$ has precisely $2.\binom{m}{2} = m(m-1)$ edges of length λ, the length 0 is found $\binom{m}{2}$ times in G_5, and the length $n/2$ is found $\binom{m}{2}$ times in G_5 if n is even. Hence, $K_{\underbrace{n,n,\cdots,n}_{m}}$ can be decomposed by G_5. □

Theorem 10. *Let $n \geq 5, m \geq 2$ be integers. Then, there is an orthogonal labelling for $G_6 \cong \bigcup_{0 \leq i < j \leq m-1} (2K_2 \cup K_{1,n-2})^{i,j}$.*

Proof. Suppose $V((2K_{1,1} \cup K_{1,n-2})^{i,j}) = \{v_s : s \in \{0, 1, 2, \cdots, n+2\}\}$. The mapping ψ_k can be used to define an orthogonal labelling for the subgraph G_6, which can be defined by $\psi_k : V((2K_{1,1} \cup K_{1,n-2})^{i,j}) \longrightarrow \mathbb{Z}_n \times \{i, j\}$ which is defined by

$$\psi_k(v_0) = 0_i, \psi_k(v_1) = 1_i, \psi_k(v_2) = (n-1)_i, \psi_k(v_{s+3}) = (s)_j, s \in \{0, \cdots, n-1\}, \quad (13)$$

and the edge set of $(2K_{1,1} \cup K_{1,n-2})^{i,j}$ is

$$E((2K_{1,1} \cup K_{1,n-2})^{i,j}) = \{\{0_i, s_j\}: s \in \{0, 1, \cdots, n-3\}\} \cup \left\{\left\{1_i, (n-1)_j\right\}, \left\{(n-1)_i, (n-2)_j\right\}\right\}, \quad (14)$$

see Figure 9. From the edge set of G_6, the following conditions are verified: Each graph $(2K_{1,1} \cup K_{1,n-2})^{i,j}$ has

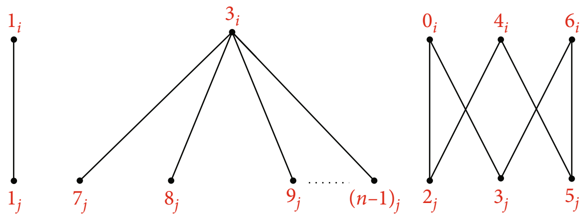

FIGURE 8: The labelling for $(K_{1,1} \cup C_6 \cup K_{1,n-7})^{i,j}$.

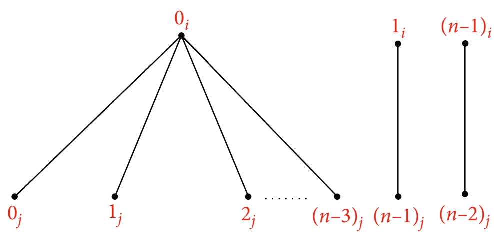

FIGURE 9: The labelling for $(2K_{1,1} \cup K_{1,n-2})^{i,j}$.

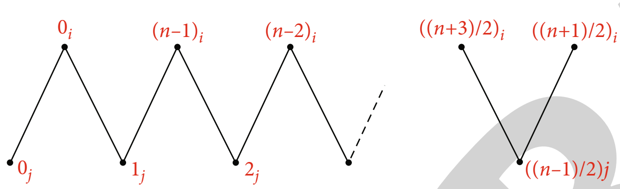

FIGURE 10: The labelling for $(P_{n+1})^{i,j}$.

precisely two edges of length $\lambda \in \{1, 2, \cdots, \lfloor (n-1)/2 \rfloor\}$, the length 0 is found once in $(2K_{1,1} \cup K_{1,n-2})^{i,j}$, the length $n/2$ is found once in $(2K_{1,1} \cup K_{1,n-2})^{i,j}$ if n is even, for every $\lambda \in \{1, 2, \cdots, \lfloor (n-1)/2 \rfloor\}$, G_6 has precisely $2.\binom{m}{2} = m(m-1)$ edges of length λ, the length 0 is found $\binom{m}{2}$ times in G_6, and the length $n/2$ is found $\binom{m}{2}$ times in G_6 if n is even. Hence, $K_{\underbrace{n,n,\cdots,n}_{m}}$ can be decomposed by G_6. □

Theorem 11. *Let $n > 1, m \geq 2$ be integers. Then, there is an orthogonal labelling for $G_7 \cong \bigcup_{0 \leq i < j \leq m-1} (P_{n+1})^{i,j}$.*

Proof. Suppose $V((P_{n+1})^{i,j}) = \{v_s : s \in \{0, 1, 2, \cdots, n\}\}$. The mapping ψ_k can be used to define an orthogonal labelling for the subgraph G_7, which can be defined by $\psi_k : V((P_{n+1})^{i,j}) \longrightarrow \mathbb{Z}_n \times \{i, j\}$ which is defined by

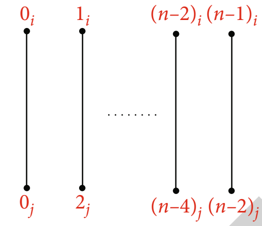

FIGURE 11: The labelling for $(nK_{1,1})^{i,j}$.

$$\psi_k(v_s) = ((n-s)(\text{mod } n))_i, s \in \left\{0, 1, \cdots, \frac{n-3}{2}\right\}, \psi_k(v_{n-1/2}) = \left(\frac{n+1}{2}\right)_i, \psi_k(v_{n-3/2+s+2}) = s_j, \tag{15}$$

$s \in \{0, 1, \cdots, (n-1)/2\}$, and the edge set of $(P_{n+1})^{i,j}$ is $E((P_{n+1})^{i,j}) = \{\{((n+1)/2)_i, ((n-1)/2)_j\}\} \cup$
.-
$\{\{((n-s)(\text{mod } n))_i, (s+\alpha)_j\}: s \in \{0, 1, \cdots, (n-3)/2\}, \alpha \in \{0, 1\}\}$,see Figure 10. From the edge set of G_7, the following conditions are verified: Each graph $(P_{n+1})^{i,j}$ has precisely two edges of length $\lambda \in \{1, 2, \cdots, \lfloor (n-1)/2 \rfloor\}$, the length 0 is only present once in $(P_{n+1})^{i,j}$, the length $n/2$ is found once in $(P_{n+1})^{i,j}$ if n is even, for every $\lambda \in \{1, 2, \cdots, \lfloor (n-1)/2 \rfloor\}$, G_7 has precisely $2.\binom{m}{2} = m(m-1)$ edges of length λ, the length 0 is found $\binom{m}{2}$ times in G_7, and the length $n/2$ is found $\binom{m}{2}$ times in G_7 if n is even. Hence, $K_{\underbrace{n,n,\cdots,n}_{m}}$ can be decomposed by G_7. □

Theorem 12. *Let $n \equiv 1 \bmod 6, n \equiv 5 \bmod 6, m \geq 2$ be an integer. Then, there is an orthogonal labelling for $K_{\underbrace{n,n,\cdots,n}_{m}}$ by $G_8 \cong \bigcup_{0 \leq i < j \leq m-1} (nK_{1,1})^{i,j}$.*

Proof. Suppose $V((nK_{1,1})^{i,j})$ is $V(nK_{1,1})^{i,j} = \{v_s : s \in \{0, 1, 2, \cdots, 2n-1\}\}$. The mapping ψ_k can be used to define an orthogonal labelling for the subgraph G_8, which can be defined by $\psi_k : V((nK_{1,1})^{i,j}) \longrightarrow \mathbb{Z}_n \times \{i, j\}$ which is defined by $\psi_k(v_s) = s_i, s \in \{0, 1, \cdots, n-1\}, \psi_k(v_{n+s-1}) = ((2(s-1)) \bmod n)_j$, $s \in \{1, 2, \cdots, n\}$, and the edge set of $(nK_{1,1})^{i,j}$ is $E((nK_{1,1})^{i,j}) = \{\{s_i, ((2s)(\text{mod } n))_j\}: s \in \{0, 1, \cdots, n-1\}\}$, see Figure 11. From the edge set of G_8, the following conditions are verified: Each graph $(nK_{1,1})^{i,j}$ has precisely two edges of length $\lambda \in \{1, 2, \cdots, \lfloor (n-1)/2 \rfloor\}$, the length 0 is found once in $(nK_{1,1})^{i,j}$, the length $n/2$ is found once in $(nK_{1,1})^{i,j}$ if n is even, for every $\lambda \in \{1, 2, \cdots, \lfloor (n-1)/2 \rfloor\}$,$G_8$ has precisely $2.\binom{m}{2} = m(m$

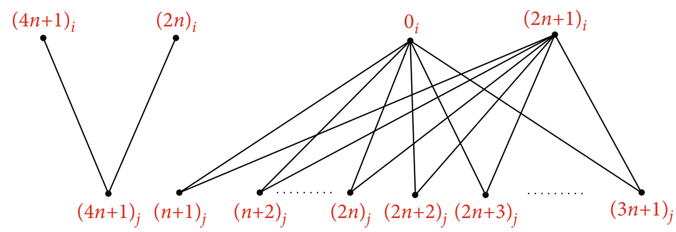

Figure 12: The labelling for $(K_{1,2} \cup K_{2,2n})^{i,j}$.

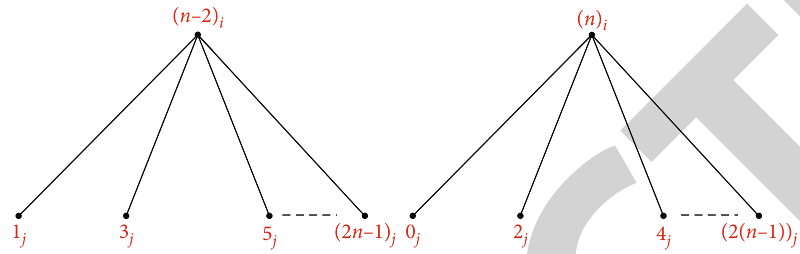

Figure 13: The labelling for $(2K_{1,n})^{i,j}$.

$-\,1)$ edges of length λ, the length 0 is found $\binom{m}{2}$ times in G_8, and the length $n/2$ is found $\binom{m}{2}$ times in G_8 if n is even. Hence, $K_{\underbrace{n,n,\cdots,n}_{m}}$ can be decomposed by G_8. □

Theorem 13. *Let $n \geq 1, m \geq 2$ be integers. Then, there is an orthogonal labelling for*

$$G_9 \cong \bigcup_{0 \leq i < j \leq m-1} (K_{1,2} \cup K_{2,2n})^{i,j}. \tag{16}$$

Proof. Suppose $V((K_{1,2} \cup K_{2,2n})^{i,j}) = \{v_s : s \in \{0, 1, 2, \cdots, 2n + 4\}\}$. The mapping ψ_k can be used to define an orthogonal labelling for the subgraph G_9, which can be defined by $\psi_k : V((K_{1,2} \cup K_{2,2n})^{i,j}) \longrightarrow \mathbb{Z}_{4n+2} \times \{i, j\}$ which is defined by

$$\psi_k(v_0) = (4n+1)_i, \psi_k(v_1) = (2n)_i, \psi_k(v_2) = 0_i, \psi_k(v_3) = (2n+1)_i, \psi_k(v_4) = (4n+1)_j, \tag{17}$$

$\psi_k(v_s) = (n+s-4)_j, s \in \{5, \cdots, n+4\}$, $\psi_k(v_{n+s}) = (2n+s-3)_j, s \in \{5, \cdots, n+4\}$, and the edge set of $(K_{1,2} \cup K_{2,2n})^{i,j}$ is

$$\begin{aligned} E((K_{1,2} \cup K_{2,2n})^{i,j}) = &\left\{\left\{(4n+1)_i, (4n+1)_j\right\}, \left\{(2n)_i, (4n+1)_j\right\}\right. \\ &\cup \left\{\{0_i, s_j\}, \{(2n+1)_i, s_j\}: s \in \{n+1, \cdots, 2n\}\right\} \\ &\cup \left\{\{0_i, s_j\}, \{(2n+1)_i, s_j\}, s \in \{2n+2, \cdots, 3n+1\}\right\}, \end{aligned} \tag{18}$$

see Figure 12. From the edge set of G_9, the following conditions are verified: Each graph $(K_{1,2} \cup K_{2,2n})^{i,j}$ has precisely two edges of length $\lambda \in \{1, 2, \cdots, \lfloor (4n+1)/2 \rfloor\}$, the length 0 is found once in $(K_{1,2} \cup K_{2,2n})^{i,j}$, the length $2n+1$ is found once in $(K_{1,2} \cup K_{2,2n})^{i,j}$, for every $\lambda \in \{1, 2, \cdots, \lfloor (4n+1)/2 \rfloor\}$,G_9 has precisely $2.\binom{m}{2} = m(m-1)$ edges of length λ, the length 0 is found $\binom{m}{2}$ times in G_9, and the length $2n+1$ is found $\binom{m}{2}$ times in G_9. Hence, $K_{\underbrace{(4n+2),(4n+2),\cdots,(4n+2)}_{m}}$ can be decomposed by G_9. □

Theorem 14. *Let $n \geq 2, m \geq 2$ be integers. Then, there is an orthogonal labelling for*

$$G_{10} \cong \bigcup_{0 \leq i < j \leq m-1} (2K_{1,n})^{i,j}. \tag{19}$$

Proof. Suppose $V((2K_{1,n})^{i,j}) = \{v_s : s \in \{0, 1, 2, \cdots, 2n+1\}\}$. The mapping ψ_k can be used to define an orthogonal labelling for the subgraph G_{10}, which can be defined by $\psi_k : V((2K_{1,n})^{i,j}) \longrightarrow \mathbb{Z}_{2n} \times \{i, j\}$ which is defined by $\psi_k(v_0) = (n-2)_i, \psi_k(v_1) = n_i, \psi_k(v_{s+2}) = s_j, s \in \{0, \cdots, 2n-1\}$, and the edge set of $(2K_{1,n})^{i,j}$ is $E((2K_{1,n})^{i,j}) = \{\{n_i, (2s+1)_j\}, \{(n-2)_i, (2s)_j\}: s \in \{0, 1, \cdots, n-1\}\}$, see Figure 13. From the edge set of G_{10}, the following conditions are verified: Each graph $(2K_{1,n})^{i,j}$ has precisely two edges of length $\lambda \in \{1, 2, \cdots, \lfloor (2n-1)/2 \rfloor\}$, the length 0 is found once in

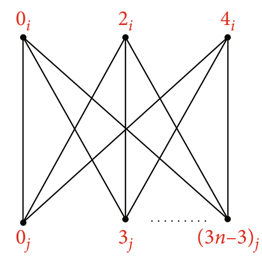

Figure 14: The labelling for $(K_{3,n})^{i,j}$.

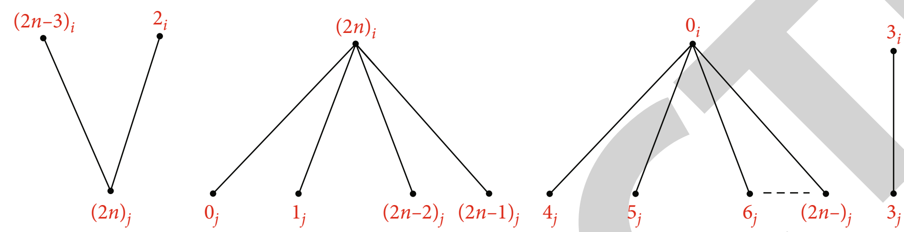

Figure 15: The labelling for $(K_{1,1} \cup K_{1,2} \cup K_{1,4} \cup K_{1,2n\text{-}6})^{i,j}$.

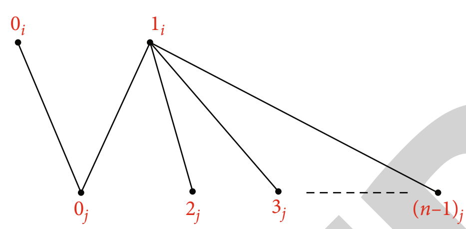

Figure 16: The labelling for $(C_2(1, n-2))^{i,j}$.

$(2K_{1,n})^{i,j}$, the length n is found once in $(2K_{1,n})^{i,j}$, for every $\lambda \in \{1, 2, \cdots, \lfloor (2n-1)/2 \rfloor\}$, G_{10} has precisely $2 \cdot \binom{m}{2} = m(m-1)$ edges of length λ, the length n is found $\binom{m}{2}$ times in G_{10}, and the length 0 is found $\binom{m}{2}$ times in G_{10}. Hence, $K_{\underbrace{2n,2n,\cdots,2n}_{m}}$ can be decomposed by G_{10}. □

Theorem 15. *For all positive integers n with gcd $(n, 3) = 1$, $m \geq 2$. Then, there is an orthogonal labelling for*

$$G_{11} \cong \bigcup_{0 \leq i<j \leq m-1} (nK_{2,2})^{i,j}. \tag{20}$$

Proof. Suppose $V((nK_{2,2})^{i,j})$ is $V((nK_{2,2})^{i,j}) = \{v_s : s \in \{0, 1, 2, \cdots, 4n-1\}\}$. The mapping ψ_k can be used to define an orthogonal labelling for the subgraph G_{11}, which can be defined by $\psi_k : V((nK_{2,2})^{i,j}) \longrightarrow \mathbb{Z}_{4n} \times \{i, j\}$ which is defined by

$$\psi_k(v_s) = s_i, s \in \{0, 1, \cdots, 2n-1\}, \psi_k(v_{2n+s}) = ((2s)(\text{mod } 4n))_j, s \in \{0, 1, \cdots, 2n-1\}, \tag{21}$$

and the edge set of $(nK_{2,2})^{i,j}$ is

$$E((nK_{2,2})^{i,j}) = \left\{\left\{s_i, (2s)_j\right\}: s \in \{0, 1, \cdots, 2n-1\}\right\} \cup \left\{\left\{(s-2n)_i, ((2s-2n)(\text{mod } 4n))_j\right\}: s \in \{2n, \cdots, 4n-1\}\right\}. \tag{22}$$

From the edge set of G_{11}, the following conditions are verified: Each graph $(nK_{2,2})^{i,j}$ has precisely two edges of length $\lambda \in \{1, 2, \cdots, \lfloor (4n-1)/2 \rfloor\}$, the length 0 is found once in $(nK_{2,2})^{i,j}$, the length $2n$ is found once in $(nK_{2,2})^{i,j}$, for every $\lambda \in \{1, 2, \cdots, \lfloor (4n-1)/2 \rfloor\}$, G_{11} has precisely $2 \cdot \binom{m}{2} = m(m-1)$ edges of length λ, the length 0 is found $\binom{m}{2}$ times in G_{11}, and the length $2n$ is found $\binom{m}{2}$ times in G_{11}. Hence, $K_{\underbrace{4n,4n,\cdots,4n}_{m}}$ can be decomposed by G_{11}. □

Theorem 16. *Let $n \geq 3, m \geq 2$ be integers. Then, there is an orthogonal labelling for*

$$G_{12} \cong \bigcup_{0 \leq i<j \leq m-1} (K_{3,n})^{i,j}. \tag{23}$$

Proof. Suppose $V((K_{3,n})^{i,j}) = \{v_s : s \in \{0, 1, 2, \cdots, 2n+4\}\}$. The mapping ψ_k can be used to define an orthogonal labelling for the subgraph G_{12}, which can be defined by $\psi_k : V((K_{3,n})^{i,j}) \longrightarrow \mathbb{Z}_{3n} \times \{i, j\}$ which is defined by $\psi_k(v_0) = 0_i$, $\psi_k(v_1) = 2_i$, $\psi_k(v_2) = 4_i$, $\psi_k(v_s) = (3(s-3))_j$, $s \in \{3, \cdots, n+2\}$, and the edge set of $(K_{3,n})^{i,j}$ is $E((K_{3,n})^{i,j}) = \{\{a_i, b_j\}: a \in \{0, 2, 4\}, b \in \{0, 3, 6, \cdots, 3n-3\}\}$, see Figure 14. From the

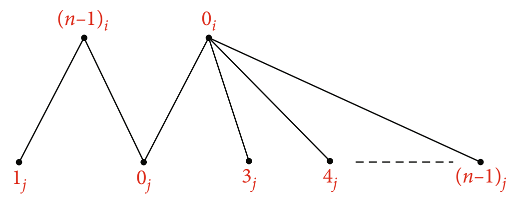

Figure 17: The labelling for $(C_3(1,0,n-3))^{i,j}$.

edge set of G_{12}, the following conditions are verified: Each graph $(K_{3,n})^{i,j}$ has precisely two edges of length $\lambda \in \{1,2,\cdots,\lfloor (3n-1)/2 \rfloor\}$, the length 0 is only present once in $(K_{3,n})^{i,j}$, the length $3n/2$ is found once in $(K_{3,n})^{i,j}$ if n is even, for every $\lambda \in \{1,2,\cdots,\lfloor (3n-1)/2 \rfloor\}$,$G_{12}$ has precisely $2.\binom{m}{2} = m(m-1)$ edges of length λ, the length 0 is found $\binom{m}{2}$ times in G_{12}, and the length $3n/2$ is found $\binom{m}{2}$ times in G_{12} if n is even. Hence, $K_{\underbrace{3n,3n,\cdots,3n}_{m}}$ can be decomposed by G_{12}. □

Theorem 17. *Let $n \geq 4, m \geq 2$ be integers. Then, there is an orthogonal labelling for*

$$G_{13} \cong \bigcup_{0 \leq i < j \leq m-1} (K_{1,1} \cup K_{1,2} \cup K_{1,4} \cup K_{1,2n-6})^{i,j}. \quad (24)$$

Proof. Suppose $V((K_{1,1} \cup K_{1,2} \cup K_{1,4} \cup K_{1,2n-6})^{i,j}) = \{v_s : s \in \{0,1,2,\cdots,2n+4\}\}$. The mapping ψ_k can be used to define an orthogonal labelling for the subgraph G_{13}, which can be defined by $\psi_k : V((K_{1,1} \cup K_{1,2} \cup K_{1,4} \cup K_{1,2n-6})^{i,j}) \longrightarrow \mathbb{Z}_{2n+1} \times \{i,j\}$ which is defined by

$$\begin{aligned} \psi_k(v_0) &= 3_i, \psi_k(v_1) = (2n-3)_i, \psi_k(v_2) = 2_i, \psi_k(v_3) \\ &= 0_i, \psi_k(v_4) = (2n)_i, \psi_k(v_5) = 3_j, \psi_k(v_6) \\ &= (2n)_j, \psi_k(v_7) = 0_j, \psi_k(v_8) = 1_j, \psi_k(v_9) \\ &= (2n-2)_j, \psi_k(v_{10}) = (2n-1)_j, \psi_k(v_{s+1}) \\ &= (s-6)_j, s \in \{10,\cdots,2n+3\}, \end{aligned} \quad (25)$$

and the edge set of $(K_{1,1} \cup K_{1,2} \cup K_{1,4} \cup K_{1,2n-6})^{i,j}$ is

$$\begin{aligned} E((K_{1,1} \cup K_{1,2} \cup K_{1,4} \cup K_{1,2n-6})^{i,j}) = \{&\{3_i,3_j\},\{(2n-3)_i,(2n)_j\},\{2_i,(2n)_j\},\{(2n)_i,0_j\},\{(2n)_i,1_j\},\{(2n)_i,(2n-2)_j\},\{(2n)_i,(2n-1)_j\}\} \\ &\cup \{\{0_i,a_j\}: a \in \{4,5,\cdots,2n-3\}\}, \end{aligned} \quad (26)$$

see Figure 15. From the edge set of G_{13}, the following conditions are verified: Each graph $(K_{1,1} \cup K_{1,2} \cup K_{1,4} \cup K_{1,2n-6})^{i,j}$ has precisely two edges of length $\lambda \in \{1,2,\cdots,n\}$, the length 0 is found once in $(K_{1,1} \cup K_{1,2} \cup K_{1,4} \cup K_{1,2n-6})^{i,j}$, for every $\lambda \in \{1,2,\cdots,n\}$,$G_{13}$ has precisely $2.\binom{m}{2} = m(m-1)$ edges of length λ, and the length 0 is found $\binom{m}{2}$ times in G_{13}. Hence, $K_{\underbrace{(2n+1)n,(2n+1)n,\cdots,(2n+1)n}_{m}}$ can be decomposed by G_{13}. □

4. Decompositions of $K_{\underbrace{n,n,\cdots,n}_{m}}$ by Connected Caterpillars

Theorem 18. *Let $n \geq 2, m \geq 2$ be integers. Then, there is an orthogonal labelling for*

$$G_{14} \cong \bigcup_{0 \leq i < j \leq m-1} (C_2(1,n-2))^{i,j}, \quad (27)$$

Proof. Suppose $V((C_2(1,n-2))^{i,j}) = \{v_s : s \in \{0,1,\cdots,n\}\}$. The mapping ψ_k can be used to define an orthogonal labelling for the subgraph G_{14}, which can be defined by $\psi_k : V((C_2(1,n-2))^{i,j}) \longrightarrow \mathbb{Z}_n \times \{i,j\}$ which is defined by

$$\begin{aligned} \psi_k(v_0) &= 0_i, \psi_k(v_1) = 1_i, \psi_k(v_2) = 0_j, \psi_k(v_s) \\ &= (s-1)_j, s \in \{3,4,\cdots,n\}, \end{aligned} \quad (28)$$

and the edge set of $(C_2(1,n-2))^{i,j}$ is

$$\begin{aligned} E((C_2(1,n-2))^{i,j}) &= \{\{0_i,0_j\},\{1_i,0_j\}\} \\ &\cup \{\{1_i,s_j\}: s \in \{2,3,\cdots,n-1\}\}, \end{aligned} \quad (29)$$

see Figure 16. From the edge set of G_{14}, the following conditions are verified: Each graph $(C_2(1,n-2))^{i,j}$ has precisely two edges of length $\lambda \in \{1,2,\cdots,\lfloor (n-1)/2 \rfloor\}$, the length 0 is found once in $(C_2(1,n-2))^{i,j}$, the length $n/2$ is found once in $(C_2(1,n-2))^{i,j}$, for every $\lambda \in \{1,2,\cdots,\lfloor (n-1)/2 \rfloor\}$,$G_{14}$ has precisely $2.\binom{m}{2} = m(m-1)$ edges of length λ, the length $n/2$ is found $\binom{m}{2}$ times in G_{14}, and the length 0 is found $\binom{m}{2}$ times in G_{14}. Hence, $K_{\underbrace{n,n,\cdots,n}_{m}}$ can be decomposed by G_{14}. □

Theorem 19. *Let $n \geq 3, m \geq 2$ be integers. Then, there is an orthogonal labelling for*

$$G_{15} \cong \bigcup_{0 \leq i<j \leq m-1}\left(C_3(1,0,n-3)\right)^{i,j}. \quad (30)$$

Proof. Suppose $V\left(\left(C_3(1,0,n-3)\right)^{i,j}\right)=\{v_s : s \in\{0,1,\cdots,n\}\}$. The mapping ψ_k can be used to define an orthogonal labelling for the subgraph G_{15}, which can be defined by $\psi_k : V\left(\left(C_3(1,0,n-3)\right)^{i,j}\right) \longrightarrow \mathbb{Z}_n \times\{i,j\}$ which is defined by $\psi_k(v_0) = 0_i, \psi_k(v_1)=(n-1)_i, \psi_k(v_2)=0_j$, $\psi_k(v_3)=1_j, \psi_k(v_s)=(s-1)_j, s \in\{4,5,\cdots,n\}$, and the edge set of $\left(C_3(1,0,n-3)\right)^{i,j}$ is

$$E\left(\left(C_3(1,0,n-3)\right)^{i,j}\right)=\left\{\left\{0_i, 0_j\right\},\left\{(n-1)_i, 0_j\right\},\left\{(n-1)_i, 1_j\right\}\right\} \cup\left\{\left\{0_i, s_j\right\}: s \in\{3,4,\cdots,n-1\}\right\}, \quad (31)$$

see Figure 17. From the edge set of G_{15}, the following conditions are verified: Each graph $\left(C_3(1,0,n-3)\right)^{i,j}$ has precisely two edges of length $\lambda \in\{1,2,\cdots,\lfloor(n-1)/2\rfloor\}$, the length 0 is found once in $\left(C_3(1,0,n-3)\right)^{i,j}$, the length $n/2$ is found once in $\left(C_3(1,0,n-3)\right)^{i,j}$ if n is even, for every $\lambda \in\{1,2,\cdots,\lfloor(n-1)/2\rfloor\}$, G_{15} has precisely $2.\binom{m}{2}=m(m-1)$ edges of length λ, the length $n/2$ is found $\binom{m}{2}$ times in G_{15} if n is even, and the length 0 is found $\binom{m}{2}$ times in G_{15}. Hence, $K_{\underbrace{n,n,\cdots,n}_{m}}$ can be decomposed by G_{15}. □

Theorem 20. *Let $n \geq 4, m \geq 2$ be integers. Then, there is an orthogonal labelling for*

$$G_{16} \cong \bigcup_{0 \leq i<j \leq m-1}\left(C_4(1,0,0,n-4)\right)^{i,j}. \quad (32)$$

Proof. Suppose $V\left(\left(C_4(1,0,0,n-4)\right)^{i,j}\right)=\{v_s : s \in\{0,1,\cdots,n\}\}$. The mapping ψ_k can be used to define an orthogonal labelling for the subgraph G_{16}, which can be defined by $\psi_k : V\left(\left(C_4(1,0,0,n-4)\right)^{i,j}\right) \longrightarrow \mathbb{Z}_n \times\{i,j\}$ which is defined by

$$\psi_k(v_0)=0_i, \psi_k(v_1)=(n-1)_i, \psi_k(v_2)=0_j, \psi_k(v_3)=1_j, \psi_k(v_4)=(n-2)_j, \psi_k(v_{s+3})=s_j, \quad (33)$$

$s \in\{2,3, \quad \cdots, n-3\}$, and the edge set of $\left(C_4(1,0,0,n-4)\right)^{i,j}$ is

$$E\left(\left(C_4(1,0,0,n-4)\right)^{i,j}\right)=\left\{\left\{0_i, 0_j\right\},\left\{0_i, 1_j\right\},\left\{(n-1)_i,(n-2)_j\right\},\left\{(n-1)_i, 1_j\right\}\right\} \cup\left\{\left\{s_i, 0_j\right\}: s \in\{2,3,\cdots,n-3\}\right\}, \quad (34)$$

see Figure 18. From the edge set of G_{16}, the following conditions are verified: Each graph $\left(C_4(1,0,0,n-4)\right)^{i,j}$ has precisely two edges of length $\lambda \in\{1,2,\cdots,\lfloor(n-1)/2\rfloor\}$, the length 0 is found once in $\left(C_4(1,0,0,n-4)\right)^{i,j}$, the length $n/2$ is found once in $\left(C_4(1,0,0,n-4)\right)^{i,j}$ if n is even, for every $\lambda \in\{1,2,\cdots,\lfloor(n-1)/2\rfloor\}$, G_{16} has precisely $2.\binom{m}{2}=m(m-1)$ edges of length λ, the length $n/2$ is found $\binom{m}{2}$ times in G_{16} if n is even, and the length 0 is found $\binom{m}{2}$ times in G_{16}. Hence, $K_{\underbrace{n,n,\cdots,n}_{m}}$ can be decomposed by G_{16}. □

Theorem 21. *Let $n \geq 5, m \geq 2$ be integers. Then, there is an orthogonal labelling for*

$$G_{17} \cong \bigcup_{0 \leq i<j \leq m-1}\left(C_5(1,0,0,0,n-5)\right)^{i,j}. \quad (35)$$

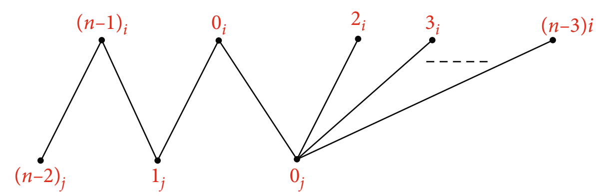

Figure 18: The labelling for $(C_4(1,0,0,n-4))^{i,j}$.

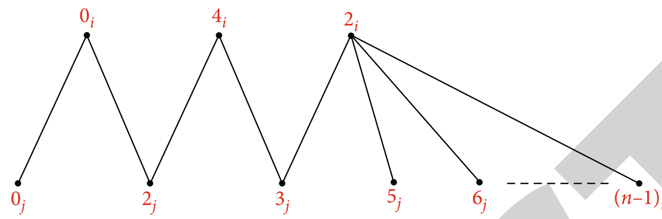

Figure 19: The labelling for $(C_5(1,0,0,0,n-5))^{i,j}$.

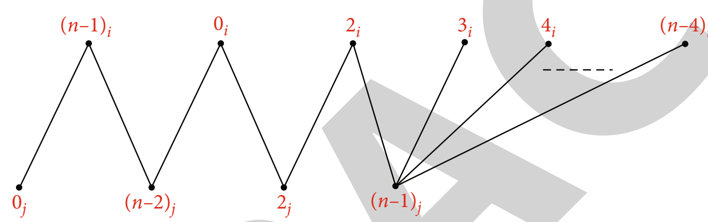

Figure 20: The labelling for $(C_6(1,0,0,0,0,n-6))^{i,j}$.

Proof. Suppose $V((C_5(1,0,0,0,n-5))^{i,j}) = \{v_s : s \in \{0,1,\cdots,n\}\}$. The mapping ψ_k can be used to define an orthogonal labelling for the subgraph G_{17}, which can be defined by $\psi_k : V((C_5(1,0,0,0,n-5))^{i,j}) \longrightarrow \mathbb{Z}_n \times \{i,j\}$ which is defined by

$$\begin{gathered}\psi_k(v_0) = 0_j, \psi_k(v_1) = 0_i, \psi_k(v_2) = 2_j, \psi_k(v_3) = 4_i, \psi_k(v_4) = 3_j,\\ \psi_k(v_5) = 2_i, \psi_k(v_{s+1}) = s_j, s \in \{5,6,\cdots,n-1\},\end{gathered} \tag{36}$$

and the edge set of $(C_5(1,0,0,0,n-5))^{i,j}$ is

$$E\left((C_5(1,0,0,0,n-5))^{i,j}\right) = \{\{0_i,0_j\},\{2_i,3_j\},\{0_i,2_j\},\{4_i,2_j\},\{4_i,3_j\}\} \cup \{\{2_i,s_j\}: s \in \{5,6,\cdots,n-1\}\}, \tag{37}$$

see Figure 19. From the edge set of G_{17}, the following conditions are verified: Each graph $(C_5(1,0,0,0,n-5))^{i,j}$ has precisely two edges of length $\lambda \in \{1,2,\cdots,\lfloor (n-1)/2 \rfloor\}$, the length 0 is found once in $(C_5(1,0,0,0,n-5))^{i,j}$, the length $n/2$ is found once in $(C_5(1,0,0,0,n-5))^{i,j}$ if n is even, for every $\lambda \in \{1,2,\cdots,\lfloor (n-1)/2 \rfloor\}$,$G_{17}$ has precisely $2.\binom{m}{2} = m(m-1)$ edges of length λ, the length $n/2$ is found $\binom{m}{2}$ times in G_{17} if n is even, and the length 0 is found $\binom{m}{2}$ times in G_{17}. Hence, $K_{\underbrace{n,n,\cdots,n}_{m}}$ can be decomposed by G_{17}. □

Theorem 22. *Let $n \geq 6, m \geq 2$ be integers. Then, there is an orthogonal labelling for*

$$G_{18} \cong \bigcup_{0 \leq i < j \leq m-1} \left(C_6(1,0,0,0,0,n-6)\right)^{i,j}. \quad (38)$$

Proof. Suppose $V\left(\left(C_6(1,0,0,0,0,n-6)\right)^{i,j}\right) = \{v_s : s \in \{0, 1, \cdots, n\}\}$. The mapping ψ_k can be used to define an orthogonal labelling for the subgraph G_{18}, which can be defined by $\psi_k : V\left(\left(C_6(1,0,0,0,0,n-6)\right)^{i,j}\right) \longrightarrow \mathbb{Z}_n \times \{i, j\}$ which is defined by

$$\begin{aligned}\psi_k(v_0) = 0_j, \psi_k(v_1) = (n-1)_i, \psi_k(v_2) = (n-2)_j, \psi_k(v_3) \\ = 0_i, \psi_k(v_4) = 2_j, \psi_k(v_5) = 2_i,.\end{aligned} \quad (39)$$

$\psi_k(v_6) = (n-1)_j, \psi_k(v_{s+4}) = s_i, s \in \{3, 4, \cdots, n-4\}$, and the edge set of $\left(C_6(1,0,0,0,0,n-6)\right)^{i,j}$ is

$$E\left(\left(C_6(1,0,0,0,0,n-6)\right)^{i,j}\right) = \left\{\left\{(n-1)_i, 0_j\right\}, \left\{(n-1)_i, (n-2)_j\right\}, \left\{0_i, (n-2)_j\right\}, \left\{0_i, 2_j\right\}, \left\{2_i, 2_j\right\}, \left\{2_i, (n-1)_j\right\}\right\} \cup \left\{\left\{s_i, (n-1)_j\right\} : s \in \{3, 4, \cdots, n-4\}\right\}, \quad (40)$$

see Figure 20. From the edge set of G_{18}, the following conditions are verified: Each graph $\left(C_6(1,0,0,0,0,n-6)\right)^{i,j}$ has precisely two edges of length $\lambda \in \{1, 2, \cdots, \lfloor (n-1)/2 \rfloor\}$, the length 0 is only present once in $\left(C_6(1,0,0,0,0,n-6)\right)^{i,j}$, the length $n/2$ is found once in $\left(C_6(1,0,0,0,0,n-6)\right)^{i,j}$ if n is even, for every $\lambda \in \{1, 2, \cdots, \lfloor (n-1)/2 \rfloor\}$,$G_{18}$ has precisely $2.\binom{m}{2} = m(m-1)$ edges of length λ, the length $n/2$ is found $\binom{m}{2}$ times in G_{18} if n is even, and the length 0 is found $\binom{m}{2}$ times in G_{18}. Hence, $K_{\underbrace{n,n,\cdots,n}_{m}}$ can be decomposed by G_{18}. □

Theorem 23. *Let $n \geq 7, m \geq 2$ be integers. Then, there is an orthogonal labelling for*

$$G_{19} \cong \bigcup_{0 \leq i < j \leq m-1} \left(C_7(1,0,0,0,0,0,n-7)\right)^{i,j}. \quad (41)$$

Proof. Suppose $V\left(\left(C_7(1,0,0,0,0,0,n-7)\right)^{i,j}\right) = \{v_s : s \in \{0, 1, \cdots, n\}\}$. The mapping ψ_k can be used to define an orthogonal labelling for the subgraph G_{19}, which can be defined by $\psi_k : V\left(\left(C_7(1,0,0,0,0,0,n-7)\right)^{i,j}\right) \longrightarrow \mathbb{Z}_n \times \{i, j\}$ which is defined by $\psi_k(v_0) = 2_j, \psi_k(v_1) = 1_i, \psi_k(v_2) = 0_j, \psi_k(v_3) = 0_i$, $\psi_k(v_4) = 3_j, \psi_k(v_5) = 6_i, \psi_k(v_6) = 4_j$, $\psi_k(v_7) = 2_i, \psi_k(v_{s+2}) = s_j, s \in \{6, 7, \cdots, n-2\}$, and the edge set of $\left(C_7(1,0,0,0,0,0,n-7)\right)^{i,j}$ is

$$\begin{aligned}E\left(\left(C_7(1,0,0,0,0,0,n-7)\right)^{i,j}\right) = \left\{\left\{1_i, 2_j\right\}, \left\{1_i, 0_j\right\}, \left\{0_i, 0_j\right\}, \left\{0_i, 3_j\right\}, \left\{6_i, 3_j\right\}, \left\{6_i, 4_j\right\}, \left\{2_i, 4_j\right\}\right\} \\ \cup \left\{\left\{2_i, s_j\right\} : s \in \{6, 7, \cdots, n-2\}\right\},\end{aligned} \quad (42)$$

see Figure 21. From the edge set of G_{19}, the following conditions are verified: Each graph $\left(C_7(1,0,0,0,0,0,n-7)\right)^{i,j}$ has precisely two edges of length $\lambda \in \{1, 2, \cdots, \lfloor (n-1)/2 \rfloor\}$, the length 0 is only present once in $\left(C_7(1,0,0,0,0,0,n-7)\right)^{i,j}$, the length $n/2$ is found once in $\left(C_7(1,0,0,0,0,0,n-7)\right)^{i,j}$ if n is even, for every $\lambda \in \{1, 2, \cdots, \lfloor (n-1)/2 \rfloor\}$,$G_{19}$ has precisely $2.\binom{m}{2} = m(m-1)$ edges of length λ, the length $n/2$ is found $\binom{m}{2}$ times in G_{19} if n is even, and the length 0 is found $\binom{m}{2}$ times in G_{19}. Hence, $K_{\underbrace{n,n,\cdots,n}_{m}}$ can be decomposed by G_{19}. □

Theorem 24. *Let $n \geq 8, m \geq 2$ be integers. Then, there is an orthogonal labelling for*

$$G_{20} \cong \bigcup_{0 \leq i < j \leq m-1} \left(C_8(1,0,0,0,0,0,0,n-8)\right)^{i,j}. \quad (43)$$

Proof. Suppose $V\left(\left(C_8(1,0,0,0,0,0,0,n-8)\right)^{i,j}\right) = \{v_s : s \in \{0, 1, \cdots, n\}\}$. The mapping ψ_k can be used to define an orthogonal

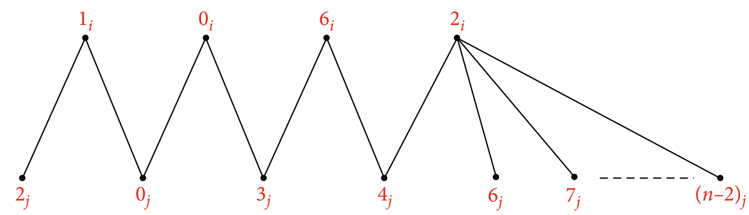

Figure 21: The labelling for $(C_7(1,0,0,0,0,0,n-7))^{i,j}$.

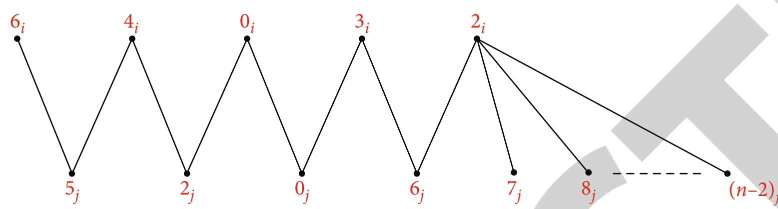

Figure 22: The labelling for $(C_8(1,0,0,0,0,0,0,n-8))^{i,j}$.

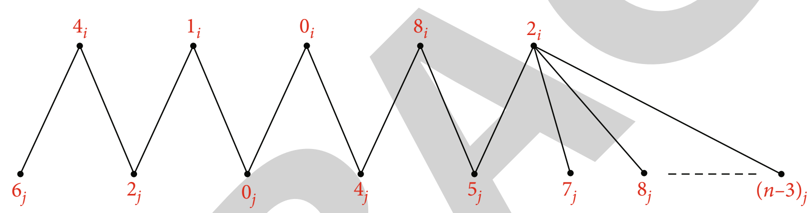

Figure 23: The labelling for $(C_9(1,0,0,0,0,0,0,0,n-9))^{i,j}$.

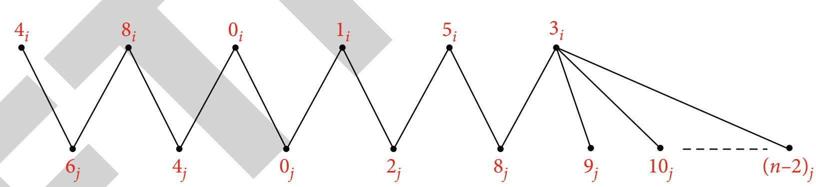

Figure 24: The labelling for $(C_{10}(1,0,0,0,0,0,0,0,0,n-10))^{i,j}$.

labelling for the subgraph G_{20}, which can be defined by $\psi_k : V((C_8(1,0,0,0,0,0,0,n-8))^{i,j}) \longrightarrow \mathbb{Z}_n \times \{i,j\}$ which is defined by-
$\psi_k(v_0) = 6_i, \psi_k(v_1) = 5_j, \psi_k(v_2) = 4_i, \psi_k(v_3) = 2_j, \psi_k(v_4) = 0_i,$
$\psi_k(v_5) = 0_j, \psi_k(v_6) = 3_i, \psi_k(v_7) = 6_j$,.

$\psi_k(v_8) = 2_i, \varphi(v_{i+2}) = i_1, i \in \{7,8,\cdots,n-2\}$, and the edge set of $(C_8(1,0,0,0,0,0,0,n-8))^{i,j}$ is

$$E((C_8(1,0,0,0,0,0,0,n-8))^{i,j}) = \{\{6_i,5_j\},\{4_i,2_j\},\{0_i,2_j\},\{0_i,0_j\},\{3_i,0_j\},\{3_i,6_1\},\{2_i,6_j\}\} \cup \{\{2_i,s_j\}: s \in \{7,8,\cdots,n-2\}\}, \tag{44}$$

see Figure 22. From the edge set of G_{20}, the following conditions are verified: Each graph $(C_8(1,0,0,0,0,0,0,n-8))^{i,j}$ has precisely two edges of length $\lambda \in \{1,2,\cdots,\lfloor (n-1)/2 \rfloor\}$, the length 0 is found once in $(C_8(1,0,0,0,0,0,0,n-8))^{i,j}$, the length $n/2$ is found once in $(C_8(1,0,0,0,0,0,0,n-8))^{i,j}$ if n is even, for every $\lambda \in \{1,2,\cdots,\lfloor (n-1)/2 \rfloor\}$,$G_{20}$ has precisely $2.\binom{m}{2} = m(m-1)$ edges of length λ, the length $n/2$ is found $\binom{m}{2}$ times in G_{20} if n is even, and the length 0 is found $\binom{m}{2}$ times in G_{20}. Hence, $K_{\underbrace{n,n,\cdots,n}_{m}}$ can be decomposed by G_{20}. □

Theorem 25. *Let $n \geq 9, m \geq 2$ be integers. Then, there is an orthogonal labelling for*

$$G_{21} \cong \bigcup_{0 \leq i < j \leq m-1} (C_9(1,0,0,0,0,0,0,0,n-9))^{i,j}. \quad (45)$$

Proof. Suppose $V((C_9(1,0,0,0,0,0,0,0,n-9))^{i,j}) = \{v_s : s \in \{0,1,\cdots,n\}\}$. The mapping ψ_k can be used to define an orthogonal labelling for the subgraph G_{21}, which can be defined by $\psi_k : V((C_9(1,0,0,0,0,0,0,0,n-9))^{i,j}) \longrightarrow \mathbb{Z}_n \times \{i,j\}$ which is defined by-
$\psi_k(v_0) = 6_j, \psi_k(v_1) = 4_i, \psi_k(v_2) = 2_j, \psi_k(v_3) = 1_i, \psi_k(v_4) = 0_j, \psi_k(v_5) = 0_i, \psi_k(v_6) = 4_j, \psi_k(v_7) = 8_i$,.
$\psi_k(v_8) = 5_j, \psi_k(v_9) = 2_i, \psi_k(v_{s+3}) = s_j, s \in \{7,8,\cdots,n-3\}$, and the edge set of $(C_9(1,0,0,0,0,0,0,0,n-9))^{i,j}$ is

$$E((C_9(1,0,0,0,0,0,0,0,n-9))^{i,j}) = \{\{4_i,6_j\},\{4_i,2_j\},\{1_i,2_j\},\{1_i,0_j\},\{0_i,0_j\},\{0_i,4_j\},\{8_i,4_j\},\{8_i,5_j\},\{2_i,5_j\}\} \cup \{\{2_i,s_j\} : s \in \{7,8,\cdots,n-3\}\}, \quad (46)$$

see Figure 23. From the edge set of G_{21}, the following conditions are verified: Each graph $(C_9(1,0,0,0,0,0,0,0,n-9))^{i,j}$ has precisely two edges of length $\lambda \in \{1,2,\cdots,\lfloor (n-1)/2 \rfloor\}$, the length 0 is found once in $(C_9(1,0,0,0,0,0,0,0,n-9))^{i,j}$, the length $n/2$ is found once in $(C_9(1,0,0,0,0,0,0,0,n-9))^{i,j}$ if n is even, for every $\lambda \in \{1,2,\cdots,\lfloor (n-1)/2 \rfloor\}$,$G_{21}$ has precisely $2.\binom{m}{2} = m(m-1)$ edges of length λ, the length $n/2$ is found $\binom{m}{2}$ times in G_{21} if n is even, and the length 0 is found $\binom{m}{2}$ times in G_{21}. Hence, $K_{\underbrace{n,n,\cdots,n}_{m}}$ can be decomposed by G_{21}. □

Theorem 26. *Let $n \geq 10, m \geq 2$ be integers. Then, there is an orthogonal labelling for*

$$G_{22} \cong \bigcup_{0 \leq i < j \leq m-1} (C_{10}(1,0,0,0,0,0,0,0,0,n-10))^{i,j}. \quad (47)$$

Proof. Suppose $V((C_{10}(1,0,0,0,0,0,0,0,0,n-10))^{i,j}) = \{v_s : s \in \{0,1,\cdots,n\}\}$. The mapping ψ_k can be used to define an orthogonal labelling for the subgraph G_{22}, which can be defined by $\psi_k : V((C_{10}(1,0,0,0,0,0,0,0,0,n-10))^{i,j}) \longrightarrow \mathbb{Z}_n \times \{i,j\}$ which is defined by

$$\begin{aligned} &\psi_k(v_0) = 4_i, \\ &\psi_k(v_1) = 6_j, \psi_k(v_2) = 8_i, \psi_k(v_3) = 4_j, \psi_k(v_4) = 0_i, \psi_k(v_5) = 0_j, \psi_k(v_6) = 1_i, \psi_k(v_7) \\ &= 2_j, \psi_k(v_8) = 5_i, \psi_k(v_9) = 8_j, \psi_k(v_{10}) = 3_i, \psi_k(v_{s+2}) = s_j, s \in \{9,10,\cdots,n-2\}, \end{aligned} \quad (48)$$

and the edge set of $(C_{10}(1,0,0,0,0,0,0,0,0,n-10))^{i,j}$ is

$$E((C_{10}(1,0,0,0,0,0,0,0,0,n-10))^{i,j}) = \{\{4_i,6_j\},\{8_i,6_j\},\{8_i,4_j\},\{0_i,4_j\},\{0_i,0_j\},\{1_i,0_j\},\{1_i,2_j\},\{5_i,2_j\},\{5_i,8_j\}\} \cup \{\{3_i,s_j\} : s \in \{8,9,\cdots,n-2\}\}, \quad (49)$$

see Figure 24. From the edge set of G_{22}, the following conditions are verified: Each graph $(C_{10}(1,0,0,0,0,0,0,0,0,n-10))^{i,j}$ has precisely two edges of length $\lambda \in \{1,2,\cdots,\lfloor (n-1)/2 \rfloor\}$, the length 0 is found once in $(C_{10}(1,0,0,0,0,0,0,0,0,n-10))^{i,j}$, the length $n/2$ is found once in $(C_{10}(1,0,0,0,0,0,0,0,0,n-10))^{i,j}$ if n is even, for every $\lambda \in \{1,2,\cdots,\lfloor (n-1)/2 \rfloor\}$,$G_{22}$ has precisely $2.\binom{m}{2} = m(m-1)$ edges of length λ, the length $n/2$ is found $\binom{m}{2}$ times in G_{22} if n is even, and the length 0 is found $\binom{m}{2}$ times in G_{22}. Hence, $K_{\underbrace{n,n,\cdots,n}_{m}}$ can be decomposed by G_{22}. □

5. Conclusion

As known, there are several types of graphs labelling. Herein, we are concerned with orthogonal labelling notion. As a generalization to the orthogonal labelling approach provided in the literature for finding the decomposition of circulant-balanced complete bipartite graphs $K_{n,n}$, we have developed a generalized orthogonal labelling approach for decomposing the circulant-balanced complete multipartite graphs $K_{\underbrace{n,n,\cdots,n}_{m}}$; $m, n \geq 2$, in this study. In the future, we will work to improve the orthogonal labelling approach so that it may be used with all types of circulant graphs.

Nomenclatures

K_m: Complete graph having m vertices
kH: k disjoint unions of graph H
$K_{m,n}$: Complete bipartite graph with size $m+n$, where the vertex set is divided into two sets with sizes m and n
C_x: Cycle graph on x vertices
P_m: Path graph on m vertices
$V(G)$: Vertex set of graph G
$E(G)$: Edge set of graph G
$G \cup H$: Disjoint union of graphs G and H.

Data Availability

The data used to support the findings of this study are available from the corresponding author on request.

Conflicts of Interest

The authors declare no conflict of interest.

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