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Research Article

The Long-Time Behavior of 2D Nonautonomous g-Navier-Stokes Equations with Weak Dampness and Time Delay

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In this paper, we discuss the long-time behavior of g-Navier-Stokes equations with weak dampnesss and time delay. The uniformly attracting sets of processes are obtained. On the basis of the method with asymptotic compactness, the existence of the uniform attractor for the equation is proved with the restriction of the forcing term belonging to translational compacted function space.

1. Introduction

The understanding of the behavior with dynamical systems was one of the most important problems of modern mathematical physics (see [1–17]). In the last decades, g-Navier-Stokes equations have received increasing attention due to their importance in the fluid motion. In [2–4], the existence of weak solution and strong solution for the 2D g-Navier-Stokes equation on some bounded domain was studied. The Hausdorff and fractal dimension of the global attractor about the 2D g-Navier-Stokes equation for the periodic and Dirichlet boundary conditions and the global attractor of the 2D g-Navier-Stokes equation on some unbounded domains were researched in [5]. In [6–10], the finite dimensional global attractor and the pullback attractor for g-Navier-Stokes equation were studied. Moreover, Anh et al. studied long-time behavior for 2D nonautonomous g-Navier-Stokes equations and the stability of solutions to stochastic 2D g-Navier-Stokes equation with finite delays in [11, 12]; Quyee researched the stationary solutions to 2D g-Navier-Stokes equation and pullback attractor for 2D g-Navier-Stokes equation with infinite delays in [13]. Recently, the random attractors for the 2D stochastic g-Navier-Stokes equation were researched in [14]. From these researches, we can see that the attractor of 2D g-Navier-Stokes equation is still important. We would like to use the theory of uniform attractors to study it. So, the present research is necessary and has a theoretical basis.

In this paper, we study the existence of the uniform attractor of the g-Navier-Stokes equation with weak dampness and time delay which have the following form:

$$\begin{gathered}\frac{\partial u}{\partial t}-\nu\Delta u+(u\cdot\nabla)u+\alpha u+\nabla p=f(x,t)+h(t,u_t),\ \text{on}\ (\tau,+\infty)\times\Omega,\\ \nabla\cdot(gu)=0,\ \text{on}\ (\tau,+\infty)\times\Omega,\\ u(x,t)=0,\ \text{on}\ (\tau,+\infty)\times\partial\Omega,\\ u(\tau,x)=u_0(x)\ x\in\Omega,\end{gathered}\tag{1}$$

where $u(t,x)\in R^2$ and $p(t,x)\in R$ denote the velocity and pressure, respectively. $\nu>0$ is the viscosity coefficient, αu denotes linear dampness, and $\alpha>0$ is positive constant. $f=f(x,t)\in(L^2(\Omega))^2$ is the time-dependent external force term, $h(t,u_t)$ is another external force term with time delay. $0<m_0\le g=g(x_1,x_2)\le M_0$ and $g=g(x_1,x_2)$ are suitable real-valued smooth functions; when $g=1$, Equation (1) becomes the usual 2D Navier-Stokes equations.

This paper is organized as follows. In Section 2, we first introduce some notations and preliminary results for the g

-Navier-Stokes equation. In Section 3, we prove existence of the uniform attractor of 2D g-Navier-Stokes equation with weak dampness and time delay on the bounded domains.

2. Preliminaries

We assume that the Poincare inequality holds on Ω, i.e., there exists $\lambda_1 > 0$, such that

$$\int_\Omega \phi^2 g dx \le \frac{1}{\lambda_1}\int_\Omega |\nabla\phi|^2 g dx, \forall \phi \in H_0^1(\Omega). \quad (2)$$

Let $L^2(g) = (L^2(\Omega))^2$ with the inner products $(u, v) = \int_\Omega u \cdot v g dx$ and the norms $|\cdot| = (\cdot, \cdot)^{1/2}$, $u, v \in L^2(g)$. Let $H_0^1(g) = (H_0^1(\Omega))^2$, which is endowed with the inner products $((u, v)) = \int_\Omega \Sigma_{j=1}^2 \nabla u_j \cdot \nabla v_j g dx$ and the norms $\|\cdot\| = ((\cdot, \cdot))^{1/2}$, where $u = (u_1, u_2), v = (v_1, v_2) \in H_0^1(g)$.

Let $D(\Omega)$ be the space of $\mathscr{C}^\infty$ function with the compact support contained in Ω, and let $\aleph = \{v \in (D(\Omega))^2 : \nabla \cdot g v = 0 \text{ on } \Omega\}$; the closure of $\aleph$ in $L^2(g)$ is H_g; the closure of $\aleph$ in $H_0^1(g)$ is V_g. H_g has the inner product and norm of $L^2(g)$, And V_g has the inner product and norm of $H_0^1(g)$.

It follows from (2) that

$$|u|^2 \le \frac{1}{\lambda_1}\|u\|^2, \forall u \in V_g. \quad (3)$$

We define a g-Laplacian operator as follows: $-\Delta_g u = -(1/g)(\nabla \cdot g \nabla)u = -\Delta u - (1/g)\nabla g \cdot \nabla u$.

Using the g-Laplacian operator, we rewrite the first Equation (1) as follows:

$$\frac{\partial u}{\partial t} - \nu\Delta_g u + \nu\frac{\nabla g}{g}\cdot\nabla u + (u, \nabla)u + \alpha u + \nabla p = f + h(t, u_t). \quad (4)$$

From [2], we can define a g-orthogonal projection $P_g : L^2(g) \longrightarrow H_g$ and a g-Stokes operator $A_g u = -P_g((1/g)(\nabla\cdot(g\nabla u)))$.

Applying the projection P_g into (4), we can obtain the following weak formulation of (1): let $f \in V_g$ and $u_0 \in H_g$, we find that

$$u \in L^\infty(0, T; H_g) \cap L^2(0, T; V_g), T > 0, \quad (5)$$

such that $\forall v \in V_g, \forall t > 0$.

$$\frac{d}{dt}(u, v) + \nu((u, v)) + b_g(u, u, v) + \alpha(u, v) + \nu(Ru, v) = \langle f, v\rangle + \langle h(t, u_t), v\rangle, \quad (6)$$

$$u(0) = u_0, \quad (7)$$

where $b_g : V_g \times V_g \times V_g \longrightarrow R$ is given by

$$b_g(u, v, w) = \sum_{i,j=1}^2 \int u_i \frac{\partial v_j}{\partial x} w_j g dx, \quad (8)$$

and $Ru = P_g[(1/g)(\nabla g \cdot \nabla)u]$, such that $(Ru, v) = b(\nabla g/g, u, v), \forall u, v \in V_g$. Then, the weak formulation of (6) and (7) is equivalent to the functional equations

$$\frac{du}{dt} + \nu A_g u + Bu + \alpha u + \nu Ru = f + h, \quad (9)$$

$$u(0) = u_0, \quad (10)$$

where $A_g : V_g \longrightarrow V_{g'}$ is the g-Stokes operator defined by $\langle A_g u, v\rangle = ((u, v)), \forall u, v \in V_g$.$B(u) = B(u, u) = P_g(u \cdot \nabla)u$ is bilinear operator and $B : V_g \times V_g \longrightarrow V'_g$, $\langle B(u, v), w\rangle = b_g(u, v, w), \forall u, v, w \in V_g$, where B and R satisfy the following inequalities [2, 4]:

$$\|B(u)\|_{V'_g} \le c|u|\|u\|, \|Ru\|_{V'_g} \le \frac{|\nabla g|_\infty}{m_0 \lambda_1^{1/2}}\|u\|, \forall u \in V_g. \quad (11)$$

Let $T > \tau$, $u : (\tau - r, T) \longrightarrow (L^2(\Omega))^2$. For every $t \in (\tau, T)$, we define $u_t(s) = u(t + s), s \in (-h, 0)$. For convenience, we denote $C_{H_g} = C^0([-h, 0]; H_g), C_{V_g} = C^0([-h, 0]; V_g)$, $L^2_{H_g} = L^2(-h, 0; H_g), L^2_{V_g} = L^2(-h, 0; V_g)$.

Let $h : R \times C_{H_g} \longrightarrow (L^2(\Omega))^2$ satisfy the following assumptions:

(I) $\forall \xi \in C_{H_g}, t \in R \longrightarrow h(t, \xi) \in (L^2(\Omega))^2$ is measureable,

(II) $\forall t \in R, h(t, 0) = 0$,

(III) $\exists L_g > 0$, such that $\forall t \in R, \forall \xi, \eta \in C_{H_g}$, there is $|h(t, \xi) - h(t, \eta)| \le L_g\|\xi - \eta\|_{C_{H_g}}$.

(IV) $\exists m_0 \ge 0, C_g > 0, \forall m \in [0, m_0], \tau \le t, u, v \in C^0([\tau - r, t]; H_g)$, such that

$$\int_\tau^t e^{ms}|h(s, u_s) - h(s, v_s)|^2 ds \le C_g^2\int_{\tau - r}^t e^{ms}|u(s) - v(s)|^2 ds. \quad (12)$$

$\forall t \in [\tau, T], \forall u, v \in L^2(\tau - r, T; H_g)$, from (IV), we have

$$\int_\tau^t |h(s, u_s) - h(s, v_s)|^2_{(L^2(\Omega))^2} ds \le C_g^2\int_{\tau - r}^t |u(s) - v(s)|^2 ds. \quad (13)$$

Definition 1. Let $u_0 \in H_g, \phi \in L^2_{H_g}, f \in L^2_{\text{Loc}}(R; V'_g)$ and $h : R \times C_{H_g} \longrightarrow (L^2(\Omega))^2$ satisfy the hypotheses (I)-(IV). For every $\tau \in \mathrm{R}$, a function $u \in L^2(\tau, T; V_g) \cap L^\infty(\tau, T; H_g), \forall T > \tau$ is called a weak solution of problem (1) if it fulfils

$$\frac{d}{dt}u(t) + \nu A_g u(t) + B(u(t)) + \alpha u(t) + \nu R(u(t)) = f(t) + h(t, u_t) \text{on} \mathscr{D}'\left(\tau, +\infty; V_{g'}\right) u(\tau) = u_0. \tag{14}$$

We can obtain the following theorem by the standard Faedo-Galerkin methods, where we let $T > \tau > 0$. Other cases can be similarly proved.

Theorem 2. *Let $f \in L^2_{Loc}(R; V'_g)$, $u_0(x) \in H_g$, $h \in (L^2(\Omega))^2$ satisfies the assumptions (I)-(IV), there exists a unique solution*

$$u(x,t) \in L^\infty\left(0, T; H_g\right) \cap L^2\left(0, T; V_g\right), (\forall T > 0), \tag{15}$$

such that (6) and (7) holds.

Proof. We apply the Faedo-Galerkin methods. Since V_g is separable and $\aleph$ is dense in V_g, there exists a sequence $\{w_i\}_{i\in N} \in \aleph$, which forms a complete orthonormal system in H_g and a basic for V_g. Let m be a positive integer, for each m, we define an approximate solution u_m of (6) as $u_m = \sum_{i=1}^m \phi_{im}(t) w_i$, which satisfies

$$\begin{aligned}&\left(u'_m(t), w_j\right) + \nu\left(\left(u_m(t), w_j\right)\right) + b\left(u_m(t), u_m(t), w_j\right) \\ &\quad + \alpha\left(u_m(t), w_j\right) + b\left(\frac{\nabla g}{g}, u_m(t), w_j\right) \\ &= \left\langle f(t), w_j\right\rangle + \left\langle H(t, u_t), w_j\right\rangle,\end{aligned} \tag{16}$$

for $t \in [0, T]$, $j = 1, \cdots, m$ and $u_m(0) = u_{0m}$, where u_{0m} is the orthegonal projection in H_g of u_0 onto the space spanned by $w_1, \cdots, w_m$. Then, we can obtain

$$\begin{aligned}&\sum_{i=1}^m (w_i . w_j)\phi'_{im}(t) + \nu\sum_{i=1}^m ((w_i, w_j))\phi_{im}(t) \\ &\quad + \sum_{i,l=1}^m b(w_i, w_l, w_j)\phi_{im}(t)\phi_{lm}(t) + \alpha\sum_{i=1}^m (w_i, w_j)\phi_{im}(t) \\ &\quad + \sum_{i=1}^m b\left(\frac{\nabla g}{g}, w_i, w_j\right)\phi_{im}(t) = \left\langle f(t), w_j\right\rangle + \left\langle h(t, u_t), w_j\right\rangle.\end{aligned} \tag{17}$$

We can write the differential equations in the usual form

$$\begin{aligned}&\sum_{i=1}^m \phi'_{im}(t) + \sum_{j=1}^m \alpha_{ij}\phi_{jm}(t) + \sum_{j,k=1}^m \alpha_{ijk}\phi_{jm}(t)\phi_{km}(t) \\ &= \sum_{j=1}^m \beta_{ij}\left\langle f(t), w_j\right\rangle + \sum_{j=1}^m \gamma_{ij}\left\langle h(t, u_t), w_j\right\rangle,\end{aligned} \tag{18}$$

where $\alpha_{ij}, \alpha_{ijk}, \beta_{ij} \in \mathrm{R}$.

Let $\phi_{im}(0)$ be the ith component of u_{0m}. The nonlinear ordinary differential system (18) has a maximal solution defined on some interval $[0, t_m]$. If $t_m < T$, then $|u_m(t)| \longrightarrow \infty$ as $t \longrightarrow t_m$. The following we will prove $t_m = T$. We need several estimates to do.

We multiply (16) by $\phi_{jm}(t)$ and add these equations for $j = 1, \cdots, m$ to obtain

$$\begin{aligned}&\left(u'_m(t), u_m(t)\right) + \nu\|u_m(t)\|^2 = \langle f(t), u_m(t)\rangle \\ &\quad + \langle h(t, u_t), u_m(t)\rangle - \alpha|u_m(t)|^2 - b\left(\left(\frac{\nabla g}{g}\cdot\nabla\right)u_m(t), u_m(t)\right).\end{aligned} \tag{19}$$

Then, we have

$$\begin{aligned}&\frac{d}{dt}|u_m(t)|^2 + 2\nu\|u_m(t)\|^2 = 2\langle f(t), u_m(t)\rangle + 2\langle h(t, u_t), u_m(t)\rangle \\ &\quad - 2b\left(\left(\frac{\nabla g}{g}\cdot\nabla\right)u_m(t), u_m(t)\right) - 2\alpha|u_m(t)|^2 \\ &\le 2\|f(t)\|_{V'_g}\|u_m(t)\|_{V_g} + 2\|h(t, u_t)\|_{V'_g}\|u_m(t)\|_{V_g} \\ &\quad + \frac{2}{m}|\nabla g|_\infty|u_m(t)|\|u_m(t)\| + \|u_m(t)\|^2 + 2\alpha|u_m(t)|^2 \\ &\le \nu\|u_m(t)\|^2 + \frac{8}{\nu}\|f(t)\|^2 + \nu\|u_m(t)\|^2 + \frac{8}{\nu}\|h(t, u_t)\|^2 \\ &\quad + \frac{2}{\nu m^2}|\nabla g|^2_\infty|u_m(t)|^2 + \nu|u_m(t)|^2 + 2\alpha|u_m(t)|^2,\end{aligned} \tag{20}$$

so that

$$\begin{aligned}\frac{d}{dt}|u_m(t)|^2 &\le \frac{8}{\nu}\left(\|f(t)\|^2 + \|h(t, u_t)\|^2\right) \\ &\quad + \left(\frac{2}{\nu m^2}|\nabla g|^2_\infty + \nu + 2\alpha\right)|u_m(t)|^2.\end{aligned} \tag{21}$$

Let $K = 2/\nu m^2|\nabla g|^2_\infty + \nu + 2\alpha$, then

$$\frac{d}{dt}|u_m(t)|^2 \le K|u_m(t)|^2 + \frac{8}{\nu}\left(\|f(t)\|^2 + \|h(t, u_t)\|^2\right). \tag{22}$$

By the Gronwall inequality, we have

$$|u_m(t)|^2 \le e^{Kt}\left(|u_m(0)|^2 + \frac{8}{\nu}\int_0^t\left(\|f(s)\|^2_{V'_g} + \|h(t, u_s)\|^2\right)ds.\right. \tag{23}$$

Hence,

$$\sup_{s\in[0,T]}|u_m(s)|^2 \le e^{KT}\left(|u_m(0)|^2 + \frac{8}{\nu}\int_0^t\left(\|f(s)\|^2_{V'_g} + \|h(t, u_s)\|^2\right)ds,\right. \tag{24}$$

which implies that the sequence u_m remains in bounded set of $L^\infty(0,T;H_g)$. From (22), we have

$$\frac{d}{dt}|u_m(t)|^2 \le \frac{K}{\lambda_1}\|u_m(t)\|^2 + \frac{8}{\nu}\left(\|f(t)\|^2 + \|h(t,u_t)\|^2\right). \tag{25}$$

Then,

$$\frac{d}{dt}|u_m(t)|^2 - \frac{K}{\lambda_1}\|u_m(t)\|^2 \le \frac{8}{\nu}\left(\|f(t)\|^2 + \|h(t,u_t)\|^2\right). \tag{26}$$

We intergrate (26) from 0 to T; we have

$$|u_m(t)|^2 - \frac{K}{\lambda_1}\int_0^T \|u_m(t)\|^2 dt \le |u_{0m}|^2 + \frac{8}{\nu}\int_0^T \left(\|f(t)\|^2 + \|h(t,u_t)\|^2\right)dt. \tag{27}$$

So, the u_m remains in a bounded set of $L^2(0,T;V_g)$.

Let $\tilde{u}_m$ denotes the function from R into V_g, which is equal to u_m on $[0,T]$ and to 0 on the complement of this interval. The Fourier transform of $\tilde{u}_m$ is denoted by $\hat{u}_m$. Then, we will show that there exist a positive constant C and γ such that

$$\int_{-\infty}^{+\infty} |\tau|^{2\gamma}|\hat{u}_m(\tau)|^2 d\tau < C. \tag{28}$$

Since the u_m remains in a bounded set of $L^2(0,T;V_g)$, the $\tilde{u}_m$ remains in a bounded set of $H^\gamma(\mathrm{R};V_g,H_g)$. Since $\tilde{u}_m$ has two discontinuities at 0 and T, the distribute derivative of $\tilde{u}_m$ is given by

$$\frac{d}{dt}\tilde{u}_m = \widetilde{\phi}_m + u_m(0)\delta_0 - u_m(T)\delta_T, \tag{29}$$

where δ_0 and δ_T are the dirac distributions at 0 and T, and $\phi_m = u_m'$ is the derivative of u_m on $[0,T]$. We obtain that

$$\frac{d}{dt}\langle \hat{u}_m, w_j\rangle = \left\langle \hat{f}_m, w_j\right\rangle + \langle u_{0m}, w_j\rangle\delta_0 - \langle u_m(T), w_j\rangle\delta_T, \tag{30}$$

for $j = 1, \cdots, m$, where δ_0 and δ_T are distributions at 0 and T, $f_m = f + h - \nu A u_m - b u_m - \nu R u_m - \alpha u_m$ and $\tilde{f}_m = f_m$ on $[0,T]$. By the Fourier transform, we have

$$2i\pi\tau\langle \hat{u}_m, w_j\rangle = \left\langle \hat{f}_m, w_j\right\rangle + \langle u_{0m}, w_j\rangle - \langle u_m(T), w_j\rangle \exp(-2i\pi T_\tau), \tag{31}$$

where $\hat{u}_m$ and $\hat{f}_m$ denoting the Fourier transforms of $\tilde{u}_m$ and $\tilde{f}_m$, respectively. We multiply (31) by $\widehat{\phi}_{jm}(\tau)$ and add the resulting equations for $j = 1, \cdots, m$; we get

$$2i\pi\tau|\hat{u}_m(\tau)|^2 = \left\langle \hat{f}_m(\tau), \hat{u}_m(\tau)\right\rangle + \langle u_{0m}, \hat{u}_m(\tau)\rangle - \langle u_m(T), \hat{u}_m(\tau)\rangle \exp(-2i\pi T_\tau). \tag{32}$$

We obtain

$$\int_0^T \|f_m(t)\|dt \le \int_0^T (\|f(t)\| + \|h(t,u_t)\| + \nu\|u_m(t)\| + c\|\nabla g\|_\infty\|u_m\| + \frac{\alpha}{\lambda_1}\|u_m\|^2 + c\|u_m(t)\|^2)dt. \tag{33}$$

So, $f_m(t)$ belongs to a bounded set in the space $L^1(0,T;V_g')$. For $\forall m$, we have $\sup_{\tau\in\mathrm{R}}\|\hat{f}_m(\tau)\| \le C$. Since $u_m(0)|$ and $|u_m(T)|$ are bounded, from (31), we obtain

$$|\tau||\hat{u}_m(\tau)|^2 \le C_1\|\hat{u}_m(\tau)\| + C_2|\hat{u}_m(\tau)| \le C_3\|\hat{u}_m(\tau)\|. \tag{34}$$

Let $\gamma < (1/4)$, we have

$$|\tau|^2\gamma \le C_4(\gamma)\frac{1+|\tau|}{1+|\tau|^{1-2\gamma}}, \forall\tau\in\mathrm{R}, \tag{35}$$

then

$$\int_{-\infty}^{+\infty}|\tau|^2\gamma|\hat{u}_m(\tau)|^2 d\tau \le C_4(\gamma)\int_{-\infty}^{+\infty}\frac{1+|\tau|}{1+|\tau|^{1-2\gamma}}|\hat{u}_m(\tau)|^2 d\tau. \le C_5\int_{-\infty}^{+\infty}\frac{\|\hat{u}_m(\tau)\|_{V_g}}{1+|\tau|^{1-2\gamma}}d\tau + C_6\int_{-\infty}^{+\infty}\|\hat{u}_m(\tau)\|^2 d\tau. \tag{36}$$

Since $u_m \in L^2(0,T;V_g)$, by the Parseval equality $\int_{-\infty}^{+\infty}\|\hat{u}_m(\tau)\|_{V_g}^2 d\tau < C$ and by the Schwarz inequality and the Parseval equality, we obtain

$$\int_{-\infty}^{+\infty}\frac{\|\hat{u}_m(\tau)\|_{V_g}}{1+|\tau|^{1-2\gamma}}d\tau \le \left(\int_{-\infty}^{+\infty}\frac{1}{\left(1+|\tau|^{1-2\gamma}\right)^2}d\tau\right)^{1/2} \cdot \left(\int_{-\infty}^{+\infty}\|u_m(\tau)\|^2 d\tau\right)^{1/2} < C. \tag{37}$$

So, $u_m \in H^\gamma(\mathrm{R};V_g,H_g)$, and u_m remains in a bounded set of $L^\infty(0,T;H_g), L^2(0,T;V_g)$ and $H^\gamma(\mathrm{R};V_g,H_g)$. There exists an element $u \in L^2(0,T;V_g)\cap L^\infty(0,T;H_g)$ and a subsequence $u_{m'}$ such that $u_{m'} \longrightarrow u$ in $L^2(0,T;V_g)$ weakly and $u_{m'} \longrightarrow u$ in $L^\infty(0,T;V_g)$ weak-star as $m' \longrightarrow \infty$. For any $\mathscr{B} \in \mathrm{R}^n$, we have $u_{m'}|_{\mathscr{B}} \longrightarrow u|_{\mathscr{B}}$ strongly in $L^2(0,T;H_g(\mathscr{B}))$.

For any support $\mathscr{B}_j$ of w_j, we have $u_{m'}|_{\mathscr{B}_j} \longrightarrow u|_{\mathscr{B}_j}$ strongly in $L^2(0,T;H_g(\mathscr{B}_j))$. Let ψ be a continuously differentiable function on $[0,T]$ with $\psi(T)=0$, we multiply (16) by $\psi(t)$, then integrate by parts,

$$-\int_0^T \left(u_m(t), \psi'(t)w_j\right)dt + \nu\int_0^T \left(\left(u_m(t), w_j\psi(t)\right)\right)dt + \int_0^T b\left(u_m(t), u_m(t), w_j\psi(t)\right)dt + \int_0^T \alpha\left(u_m(t), w_j\psi(t)\right)dt + \int_0^T b\left(\frac{\nabla g}{g}, u_m(t), w_j\psi(t)\right)dt = \left(u_{0m}, w_j\right)\psi(0) + \int_0^T \left\langle f(t) + h(t, u_t), w_j\psi(t)\right\rangle dt. \tag{38}$$

We have

$$-\int_0^T \left(u(t), \nu\psi'(t)\right)dt + \nu\int_0^T ((u(t), \nu\psi(t)))dt + \int_0^T b(u(t), u(t), \nu\psi(t))dt + \int_0^T \alpha(u(t), \nu\psi(t))dt + \int_0^T b\left(\frac{\nabla g}{g}, u(t), \nu\psi(t)\right)dt = (u_0, \nu)\psi(0) + \int_0^T \langle f(t) + h(t, u_t), \nu\psi(t)\rangle dt. \tag{39}$$

where $\forall \nu \in V_g$.

Finally, we prove that u satisfies (7). We multiply (6) by ψ and integrate

$$-\int_0^T \left(u(t), \nu\psi'(t)\right)dt + \nu\int_0^T ((u(t), \nu\psi(t)))dt + \int_0^T b(u(t), u(t), \nu\psi(t))dt + \int_0^T \alpha(u(t), \nu\psi(t))dt + \int_0^T b\left(\frac{\nabla g}{g}, u(t), \nu\psi(t)\right)dt = (u(0), \nu)\psi(0) + \int_0^T \langle f(t) + h(t, u_t), \nu\psi(t)\rangle dt. \tag{40}$$

We compare (39) with (40) to obtain $(u(0) - u_0, \nu)\psi(0) = 0$. Let $\psi(0) = 1$, then we have $(u(0) - u_0, \nu) = 0, \forall \nu \in V_g$. So, $u(0) = u_0$.

Now, we will prove the solution of (6) and (7) is unique. We let u_1 and u_2 be the solutions of (9) and $u = u_1 - u_2$. We have

$$\frac{\partial u}{\partial t} + \nu Au + Bu + \nu Ru + \alpha u = -Bu_1 + Bu_2, \tag{41}$$

$$u(0) = 0. \tag{42}$$

We take the scalar product of (41) with $u(t)$, then

$$\begin{aligned} &\frac{d}{dt}|u(t)|^2 + 2\nu\|u(t)\|^2 + 2b\left(\frac{\nabla g}{g}, u(t), u(t)\right) + 2\alpha|u(t)|^2 \\ &= 2b(u_2(t), u_2(t), u(t)) - 2b(u_1(t), u_1(t), u(t)) \\ &= -2b(u(t), u_2(t), u), \\ &|2b(u(t), u_2(t), u)| \le C|u(t)|\|u(t)\|\|u_2(t)\| \\ &\le \nu\|u(t)\|^2 + \frac{C^2}{\nu}|u(t)|^2\|u_2(t)\|^2, \\ &\left|2b\left(\frac{\nabla g}{g}, u(t), u(t)\right)\right| \le 2C\|\nabla g\|_\infty\|u\||u| \le \nu\|u(t)\|^2 \\ &\quad + \frac{C^2}{\nu}\|\nabla g\|_\infty^2|u(t)|^2. \end{aligned} \tag{43}$$

Therefore,

$$\frac{d}{dt}|u(t)|^2 + 2\nu\|u(t)\|^2 \le \nu\|u(t)\|^2 + \frac{C^2}{\nu}\|\nabla g\|_\infty^2|u(t)|^2 + \nu\|u(t)\|^2 + \frac{C^2}{\nu}|u(t)|^2\|u_2(t)\|^2 + \alpha|u(t)|^2. \tag{44}$$

Then,

$$\frac{d}{dt}|u(t)|^2 \le \left(\frac{C^2}{\nu}\|\nabla g\|_\infty^2 + \frac{C^2}{\nu}\|u_2(t)\|^2 + \alpha\right)|u(t)|^2. \tag{45}$$

We have

$$|u(t)|^2 \le |u(0)|^2 \exp\left(\int_0^t \left(\frac{C^2}{\nu}\|\nabla g\|_\infty^2 + \frac{C^2}{\nu}\|u_2(t)\|^2 + \alpha\right)ds\right). \tag{46}$$

Hence, $|u(t)|^2 = 0, \forall t \in [0, T]$. So, $u_1 = u_2$.

From [15], we can define a family of two parametric maps $\{U_f(t, \tau)\} = \{U_f(t, \tau) \mid t \ge \tau, \tau \in \mathrm{R}\}$ in H_g,

$$U_f(t, \tau)\colon E \longrightarrow E, t \ge \tau, \tau \in \mathrm{R}. \tag{47}$$

Here, $f \in L^\infty(\mathrm{R}^+; V_{g'})$ is called the time symbol of the system. We have the following concepts and conclusions from [15]. □

Definition 3. For the given time symbol $f \in L^\infty(R^+; V_g')$, a family of two-parametric maps $\{U(t, \tau)\}$ with $t \ge \tau \ge 0$ is called a process in H_g, if

$$\begin{aligned} &U_f(t, s)U(s, \tau) = U_f(t, \tau), \forall t \ge s \ge \tau, \tau \in \mathrm{R}, \\ &U_f(\tau, \tau) = Id, \tau \in \mathrm{R}. \end{aligned} \tag{48}$$

Now, we define translation operator in $L^\infty(\mathrm{R}^+; V_g')$. $\forall f \in L^\infty(\mathrm{R}^+; V_g')$.

$$T(h)f(s) = f(s+h), \forall h \geq 0, s \in \mathrm{R}. \tag{49}$$

We have

$$\|T(h)f\|_{L^\infty\left(\mathscr{R}^+;V_g'\right)} \leq \|f\|_{L^\infty\left(\mathscr{R}^+;V_g'\right)}, \forall h \geq 0, f \in L^\infty\left(\mathrm{R}^+ ; V_g'\right). \tag{50}$$

Denote $\Sigma = \{T(h)f(x,s) = f(x, s+h), \forall h \in \mathrm{R}\}$, where $T(\cdot)$ is the positive invariant semigroups acting on Σ and satifying $T(h)\Sigma \subset \Sigma, \forall h \geq 0$ and

$$U_{T(h)f}(t,\tau) = U_f(t+h, \tau+h), \forall h \geq 0, t \geq \tau \geq 0. \tag{51}$$

Let $\rho_{\mathscr{F}} > 0$ be a constant, obviously

$$\Sigma \subset \left\{f \in L^\infty\left(\mathrm{R}^+ ; V_g'\right) : \|f\|_{L^\infty\left(\mathrm{R}^+;V_g'\right)} \leq \rho_{\mathscr{F}}\right\}. \tag{52}$$

Let E be the Banach space; we use $\mathscr{B}(E)$ to denote the set of all bounded sets on E and consider a family of processes $\{U_f(t,\tau)\}$ with $f \in \Sigma$, the parameter f is called the symbols of the process family $\{U_f(t,\tau)\}$, Σ is called the symbol space, and we assume that Σ is a complete metric space.

Definition 4. A family of processes $\{U_f(t,\tau)\}, f \in \Sigma$ is called uniformly bounded ($w.r.t. f \in \Sigma$), if any $B \in \mathscr{B}(E)$, both

$$\bigcup_{f\in\Sigma}\bigcup_{\tau\in R}\bigcup_{t\geq\tau} U_f(t,\tau)B \in \mathscr{B}(E). \tag{53}$$

Definition 5. A set $B_0 \subset E$ is said to be uniformly absorbing for the family of processes $\{U_f(t,\tau)\}, f \in \Sigma\}$, if for any $\tau \in R$ and each $B \in \mathscr{B}(E)$, there exists $t_0 = t_0(\tau, B) \geq \tau$, such that for all $t \geq t_0$,

$$\bigcup_{f\in\Sigma} U_f(t,\tau)B \subseteq B_0. \tag{54}$$

Definition 6. A set $P \subset E$ is said uniformly atttracting set of $\{U_f(t,\tau)\}, f \in \Sigma\}$, if for any $\tau \in R$, there is

$$\lim_{t\longrightarrow+\infty}\left(\sup_{f\in\Sigma}\mathrm{dist}_E\left(U_f(t,\tau)B, P\right)\right) = 0. \tag{55}$$

A family of processes $\{U_f(t,\tau)\}, f \in \Sigma\}$ is said to uniformly compact, if there exists a compacted uniformly absorbed set in $\{U_f(t,\tau)\}, f \in \Sigma\}$. A family of processes $\{U_f(t,\tau)\}, f \in \Sigma\}$ is said to uniformly asymptotic compact, if there exists a compacted uniformly atttracting set in $\{U_f(t,\tau)\}, f \in \Sigma\}$.

Definition 7. A closed set $\mathscr{A}_\Sigma \subset \mathrm{E}$ is said to be the uniform attractor of the family of processes $\{U_f(t,\tau)\}, f \in \Sigma\}$, if

(1) $\mathscr{A}_\Sigma \subset E$ is uniformly attractive

(2) $\mathscr{A}_\Sigma \subset E$is included in any uniformly attracting set of $\{U_f(t,\tau)\}, f \in \Sigma\}$, that is $\mathscr{A}_\Sigma \subset \mathscr{A}'$.

Theorem 8. *Let $\{f_\gamma(\theta) : \gamma \in \Gamma\} \subset C = C([-r, 0] ; X)$ be equicontinuous and for any $\forall\theta \in [-r, 0]$, $\{f_\gamma(\theta) : \gamma \in \Gamma\}$ is quasi-compact in X, then $\{f_\gamma(\theta) : \gamma \in \Gamma\}$ is relatively compact in $C([-r, 0] ; X)$.*

Lemma 9 (Uniform Gronwall lemma). *Let g, h, y be local integrable function on (t_0, ∞), y' is also local integrable on (t_0, ∞), and $y'(t) \leq g(t)y(t) + h(t), \forall t \geq t_0$. $\int_t^{t+r} g(s)ds \leq a_1$, $\int_t^{t+r} h(s)ds \leq a_2$, $\int_t^{t+r} y(s)ds \leq a_3$, where r, a_1, a_2, a_3 is positive constant. Then,*

$$y(t) \leq \left(\frac{a_3}{r} + a_2\right) \exp\,(a_1), \forall t \geq t_0. \tag{56}$$

3. The Existence of Uniform Attractor for 2D g-Navier-Stokes Equations in Bounded Domain

First, we prove the existence of uniformly absorbing set in C_{H_g} and C_{V_g}; we define $u(\cdot) = u(\cdot;\tau, (u_0, \phi), f)$, where f is translation compact function. That is,

$$|f|_b^2 = \|f\|^2_{L_b^2(\mathrm{R},H_g)} = \sup_{t\in\mathrm{R}}\int_t^{t+1} |f(s)|^2 ds < \infty. \tag{57}$$

The following we use $L_c^2(\mathrm{R}, H_g)$ to represent the translation compact function class.

Lemma 10. *Let for any $\tau \leq t$, $m_0 > 0, f \in L_c^2(R, H_g)$, assume that (I)-(IV) hold, then there exist bounded absorbing sets $\{B_t\}_{t\in R}$ of process family $\{U_f(t,\tau) : t \geq \tau\}$ in C_{H_g}.*

Proof. Since $\tilde{D} \subset M^2_{H_g} = H_g \times C_{H_g}$ is bounded, then there exists $\tilde{d} \geq 0$, such that

$$|u_0|^2 + \|\phi\|^2_{L^2_{H_g}} \leq \tilde{d}^2, \forall (u_0, \phi) \in \tilde{D}. \tag{58}$$

For any $(u_0, \phi) \in \tilde{D}, \tau \in \mathrm{R}$, we define $u(\cdot) = u(\cdot;\tau, (u_0, \phi))$, then taking the inner product of (9) with $u(t)$, we have

$$\frac{1}{2}\frac{d}{dt}|u|^2 + \nu\|u\|^2 + \alpha(u,u) + \nu(Ru,u) = (f,u) + (h(t,u_t),u)$$

$$\begin{aligned}\frac{d}{dt}|u|^2 + 2\nu\|u\|^2 + 2\alpha|u|^2 &= 2(f,u) + 2(h(t,u_t),u) - 2\nu(Ru,u) \\ &\le \frac{|f|^2}{\sigma} + \sigma|u|^2 + \frac{1}{C_g}|h(t,u_t)|^2 + C_g|u|^2 + \frac{2\nu|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\|u\|^2\end{aligned}$$

$$\frac{d}{dt}|u|^2 + 2\nu\left(1 - \frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\right)\|u\|^2 + 2\alpha|u|^2 \le \frac{|f|^2}{\sigma} + \sigma|u|^2 + \frac{1}{C_g}|h(t,u_t)|^2 + C_g|u|^2$$

$$\frac{d}{dt}|u|^2 + 2\nu\lambda_1\left(1 - \frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}} + 2\alpha\right)|u|^2 \le \frac{|f|^2}{\sigma} + \sigma|u|^2 + \frac{1}{C_g}|h(t,u_t)|^2 + C_g|u|^2$$

$$\begin{aligned}\frac{d}{dt}\left(e^{mt}|u(t)|^2\right) &= me^{mt}|u(t)|^2 + e^{mt}\frac{d}{dt}|u(t)|^2 \\ &\le me^{mt}|u(t)|^2 + e^{mt}\left[\frac{|f|^2}{\sigma} + \frac{1}{C_g}|h(t,u_t)|^2\right. \\ &\quad \left. + \left(\sigma + C_g\right)|u|^2 - 2\nu\lambda_1\left(1 - \frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}} + 2\alpha\right)|u|^2\right].\end{aligned} \tag{59}$$

Let $\beta = 1 - (|\nabla g|_\infty/m_0\lambda_1^{1/2}) + 2\alpha$, then

$$\begin{aligned}\frac{d}{dt}\left(e^{mt}|u(t)|^2\right) &\le me^{mt}|u(t)|^2 \\ &\quad + e^{mt}\left[\frac{|f|^2}{\sigma} + \frac{1}{C_g}|h(t,u_t)|^2 + \left(\sigma + C_g - 2\nu\lambda_1\beta\right)|u|^2\right].\end{aligned} \tag{60}$$

Integrating both sides from τ to t, then

$$\begin{aligned}e^{mt}|u(t)|^2 - e^{m\tau}|u(\tau)|^2 &\le \int_\tau^t \frac{e^{ms}|f|^2}{\sigma}ds + \int_\tau^t \frac{e^{ms}}{C_g}|h(s,u_s)|^2 ds \\ &\quad + \int_\tau^t e^{ms}\left[m + \left(\sigma + C_g - 2\nu\lambda_1\beta\right)\right]|u(s)|^2 ds.\end{aligned} \tag{61}$$

Then,

$$\begin{aligned}e^{mt}|u(t)|^2 &\le \int_\tau^t \frac{e^{ms}|f|^2}{\sigma}ds + \int_\tau^t \frac{e^{ms}}{C_g}|h(t,u_t)|^2 ds \\ &\quad + \int_\tau^t e^{ms}\left[m + \left(\sigma + C_g - 2\nu\lambda_1\beta\right]|u(s)|^2 ds \\ |u(t)|^2 &\le e^{-mt}\int_\tau^t \frac{e^{ms}|f|^2}{\sigma}ds + e^{-mt}\int_\tau^t \frac{e^{ms}}{C_g}|h(t,u_t)|^2 ds \\ &\quad + e^{-mt}\int_\tau^t e^{ms}\left[m + \left(\sigma + C_g - 2\nu\lambda_1\beta\right)\right]|u(s)|^2 ds,\end{aligned} \tag{62}$$

for

$$e^{-mt}\int_\tau^t \frac{e^{ms}}{C_g}|h(t,u_t)|^2 ds \le e^{-mt}C_g\int_{\tau-h}^{\tau} e^{ms}|\phi(s-\tau)|^2 ds. \tag{63}$$

Let $s - \tau = \theta$, then

$$\begin{aligned}e^{-mt}\int_\tau^t \frac{e^{ms}}{C_g}|h(t,u_t)|^2 ds &\le e^{-mt}C_g e^{m\tau}\int_{-h}^{0}|\phi(\theta)|^2 d\theta \\ e^{-mt}\int_\tau^t \frac{e^{ms}|f|^2}{2}ds &= \frac{1}{\sigma}\int_\tau^t e^{-m(t-s)}|f|^2 ds \\ &\le \frac{1}{\sigma}\left[\int_{t-1}^{t} e^{-m(t-s)}|f(s)|^2 ds + \int_{t-2}^{t-1} e^{-m(t-s)}|f(s)|^2 ds + \cdots\right] \\ &\le \frac{1}{\sigma}\left(1 + e^{-m} + e^{-2m} + \cdots\right)\sup_{t\in R}\int_t^{t+1}|f|^2 ds \\ &= \frac{1}{\sigma(1-e^{-m})}|f|_b^2.\end{aligned} \tag{64}$$

Taking $m \in (0, m_0)$, such that $m + \sigma + C_g - 2\nu\lambda_1\beta < 1$, then $|\nabla g|_\infty < ((m_0[2\nu\lambda_1(1+2\alpha) - \sigma - C_g + 1])/2\nu\lambda_1^{1/2})$, so

$$|u(t)|^2 \le \frac{1}{\sigma(1-e^{-m})}|f|_b^2 + \tilde{d}^2\left(1 + C_g\right)e^{-mt}\cdot e^{m\tau}(t \ge \tau). \tag{65}$$

Let $t \ge \tau + h, \forall\theta \in [-h, 0]$, then

$$\begin{aligned}|u(t+\theta)|^2 &\le \frac{|f|_b^2}{\sigma(1-e^{-m})} + \tilde{d}^2\left(1 + C_g\right)e^{-m(t+\theta)}\cdot e^{m\tau} \\ &\le \frac{|f|_b^2}{\sigma(1-e^{-m})} + \tilde{d}^2 e^{mh}\left(1 + C_g\right)e^{-mt}\cdot e^{m\tau}.\end{aligned} \tag{66}$$

Then,

$$\|u_t\|_{C_{H_g}}^2 \le \frac{|f|_b^2}{\sigma(1-e^{-m})} + \tilde{d}^2 e^{mh}\left(1 + C_g\right)e^{-mt}\cdot e^{m\tau}(t \ge \tau + h). \tag{67}$$

Let $B_1 = \{u_t \mid \|u_t\|_{C_{H_g}}^2 \le (|f|_b^2/\sigma(1-e^{-m}))\}$, we will prove the existence of the uniformly absorbing bounded set in C_{V_g}. First, we must prove the boundedness of $\int_t^{t+1}\|u(s)\|^2 ds$. □

Lemma 11. *Given that $D \in \mathscr{B}(M_{H_g}^2)$, then there exist $T_{H_g}(D)$ and constant I_{V_g}, such that*

$$\int_t^{t+1}\|u(s)\|^2 ds \le I_{V_g}.\forall t \ge T_{H_g}(D) + r + 1, \tag{68}$$

where $\mathscr{B}(M_{H_g}^2)$ denotes any bounded set on the $M_{H_g}^2$.

Proof. Taking the inner product of (9) with $u(t)$,

$$\frac{d}{dt}|u|^2 + 2\nu\|u\|^2 + 2\alpha|u|^2 = 2(f,u) + 2(h(t,u_t),u) - 2\nu(Ru,u) \le \frac{|f|^2}{\sigma} + \sigma|u|^2 + \frac{1}{C_g}|h(t,u_t)|^2 + C_g|u|^2 + \frac{2\nu|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\|u\|^2 \le \frac{|f|^2}{\sigma} + \frac{\sigma}{\lambda_1}\|u\|^2 + \frac{1}{C_g}|h(t,u_t)|^2 + \frac{C_g}{\lambda_1}\|u\|^2 + \frac{2\nu|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\|u\|^2. \tag{69}$$

Then,

$$\frac{d}{dt}|u|^2 + \left(2\nu - \frac{\sigma}{\lambda_1} - \frac{C_g}{\lambda_1} - \frac{2\nu|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\right)\|u\|^2 + 2\alpha|u|^2 \le \frac{|f|^2}{\sigma} + \frac{1}{C_g}|h(t,u_t)|^2. \tag{70}$$

Integrating on both sides in $[t, t+1]$, we have

$$|u(t+1)|^2 - |u(t)|^2 + \left(2\nu - \frac{\sigma}{\lambda_1} - \frac{C_g}{\lambda_1} - \frac{2\nu|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\right) \cdot \int_t^{t+1}\|u\|^2 ds \le \int_t^{t+1}\frac{|f|^2}{\sigma}ds + \frac{1}{C_g}\int_t^{t+1}|h(t,u_t)|^2 ds \le |f|_b^2 + C_g\int_{t-h}^{t+1}|u(s)|^2 ds \le |f|_b^2 + C_g\int_{t-h}^{t}|u(s)|^2 ds + C_g\int_t^{t+1}|u(s)|^2 ds \le |f|_b^2 + C_g\int_{t-h}^{t}|u(s)|^2 ds + \frac{C_g}{\lambda_1}\int_t^{t+1}\|u(s)\|^2 ds. \tag{71}$$

Then,

$$\left(2\nu - \frac{\sigma}{\lambda_1} - \frac{2C_g}{\lambda_1} - \frac{2\nu|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\right)\int_t^{t+1}\|u\|^2 ds \le |f|_b^2 + C_g\int_{t-h}^{t}|u(s)|^2 ds + |u(t)|^2 \le |f|_b^2 + C_g\int_{t-h}^{t}\|u_s\|_{C_{H_g}}^2 ds + \rho_{H_g}^2 \le |f|_b^2 + \left(1 + hC_g\right)\rho_{H_g}^2. \tag{72}$$

When $2\nu - (\sigma/\lambda_1) - (2C_g/\lambda_1) - (2\nu|\nabla g|_\infty/m_0\lambda_1^{1/2}) > 0$, that is $|\nabla g|_\infty < ((2\nu\lambda_1 - \sigma - 2C_g)/2\nu\lambda_1^{1/2})$, we have

$$\int_t^{t+1}\|u(s)\|^2 ds \le I_{V_g}. \forall t \ge T_{H_g}(D) + r + 1, \tag{73}$$

where

$$I_{V_g} = \frac{1}{2\nu - (\sigma/\lambda_1) - \left(2C_g/\lambda_1\right) - \left(2\nu|\nabla g|_\infty/m_0\lambda_1^{1/2}\right)} \cdot \left(|f|_b^2 + \left(1 + hC_g\right)\rho_{H_g}^2\right).\rho_{H_g} = \frac{|f|_b^2}{\sigma(1 - e^{-m})}. \tag{74}$$

□

Lemma 12. *For any $\tau \le t$, $m_0 > 0$, $f \in L_c^2(R, H_g)$. Assume that (I)-(IV) hold, then there exists uniformly bounded absorbing set $B_2 \subset C_{V_g}$ of process family $\{U_f(t,\tau): t \ge \tau\}$ in C_{V_g}.*

Proof. Let $D \in \mathscr{B}(M_{H_g}^2)$, taking the inner product of (9) with $A_g u$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + \nu\left|A_g u\right|^2 + \alpha\|u\|^2 + B\left(u,u,A_g u\right) + \nu\left(Ru, A_g u\right) = \left(f, A_g u\right) + \left(h(t,u_t), A_g u\right),$$
$$\frac{d}{dt}\|u\|^2 + 2\nu\left|A_g u\right|^2 + 2\alpha\|u\|^2 \le 2\left(f, A_g u\right) + 2\left(h(t,u_t), A_g u\right) - 2B\left(u,u,A_g u\right) - 2\nu\left(Ru, A_g u\right), \tag{75}$$

for

$$2\left(f, A_g u\right) + 2\left(h(t,u_t), A_g u\right) \le 2\left|A_g u\right|(|f| + |h(t,u_t)|) \le \frac{\nu}{2}\left|A_g u\right|^2 + \frac{4}{\nu}\left(|f|^2 + |h(t,u_t)|^2\right),$$
$$2\left|B\left(u,u,A_g u\right)\right| \le 2c_1|u|^{1/2}\|u\|\left|A_g u\right|^{3/2} \le \frac{\nu}{2}\left|A_g u\right|^2 + \frac{2c_1'}{\nu^3}|u|^2\|u\|^4,$$
$$2\nu\left|Ru, A_g u\right| \le 2\nu|Ru| \cdot \left|A_g u\right| \le 2\nu\frac{|\nabla g|_\infty}{m_0}\|u\| \cdot \left|A_g u\right| \le 2\nu\frac{|\nabla g|_\infty}{m_0}\frac{1}{\sqrt{\lambda_1}}\left|A_g u\right|^2, \forall u \in D\left(A_g u\right). \tag{76}$$

Then,

$$\frac{d}{dt}\|u\|^2 + 2\nu\left|A_g u\right|^2 \le \frac{\nu}{2}\left|A_g u\right|^2 + \frac{4}{\nu}\left(|f|^2 + |h(t,u_t)|^2\right) + \frac{\nu}{2}\left|A_g u\right|^2 + \frac{2c_1'}{\nu^3}|u|^2\|u\|^4 + \frac{2\nu|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\left|A_g u\right|^2 - 2\alpha\|u\|^2,$$
$$\frac{d}{dt}\|u\|^2 + \left(\nu - \frac{2\nu|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\right)\left|A_g u\right|^2 \le \frac{4}{\nu}\left(|f|^2 + |h(t,u_t)|^2\right) + \frac{2c_1'}{\nu^3}|u|^2\|u\|^4 - 2\alpha\|u\|^2,$$
$$\frac{d}{dt}\|u\|^2 + \nu\left(1 - \frac{2|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\right)\lambda_1\|u\|^2 \le \frac{4}{\nu}\left(|f|^2 + L_g^2\|u_s\|_{C_{H_g}}^2\right) + \frac{2c_1'}{\nu^3}|u|^2\|u\|^4 - 2\alpha\|u\|^2 \le \frac{4}{\nu}\left(|f|^2 + L_g^2\rho_{H_g}^2\right) + \frac{2c_1'}{\nu^3}|u|^2\|u\|^4. \tag{77}$$

Applying Lemma 9,

$$\|u(r)\|^2 \le (a_3 + a_2)e^{a_1} \forall r \ge t_0 + 1, s \ge \tilde{T}_{\tilde{D}}, \quad (78)$$

where $a_3 = I_{V_g}$, $a_2 = 4/\nu(|f|^2 + L_g^2\rho_{H_g}^2)$, $a_1 = (2c_1'/\nu^3)\rho_{H_g}^2 I_{V_g}$. If taking $s \ge \tilde{T}_{\tilde{D}} + 1 + h$, then

$$\sup_{\theta\in[-h,0]} \|u(t_0+\theta)\|^2 \le (a_3 + a_2)e^{a_1} = \rho_{V_g}^2. \quad (79)$$

Let $u(\cdot) = u(\cdot; t-s, (u_0, \phi))$, so $u_t(\cdot) \in C_{V_g}$, $\forall s > h$. Then,

$$B_2 = \left\{ u_t \mid \|u_t\|_{C_{V_g}} \le \rho_{V_g}, \forall t \in \mathrm{R}, s \ge \tilde{T}_{\tilde{D}} + 1 + h \right\}. \quad (80)$$

□

From [16], we have the following definition.

Definition 13. Let E be Banach space, if $\forall \varepsilon > 0$, there exists $\eta > 0$, such that

$$\sup \int_t^{t+\eta} \|f\|_E^2 ds < \varepsilon. \quad (81)$$

Then, $f \in L^2_{\mathrm{loc}}(\mathrm{R}, E)$ is called normal function.

We will take the sets of all normal function classes in $L^2_{\mathrm{loc}}(\mathrm{R}, E)$ as $L^2_n(\mathrm{R}; E)$. From [17], we can see that $L^2_n(\mathrm{R}; E)$ is the true subspace of $L^2_c(\mathrm{R}; E)$. Therefore, the translation compact function must be a normal function.

Theorem 14. *Suppose that nonlinear term h satisfies (I)-(III), f is translation compact function in $L^2_{loc}(R, H_g)$, then process family $\{U_f(\cdot,\cdot) \mid f \in \Sigma\}$ exist uniform attractor $\mathscr{A}_\Sigma$, and $\mathscr{A}_\Sigma \subset H_g \times C_{H_g}$.*

Proof. Since B_2 is bounded set in C_{V_g} and uniform absorbed set of $\{U_f(\cdot,\cdot) \mid f \in \Sigma\}$. For each $\tau \in \mathrm{R}$, we take a set

$$B_3 = \bigcup_{f\in\Sigma} U_f(\tau + r, \tau) j(B_2), \quad (82)$$

where j denotes any compact self-adjoint operator, then $B_3 \subset B_2 \subset B_1$, and B_3 is another uniform absorbing bounded set of $\{U_f(\cdot,\cdot) \mid f \in \Sigma\}$ in C_{V_g}. Now, we will prove B_3 is relatively compact in C_{H_g}. From Theorem 8, we only need to prove B_3 is equicontinuous and uniform bounded in C_{H_g}. From the definition of B_3, we can obtain it is uniformly bounded. Now, we will prove B_3 is equicontinuous. For any $\theta_1, \theta_2 \in [-r, 0]$, $\phi \in B_2$, $f \in \Sigma$,

$$\begin{aligned} &\left|U_f(\tau+r,\tau)(j(\phi))(\theta_1) - U_f(\tau+r,\tau)(j(\phi))(\theta_2)\right| \\ &= |u(\tau+r+\theta_1; \tau, (j(\phi)), f) - u(\tau+r+\theta_2; \tau, (j(\phi)), f)|. \end{aligned} \quad (83)$$

Let $\theta_2 > \theta_1$, and denote $u(\cdot) = u(\cdot; \tau, j(\phi), f)$ as $u(\cdot)$, then

$$\begin{aligned} &|u(\tau+r+\theta_1) - u(\tau+r+\theta_2)| = \left| \int_{\tau+r+\theta_1}^{\tau+r+\theta_2} \frac{du(s)}{dt} ds \right| \\ &\le \int_{\tau+r+\theta_1}^{\tau+r+\theta_2} \frac{du(s)}{dt} ds \le \int_{\tau+r+\theta_1}^{\tau+r+\theta_2} (\nu|A_g u(s)| + \alpha|u(s)| \\ &\quad + |B(u(s))| + |f| + |h(s,u_s)| + \nu|R(u(s))|) ds \\ &\le \int_{\tau+r+\theta_1}^{\tau+r+\theta_2} (\nu|A_g u(s)| + c_1|A_g u(s)|\|u(s)\| + \alpha|u(s)| \\ &\quad + |f| + |h(s,u_s)| + \nu|R(u(s))|) ds \\ &\le \int_{\tau+r+\theta_1}^{\tau+r+\theta_2} \left[(\nu + c_1\|u(s)\|)|A_g u(s)| + L_g\|u_s\|_{C_{H_g}} + \frac{\nu|\nabla g|_\infty}{m_0}\|u\| \right] \\ &\quad \cdot ds + \int_{\tau+r+\theta_1}^{\tau+r+\theta_2} |f| ds + \int_{\tau+r+\theta_1}^{\tau+r+\theta_2} \alpha|u(s)| ds. \end{aligned} \quad (84)$$

We estimate the items on the right end of the above formula, let $\theta_1 \longrightarrow \theta_2$,

$$\begin{aligned} &(i) \int_{\tau+r+\theta_1}^{\tau+r+\theta_2} |f| ds \le \sup_{t\in\mathrm{R}} \int_\tau^{t+\theta_2-\theta_1} |f| ds \longrightarrow 0, \forall f \in \Sigma, \\ &(ii) \int_{\tau+r+\theta_1}^{\tau+r+\theta_2} \left[(\nu + c_1\|u(s)\|)|A_g u(s)| + L_g\|u_s\|_{C_{H_g}} \right] ds \\ &\le \int_{\tau+r+\theta_1}^{\tau+r+\theta_2} (\nu + c_1\|u(s)\|)|A_g u(s)| ds + \rho_{H_g} L_g |\theta_1 - \theta_2| \\ &\le \int_{\tau+r+\theta_1}^{\tau+r+\theta_2} \left(\nu + c_1\rho_{V_g}\right)|A_g u(s)| ds + \rho_{H_g} L_g |\theta_1 - \theta_2| \\ &\le \left(\nu + c_1\rho_{V_g}\right)|\theta_1 - \theta_2|^{1/2} \left(\int_{\tau+r+\theta_1}^{\tau+r+\theta_2} |A_g u(s)|^2 ds \right)^{1/2} \\ &\quad + \rho_{H_g} L_g |\theta_1 - \theta_2|. \end{aligned} \quad (85)$$

Since

$$\begin{aligned} &\frac{d}{dt}\|u\|^2 + \nu\left(1 - \frac{2|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\right)|A_g u|^2 \le \frac{4}{\nu}\left(|f|^2 + L_g^2\|u_s\|_{L_{H_g}}^2\right) \\ &\quad + \frac{2c_1'}{\nu^3}|u|^2\|u\|^4 - 2\alpha\|u\|^2, \end{aligned} \quad (86)$$

we let $\alpha_1 = 4/\nu(|f|^2 + L_g^2\|u_s\|_{L_{H_g}}^2)$, $\alpha_2 = 2c_1'/\nu^3$, $\alpha_3 = 1 - (2|\nabla g|_\infty/m_0\lambda_1^{1/2})$, $\alpha_4 = 2\alpha$.

When $s \geq \tilde{T}_{\tilde{D}} + 1 + h, \theta_1, \theta_2 \in [-h, 0]$, and $\theta_2 > \theta_1$, then

$$\begin{aligned}
|A_g u|^2 &\leq \frac{\alpha_1}{\alpha_3} + \frac{\alpha_2}{\alpha_3}|u|^2\|u\|^4 - \frac{1}{\alpha_3}\frac{d}{dt}\|u\|^2 - \frac{\alpha_4}{\alpha_3}\|u\|^2, \\
\int_{t+\theta_1}^{t+\theta_2}|A_g u|^2 ds &\leq \frac{\alpha_1}{\alpha_3}|\theta_2 - \theta_1| + \frac{\alpha_2}{\alpha_3}\int_{t+\theta_1}^{t+\theta_2}|u(s)|^2\|u(s)\|^4 ds \\
&- \frac{1}{\alpha_3}\|u(t+\theta_2)\|^2 + \frac{1}{\alpha_3}\|u(t+\theta_1)\|^2 - \frac{\alpha_4}{\alpha_3}\int_{t+\theta_1}^{t+\theta_2}\|u(s)\|^2 \\
&\cdot ds \leq \left(\frac{\alpha_1}{\alpha_3} + \frac{\alpha_2}{\alpha_3}\rho_{H_g}^2\rho_{V_g}^4 - \frac{\alpha_4}{\alpha_3}\rho_{V_g}^2\right)|\theta_2 - \theta_1| \\
&+ \frac{1}{\alpha_3}\|u(t+\theta_1)\|^2.
\end{aligned} \tag{87}$$

Let

$$\beta_1 = \frac{\alpha_1}{\alpha_3} + \frac{\alpha_2}{\alpha_3}\rho_{H_g}^2\rho_{V_g}^4 - \frac{\alpha_4}{\alpha_3}\rho_{V_g}^2, \beta_2 = \frac{1}{\alpha_3}\rho_{V_g}^2. \tag{88}$$

Then,

$$\int_{t+\theta_1}^{t+\theta_2}|A_g u|^2 ds < \beta_1|\theta_2 - \theta_1| + \beta_2. \tag{89}$$

So,

$$\begin{aligned}
&\int_{t+r+\theta_1}^{t+r+\theta_2}\left[(\nu + c_1\|u(s)\|)|A_g u(s)| + L_g\|u_s\|_{C_{H_g}}\right]ds \\
&\leq \left(\nu + c_1\rho_{V_g}\right)|\theta_1 - \theta_2|^{1/2}(\beta_1|\theta_1 - \theta_2| + \beta_2)^{1/2} + \rho_{V_g}L_g|\theta_1 - \theta_2|.
\end{aligned} \tag{90}$$

And $\alpha_4/\alpha_3\int_{t+r+\theta_1}^{t+r+\theta_2}\alpha\|u(s)\|ds \leq \alpha\rho_{V_g}|\theta_2 - \theta_1|$. When $\theta_1 \longrightarrow \theta_2, \forall\phi \in B_2, f \in \Sigma$, we have

$$\begin{aligned}
&|u(t+r+\theta_1) - u(t+r+\theta_2)| \\
&\leq \int_{t+r+\theta_1}^{t+r+\theta_2}|f|ds + \int_{t+r+\theta_1}^{t+r+\theta_2}\frac{\nu|\nabla g|_\infty}{m_0}\|u\|ds \\
&+ \int_{t+r+\theta_1}^{t+r+\theta_2}\left[(\nu + c_1\|u(s)\|)|A_g u(s)| + L_g\|u_s\|_{C_{H_g}}\right]ds \\
&+ \int_{t+r+\theta_1}^{t+r+\theta_2}\alpha\|u(s)\|ds \leq \sup_{t\in\mathbb{R}}\int_t^{t+\theta_2-\theta_1}|f(s)|ds + \frac{\nu|\nabla g|_\infty}{m_0}\rho_{V_g}|\theta_2 - \theta_1| \\
&+ \alpha\rho_{V_g}|\theta_2 - \theta_1| + \left(\nu + c_1\rho_{V_g}\right)|\theta_1 - \theta_2|^{1/2}(\beta_1|\theta_1 - \theta_2| \\
&+ \beta_2)^{1/2} + \rho_{V_g}L_g|\theta_1 - \theta_2| \longrightarrow 0.
\end{aligned} \tag{91}$$

Then, B_3 is equicontinuous, and B_3 is relatively compact in C_{H_g}, so $\bar{B}_3$ is compacted uniformly absorbing set of $\{U_f(\cdot, \cdot) \mid f \in \Sigma\}$ in C_{H_g}, Let $\tilde{B}_3 = j(\bar{B}_3)$, since $H_g \times C_{H_g} = M_{H_g}^2$, $V_g \subset H_g$ and the embedding mapping is continuous, so $\tilde{B}_3$ is compact in $M_{H_g}^2$; $\tilde{B}_3$ is also compacted uniformly absorbing set of $\{U_f(\cdot, \cdot) \mid f \in \Sigma\}$ in $M_{H_g}^2$. Then, process family $\{U_f(\cdot, \cdot) \mid f \in \Sigma\}$ exists uniform attractor $\mathscr{A}_\Sigma \subset H_g \times C_{H_g}$. □

Data Availability

The (data type) data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

This work does not have any conflicts of interest.

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