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Research Article

Almost Periodic Solutions for a Second-Order Nonlinear Equation with Mixed Delays

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This paper is devoted to studying a second-order nonlinear equation with mixed delays. Some sufficient conditions for the existence and exponential stability of the almost periodic solutions are established. The results of this paper extend the existing ones.

1. Introduction

Consider the following nonlinear second-order equation with mixed delays

$$x''(t)+f\left(x'(t)\right)+f(x(t))x'(t)+g_0(x(t))+\sum_{j=1}^{m}g_j\left(x\left(t-\tau_j(t)\right)\right)+\int_0^\infty K(s)h(x(t-s))ds=e(t), \quad (1)$$

where f, h and $g_j(j=0,1,\cdots,m)$ are continuous functions on $\mathbb{R}$, $K(s)$ is a continuous and integrable function on $[0,\infty)$, and $e(t)$ and $\tau_j(t)>0$ are almost periodic functions on $\mathbb{R}$, $j=1,\cdots,m$. When $f(x'(t))=0$ and $\sum_{j=1}^m g_j(x(t-\tau_j(t)))=0$, Equation (1) is changed into the following Liénard equation with distributed delays

$$x''(t)+f(x(t))x'(t)+g_0(x(t))+\int_0^\infty K(s)h(x(t-s))ds=e(t). \quad (2)$$

In 2007, Gao and Liu [1] studied the existence and exponential stability of the almost periodic solution of Equation (2). When $f(x'(t))=0$ and $\int_0^\infty K(s)h(x(t-s))ds=0$, Equation (1) is changed into the following Liénard equation with multiple time-varying delays

$$x''(t)+f(x(t))x'(t)+g_0(x(t))+\sum_{j=1}^{m}g_j\left(x\left(t-\tau_j(t)\right)\right)=e(t). \quad (3)$$

In 2010, Gao and Liu [1] studied the existence and exponential stability of the almost periodic solution of Equation (3). It is easy to see that Equation (1) is a generalization of Equations (2) and (3).

The concept of almost periodicity is with deep historical roots. Some problems in astronomy were to explain some curious behavior of the moon, sun, and the planets by using almost periodicity. Bohr [2] firstly introduced the formal theory of almost periodic functions. Almost periodic functions are functions that are periodic up to a small error. After that, some remarkable results in the area of almost periodicity have been obtained by many authors. Almost periodic solutions of higher order differential equations have a wide range of applications, and many researchers have done a lot of research. In 2009, Xiao and Meng [3] studied the existence and exponential stability of positive almost periodic solutions of high-order Hopfield neural networks with time-varying delays. Almost periodic solutions of quaternion-valued neutral type high-order Hopfield neural networks with state-dependent delays and leakage delays were considered in [4]. Lian et al. [5] studied the stability and almost periodicity for delayed high-order Hopfield neural networks with discontinuous activations. Dads and Lhachimi [6] obtained some necessary and sufficient conditions in order to ensure the existence and uniqueness of pseudo almost periodic solutions for a second-order differential equation with piecewise constant argument. For more results about almost periodic solution of differential equations and dynamic system, see [7–9]. In [10–12], almost periodic solutions in Banach spaces have been studied. For positive almost periodic solutions, see [13–15]; for pseudo almost periodic solutions and almost periodic solutions, see [16–19]. We give the following definition for almost periodic function.

Definition 1 ([20, 21]). Let $f(t) \in C(\mathbb{R}, \mathbb{R}^n)$. $f(t)$ is said to be almost periodic on $\mathbb{R}$; if for any $\varepsilon > 0$,

$$T(f, \varepsilon) = \{\tau \in \mathbb{R} : |f(t+\tau) - f(t)| < \varepsilon \text{ for all } t \in \mathbb{R}\}, \tag{4}$$

is relatively dense in $\mathbb{R}$. That is, there is a $l_\varepsilon > 0$ such that any interval of the length l_ε contains at least one point of $T(f, \varepsilon)$.

The main contributions of our study are as follows:

(1) We introduce a more complicated second-order nonlinear equation with mixed delays which is different from the existing second-order nonlinear equations (see [1, 22])

(2) We use innovative mathematical analysis technology and Lyapunov functional method for studying the existence and exponential stability of almost periodic solutions for the second-order nonlinear equation

(3) In the present paper, use a variable transformation, and a second-order equation is changed into a first-order system; thus, we can easily consider the second-order equation

The following sections are organized as follows: Section 2 gives some preliminary results. In Section 3, we give some sufficient conditions for the existence and exponential stability of almost periodic solutions to Equation (1). In Section 4, an example is given to show the feasibility of our results. Finally, Section 5 concludes the paper.

2. Preliminary Results

Let d_0 be a positive constant and

$$\begin{aligned} r(x) &= \int_0^x [f(x) - d_0]dx, \\ y(t) &= x'(t) + r(x). \end{aligned} \tag{5}$$

Then, (1) can be rewritten by

$$\begin{cases} x'(t) = y(t) - r(x), \\ y'(t) = -d_0 y(t) + d_0 r(x) - f(y(t) - r(x)), \\ -g_0(x(t)) - \sum_{j=1}^{m} g_j(x(t - \tau_j(t))) - \int_0^\infty K(s)h(x(t-s))ds + e(t). \end{cases} \tag{6}$$

Since $\tau_j(t)(j = 1, 2, \cdots, m)$ and $e(t)$ are almost periodic functions, based on Definition 1, for $\forall \varepsilon > 0$, there is a $l_\varepsilon > 0$ such that any interval of the length l_ε; there exists a number $\delta = \delta(\varepsilon)$ such that

$$\begin{aligned} |\tau_j(t+\delta) - \tau_j(t)| &< \varepsilon, \\ |e(t+\delta) - e(t)| &< \varepsilon. \end{aligned} \tag{7}$$

Let $BC((-\infty, 0), \mathbb{R})$ be the space of bounded continuous functions ψ with the supremum norm $\|\bullet\|$. From the basic theory of functional differential equation in [23], system (6) exists a solution $(x(t), y(t))^T$ with initial conditions

$$\begin{aligned} x(s) &= \psi(s), \\ y(0) &= y_0, \\ s &\in (-\infty, 0]. \end{aligned} \tag{8}$$

Now, we give the definition of exponential stability for system (6).

Definition 2. Let $z^*(t) = (x^*(t), y^*(t))^T$ be an almost periodic solution of system (6) with initial value $(\psi^*(s), y_0^*) \in BC \times \mathbb{R}$. Assume that there exist constants $\mu > 0$ and $M > 1$ such that for every solution $z(t) = (x(t), y(t))^T$ of system (6) with initial value $(\psi(s), y_0) \in BC \times \mathbb{R}$,

$$\begin{aligned} \max\{|x(t) - x^*(t)|, |y(t) - y^*(t)|\} &\le M, \\ \max\{\|\psi(s) - \psi^*(s)\|, |y_0 - y_0^*|\}e^{-\mu t}, \\ \forall t > 0, \end{aligned} \tag{9}$$

where $\|\psi(s) - \psi^*(s)\| = \max_{s\in(-\infty,0]}\{|\psi(s) - \psi^*(s)|\}$. Then, $z^*(t)$ is said to be globally exponentially stable.

We need the following assumptions:
H1. There exists a constant $d_1 > 1$ such that

$$d_1|u - v| \le \operatorname{sgn}(u - v)(r(u) - r(v)) \text{ for } u, v \in \mathbb{R}. \tag{10}$$

H2. For $j = 0, 1, \cdots, m$, assume $g_j(0) = 0$ and

$$\begin{aligned} &|(g_0(u) - d_0 r(u)) - (g_0(v) - d_0 r(v))| \le L_0 |u - v|, \\ &\left|g_j(u) - g_j(v)\right| \le L_j |u - v|, \\ &\text{for } u, v \in \mathbb{R}, \end{aligned} \quad (11)$$

where d_0 is defined by (5) and L_j is positive constant.

H3. Assume $f(0) = h(0) = 0$, and there exist positive constants L_f and L_h such that

$$\begin{aligned} &|f(u) - f(v)| \le L_f |u - v|, \\ &|h(u) - h(v)| \le L_h |u - v|, \\ &\text{for } u, v \in \mathbb{R}. \end{aligned} \quad (12)$$

Lemma 3. *Suppose that assumptions H1–H3 hold. If $(\tilde{x}(t), \tilde{y}(t))^T$ is a solution of system (6) with initial conditions*

$$\begin{aligned} &\tilde{x}(s) = \tilde{\psi}(s), \\ &\tilde{y}(0) = \tilde{y}_0, \\ &\max\{|\tilde{\psi}(s)|, |\tilde{y}_0|\} \le \Gamma s \in (-\infty, 0], \end{aligned} \quad (13)$$

where $\Gamma > 0$ satisfies

$$\left(d_0 - \sum_{j=0}^{m} L_j - L_h \int_0^\infty |K(s)| ds - L_f\left(1 + \frac{1}{2}\Gamma + d_0\right)\right)\Gamma > \sup_{t \in \mathbb{R}} |e(t)|, \quad (14)$$

then

$$\max\{|\tilde{x}(t)|, |\tilde{y}(t)|\} \le \Gamma \text{ for all } t \ge 0. \quad (15)$$

Proof. Assume that (15) does not holds. Then, one of the following cases must occur:

Case 1. There exists $t_1 > 0$ such that

$$\begin{aligned} &\max\{|\tilde{x}(t_1)|, |\tilde{y}(t_1)|\} = |\tilde{x}(t_1)| = \Gamma, \\ &\max\{|\tilde{x}(t)|, |\tilde{y}(t)|\} < \Gamma, \\ &\text{for all } t \in (-\infty, t_1). \end{aligned} \quad (16)$$

Case 2. There exists $t_2 > 0$ such that

$$\begin{aligned} &\max\{|\tilde{x}(t_2)|, |\tilde{y}(t_2)|\} = |\tilde{y}(t_2)| = \Gamma, \\ &\max\{|\tilde{x}(t)|, |\tilde{y}(t)|\} < \Gamma, \\ &\text{for all } t \in (-\infty, t_2). \end{aligned} \quad (17)$$

If Case 1 holds, calculating the upper right derivative of $D^+(|\tilde{x}(t)|)$, together with assumption H1, (6) and (16) imply that

$$0 \le D^+(|\tilde{x}(t_1)|) \le -d_1 |\tilde{x}(t_1)| + |\tilde{y}(t_1)| \le -(d_1 - 1)\Gamma < 0, \quad (18)$$

which is a contradiction and implies that (15) holds.

If Case 2 holds, calculating the upper right derivative of $D^+(|\tilde{y}(t)|)$, together with H1 and H3, (6), (14), and (17) imply that

$$\begin{aligned} 0 &\le D^+(|\tilde{y}(t_2)|) \\ &\le -d_0 |y(t_2)| + |g_0(x(t_2)) - d_0 r(x(t_2))| \\ &\quad + L_h \int_0^\infty |K(s)||(x(t_2 - s))| ds + L_f(|y(t_2)| + |r(x(t_2))|) \\ &\quad + \sum_{j=1}^{m} L_j \left|x\left(t_2 - \tau_j(t_2)\right)\right| + |e(t_2)| \\ &\le -d_0 \Gamma + \sum_{j=0}^{m} L_j \Gamma + |e(t_2)| + + L_h \Gamma \int_0^\infty |K(s)| ds \\ &\quad + L_f\left(\Gamma + \frac{1}{2}\Gamma^2 + d_0 \Gamma\right) \\ &= -\left(d_0 - \sum_{j=0}^{m} L_j - L_h \int_0^\infty |K(s)| ds - L_f\left(1 + \frac{1}{2}\Gamma + d_0\right)\right)\Gamma \\ &\quad + |e(t_2)| < 0, \end{aligned} \quad (19)$$

which is a contradiction and implies that (15) holds. □

Lemma 4. *Suppose that assumptions H1–H3 hold and there exists a constant $\mu_0 > 0$ such that*

$$\int_0^\infty |K(u)| e^{\mu_0 u} du < \infty. \quad (20)$$

If $(x(t), y(t))^T$ is a solution of system (6) with initial conditions

$$\begin{aligned} &x(s) = \psi(s), \\ &y(0) = y_0, \\ &\max\{|\psi(s)|, |y_0|\} \le \Gamma s \in (-\infty, 0], \end{aligned} \quad (21)$$

then for any $\varepsilon > 0$ and $\alpha \in \mathbb{R}$, there exists $l(\varepsilon) > 0$ such that every interval $[\alpha, \alpha + l]$ contains at least one number δ for which there exists $N > 0$ such that

$$\max\{|x(t + \delta) - x(t)|, |y(t + \delta) - y(t)|\} \le \varepsilon \text{ for all } t > N. \quad (22)$$

Proof. Let

$$\vartheta(\delta, t) = \sum_{j=1}^{m}\left[g_j\left(x\left(t-\tau_j(t+\delta)+\delta\right)\right) - g_j\left(x\left(t-\tau_j(t)+\delta\right)\right)\right] + e(t+\delta) - e(t). \tag{23}$$

In view of Lemma 3, the solution $(x(t), y(t))^T$ is bounded and

$$\max\{|x(t)|, |y(t)|\} \le \Gamma \text{ for all } t \ge 0. \tag{24}$$

Thus, $x(t)$ and $y(t)$ are uniformly continuous on $\mathbb{R}$. In view of (7), for any $\varepsilon > 0$, there exists $l(\varepsilon) > 0$ such that every interval $[\alpha, \alpha + l]$, $\alpha \in \mathbb{R}$ contains at least one number δ for which there exists $N_0 > 0$ such that

$$|\vartheta(\delta, t)| \le \frac{1}{2}\gamma\varepsilon \text{ for } t \ge N_0, \tag{25}$$

where $\gamma > 0$ satisfies

$$\gamma \le \min\left\{d_1 - 1 - \mu, d_0 - L_f - L_r - \sum_{j=0}^{m} L_j e^{\mu\tau} - L_h\int_0^{\infty}|K(s)|e^{\mu s}ds - \mu\right\}, \tag{26}$$

$\mu > 0$ is a constant, $\tau = \max_{t\in\mathbb{R}}\tau_j(t), j = 1, 2, \cdots, m$. Let $u(t) = x(t+\delta) - x(t)$ and $v(t) = y(t+\delta) - y(t)$. Let $N_1 > \max\{N_0, -\delta\}$. For $t \ge K_1$, we obtain

$$u'(t) = -[r(x(t+\delta)) - r(x(t))] + y(t+\delta) - y(t), \tag{27}$$

$$\begin{aligned} v'(t) = &-d_0[y(t+\delta) - y(t)] - [g_0(x(t+\delta)) - d_0 r(x(t+\delta))] \\ &+ [g_0(x(t)) - d_0 r(x(t))] \\ &+ \sum_{j=1}^{m}\left[g_j\left(x\left(t-\tau_j(t)+\delta\right)\right) - g_j\left(x\left(t-\tau_j(t)\right)\right)\right] \\ &+ \vartheta(\delta, t) - [f(y(t+\delta) - r(x(t+\delta))) - f(y(t) - r(t))] \\ &- \int_0^{\infty}K(s)[h(x(t-s+\delta)) - h(x(t-s))]ds. \end{aligned} \tag{28}$$

Calculating the upper right derivative of $e^{\mu s}|u(s)|$ and $e^{\mu s}|v(s)|$, due to (27), (28), and assumptions H1–H3, for $t \ge N_1$, we have

$$\begin{aligned} D^+(e^{\mu s}|u(s)|)|_{s=t} &= \mu e^{\mu t}u(t) + e^{\mu t}\operatorname{sign}(u(t))(-[r(x(t+\delta)) - r(x(t))] \\ &\quad + y(t+\delta) - y(t)) \\ &< e^{\mu t}((\mu - d_1)u(t) + v(t)) + \frac{1}{2}\gamma e^{\mu t}, \end{aligned} \tag{29}$$

$$\begin{aligned} D^+(e^{\mu s}|v(s)|)|_{s=t} &= \mu e^{\mu t}v(t) + e^{\mu t}\operatorname{sign}(v(t))\Bigg(-d_0[y(t+\delta) - y(t)] \\ &\quad - [g_0(x(t+\delta)) - d_0 r(x(t+\delta))] + [g_0(x(t)) - d_0 r(x(t))] \\ &\quad + \sum_{j=1}^{m}\left[g_j\left(x\left(t-\tau_j(t)+\delta\right)\right) - g_j\left(x\left(t-\tau_j(t)\right)\right)\right] \\ &\quad + \vartheta(\delta, t) - [f(y(t+\delta) - r(x(t+\delta))) - f(y(t) - r(t))] \\ &\quad - \int_0^{\infty}K(s)[h(x(t-s+\delta)) - h(x(t-s))]ds\Bigg) \\ &\le e^{\mu t}\Bigg(\left(\mu - d_0 + L_f\right)|v(t)| + (L_0 + L_r)|u(t)| \\ &\quad + \sum_{j=1}^{m}L_j u\left(t-\tau_j(t)\right) + L_h\int_0^{\infty}K(s)u(t-s)ds\Bigg) + \frac{1}{2}\gamma e^{\mu t}. \end{aligned} \tag{30}$$

Let

$$\mathscr{M}(t) = \max_{s\le t}\{e^{\mu s}\max\{|u(s)|, |v(s)|\}\}. \tag{31}$$

Obviously, $\mathscr{M}(t) \ge e^{\mu t}\max\{|u(t)|, |v(t)|\}$. Now, we consider two cases.

Case 1

$$\mathscr{M}(t) > e^{\mu t}\max\{|u(t)|, |v(t)|\} \text{ for } t \ge N_1. \tag{32}$$

In this case, we claim that

$$\mathscr{M}(t) = \mathscr{M}(N_1) \text{ for } t \ge N_1. \tag{33}$$

If (32) does not holds, then there exists $t_3 > N_1$ such that $\mathscr{M}(t_3) > \mathscr{M}(N_1)$. Since $e^{\mu t}\max\{|u(t)|, |v(t)|\} \le \mathscr{M}(N_1)$ for all $t \le N_1$, there must exist $\alpha \in (N_1, t_3)$ such that

$$e^{\mu\alpha}\max\{|u(\alpha)|, |v(\alpha)|\} = \mathscr{M}(t_3) \ge \mathscr{M}(\alpha), \tag{34}$$

which contradicts (32) and implies that (33) holds. It follows that there exists $t_4 \ge N_1$ such that

$$\max\{|u(t)|, |v(t)|\} \le e^{-\mu t}\mathscr{M}(t) = e^{-\mu t}\mathscr{M}(N_1) < \varepsilon \text{ for } t \ge t_4. \tag{35}$$

Case 2. There is a point $t_5 \ge N_1$ such that $\mathscr{M}(t_5) = e^{\mu t_5}\max\{|u(t_5)|, |v(t_5)|\}$. If $\mathscr{M}(t_5) = e^{\mu t_5}|u(t_5)|$, by (29), we have

$$\begin{aligned} D^+(e^{\mu s}|u(s)|)|_{s=t_5} &= \mu e^{\mu t_5}u(t_5) + e^{\mu t_5}\operatorname{sign}(u(t))(-[r(x(t_5+\delta)) \\ &\quad - r(x(t_5))] + y(t_5+\delta) - y(t_5)) \\ &< e^{\mu t_5}(\mu - d_1)|u(t)| + e^{\mu t_5}|v(t_5)| + \frac{1}{2}\gamma e^{\mu t_5} \\ &\le [\mu - (d_1 - 1)]\mathscr{M}(t_5) + \frac{1}{2}\gamma\varepsilon e^{\mu t_5} \\ &\le -\gamma\mathscr{M}(t_5) + \gamma\varepsilon e^{\mu t_5}. \end{aligned} \tag{36}$$

On the other hand, if $\mathscr{M}(t_5) = e^{\mu t_5}|v(t_5)|$, by (30), we have

$$\begin{aligned}
&D^+(e^{\mu s}|v(s)|)|_{s=t_5} \\
&\leq e^{\mu t_5}\Bigg((\mu - d_0 + L_f)|v(t_5)| + (L_0 + L_r)|u(t_5)| \\
&\quad + \sum_{j=1}^{m} L_j u(t_5 - \tau_j(t_5)) + L_h \int_0^\infty |K(s)u(t_5 - s)|ds \Bigg) + \frac{1}{2}\gamma e^{\mu t_5} \\
&\leq \left[\mu - \left(d_0 - L_f - L_r - \sum_{j=0}^{m} L_j e^{\mu\tau} - L_h \int_0^\infty |K(s)|e^{\mu s}ds\right)\right] \\
&\quad \cdot \mathscr{M}(t_5) + \frac{1}{2}\gamma\varepsilon e^{\mu t_5} \\
&\leq -\gamma\mathscr{M}(t_5) + \gamma\varepsilon e^{\mu t_5}.
\end{aligned} \tag{37}$$

If $\mathscr{M}(t_5) \geq \varepsilon e^{\mu t_5}$, in view of (35) and (36), $\mathscr{M}(t)$ is strictly decreasing in a small neighborhood of t_5 which contradicts that $\mathscr{M}(t)$ is nondecreasing. Hence,

$$e^{\mu t_5} \max\{|u(t_5)|, |v(t_5)|\} = \mathscr{M}(t_5) < \varepsilon e^{\mu t_5}, \tag{38}$$

$$\max\{|u(t_5)|, |v(t_5)|\} < \varepsilon. \tag{39}$$

For $\forall t > t_5$, by the same approach as was used in the proof of (38), we have

$$\max\{|u(t)|, |v(t)|\} < \varepsilon \text{ if } \mathscr{M}(t) = e^{\mu t} \max\{|u(t)|, |v(t)|\}. \tag{40}$$

On the other hand, if $\mathscr{M}(t) > e^{\mu t} \max\{|u(t)|, |v(t)|\}$ for $t > t_5$. We can choose $t_5 \leq t_6 < t$ such that

$$\begin{aligned}
&\mathscr{M}(t_6) = e^{\mu t_6} \max\{|u(t_6)|, |v(t_6)|\} < e^{\mu t_6}\varepsilon, \\
&\mathscr{M}(s) > e^{\mu s} \max\{|u(s)|, |v(s)|\}, \\
&\text{for } s \in (t_6, t].
\end{aligned} \tag{41}$$

Using an argument similar to that in the proof of Case 1, we can show that

$$\mathscr{M}(s) = \mathscr{M}(t_6) \text{ for } s \in (t_6, t], \tag{42}$$

which implies that

$$\begin{aligned}
\max\{|u(t)|, |v(t)|\} &< e^{-\mu t}\mathscr{M}(t) = e^{-\mu t}\mathscr{M}(t_6) \\
&= \max\{|u(t_6)|, |v(t_6)|\} e^{-(t-t_6)} < \varepsilon.
\end{aligned} \tag{43}$$

Thus, there exists $N > 0$ such that

$$\max\{|u(t)|, |v(t)|\} \leq \varepsilon \text{ for all } t > N. \tag{44}$$

We complete the proof of Lemma 4. □

3. Almost Periodic Solution of Equation (1)

Theorem 5. *Suppose that H1–H3 hold. Then, system (6) has exactly one almost periodic solution $z^*(t) = (x^*(t), y^*(t))^T$ which is globally exponentially stable.*

Proof. Let $z(t) = (x(t), y(t))^T$ be a solution of system (6) with initial conditions

$$\begin{aligned}
&x(s) = \psi(s), \\
&y(0) = y_0, \\
&\max\{|\psi(s)|, |y_0|\} \leq \Gamma s \in (-\infty, 0].
\end{aligned} \tag{45}$$

Let

$$\begin{aligned}
\vartheta_k(t) = &\sum_{j=1}^{m} \left[g_j(x(t - \tau_j(t + t_k) + t_k)) - g_j(x(t - \tau_j(t) + t_k))\right] \\
&+ e(t + t_k) - e(t),
\end{aligned} \tag{46}$$

where t_k is any sequence of real numbers. In view of Lemmas 3 and 4, the solution $z(t) = (x(t), y(t))^T$ is bounded and (15) holds. Using (7) and (45), we can select a sequence $t_k \longrightarrow +\infty$ such that

$$|\vartheta_k(t)| \leq \frac{1}{k} \text{ for } t \geq 0. \tag{47}$$

Since $\{(x(t + t_k), y(t + t_k))^T\}_{k=1}^{\infty}$ is uniformly bounded and equiuniformly continuous, using Arzela-Ascoli lemma and diagonal selection principle, we can choose a subsequence $\{t_{k_j}\}$ of $\{t_k\}$ such that $(x(t + t_{k_j}), y(t + t_{k_j}))^T$ (for convenience, we still denote $(x(t + t_k), y(t + t_k))^T$) uniformly converges to a continuous function $z^*(t) = (x^*(t), y^*(t))^T$ on any compact set of $\mathbb{R}$, and

$$\max\{|x^*(t)|, |y^*(t)|\} \leq \Gamma \text{ for } t \in \mathbb{R}. \tag{48}$$

Now, we show that $z^*(t)$ is a solution of (6). In fact, for $t > 0$ and $\Delta t \in \mathbb{R}$, by (46), we have

$$\begin{aligned}
x^*(t + \Delta t) - x^*(t) &= \lim_{k \longrightarrow +\infty} [x(t + \Delta t + t_k) - x^*(t + t_k)] \\
&= \lim_{k \longrightarrow +\infty} \int_t^{t+\Delta t} (-r(x(s + t_k)) + y(s + t_k))ds \\
&= \int_t^{t+\Delta t} (-r(x^*(s)) + y^*(s))ds,
\end{aligned} \tag{49}$$

$$
\begin{aligned}
y^*(t+\Delta t) - y^*(t) &= \lim_{k\longrightarrow+\infty}\left[y(t+\Delta t+t_k) - y^*(t+t_k)\right] \\
&= \lim_{k\longrightarrow+\infty}\int_t^{t+\Delta t}\Big\{-d_0 y(s+t_k) + d_0 r(x(s+t_k)) \\
&\quad - f(y(s+t_k) - r(x(s+t_k))) - g_0(x(s+t_k)) \\
&\quad - \sum_{j=1}^m g_j\left(x\left(s+t_k-\tau_j(s+t_k)\right)\right) \\
&\quad - \int_0^\infty K(\mu)h(x(s+t_k-\mu))d\mu + e(s+t_k)\Big\}ds \\
&= \int_t^{t+\Delta t}\Big\{-d_0 y^*(s) + d_0 r(x^*(s)) - f(y^*(s) - r(x^*(s))) \\
&\quad - g_0(x^*(s)) - \sum_{j=1}^m g_j\left(x^*\left(s-\tau_j(s)\right)\right) \\
&\quad - \int_0^\infty K(\mu)h(x^*(s-\mu))d\mu + e(s)\Big\}ds \\
&\quad + \lim_{k\longrightarrow+\infty}\int_t^{t+\Delta t}\vartheta_k(s)ds \\
&= \int_t^{t+\Delta t}\Big\{-d_0 y^*(s) + d_0 r(x^*(s)) - f(y^*(s) - r(x^*(s))) \\
&\quad - g_0(x^*(s)) - \sum_{j=1}^m g_j\left(x^*\left(s-\tau_j(s)\right)\right) \\
&\quad - \int_0^\infty K(\mu)h(x^*(s-\mu))d\mu + e(s)\Big\}ds.
\end{aligned}
\tag{50}
$$

From (47) and (48), we have

$$
\begin{cases}
x^{*'}(t) = y^*(t) - r(x^*), \\
y^{*'}(t) = -d_0 y^*(t) + d_0 r(x^*) - f(y^*(t) - r(x^*)), \\
-g_0(x^*(t)) - \sum_{j=1}^m g_j\left(x^*\left(t-\tau_j(t)\right)\right) - \int_0^\infty K(s)h(x^*(t-s))ds + e(t).
\end{cases}
\tag{51}
$$

Thus, $z^*(t)$ is a solution of (6).

Now, we show that $z^*(t)$ is an almost periodic solution of (6). From Lemma 4, for any $\varepsilon > 0$, there exists $l(\varepsilon) > 0$ such that every interval $[\alpha, \alpha + l]$ contains at least one number δ for which there exists $N > 0$ such that

$$
\max\left\{|x(t+\delta) - x(t)|, |y(t+\delta) - y(t)|\right\} \leq \varepsilon \text{ for all } t > N.
\tag{52}
$$

For any $s \in \mathbb{R}$, there exists a sufficient large positive integer $N_0 > N$ such that for any $k > N_0$,

$$
\begin{aligned}
&s + t_k > N_0, \\
&\max\left\{|x(s+t_k+\delta) - x(s+t_k)|, |y(s+t_k+\delta) - y(s+t_k)|\right\} \leq \varepsilon.
\end{aligned}
\tag{53}
$$

Let $k \longrightarrow +\infty$ in (51), then

$$
\begin{aligned}
&|x^*(s+\delta) - x^*(s)| \leq \varepsilon, \\
&|y^*(s+\delta) - y^*(s)| \leq \varepsilon,
\end{aligned}
\tag{54}
$$

which imply that $z^*(t)$ is an almost periodic solution of (6).

Finally, we show that $z^*(t)$ is globally exponentially stable. Let $z^*(t) = (x^*(t), y^*(t))^T$ be an almost periodic solution of (6) with initial value $(\psi^*(s), y_0^*) \in C((-\infty,0), \mathbb{R}) \times \mathbb{R}$. Let $z(t) = (x(t), y(t))^T$ be an arbitrary solution of (6) with initial value $(\psi(s), y_0) \in C((-\infty,0), \mathbb{R}) \times \mathbb{R}$. Let $\tilde{u}(t) = x(t) - x^*(t)$ and $\tilde{v}(t) = y(t) - y^*(t)$, and then,

$$
\begin{cases}
\tilde{u}'(t) = \tilde{v}(t) - [r(x(t)) - r(x^*(t))], \\
\tilde{v}'(t) = -d_0\tilde{v}(t) + d_0[r(x(t)) - r(x^*(t))] - [f(y(t) - r(x)) - f(y^*(t) - r(x^*))], \\
-[g_0(x(t)) - g_0(x^*(t))] - \sum_{j=1}^m\left[g_j\left(x\left(t-\tau_j(t)\right)\right) - g_j\left(x^*\left(t-\tau_j(t)\right)\right)\right] - \int_0^\infty K(s)[h(x(t-s)) - h(x^*(t-s))]ds.
\end{cases}
\tag{55}
$$

Construct the following Lyapunov functionals:

$$
\begin{aligned}
V_1(t) &= |\tilde{u}(t)|e^{\mu t}, \\
V_2(t) &= |\tilde{v}(t)|e^{\mu t}.
\end{aligned}
\tag{56}
$$

Calculate the upper right derivative of $V_1(t)$ and $V_2(t)$ along the solution of (53) with the initial conditions $(\psi(s) - \psi^*(s), y_0 - y_0^*)$, and then,

$$
\begin{aligned}
D^+(V_1(t)) &= \mu e^{\mu t}\tilde{u}(t) + e^{\mu t}\operatorname{sign}(\tilde{u}(t))(-[r(x(t)) - r(x^*(t))] + \tilde{v}(t)) \\
&< e^{\mu t}(\mu - d_1)|\tilde{u}(t)| + e^{\mu t}|\tilde{v}(t)|,
\end{aligned}
\tag{57}
$$

$$D^{+}(V_2(t)) \le e^{\mu t}\Bigg(\left(\mu - d_0 + L_f\right)|\tilde{v}(t)| + (L_0 + L_r)|\tilde{u}(t)| + \sum_{j=1}^{m} L_j \left|\tilde{u}\left(t-\tau_j(t)\right)\right| + L_h \int_0^{\infty} |K(s)\tilde{u}(t-s)|ds \Bigg). \tag{58}$$

Let $M > 1$ be an arbitrary real number and

$$\Xi = \max\{\|\psi - \psi^*\|, |y_0 - y_0^*|\} > 0. \tag{59}$$

It is easy to see that

$$\begin{aligned} V_1(t) &= |\tilde{u}(t)|e^{\mu t} < M\Xi, \\ V_2(t) &= |\tilde{v}(t)|e^{\mu t} < M\Xi, \\ &\text{for } t \in (-\infty, 0]. \end{aligned} \tag{60}$$

We claim that

$$\begin{aligned} V_1(t) &= |\tilde{u}(t)|e^{\mu t} < M\Xi, \\ V_2(t) &= |\tilde{v}(t)|e^{\mu t} < M\Xi, \\ &\text{for } t > 0. \end{aligned} \tag{61}$$

If not, one of the following two cases must occur.
Case 1. There exists $T_1 > 0$ such that

$$\begin{aligned} V_1(T_1) &= M\Xi, \\ V_i(t) &< M\Xi, \\ &\text{for all } t \in (-\infty, T_1),\ i = 1, 2. \end{aligned} \tag{62}$$

Case 2. There exists $T_2 > 0$ such that

$$\begin{aligned} V_2(T_2) &= M\Xi, \\ V_i(t) &< M\Xi, \\ &\text{for all } t \in (-\infty, T_2),\ i = 1, 2. \end{aligned} \tag{63}$$

If Case 1 holds, by (55) and (60), we have

$$\begin{aligned} 0 \le D^{+}(V_1(T_1)) &< e^{\mu T_1}(\mu - d_1)|\tilde{u}(T_1)| + e^{\mu T_1}|\tilde{v}(T_1)| \\ &\le [\mu - (d_1 - 1)]M\Xi, \end{aligned} \tag{64}$$

which contradicts $\mu - (d_1 - 1) < 0$. Hence, (60) holds.

If Case 2 holds, by (56) and (61), we have

$$\begin{aligned} 0 &\le D^{+}(V_2(T_2)) \\ &\le e^{\mu T_2}\left(\mu - d_0 + L_f\right)|\tilde{v}(T_2)| + e^{\mu T_2}(L_0 + L_r)|\tilde{u}(T_2)| \\ &\quad + \sum_{j=1}^{m} L_j \left|\tilde{u}\left(t-\tau_j(t)\right)\right| e^{\mu\left(T_2 - \tau_j(T_2)\right)} e^{\mu\tau_j(T_2)} \\ &\quad + L_h \int_0^{\infty} |K(s)\tilde{u}(t-s)| e^{\mu(T_2 - s)} e^{\mu s} ds. \\ &\le \left(\mu - d_0 + L_f + L_r + \sum_{j=0}^{m} L_j e^{\mu\tau} + L_h \int_0^{\infty} |K(s)| e^{\mu s} ds - \mu\right) M\Xi, \end{aligned} \tag{65}$$

which contradicts

$$\mu - d_0 + L_f + L_r + \sum_{j=0}^{m} L_j e^{\mu\tau} + L_h \int_0^{\infty} |K(s)| e^{\mu s} ds < 0. \tag{66}$$

Hence, (61) holds. Hence, (59) holds, and $z^*(t)$ is globally exponentially stable. □

Remark 6. In general, constructing Lyapunov functional is a main research method for studying stability problems of nonlinear systems (see [10–15]). However, constructing a proper Lyapunov functional is very difficult for a complicated system. Hence, it is necessary to develop new research methods, such as matrix measure approach, comparison theorem, and special inequality technique. We hope to use some new methods to study second-order nonlinear equation with mixed delays in the follow-up research work.

4. Examples

As applications, consider the following second-order nonlinear equation:

$$\begin{aligned} &x''(t) + 5x'(t) + 5x(t)x'(t) - \frac{1}{4}|x(t)| + x^3(t) \\ &\quad - \frac{1}{8}|x(t - \sin t)| - \frac{1}{8}\left|\arctan x(t - \cos t) - \int_0^{\infty} e^{-u}\cos u\right| \\ &\quad \cdot \frac{1}{5}|x(t-u)|du = 5\cos t. \end{aligned} \tag{67}$$

From (1) and (65), it is easy to see that

$$\begin{aligned} f(x(t)) &= 5x(t), \\ g_0(x) &= -\frac{1}{4}|x(t)| + x^3(t), \\ g_1(x(t-\tau_1(t))) &= -\frac{1}{8}|x(t - \sin t)|, \\ g_2(x(t-\tau_2(t))) &= -\frac{1}{8}\arctan x(t - \cos t), \end{aligned}$$

$$K(t) = e^{-t}\cos t,$$
$$h(x(t)) = \frac{1}{5}|x(t)|, \tag{68}$$
$$e(t) = 5\cos t.$$

Let

$$r(x) = \int_0^x [f(x) - d_0]dx,$$
$$y(t) = x'(t) + r(x), \tag{69}$$

then (65) can be rewritten by

$$\begin{cases} x'(t) = y(t) - r(x), \\ y'(t) = -d_0 y(t) + d_0 r(x) - f(y(t) - r(x)), \\ -g_0(x(t)) - \sum_{j=1}^{2} g_j(x(t-\tau_j(t))) - \int_0^\infty K(s)h(x(t-s))ds + e(t). \end{cases} \tag{70}$$

Choose proper d_0; it is easy to see all assumptions of Theorem 5. Hence, system (68) has exactly one almost periodic solution which is globally exponentially stable. Thus, system (65) has exactly one almost periodic solution which is globally exponentially stable.

Remark 7. System (65) is a more general nonlinear system than the ones of [1, 13–15, 22], and the criterion of system (65) can be applicable for proving that the corresponding ones of [1, 13–15, 22]. In this paper, we study a more general and complicated second-order nonlinear system and obtain dynamic behaviors of almost periodic solution, and the results in the present paper are new and have wide applications for delay differential equations.

5. Conclusions

In this paper, we obtain existence and exponential stability of almost periodic solution for a second-order nonlinear equation with mixed delays by using mathematic analysis technique and Lyapunov functional method. Since there exist mixed delays in the second-order nonlinear equation and almost periodic solutions have particular properties, the existing methods are no longer applicable; we introduce a variable substitution and change the second-order nonlinear equation into a first-order two-dimensional system for overcoming the above difficulties. Finally, an example has been given at the end of this paper to illustrate the effectiveness and feasibility of the proposed criterion.

It should be pointed out that the research method of this paper is inspired by literatures [1, 22], but the equations studied in this paper are more complex, so more mathematical analysis skills are needed. The methods of this article can also be used to study other types of high-order nonlinear equations. However, we cannot obtain dynamic behaviors of almost periodic solution for high-order nonlinear equation with p-Laplacian operator in the present paper which is our future research direction.

Data Availability

The numerical data used to support the findings of this study are included in the article.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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