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Research Article

Computing Wiener and Hyper-Wiener Indices of Zero-Divisor Graph of $\mathbb{Z}_{\wp^3} \times \mathbb{Z}_{\mathfrak{J}_1\mathfrak{J}_2}$

Yonghong Liu[1], **Abdul Rauf**[2], **AdnanAslam**[3], **Saira Ishaq**[2] **and Abudulai Issa**[4]

[1]*School of Computer Science, Chengdu University, Chengdu, China*
[2]*Department of Mathematics, Air University Islamabad, Multan Campus, Pakistan*
[3]*Department of Natural Sciences and Humanities, University of Engineering and Technology, Lahore, Pakistan (RCET), Pakistan*
[4]*Department of Mathematics, University of Lome, 1 PoBox 1515 Lome 1, Togo*

Correspondence should be addressed to Abudulai Issa; issaabudulai13@gmail.com

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Let $S = \mathbb{Z}_{\wp^3} \times \mathbb{Z}_{\mathfrak{J}_1\mathfrak{J}_2}$ be a commutative ring where $\wp$, $\mathfrak{J}_1$ and $\mathfrak{J}_2$ are positive prime integers with $\mathfrak{J}_1 \neq \mathfrak{J}_2$. The zero-divisor graph assigned to S is an undirected graph, denoted as $Y(S)$ with vertex set $V(Y(S))$ consisting of all Zero-divisor of the ring S and for any c, d $\in V(Y(S))$, $cd \in E(Y(S))$ if and only if cd =0. A topological index/descriptor is described as a topological-invariant quantity that transforms a molecular graph into a mathematical real number. In this paper, we have computed distance-based polynomials of $Y(R)$ i-e Hosoya polynomial, Harary polynomial, and the topological indices related to these polynomials namely Wiener index, and Hyper-Wiener index.

1. Introduction

The Zero-divisor graph is the undirected graph on the set of the Zero-divisors of a commutative ring. In a Zero-divisor graph, the set of zero-divisors are considered as vertices and pairs of elements whose product is zero as its edges. The Zero-divisor graph is very helpful to study the algebraic properties of rings using graph-theoretical tools.

In 1988 Beck was the first who gives the idea of interlinking two main mathematics topics Algebra, and Graph theory [1]. First, he presented the concept of a Zero-divisor graph of Commutative Ring R, in which all the elements of ring R were considered as the vertices of a Zero-divisor graph, and those two distinct vertices c and d are connected if and only if cd =0. Beck's main objective in his work was to show the coloring of the Commutative ring. Naseer and Anderson extended the work by Beck's in [2]. In [3], Anderson-Livingston worked on the Zero-divisor graphs in which only non-zero Zero-divisors are considered as the vertices. Anderson-Livingston discussed the relations between ring theoretic properties of R and graph-theoretic properties of the $Y(R)$. Furthermore, this study presents some important results of zero-divisor graphs. Later Anderson, Frazier, Lauve, Levy, Livingston, and Shapiro [4, 5] worked on translating the algebraic properties of rings into graphical language.

Redmond developed the idea of the Zero-divisor graphs associated to non-commutative rings [6]. He defined a Zero-divisor graph associated with a non-commutative ring in many more ways, including both directed and undirected graphs. Redmond [7] carried on this work by extending the concept of Zero-divisor graph of a Commutative ring to an ideal-based Zero-divisor graph of a Commutative ring by replacing elements whose product is zero with elements whose product lies in some ideal I of ring R. Since then, many researchers have worked on it and defined graphs such as unit graphs, the equivalence class of Zero-divisor graphs, total graphs, ideal-based Zero-divisor graphs, the Jacobson graphs, and so on (see, [7–11]).

Let G be graph with vertex set V(G) and edge set E(G). The degree of a vertex v is the number of edges attached to v. For any two vertices u and v, the distance between them is denoted by d(u,v) and is defined as the length of the shortest path joining them. For instance, d (u, v) = 0 If and only if u = v and d (u, v) = ∞ if there does not exist any possible shortest path. For more details on the basic definition related to graph theory, the readers can see the book by [12].

Let Z(S) be the set of Zero-divisors of the Commutative ring 'S'. The Zero-divisor graph of S is denoted by Y (S) = (V(Y (S)), E(Y (S))), which is an undirected graph with V(Y (S)) = Z(S) \ {0} as vertex set, and for different c, d ∈ V(Y(S)), the vertices c and d have an edge $cd \in$ E(Y(S)) between them if and only if cd = 0 [13]. It is very interesting to translate the algebraic properties of algebraic graphs into numerical molecular descriptors. A topological index/descriptor is described as a topological-invariant quantity that transforms a molecular graph into a mathematical real number. QSPR/QSAR studies are majorly concerned with the application of these molecular descriptors or Topological indices. Many molecular descriptors have been introduced in the last decade, demonstrating their importance. A molecular structure is denoted as a graph with atoms as vertices and bonds as edges. Then, using various forms of topological indices, one can study both the algebraic and chemical aspects of the compounds. There are two main types of topological indices, the first based on degree [14–19] and the second based on distance. Randic Connectivity index, Zagreb Indices, the Harmonic index, Atom bond Connectivity, and Geometric Arithmetic index are degree-based topological indices. Some well-known distance-based topological indices are the Wiener index, Hosoya index, and Estrada index [20, 21]. For details related to the computation of Hosoya and Harary polynomial of zero divisor graphs associated to some commutative rings, the readers can see [22–24].

Eccentricity-based indices have been effectively used to construct a variety of mathematical models for the prediction of biological activities of various types. Several authors have examined the uses and mathematical properties of these indices. Further, the readers can see [25–28] for more details about eccentricity based-indices. In 1988 Hosoya polynomial was introduced by Haruo Hosoya [29]. With its vast application in graph theory and chemistry, it is proved to be an effective distance-based topological index [30]. The Hosoya polynomial has many chemical applications [31]. Almost all distance-based topological indices can be computed from this polynomial [32–34]. The Hosoya polynomial is related to a variety of topological indices, the most well-known of which is the Wiener, Hyper Wiener, and Harary polynomial. For further study see [21, 35–37]. Hosoya polynomial of a graph H is defined as:

$$H(H, x) = \sum_{v\epsilon V(H)} \sum_{u\epsilon V(H)} x^{d(u,v)}. \quad (1)$$

The Harary polynomial was introduced in 1985 and is denoted by $h(H, x)$ and is defined as:

$$h(H, x) = \sum_{v\epsilon V(H)} \sum_{u\epsilon V(H)} \frac{x^{d(u,v)}}{d(u, v)}. \quad (2)$$

The generalized Harary index of graph H is denoted by$h_t(H)$ and defined as:

$$h_t(H) = \sum_{v\epsilon V(H)} \sum_{u\epsilon V(H)} \frac{1}{d(u, v) + t}. \quad (3)$$

Where t= 1,2, 3, ... The Wiener index is the oldest topological index introduced by Harold Wiener in 1947 for the study of boiling points of paraffin [38]. It plays a very important role in inverse structure-property relationship problems [39]. For applications and mathematical properties of Weiner index see [40–46]. The Wiener index is defined as:

$$W(H) = \frac{1}{2} \sum_{v\epsilon V(H)} \sum_{u\epsilon V(H)} d(u, v) \quad (4)$$

In 1993 another distance-based topological index was introduced by Randic known as the Hyper-Wiener index [35]. This index is used for predicting physicochemical properties of organic compounds [47], and is defined as:

$$WW(H) = \frac{1}{2} \sum_{v\epsilon V(H)} \sum_{u\epsilon V(H)} \left(d(u, v)^2 + d(u, v)\right) \quad (5)$$

It is easy to observe that there is an effective relation between Hosoya Polynomial, Wiener Index, and Hyper-Wiener Index.

$$W(H) = H'(H : 1), WW(H) = H'(H ; 1) + \frac{1}{2} H''(H ; 1) \quad (6)$$

2. Result and Discussion

Let $\mathbb{Z}_{\wp^3} \times \mathbb{Z}_{\Im_1\Im_2}$ be a commutative ring where $\wp, \Im_1$and $\Im_2$ are positive prime integers with$\Im_1 \neq \Im_2$. The total number of elements in this ring is$\wp^3 \Im_1 \Im_2$. We are only concerned with the set of zero-divisors of $\mathbb{Z}_{\wp^3} \times \mathbb{Z}_{\Im_1\Im_2}$ denoted by $V(Y(\mathbb{Z}_{\wp^3} \times \mathbb{Z}_{\Im_1\Im_2}))$ having cardinality$(\wp^3 - \wp^2)(\Im_1 + \Im_2 - 1) + \wp^2 \Im_1 \Im_2 - 1$. We made the degree-based disjoint partition for vertices of the zero-divisor graph $Y(\mathbb{Z}_{\wp^3} \times \mathbb{Z}_{\Im_1\Im_2})$ of the commutative ring, which are given as:

$$W_1 = \{(0, \mathrm{v}): \mathrm{v} \in Z_{\Im_1\Im_2} \text{ with } v \neq 0, k_1\Im_1, k_2\Im_2, 1 \leq k_1 \leq \Im_2 - 1, 1 \leq k_2 \leq \Im_1 - 1\}, \quad (7)$$

$$W_2 = \{(0, \mathrm{v}): \mathrm{v} = k_1\mathfrak{I}_1, 1 \le k_1 \le \mathfrak{I}_2 - 1\}, \quad (8)$$

$$W_3 = \{(0, \mathrm{v}): \mathrm{v} = k_2\mathfrak{I}_2, 1 \le k_2 \le \mathfrak{I}_1 - 1\}, \quad (9)$$

$$W_4 = \left\{(u, 0): u\epsilon\mathbb{Z}_{g^3}, \quad \text{with} \quad u \ne 0, t_1 g, 1 \le t_1 \le g^2 - 1\right\}, \quad (10)$$

$$W_5 = \left\{(u, v): u \ne 0, t_1 g, 1 \le t_1 \le g^2 - 1, \mathrm{v} = k_1\mathfrak{I}_1, 1 \le k_1 \le \mathfrak{I}_2 - 1\right\}, \quad (11)$$

$$W_6 = \left\{(u, v): u \ne 0, t_1 g, 1 \le t_1 \le g^2 - 1, \mathrm{v} = k_2\mathfrak{I}_2, 1 \le k_2 \le \mathfrak{I}_1 - 1\right\}, \quad (12)$$

$$W_7 = \left\{(u, 0): u = t_1 g, 1 \le t_1 \le g^2 - 1 \quad \text{with}\, t_1 \ne l g, 1 \le t_1 \le g - 1\right\}, \quad (13)$$

$$W_8 = \left\{(u, v): u = t_1 g, 1 \le t_1 \le g^2 - 1 \quad \text{with} \quad t_1 \ne l g, 1 \le t_1 \le g - 1, \mathrm{v} \ne 0, k_1\mathfrak{I}_1, k_2\mathfrak{I}_2, 1 \le k_1 \le \mathfrak{I}_2 - 1, 1 \le k_2 \le \mathfrak{I}_1 - 1\right\}, \quad (14)$$

$$W_9 = \left\{(u, v): u = t_1 g, 1 \le t_1 \le g^2 - 1 \quad \text{with} \quad t_1 \ne l g, 1 \le t_1 \le g - 1, \mathrm{v} = k_1\mathfrak{I}_1, 1 \le k_1 \le \mathfrak{I}_2 - 1\right\}, \quad (15)$$

$$W_{10} = \left\{(u, v): u = t_1 g, 1 \le t_1 \le g^2 - 1 \quad \text{with} \quad t_1 \ne l g, 1 \le t_1 \le g - 1, \mathrm{v} = k_1\mathfrak{I}_2, 1 \le k_2 \le \mathfrak{I}_1 - 1\right\}, \quad (16)$$

$$W_{11} = \left\{(u, 0): u = t_2 g^2, 1 \le t_2 \le g - 1\right\}, \quad (17)$$

$$W_{12} = \left\{(u, v): u = t_2 g^2, 1 \le t_2 \le g - 1, \mathrm{v} \ne 0, k_1\mathfrak{I}_1, k_2\mathfrak{I}_2, 1 \le k_1 \le \mathfrak{I}_2 - 1, 1 \le k_2 \le \mathfrak{I}_1 - 1\right\}, \quad (18)$$

$$W_{13} = \left\{(u, v): u = t_2 g^2, 1 \le t_2 \le g - 1, \mathrm{v} = k_1\mathfrak{I}_1, 1 \le k_1 \le \mathfrak{I}_2 - 1,\right\}, \quad (19)$$

$$W_{14} = \left\{(u, v): u = t_2 g^2, 1 \le t_2 \le g - 1, \mathrm{v}k_2\mathfrak{I}_2, 1 \le k_2 \le \mathfrak{I}_1 - 1\right\}. \quad (20)$$

From the above, it is clear that $\mathrm{V}(Y(\mathbb{Z}_{g^3} \times \mathbb{Z}_{\mathfrak{I}_1\mathfrak{I}_2})) = \bigcup_{i=1}^{14} W_i$. Observe that $|W_1| = (\mathfrak{I}_1 - 1)(\mathfrak{I}_2 - 1)$, $|W_2| = \mathfrak{I}_2 - 1$, $|W_3| = \mathfrak{I}_1 - 1$, $|W_4| = (g^3 - g^2)$, $|W_5| = (g^3 - g^2)(\mathfrak{I}_2 - 1)$, $|W_6| = (g^3 - g^2)(\mathfrak{I}_1 - 1)$, $|W_7| = g^2 - g$, $|W_8| = (g^2 - g)(\mathfrak{I}_1\mathfrak{I}_2 - \mathfrak{I}_1 - \mathfrak{I}_2 + 1)$, $|W_9| = (g^2 - g)(\mathfrak{I}_2 - 1)$, $|W_{10}| = (g^2 - g)(\mathfrak{I}_1 - 1)$, $|W_{11}| = g - 1$, $|W_{12}| = (g - 1)(\mathfrak{I}_1\mathfrak{I}_2 - \mathfrak{I}_1 - \mathfrak{I}_2 + 1)$, $|W_{13}| = g\mathfrak{I}_2 - g - \mathfrak{I}_2 + 1$, $|W_{14}| = g\mathfrak{I}_1 - g - \mathfrak{I}_1 + 1$. Furthermore $d_v = g^3 - 1$ for $v\epsilon W_1$, $d_v = \mathfrak{I}_1 g^3 - 1$ for $v\epsilon W_2$, $d_v = \mathfrak{I}_1 g^3 - 1$ for $v\epsilon W_3$, $d_v = \mathfrak{I}_1\mathfrak{I}_2 - 1$ for $v\epsilon W_4$, $d_v = \mathfrak{I}_1 - 1$ for $v\epsilon W_5$, $d_v = \mathfrak{I}_2 - 1$ for $v\epsilon W_6$, $d_v = g\mathfrak{I}_1\mathfrak{I}_2 - 1$ for $v\epsilon W_7$, $d_v = g - 1$ for $v\epsilon W_8$, $d_v = \mathfrak{I}_1 g - 1$ for $v\epsilon W_9$, $d_v = \mathfrak{I}_1 g - 1$ for $v\epsilon W_{10}$, $d_v = g^2\mathfrak{I}_1\mathfrak{I}_2 - 1$ for $v\epsilon W_{11}$, $d_v = g^2 - 1$ for $v\epsilon W_{12}$, $d_v = \mathfrak{I}_1 g^2 - 1$ for $v\epsilon W_{13}$, $d_v = \mathfrak{I}_2 g^2 - 1$ for $v\epsilon W_{14}$.

Let $1 \le i \le 14$be a fixed integer and$u \in W_i$. From the above partition one can observe that the distance $d(u, v)$ is same for any $v \in W_j$, where $1 \le j \le 14$. This observation is depicted in Figure 1. If there is an edge between any two partition sets W_i and W_j, then it means that $d(u, v) = 1$ for any $u \in W_i$and $v \in W_j$. It can be seen that the maximum distance between any two vertices of Zero-divisor graph $Y(\mathbb{Z}_{g^3} \times \mathbb{Z}_{\mathfrak{I}_1\mathfrak{I}_2})$ is atmost three. This important fact is stated in Lemma 1.

Lemma 1. *The maximum diameter of the Zero-divisor graph $Y(S)$ of a commutative ring is 3 [3].*

We have summarized the eccentricity and degree of each vertex of Zero-divisor graph $Y(\mathbb{Z}_{g^3} \times \mathbb{Z}_{\mathfrak{I}_1\mathfrak{I}_2})$ in Table 1:

3. Hosoya and Harary Polynomial of Zero Divisor Graph $Y(\mathbb{Z}_{g^3} \times \mathbb{Z}_{\mathfrak{I}_1\mathfrak{I}_2})$

Now we compute the Harary and Hosoya polynomial of zero divisor graph $Y(\mathbb{Z}_{g^3} \times \mathbb{Z}_{\mathfrak{I}_1\mathfrak{I}_2})$. First, we find the distance $d(u, v)$ for each pair of vertices u, v of $Y(\mathbb{Z}_{g^3} \times \mathbb{Z}_{\mathfrak{I}_1\mathfrak{I}_2})$. For this, we consider the partitionsW_i, $1 \le i \le 14$ as vertices. Now we compute the distance $d(u, v)$ between any two vertices such that $u \in W_i$ and $v \in W_j$. Let $x \in \mathbb{Z}, x > 0$ and $DP_x(W_i, W_j) = |\{(u, v) \in (W_i, W_j) | d(u, v) = x\}|$. If $i = j$, we simply use the notation $DP_x(W_i)$. The cardinality of the set of ordered pair of vertices which are adjacent to each other is denoted by $DP_1 Y(\mathbb{Z}_{g^3} \times \mathbb{Z}_{\mathfrak{I}_1\mathfrak{I}_2})$.

Lemma 2. *The cardinality of the set of ordered pair of vertices in $Y(\mathbb{Z}_{g^3} \times \mathbb{Z}_{\mathfrak{I}_1\mathfrak{I}_2})$ which are adjacent to each other is:*

$$DP_1 Y(\mathbb{Z}_{g^3} \times \mathbb{Z}_{\mathfrak{I}_1\mathfrak{I}_2}) = \frac{g^3 + g^2 + g + g^3(\mathfrak{I}_1 + \mathfrak{I}_2) + (\mathfrak{I}_1\mathfrak{I}_2 + \mathfrak{I}_1 + \mathfrak{I}_2)(g^2 + g + 1) - 14}{2} \quad (21)$$

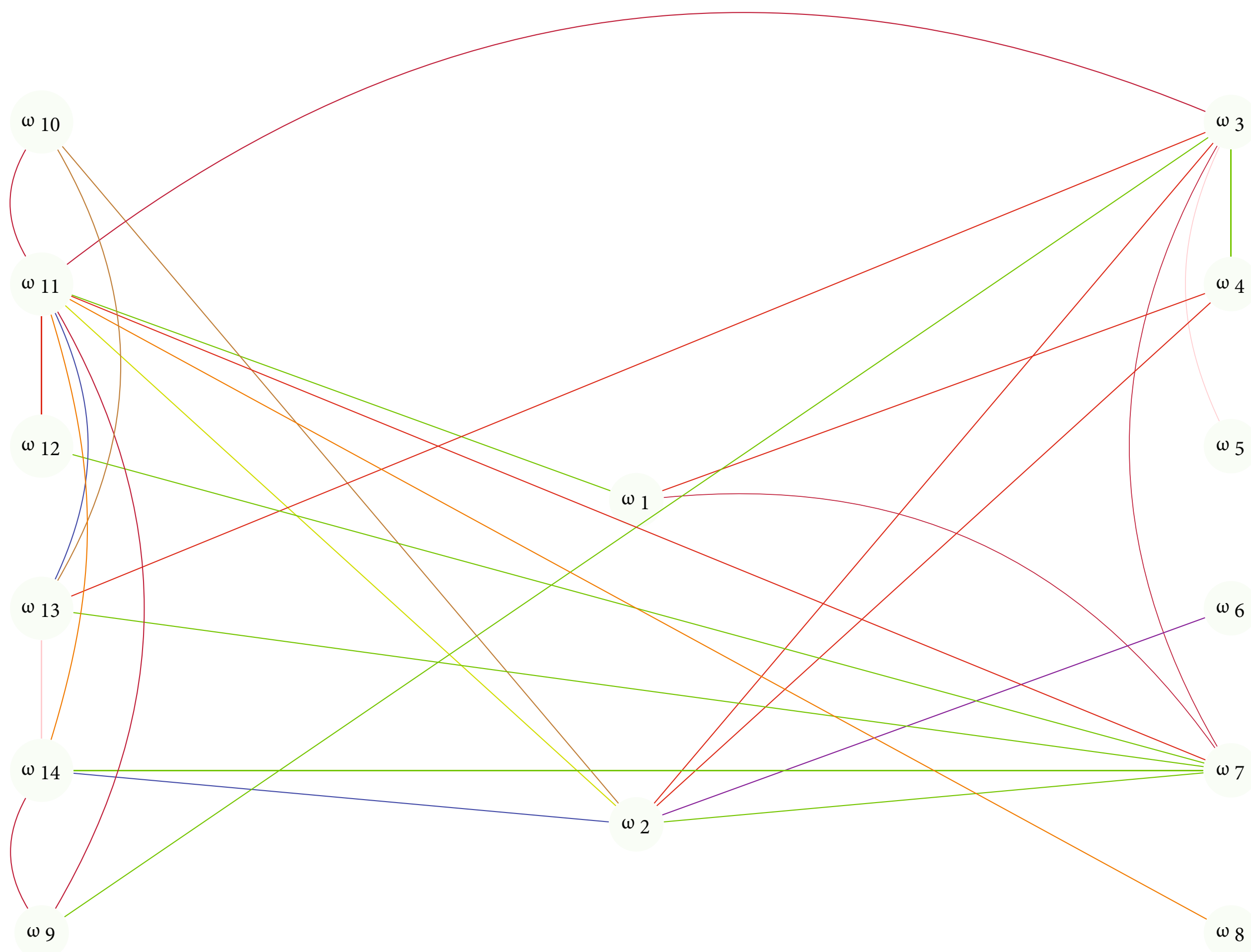

FIGURE 1: Zero-divisor graph of $\mathbb{Z}_{g^3} \times \mathbb{Z}_{\mathfrak{I}_1\mathfrak{I}_2}$.

TABLE 1: Eccentricity and degree of each vertex of Zero-divisor graph $Y(\mathbb{Z}_{g^3} \times \mathbb{Z}_{\mathfrak{I}_1\mathfrak{I}_2})$.

Vertices	Eccentricity	Degree	Frequency
W_1	3	$g^3 - 1$	$\mathfrak{I}_1\mathfrak{I}_2 - \mathfrak{I}_1 - \mathfrak{I}_2 + 1$
W_2	2	$\mathfrak{I}_1 g^3 - 1$	$\mathfrak{I}_2 - 1$
W_3	2	$\mathfrak{I}_2 g^3 - 1$	$\mathfrak{I}_1 - 1$
W_4	3	$\mathfrak{I}_1\mathfrak{I}_2 - 1$	$g^3 - g^2$
W_5	3	$\mathfrak{I}_1 - 1$	$(g^3 - g^2)(\mathfrak{I}_2 - 1)$
W_6	3	$\mathfrak{I}_2 - 1$	$(g^3 - g^2)(\mathfrak{I}_1 - 1)$
W_7	2	$g\mathfrak{I}_1\mathfrak{I}_2 - 1$	$g^2 - g$
W_8	3	$g - 1$	$(g^2 - g)(\mathfrak{I}_1 - 1)(\mathfrak{I}_2 - 1)$
W_9	3	$\mathfrak{I}_1 g - 1$	$(g^2 - g)(\mathfrak{I}_2 - 1)$
W_{10}	3	$\mathfrak{I}_2 g - 1$	$(g^2 - g)(\mathfrak{I}_1 - 1)$
W_{11}	2	$g^2\mathfrak{I}_1\mathfrak{I}_2 - 1$	$g - 1$
W_{12}	3	$g^2 - 1$	$(g - 1)(\mathfrak{I}_1\mathfrak{I}_2 - \mathfrak{I}_1 - \mathfrak{I}_2 + 1)$
W_{13}	3	$\mathfrak{I}_1 g^2 - 1$	$g\mathfrak{I}_2 - g - \mathfrak{I}_2 + 1$
W_{14}	3	$\mathfrak{I}_2 g^2 - 1$	$g\mathfrak{I}_1 - g - \mathfrak{I}_1 + 1$

Proof. By Handshaking lemma and using table 1 we concluded

$$DP_1Y(\mathbb{Z}_{g^3}\times\mathbb{Z}_{\mathfrak{I}_1\mathfrak{I}_2}) = \frac{1}{2}[g^3-1+\mathfrak{I}_1g^3-1+\mathfrak{I}_2g^3-1+\mathfrak{I}_1\mathfrak{I}_2-1+\mathfrak{I}_1-1+\mathfrak{I}_2-1+g\mathfrak{I}_1\mathfrak{I}_1-1+g-1+\mathfrak{I}_1g-1+\mathfrak{I}_2g -1+g^2\mathfrak{I}_1\mathfrak{I}_2-1+g^2-1+\mathfrak{I}_1g^2-1+\mathfrak{I}_2g^2-1] = \frac{g^3+g^2+g+g^3(\mathfrak{I}_1+\mathfrak{I}_2)+(\mathfrak{I}_1\mathfrak{I}_2+\mathfrak{I}_1+\mathfrak{I}_2)(g^2+g+1)-14}{2} \tag{22}$$

□

Lemma 3. *The cardinality of the set of ordered pair of vertices in $Y(\mathbb{Z}_{g^3}\times\mathbb{Z}_{\mathfrak{I}_1\mathfrak{I}_2})$ which are distance 2 to each other is*

$$DP_2Y(\mathbb{Z}_{g^3}\times\mathbb{Z}_{\mathfrak{I}_1\mathfrak{I}_2}) = \Omega_1+\Omega_2+\Omega_3+\Omega_4+\Omega_5+\Omega_6+\Omega_7. \tag{23}$$

Where

$$\Omega_1 = [(\mathfrak{I}_1\mathfrak{I}_2-\mathfrak{I}_1-\mathfrak{I}_2)!+(\mathfrak{I}_2-2)!+(\mathfrak{I}_1-2)!+(g^3-g^2-1)! + ((g^3-g^2)(\mathfrak{I}_2-1)-1)!+((g^3-g^2)(\mathfrak{I}_1-1)-1)! + (g^2-g-1)!+[(g^2-g)(\mathfrak{I}_1\mathfrak{I}_2-\mathfrak{I}_1-\mathfrak{I}_2+1)-1]! + [(g_1{}^2\mathfrak{I}_2-g_1{}^2-g\mathfrak{I}_2+g)-1]!+[(g^2-g)(\mathfrak{I}_1-1)-1]! + (g-2)!+[(g-1)(\mathfrak{I}_1\mathfrak{I}_2-\mathfrak{I}_1-\mathfrak{I}_2+1)-1]!+(g\mathfrak{I}_1-g-\mathfrak{I}_1)! + (g\mathfrak{I}_2-g-\mathfrak{I}_2)!], \tag{24}$$

$$\Omega_2 = (\mathfrak{I}_1\mathfrak{I}_2-\mathfrak{I}_1-\mathfrak{I}_2+1)\{(\mathfrak{I}_1\mathfrak{I}_2-1)(g^2-1)+\mathfrak{I}_1+\mathfrak{I}_2\}, \tag{25}$$

$$\Omega_3 = (\mathfrak{I}_2-1)^2\{(g^3-g^2)+(g^2-g)(\mathfrak{I}_1-1)+(g^2-g) + (g\mathfrak{I}_1-g-\mathfrak{I}_1+1)+(g-1)\}+(\mathfrak{I}_1-1)^2\{(g^3-g^2) + (g^2-g)(\mathfrak{I}_1-1)+(g^2-g)+(g\mathfrak{I}_1-g-\mathfrak{I}_1+1)+(g-1)\}, \tag{26}$$

$$\Omega_4 = (g^3-g^2)\{(\mathfrak{I}_1+\mathfrak{I}_2-2)(g^3-g^2)+\mathfrak{I}_1\mathfrak{I}_2(g^2-g) + g(-1+\mathfrak{I}_1+\mathfrak{I}_2)+1-\mathfrak{I}_1-\mathfrak{I}_2\}, \tag{27}$$

$$\Omega_5 = (g^3-g^2)(\mathfrak{I}_2-1)\{(g^2-g)+(g^2-g)(1+\mathfrak{I}_1\mathfrak{I}_2-\mathfrak{I}_1-\mathfrak{I}_2) + (g^2-g)(\mathfrak{I}_2-1)+(g-1)+(g-1)(\mathfrak{I}_2-1)\} + (g^3-g^2)(\mathfrak{I}_1-1)\{(g^2-g)+(g^2-g)(\mathfrak{I}_1-1) + (g^2-g)(\mathfrak{I}_2-1)+(g-1)+(g-1)(\mathfrak{I}_1-1)\}, \tag{28}$$

$$\Omega_6 = (g^2-g)^2(\mathfrak{I}_1\mathfrak{I}_2-1)+(g^2-g)(\mathfrak{I}_1\mathfrak{I}_2-\mathfrak{I}_1-\mathfrak{I}_2+1) \cdot\{(\mathfrak{I}_1+\mathfrak{I}_2)(g^2-g)+(g-1)(1+\mathfrak{I}_1\mathfrak{I}_2)-2g^2\} + (g^2\mathfrak{I}_2-g^2-g\mathfrak{I}_2+g) \cdot\{g^2\mathfrak{I}_1-g^2-2g\mathfrak{I}_1+g\mathfrak{I}_1\mathfrak{I}_2+g-\mathfrak{I}_1\mathfrak{I}_2+\mathfrak{I}_1\}, \tag{29}$$

$$\Omega_7 = (g^2\mathfrak{I}_1-g^2-g\mathfrak{I}_1+g)\{g\mathfrak{I}_1\mathfrak{I}_2-\mathfrak{I}_1\mathfrak{I}_2-g+1\} + (g\mathfrak{I}_1-g-\mathfrak{I}_1+1)(\mathfrak{I}_2-1)(\mathfrak{I}_1+\mathfrak{I}_2-2\}. \tag{30}$$

Proof. From Figure 1. We conclude that W_1 is at distance 2 from $W_1, W_2, W_3, W_8, W_9, W_{10}, W_{12}, W_{13}$ and W_{14}. Hence by using the values from table 1, we have:

$$DP_2(W_1) = (\mathfrak{I}_1\mathfrak{I}_2-\mathfrak{I}_1-\mathfrak{I}_2)! \tag{31}$$

$$DP_2(W_1,W_2) = (\mathfrak{I}_1-1)(\mathfrak{I}_2-1)^2 \tag{32}$$

$$DP_2(W_1,W_3) = (\mathfrak{I}_1-1)^2(\mathfrak{I}_2-1) \tag{33}$$

$$DP_2(W_1,W_8) = (g^2-g)(\mathfrak{I}_1-1)^2(\mathfrak{I}_2-1)^2 \tag{34}$$

$$DP_2(W_1,W_9) = (g^2-g)(\mathfrak{I}_1-1)(\mathfrak{I}_2-1)^2 \tag{35}$$

$$DP_2(W_1,W_{10}) = (g^2-g)(\mathfrak{I}_1-1)^2(\mathfrak{I}_1-1) \tag{36}$$

$$DP_2(W_1,W_{12}) = (g-1)(\mathfrak{I}_1-1)^2(\mathfrak{I}_2-1)^2 \tag{37}$$

$$DP_2(W_1,W_{13}) = (g-1)(\mathfrak{I}_1-1)(\mathfrak{I}_2-1)^2 \tag{38}$$

$$DP_2(W_1,W_{14}) = (g-1)(\mathfrak{I}_1-1)^2(\mathfrak{I}_2-1) \tag{39}$$

From Figure 1, we conclude that W_2 is at distance 2 from $W_2, W_5, W_8, W_9, W_{12}$ and W_{13}. Hence by using the values from table 1, we have:

$$DP_2(W_2) = (\mathfrak{I}_2-2)! \tag{40}$$

$$DP_2(W_2,W_5) = (g^3-g^2)(\mathfrak{I}_2-1)^2 \tag{41}$$

$$DP_2(W_2,W_8) = (g^2-g)(\mathfrak{I}_1-1)(\mathfrak{I}_2-1)^2 \tag{42}$$

$$DP_2(W_2,W_9) = (g^2-g)(\mathfrak{I}_2-1)^2 \tag{43}$$

$$DP_2(W_2,W_{12}) = (g-1)(\mathfrak{I}_1-1)(\mathfrak{I}_2-1)^2 \tag{44}$$

$$DP_2(W_2, W_{13}) = (\boldsymbol{g} - 1)(\Im_2 - 1)^2 \tag{45}$$

From Figure 1, we conclude that W_3 is at distance 2 from $W_3, W_6, W_8, W_{10}, W_{12}$ and W_{14}. Hence by using the values from table 1, we have:

$$DP_2(W_3) = (\Im_1 - 2)! \tag{46}$$

$$DP_2(W_3, W_6) = \left(\boldsymbol{g}^3 - \boldsymbol{g}^2\right)(\Im_1 - 1)^2 \tag{47}$$

$$DP_2(W_3, W_8) = \left(\boldsymbol{g}^2 - \boldsymbol{g}\right)(\Im_1 - 1)^2(\Im_2 - 1) \tag{48}$$

$$DP_2(W_3, W_{10}) = \left(\boldsymbol{g}^2 - \boldsymbol{g}\right)(\Im_1 - 1)^2 \tag{49}$$

$$DP_2(W_3, W_{12}) = (\boldsymbol{g} - 1)(\Im_1 - 1)^2(\Im_2 - 1) \tag{50}$$

$$DP_2(W_3, W_{14}) = (\boldsymbol{g} - 1)(\Im_1 - 1)^2 \tag{51}$$

From Figure 1. We conclude that W_4 is at distance 2 from $W_4, W_5, W_6, W_7, W_8, W_9, W_{10}, W_{11}, W_{13}$ and W_{14}. Hence by using the values from table 1, we have:

$$DP_2(W_4) = \left(\boldsymbol{g}^3 - \boldsymbol{g}^2 - 1\right)! \tag{52}$$

$$DP_2(W_4, W_5) = \left(\boldsymbol{g}^3 - \boldsymbol{g}^2\right)^2(\Im_2 - 1) \tag{53}$$

$$DP_2(W_4, W_6) = \left(\boldsymbol{g}^3 - \boldsymbol{g}^2\right)^2(\Im_1 - 1) \tag{54}$$

$$DP_2(W_4, W_7) = \left(\boldsymbol{g}^2 - \boldsymbol{g}\right)\left(\boldsymbol{g}^3 - \boldsymbol{g}^2\right) \tag{55}$$

$$DP_2(W_4, W_8) = \left(\boldsymbol{g}^3 - \boldsymbol{g}^2\right)\left(\boldsymbol{g}^2 - \boldsymbol{g}\right)(\Im_1 - 1)(\Im_2 - 1) \tag{56}$$

$$DP_2(W_4, W_9) = \left(\boldsymbol{g}^3 - \boldsymbol{g}^2\right)\left(\boldsymbol{g}^2 - \boldsymbol{g}\right)(\Im_2 - 1) \tag{57}$$

$$DP_2(W_4, W_{10}) = \left(\boldsymbol{g}^3 - \boldsymbol{g}^2\right)\left(\boldsymbol{g}^2 - \boldsymbol{g}\right)(\Im_1 - 1) \tag{58}$$

$$DP_2(W_4, W_{11}) = \left(\boldsymbol{g}^3 - \boldsymbol{g}^2\right)(\boldsymbol{g} - 1) \tag{59}$$

$$DP_2(W_4, W_{13}) = \left(\boldsymbol{g}^3 - \boldsymbol{g}^2\right)(\boldsymbol{g} - 1)(\Im_2 - 1) \tag{60}$$

$$DP_2(W_4, W_{14}) = \left(\boldsymbol{g}^3 - \boldsymbol{g}^2\right)(\boldsymbol{g} - 1)(\Im_1 - 1) \tag{61}$$

From Figure 1, we conclude that $W_5$is at distance 2 from $W_5, W_7, W_8, W_9, W_{11}$ and W_{13}. Hence by using the values from table 1, we have:

$$DP_2(W_5) = \left(\left(\boldsymbol{g}^3 - \boldsymbol{g}^2\right)(\Im_2 - 1) - 1\right)! \tag{62}$$

$$DP_2(W_5, W_7) = \left(\boldsymbol{g}^3 - \boldsymbol{g}^2\right)\left(\boldsymbol{g}^2 - \boldsymbol{g}\right)(\Im_2 - 1) \tag{63}$$

$$DP_2(W_5, W_8) = \left(\boldsymbol{g}^3 - \boldsymbol{g}^2\right)\left(\boldsymbol{g}^2 - \boldsymbol{g}\right)(\Im_1 - 1)(\Im_2 - 1)^2 \tag{64}$$

$$DP_2(W_5, W_9) = \left(\boldsymbol{g}^3 - \boldsymbol{g}^2\right)\left(\boldsymbol{g}^2 - \boldsymbol{g}\right)(\Im_2 - 1)^2 \tag{65}$$

$$DP_2(W_5, W_{11}) = \left(\boldsymbol{g}^3 - \boldsymbol{g}^2\right)(\boldsymbol{g} - 1)(\Im_2 - 1) \tag{66}$$

$$DP_2(W_5, W_{13}) = \left(\boldsymbol{g}^3 - \boldsymbol{g}^2\right)(\boldsymbol{g} - 1)(\Im_2 - 1)^2 \tag{67}$$

From Figure 1, we conclude that W_6 is at distance 2 from W_6, W_7, W_{10}, W_{11} and W_{14}. Hence by using the values from table 1, we have:

$$DP_2(W_5) = \left(\left(\boldsymbol{g}^3 - \boldsymbol{g}^2\right)(\Im_1 - 1) - 1\right)! \tag{68}$$

$$DP_2(W_5, W_7) = \left(\boldsymbol{g}^3 - \boldsymbol{g}^2\right)\left(\boldsymbol{g}^2 - \boldsymbol{g}\right)(\Im_1 - 1) \tag{69}$$

$$DP_2(W_6, W_{10}) = \left(\boldsymbol{g}^3 - \boldsymbol{g}^2\right)\left(\boldsymbol{g}^2 - \boldsymbol{g}\right)(\Im_1 - 1)^2 \tag{70}$$

$$DP_2(W_6, W_{11}) = \left(\boldsymbol{g}^3 - \boldsymbol{g}^2\right)(\boldsymbol{g} - 1)(\Im_1 - 1) \tag{71}$$

$$DP_2(W_6, W_{14}) = \left(\boldsymbol{g}^3 - \boldsymbol{g}^2\right)(\boldsymbol{g} - 1)(\Im_1 - 1)^2 \tag{72}$$

From Figure 1, we conclude that W_7 is at distance 2 from W_7, W_8, W_9 and W_{10}. Hence by using the values from table 1, we have:

$$DP_2(W_7) = \left(\boldsymbol{g}^2 - \boldsymbol{g} - 1\right)! \tag{73}$$

$$DP_2(W_7, W_8) = \left(\boldsymbol{g}^2 - \boldsymbol{g}\right)^2(\Im_1 - 1)(\Im_2 - 1) \tag{74}$$

$$DP_2(W_7, W_9) = \left(\boldsymbol{g}^2 - \boldsymbol{g}\right)^2(\Im_2 - 1) \tag{75}$$

$$DP_2(W_7, W_{10}) = \left(\boldsymbol{g}^2 - \boldsymbol{g}\right)^2(\Im_1 - 1) \tag{76}$$

From Figure 1, we conclude that W_8 is at distance 2 from $W_8, W_9, W_{10}, W_{12}, W_{13}$ and W_{14}. Hence by using the values from table 1, we have:

$$DP_2(W_8) = \left[\left(\boldsymbol{g}^2 - \boldsymbol{g}\right)(\Im_1 - 1)(\Im_2 - 1) - 1\right]! \tag{77}$$

$$DP_2(W_8, W_9) = \left(\boldsymbol{g}^2 - \boldsymbol{g}\right)^2(\Im_1 - 1)(\Im_2 - 1)^2 \tag{78}$$

$$DP_2(W_8, W_{10}) = \left(\boldsymbol{g}^2 - \boldsymbol{g}\right)^2(\Im_1 - 1)^2(\Im_2 - 1) \tag{79}$$

$$DP_2(W_8, W_{12}) = \left(\boldsymbol{g}^2 - \boldsymbol{g}\right)^2(\boldsymbol{g} - 1)(\Im_1 - 1)^2(\Im_2 - 1)^2 \tag{80}$$

$$DP_2(W_8, W_{13}) = \left(\boldsymbol{g}^2 - \boldsymbol{g}\right)^2(\boldsymbol{g} - 1)(\Im_1 - 1)(\Im_2 - 1)^2 \tag{81}$$

$$DP_2(W_8, W_{14}) = \left(\boldsymbol{g}^2 - \boldsymbol{g}\right)^2(\boldsymbol{g} - 1)(\Im_1 - 1)^2(\Im_2 - 1) \tag{82}$$

From Figure 1, we conclude that W_9 is at distance 2 from W_9, W_{10}, W_{12}, and W_{13}. Hence by using the values from table 1, we have:

$$DP_2(W_9) = \left[\left(\boldsymbol{g}^2 - \boldsymbol{g}\right)(\Im_2 - 1) - 1\right]! \tag{83}$$

$$DP_2(W_9, W_{10}) = \left(\boldsymbol{g}^2 - \boldsymbol{g}\right)^2(\Im_1 - 1)(\Im_2 - 1) \tag{84}$$

$$DP_2(W_9, W_{12}) = \left(\boldsymbol{g}^2 - \boldsymbol{g}\right)(\Im_2 - 1)^2(\boldsymbol{g} - 1)(\Im_1 - 1) \tag{85}$$

$$DP_2(W_9, W_{13}) = \left(\boldsymbol{g}^2 - \boldsymbol{g}\right)(\Im_2 - 1)^2(\boldsymbol{g} - 1) \tag{86}$$

From Figure 1, we conclude that W_{10} is at distance 2 from W_{10}, W_{12}, W_{13}, and W_{14}. Hence by using the values from table 1, we have:

$$DP_2(W_{10}) = \left[\left(\boldsymbol{g}^2 - \boldsymbol{g}\right)(\mathfrak{I}_1 - 1) - 1\right]! \tag{87}$$

$$DP_2(W_{10}, W_{12}) = \left(\boldsymbol{g}^2 - \boldsymbol{g}\right)(\mathfrak{I}_1 - 1)^2(\mathfrak{I}_2 - 1)(\boldsymbol{g} - 1) \tag{88}$$

$$DP_2(W_{10}, W_{13}) = \left(\boldsymbol{g}^2 - \boldsymbol{g}\right)(\mathfrak{I}_1 - 1)(\mathfrak{I}_2 - 1)(\boldsymbol{g} - 1) \tag{89}$$

$$DP_2(W_{10}, W_{14}) = \left(\boldsymbol{g}^2 - \boldsymbol{g}\right)(\mathfrak{I}_1 - 1)^2(\boldsymbol{g} - 1) \tag{90}$$

From Figure 1, we conclude that W_{11} is at distance 2 from W_{11}. Hence by using the values from table 1, we have:

$$DP_2(W_{11}) = (\boldsymbol{g} - 2)! \tag{91}$$

From Figure 1, we conclude that W_{12} is at distance 2 from W_{12}, W_{13}, and W_{14}. Hence by using the values from table 1, we have:

$$DP_2(W_{12}) = [(\boldsymbol{g} - 1)(\mathfrak{I}_1 - 1)(\mathfrak{I}_2 - 1) - 1]! \tag{92}$$

$$DP_2(W_{12}, W_{13}) = (\boldsymbol{g} - 1)^2(\mathfrak{I}_1 - 1)(\mathfrak{I}_2 - 1)^2 \tag{93}$$

$$DP_2(W_{12}, W_{14}) = (\boldsymbol{g} - 1)(\mathfrak{I}_1 - 1)^2(\mathfrak{I}_2 - 1) \tag{94}$$

From Figure 1, we conclude that W_{13} is at distance 2 from W_{13}. Hence by using the values from table 1, we have:

$$DP_2(W_{13}) = [(\boldsymbol{g} - 1)(\mathfrak{I}_2 - 1) - 1]! \tag{95}$$

From Figure 1, we conclude that W_{14} is at distance 2 from W_{14}. Hence by using the values from table 1, we have:

$$DP_2(W_{14}) = [(\boldsymbol{g} - 1)(\mathfrak{I}_2 - 1) - 1]! \tag{96}$$

Finally, we add all the values computed in equations (31)–(96), and we get the required formula for $DP_2Y(\mathbb{Z}_{\boldsymbol{g}^3} \times \mathbb{Z}_{\mathfrak{I}_1\mathfrak{I}_2})$:

$$\begin{aligned}
DP_2Y\left(\mathbb{Z}_{\boldsymbol{g}^3} \times \mathbb{Z}_{\mathfrak{I}_1\mathfrak{I}_2}\right) = {} & [(\mathfrak{I}_1\mathfrak{I}_2 + \mathfrak{I}_1 + \mathfrak{I}_2)! + (\mathfrak{I}_2 - 2)! \\
& + (\mathfrak{I}_1 - 2)! + \left(\boldsymbol{g}^3 - \boldsymbol{g}^2 - 1\right)! + \left(\left(\boldsymbol{g}^3 - \boldsymbol{g}^2\right)(\mathfrak{I}_2 - 1) - 1\right)! \\
& + \left(\left(\boldsymbol{g}^3 - \boldsymbol{g}^2\right)(\mathfrak{I}_1 - 1) - 1\right)! + \left(\boldsymbol{g}^2 - \boldsymbol{g} - 1\right)! \\
& + \left[\left(\boldsymbol{g}^2 - \boldsymbol{g}\right)(\mathfrak{I}_1\mathfrak{I}_2 - \mathfrak{I}_1 - \mathfrak{I}_2 + 1) - 1\right]! \\
& + \left[\left({\boldsymbol{g}_1}^2\mathfrak{I}_2 - {\boldsymbol{g}_1}^2 - \boldsymbol{g}\mathfrak{I}_2 + \boldsymbol{g}\right) - 1\right]! + \left[\left(\boldsymbol{g}^2 - \boldsymbol{g}\right)(\mathfrak{I}_1 - 1) - 1\right]! \\
& + (\boldsymbol{g} - 2)! + [(\boldsymbol{g} - 1)(\mathfrak{I}_1\mathfrak{I}_2 - \mathfrak{I}_1 - \mathfrak{I}_2 + 1) - 1]! \\
& + (\boldsymbol{g}\mathfrak{I}_1 - \boldsymbol{g} - \mathfrak{I}_1)! + (\boldsymbol{g}\mathfrak{I}_2 - \boldsymbol{g} - \mathfrak{I}_2)!] \\
& + (\mathfrak{I}_1\mathfrak{I}_2 - \mathfrak{I}_1 - \mathfrak{I}_2 + 1)\left\{(\mathfrak{I}_1\mathfrak{I}_2 - 1)\left(\boldsymbol{g}^2 - 1\right) + \mathfrak{I}_1 + \mathfrak{I}_2\right\} \\
& + (\mathfrak{I}_2 - 1)^2\left\{\left(\boldsymbol{g}^3 - \boldsymbol{g}^2\right) + \left(\boldsymbol{g}^2 - \boldsymbol{g}\right)(\mathfrak{I}_1 - 1) + \left(\boldsymbol{g}^2 - \boldsymbol{g}\right)\right. \\
& \left. + (\boldsymbol{g}\mathfrak{I}_1 - \boldsymbol{g} - \mathfrak{I}_1 + 1) + (\boldsymbol{g} - 1)\right\} + (\mathfrak{I}_1 - 1)^2\left\{\left(\boldsymbol{g}^3 - \boldsymbol{g}^2\right)\right. \\
& \left. + \left(\boldsymbol{g}^2 - \boldsymbol{g}\right)(\mathfrak{I}_1 - 1) + \left(\boldsymbol{g}^2 - \boldsymbol{g}\right) + (\boldsymbol{g} - 1)(\mathfrak{I}_1 - 1) + (\boldsymbol{g} - 1)\right\} \\
& + \left(\boldsymbol{g}^3 - \boldsymbol{g}^2\right)\left\{(\mathfrak{I}_1 + \mathfrak{I}_2 - 2)\left(\boldsymbol{g}^3 - \boldsymbol{g}^2\right) + \mathfrak{I}_1\mathfrak{I}_2\left(\boldsymbol{g}^2 - \boldsymbol{g}\right)\right. \\
& \left. + \boldsymbol{g}(\mathfrak{I}_1 + \mathfrak{I}_2 - 1) - \mathfrak{I}_1 - \mathfrak{I}_2 + 1\right\} + \left(\boldsymbol{g}^3 - \boldsymbol{g}^2\right)(\mathfrak{I}_2 - 1)\left\{\left(\boldsymbol{g}^2 - \boldsymbol{g}\right)\right. \\
& + \left(\boldsymbol{g}^2 - \boldsymbol{g}\right)(\mathfrak{I}_1 - 1)(\mathfrak{I}_2 - 1) + \left({\boldsymbol{g}_1}^2\mathfrak{I}_2 - {\boldsymbol{g}_1}^2 - \boldsymbol{g}\mathfrak{I}_2 + \boldsymbol{g}\right) + (\boldsymbol{g} - 1) \\
& \left. + (\boldsymbol{g} - 1)(\mathfrak{I}_2 - 1)\right\} + \left(\boldsymbol{g}^3 - \boldsymbol{g}^2\right)(\mathfrak{I}_1 - 1)\left\{\left(\boldsymbol{g}^2 - \boldsymbol{g}\right)\right. \\
& + \left({\boldsymbol{g}_1}^2\mathfrak{I}_1 - {\boldsymbol{g}_1}^2 - \boldsymbol{g}\mathfrak{I}_1 + \boldsymbol{g}\right) + \left({\boldsymbol{g}_1}^2\mathfrak{I}_2 - {\boldsymbol{g}_1}^2 - \boldsymbol{g}\mathfrak{I}_2 + \boldsymbol{g}\right) \\
& \left. + (\boldsymbol{g} - 1) + (\boldsymbol{g} - 1)(\mathfrak{I}_1 - 1)\right\} + \left(\boldsymbol{g}^2 - \boldsymbol{g}\right)^2(\mathfrak{I}_1\mathfrak{I}_2 - 1) \\
& + \left(\boldsymbol{g}^2 - \boldsymbol{g}\right)(1 + \mathfrak{I}_1\mathfrak{I}_2 - \mathfrak{I}_1 - \mathfrak{I}_2)\left\{(\mathfrak{I}_1 + \mathfrak{I}_2)\left(\boldsymbol{g}^2 - \boldsymbol{g}\right)\right. \\
& \left. + (\boldsymbol{g} - 1)(1 + \mathfrak{I}_1\mathfrak{I}_2) - 2\boldsymbol{g}^2\right\} + \left(\boldsymbol{g}^2\mathfrak{I}_2 - \boldsymbol{g}^2 - \boldsymbol{g}\mathfrak{I}_2 + \boldsymbol{g}\right) \\
& \cdot \left\{\boldsymbol{g}^2\mathfrak{I}_1 - \boldsymbol{g}^2 - 2\boldsymbol{g}\mathfrak{I}_1 + \boldsymbol{g}\mathfrak{I}_1\mathfrak{I}_2 + \boldsymbol{g} - \mathfrak{I}_1\mathfrak{I}_2 + \mathfrak{I}_1\right\} \\
& + \left(\boldsymbol{g}^2\mathfrak{I}_1 - \boldsymbol{g}^2 - \boldsymbol{g}\mathfrak{I}_1 + \boldsymbol{g}\right)\left\{1 - \boldsymbol{g} + \boldsymbol{g}\mathfrak{I}_1\mathfrak{I}_2 - \mathfrak{I}_1\mathfrak{I}_2\right\} \\
& + (\boldsymbol{g} - 1)(\mathfrak{I}_1 - 1)(\mathfrak{I}_2 - 1)(\mathfrak{I}_1 + \mathfrak{I}_2 - 2\} = \Omega_1 + \Omega_2 \\
& + \Omega_3 + \Omega_4 + \Omega_5 + \Omega_6 + \Omega_7
\end{aligned} \tag{97}$$

□

Lemma 4. *The cardinality of the set of ordered pair of vertices in $Y(\mathbb{Z}_{\boldsymbol{g}^3} \times \mathbb{Z}_{\mathfrak{I}_1\mathfrak{I}_2})$ which are distance 3 to each other is*

$$\begin{aligned}
& DP_3\left(Y\left(\mathbb{Z}_{\boldsymbol{g}^3} \times \mathbb{Z}_{\mathfrak{I}_1\mathfrak{I}_2}\right)\right) \\
& \quad = \left(\boldsymbol{g}^3 - \boldsymbol{g}^2\right)(\mathfrak{I}_1 - 1)(\mathfrak{I}_2 - 1)\left[\boldsymbol{g}^3 + \mathfrak{I}_1\boldsymbol{g} + \mathfrak{I}_2\boldsymbol{g}^2 - 3\right]
\end{aligned} \tag{98}$$

Proof. By using the figure, we conclude that:

$$\begin{aligned}
DP_3\left(Y\left(\mathbb{Z}_{\boldsymbol{g}^3} \times \mathbb{Z}_{\mathfrak{I}_1\mathfrak{I}_2}\right)\right) = {} & |W_1|(|W_5| + |W_6|) + |W_4||W_{12}| \\
& + |W_5|(|W_6| + |W_{10}| + |W_{12}| + |W_{14}|) \\
& + |W_6|(|W_8| + |W_9| + |W_{12}| + |W_{13}|)
\end{aligned} \tag{99}$$

Now by putting values from table 1, we obtain:

$$\begin{aligned}
&DP_3\left(Y\left(\mathbb{Z}_{g^3}\times\mathbb{Z}_{\mathfrak{I}_1\mathfrak{I}_2}\right)\right)\\
&=(\mathfrak{I}_1\mathfrak{I}_2-\mathfrak{I}_1-\mathfrak{I}_2+1)\left(\left(g^3-g^2\right)(\mathfrak{I}_2-1)\right.\\
&\quad+\left(g^3-g^2\right)(\mathfrak{I}_2-1)\big)+\left(g^3-g^2\right)(g\mathfrak{I}_1-g-\mathfrak{I}_1+1)(\mathfrak{I}_2-1)\\
&\quad+\left(g^3-g^2\right)(\mathfrak{I}_1\mathfrak{I}_2-\mathfrak{I}_1-\mathfrak{I}_2+1)\left(\left(g^3-g^2\right)\right.\\
&\quad+\left(g^2-g\right)(\mathfrak{I}_1-1)+(g\mathfrak{I}_1-g-\mathfrak{I}_1+1)(\mathfrak{I}_2-1)\\
&\quad+(g\mathfrak{I}_1-g-\mathfrak{I}_1+1)\big)+\left(g^3-g^2\right)(\mathfrak{I}_1-1)\left(\left(g^2-g\right)\right.\\
&\quad\times(\mathfrak{I}_1\mathfrak{I}_2-\mathfrak{I}_1-\mathfrak{I}_2+1)+\left(g^2-g\right)(\mathfrak{I}_2-1)\\
&\quad+(g\mathfrak{I}_1-g-\mathfrak{I}_1+1)(\mathfrak{I}_2-1)+(g\mathfrak{I}_2-g-\mathfrak{I}_2+1)\big)\\
&=\left(g^3-g^2\right)(\mathfrak{I}_1-1)(\mathfrak{I}_2-1)\left[g^3+\mathfrak{I}_1g+\mathfrak{I}_2g^2-3\right]
\end{aligned} \tag{100}$$

□

Theorem 5. *The Hosoya polynomial $H(Y(\mathbb{Z}_{g^3}\times\mathbb{Z}_{\mathfrak{I}_1\mathfrak{I}_2}))$ of the graph $Y(\mathbb{Z}_{g^3}\times\mathbb{Z}_{\mathfrak{I}_1\mathfrak{I}_2})$ is:*

$$\begin{aligned}
&H\left(Y\left(\mathbb{Z}_{g^3}\times\mathbb{Z}_{\mathfrak{I}_1\mathfrak{I}_2}\right)\right)=\left(g^3-g^2\right)(\mathfrak{I}_1+\mathfrak{I}_2-1)+g^2\mathfrak{I}_1\mathfrak{I}_2-1\\
&\quad+\left(\frac{g^3+g^2+g+g^3(\mathfrak{I}_1+\mathfrak{I}_2)+(\mathfrak{I}_1\mathfrak{I}_2+\mathfrak{I}_1+\mathfrak{I}_2)\left(g^2+g+1\right)-14}{2}\right)\\
&\quad x+(\Omega_1+\Omega_2+\Omega_3+\Omega_4+\Omega_5+\Omega_6+\Omega_7)x^2\\
&\quad+\left(\left(g^3-g^2\right)(\mathfrak{I}_1-1)(\mathfrak{I}_2-1)\left[g^3+\mathfrak{I}_1g+\mathfrak{I}_2g^2-3\right]\right)x^3
\end{aligned} \tag{101}$$

Where

$$\begin{aligned}
\Omega_1&=\left[(\mathfrak{I}_1\mathfrak{I}_2+\mathfrak{I}_1+\mathfrak{I}_2)!+(\mathfrak{I}_2-2)!+(\mathfrak{I}_1-2)!+\left(g^3-g^2-1\right)!\right.\\
&\quad+\left(\left(g^3-g^2\right)(\mathfrak{I}_2-1)-1\right)!+\left(\left(g^3-g^2\right)(\mathfrak{I}_1-1)-1\right)!\\
&\quad+\left(g^2-g-1\right)!+\left[\left(g^2-g\right)(\mathfrak{I}_1\mathfrak{I}_2-\mathfrak{I}_1-\mathfrak{I}_2+1)-1\right]!\\
&\quad+\left[\left(g_1{}^2\mathfrak{I}_2-g_1{}^2-g\mathfrak{I}_2+g\right)-1\right]!+\left[\left(g^2-g\right)(\mathfrak{I}_1-1)-1\right]!\\
&\quad+(g-2)!+[(g-1)(\mathfrak{I}_1\mathfrak{I}_2-\mathfrak{I}_1-\mathfrak{I}_2+1)-1]!\\
&\quad+(g\mathfrak{I}_1-g-\mathfrak{I}_1)!+(g\mathfrak{I}_2-g-\mathfrak{I}_2)!],
\end{aligned} \tag{102}$$

$$\Omega_2=(\mathfrak{I}_1\mathfrak{I}_2-\mathfrak{I}_1-\mathfrak{I}_2+1)\left\{(\mathfrak{I}_1\mathfrak{I}_2-1)\left(g^2-1\right)+\mathfrak{I}_1+\mathfrak{I}_2\right\}, \tag{103}$$

$$\begin{aligned}
\Omega_3&=(\mathfrak{I}_2-1)^2\left\{\left(g^3-g^2\right)+\left(g^2-g\right)(\mathfrak{I}_1-1)+\left(g^2-g\right)\right.\\
&\quad+(g\mathfrak{I}_1-g-\mathfrak{I}_1+1)+(g-1)\}+(\mathfrak{I}_1-1)^2\left\{\left(g^3-g^2\right)\right.\\
&\quad+\left(g^2-g\right)(\mathfrak{I}_1-1)+\left(g^2-g\right)+(g\mathfrak{I}_1-g-\mathfrak{I}_1+1)+(g-1)\},
\end{aligned} \tag{104}$$

$$\begin{aligned}
\Omega_4&=\left(g^3-g^2\right)\left\{(\mathfrak{I}_1+\mathfrak{I}_2-2)\left(g^3-g^2\right)+\mathfrak{I}_1\mathfrak{I}_2\left(g^2-g\right)\right.\\
&\quad+g(-1+\mathfrak{I}_1+\mathfrak{I}_2)+1-\mathfrak{I}_1-\mathfrak{I}_2\},
\end{aligned} \tag{105}$$

$$\begin{aligned}
\Omega_5&=\left(g^3-g^2\right)(\mathfrak{I}_2-1)\left\{\left(g^2-g\right)+\left(g^2-g\right)(1+\mathfrak{I}_1\mathfrak{I}_2-\mathfrak{I}_1-\mathfrak{I}_2)\right.\\
&\quad+\left(g^2-g\right)(\mathfrak{I}_2-1)+(g-1)+(g-1)(\mathfrak{I}_2-1)\}\\
&\quad+\left(g^3-g^2\right)(\mathfrak{I}_1-1)\left\{\left(g^2-g\right)+\left(g^2-g\right)(\mathfrak{I}_1-1)\right.\\
&\quad+\left(g^2-g\right)(\mathfrak{I}_2-1)+(g-1)+(g-1)(\mathfrak{I}_1-1)\},
\end{aligned} \tag{106}$$

$$\begin{aligned}
\Omega_6&=\left(g^2-g\right)^2(\mathfrak{I}_1\mathfrak{I}_2-1)+\left(g^2-g\right)(\mathfrak{I}_1\mathfrak{I}_2-\mathfrak{I}_1-\mathfrak{I}_2+1)\\
&\quad\cdot\left\{(\mathfrak{I}_1+\mathfrak{I}_2)\left(g^2-g\right)+(g-1)(1+\mathfrak{I}_1\mathfrak{I}_2)-2g^2\right\}\\
&\quad+\left(g^2\mathfrak{I}_2-g^2-g\mathfrak{I}_2+g\right)\left\{g^2\mathfrak{I}_1-g^2-2g\mathfrak{I}_1\right.\\
&\quad+g\mathfrak{I}_1\mathfrak{I}_2+g-\mathfrak{I}_1\mathfrak{I}_2+\mathfrak{I}_1\},
\end{aligned} \tag{107}$$

$$\begin{aligned}
\Omega_7&=\left(g^2\mathfrak{I}_1-g^2-g\mathfrak{I}_1+g\right)\{g\mathfrak{I}_1\mathfrak{I}_2-\mathfrak{I}_1\mathfrak{I}_2-g+1\}\\
&\quad+(g\mathfrak{I}_1-g-\mathfrak{I}_1+1)(\mathfrak{I}_2-1)(\mathfrak{I}_1+\mathfrak{I}_2-2\}.
\end{aligned} \tag{108}$$

Proof. The Hosoya polynomial of zero divisor graph $Y(\mathbb{Z}_{g^3}\times\mathbb{Z}_{\mathfrak{I}_1\mathfrak{I}_2})$ can be calculated by using lemma 2, lemma 3 and lemma 4 as follows:

$$\begin{aligned}
&H\left(Y\left(\mathbb{Z}_{g^3}\times\mathbb{Z}_{\mathfrak{I}_1\mathfrak{I}_2}\right),x\right)=DP_0\left(Y\left(\mathbb{Z}_{g^3}\times\mathbb{Z}_{\mathfrak{I}_1\mathfrak{I}_2}\right)\right)\\
&\quad+DP_1\left(Y\left(\mathbb{Z}_{g^3}\times\mathbb{Z}_{\mathfrak{I}_1\mathfrak{I}_2}\right)\right)x+DP_2\left(Y\left(\mathbb{Z}_{g^3}\times\mathbb{Z}_{\mathfrak{I}_1\mathfrak{I}_2}\right)\right)x^2\\
&\quad+DP_3\left(Y\left(\mathbb{Z}_{g^3}\times\mathbb{Z}_{\mathfrak{I}_1\mathfrak{I}_2}\right)\right)x^3=\left(g^3-g^2\right)(\mathfrak{I}_1+\mathfrak{I}_2-1)+g^2\mathfrak{I}_1\mathfrak{I}_2-1\\
&\quad+\left(\frac{g^3+g^2+g+g^3(\mathfrak{I}_1+\mathfrak{I}_2)+(\mathfrak{I}_1\mathfrak{I}_2+\mathfrak{I}_1+\mathfrak{I}_2)\left(g^2+g+1\right)-14}{2}\right)\\
&\quad x+(\Omega_1+\Omega_2+\Omega_3+\Omega_4+\Omega_5+\Omega_6+\Omega_7)x^2\\
&\quad+\left(\left(g^3-g^2\right)(\mathfrak{I}_1-1)(\mathfrak{I}_2-1)\left[g^3+\mathfrak{I}_1g+\mathfrak{I}_2g^2-3\right]\right)x^3
\end{aligned} \tag{109}$$

□

Theorem 6. *The Harary polynomial $h(Y(\mathbb{Z}_{g^3}\times\mathbb{Z}_{\mathfrak{I}_1\mathfrak{I}_2}),x)$ of the graph $Y(\mathbb{Z}_{g^3}\times\mathbb{Z}_{\mathfrak{I}_1\mathfrak{I}_2})$ is*

$$\begin{aligned}
&h\left(Y\left(\mathbb{Z}_{g^3}\times\mathbb{Z}_{\mathfrak{I}_1\mathfrak{I}_2}\right),x\right)\\
&\quad=3\left(g^3+g^2+g+g^3(\mathfrak{I}_1+\mathfrak{I}_2)+(\mathfrak{I}_1\mathfrak{I}_2+\mathfrak{I}_1+\mathfrak{I}_2)\right.\\
&\quad\quad\times\left(g^2+g+1\right)-14\big)x+3(\Omega_1+\Omega_2+\Omega_3+\Omega_4+\Omega_5\\
&\quad\quad+\Omega_6+\Omega_7)x^2+2\left(\left(g^3-g^2\right)(\mathfrak{I}_1-1)\right.\\
&\quad\quad\times(\mathfrak{I}_2-1)\left[g^3+\mathfrak{I}_1g+\mathfrak{I}_2g^2-3\right]\big)x^3
\end{aligned} \tag{110}$$

Where

$$\begin{aligned}
\Omega_1&=[(\mathfrak{I}_1\mathfrak{I}_2+\mathfrak{I}_1+\mathfrak{I}_2)!+(\mathfrak{I}_2-2)!+(\mathfrak{I}_1-2)!\\
&\quad+\left(g^3-g^2-1\right)!+\left(\left(g^3-g^2\right)(\mathfrak{I}_2-1)-1\right)!\\
&\quad+\left(\left(g^3-g^2\right)(\mathfrak{I}_1-1)-1\right)!+\left(g^2-g-1\right)!\\
&\quad+\left[\left(g^2-g\right)(\mathfrak{I}_1\mathfrak{I}_2-\mathfrak{I}_1-\mathfrak{I}_2+1)-1\right]!\\
&\quad+\left[\left(g_1{}^2\mathfrak{I}_2-g_1{}^2-g\mathfrak{I}_2+g\right)-1\right]!+\left[\left(g^2-g\right)(\mathfrak{I}_1-1)-1\right]!\\
&\quad+(g-2)!+[(g-1)(\mathfrak{I}_1\mathfrak{I}_2-\mathfrak{I}_1-\mathfrak{I}_2+1)-1]!\\
&\quad+(g\mathfrak{I}_1-g-\mathfrak{I}_1)!+(g\mathfrak{I}_2-g-\mathfrak{I}_2)!],
\end{aligned} \tag{111}$$

$$\Omega_2=(\mathfrak{I}_1\mathfrak{I}_2-\mathfrak{I}_1-\mathfrak{I}_2+1)\left\{(\mathfrak{I}_1\mathfrak{I}_2-1)\left(g^2-1\right)+\mathfrak{I}_1+\mathfrak{I}_2\right\}, \tag{112}$$

$$\Omega_3 = (\mathfrak{I}_2-1)^2\{(g^3-g^2)+(g^2-g)(\mathfrak{I}_1-1)+(g^2-g) + (g\mathfrak{I}_1-g-\mathfrak{I}_1+1)+(g-1)\} + (\mathfrak{I}_1-1)^2\{(g^3-g^2)+(g^2-g)(\mathfrak{I}_1-1)+(g^2-g) + (g\mathfrak{I}_1-g-\mathfrak{I}_1+1)+(g-1)\}, \quad (113)$$

$$\Omega_4 = (g^3-g^2)\{(\mathfrak{I}_1+\mathfrak{I}_2-2)(g^3-g^2)+\mathfrak{I}_1\mathfrak{I}_2(g^2-g) + g(-1+\mathfrak{I}_1+\mathfrak{I}_2)+1-\mathfrak{I}_1-\mathfrak{I}_2\}, \quad (114)$$

$$\Omega_5 = (g^3-g^2)(\mathfrak{I}_2-1)\{(g^2-g)+(g^2-g)(1+\mathfrak{I}_1\mathfrak{I}_2-\mathfrak{I}_1-\mathfrak{I}_2) + (g^2-g)(\mathfrak{I}_2-1)+(g-1)+(g-1)(\mathfrak{I}_2-1)\} + (g^3-g^2)(\mathfrak{I}_1-1)\{(g^2-g)+(g^2-g)(\mathfrak{I}_1-1) + (g^2-g)(\mathfrak{I}_2-1)+(g-1)+(g-1)(\mathfrak{I}_1-1)\}, \quad (115)$$

$$\Omega_6 = (g^2-g)^2(\mathfrak{I}_1\mathfrak{I}_2-1)+(g^2-g)(\mathfrak{I}_1\mathfrak{I}_2-\mathfrak{I}_1-\mathfrak{I}_2+1) \cdot \{(\mathfrak{I}_1+\mathfrak{I}_2)(g^2-g)+(g-1)(1+\mathfrak{I}_1\mathfrak{I}_2)-2g^2\} + (g^2\mathfrak{I}_2-g^2-g\mathfrak{I}_2+g)\{g^2\mathfrak{I}_1-g^2-2g\mathfrak{I}_1 + g\mathfrak{I}_1\mathfrak{I}_2+g-\mathfrak{I}_1\mathfrak{I}_2+\mathfrak{I}_1\}, \quad (116)$$

$$\Omega_7 = (g^2\mathfrak{I}_1-g^2-g\mathfrak{I}_1+g)\{g\mathfrak{I}_1\mathfrak{I}_2-\mathfrak{I}_1\mathfrak{I}_2-g+1\} + (g\mathfrak{I}_1-g-\mathfrak{I}_1+1)(\mathfrak{I}_2-1)(\mathfrak{I}_1+\mathfrak{I}_2-2\}. \quad (117)$$

Proof. The Harary polynomial of $Y(\mathbb{Z}_{g^3}\times\mathbb{Z}_{\mathfrak{I}_1\mathfrak{I}_2})$ can be calculated by using lemma 2, lemma 3 and lemma 4 as follows:

$$h(Y(\mathbb{Z}_{g^3}\times\mathbb{Z}_{\mathfrak{I}_1\mathfrak{I}_2}),x) = 6DP_1(Y(\mathbb{Z}_{g^3}\times\mathbb{Z}_{\mathfrak{I}_1\mathfrak{I}_2}))x + 3DP_2(Y(\mathbb{Z}_{g^3}\times\mathbb{Z}_{\mathfrak{I}_1\mathfrak{I}_2}))x^2 + 2DP_3(Y(\mathbb{Z}_{g^3}\times\mathbb{Z}_{\mathfrak{I}_1\mathfrak{I}_2}))x^3 = 3(g^3+g^2+g+g^3(\mathfrak{I}_1+\mathfrak{I}_2)+(\mathfrak{I}_1\mathfrak{I}_2+\mathfrak{I}_1+\mathfrak{I}_2) \cdot (g^2+g+1)-14)x + 3(\Omega_1+\Omega_2+\Omega_3+\Omega_4+\Omega_5+\Omega_6+\Omega_7)x^2 + 2((g^3-g^2)(\mathfrak{I}_1-1)(\mathfrak{I}_2-1)[g^3+\mathfrak{I}_1 g+\mathfrak{I}_2 g^2-3])x^3 \quad (118)$$

□

Corollary 7. *The Wiener index and Hyper-Wiener index for zero divisor graph $Y(\mathbb{Z}_{g^3}\times\mathbb{Z}_{\mathfrak{I}_1\mathfrak{I}_2})$ can be expressed as:*

$$W(Y(\mathbb{Z}_{g^3}\times\mathbb{Z}_{\mathfrak{I}_1\mathfrak{I}_2})) = \frac{g^3+g^2+g+g^3(\mathfrak{I}_1+\mathfrak{I}_2)+(\mathfrak{I}_1\mathfrak{I}_2+\mathfrak{I}_1+\mathfrak{I}_2)(g^2+g+1)-14}{2} + (\Omega_1+\Omega_2+\Omega_3+\Omega_4+\Omega_5+\Omega_6+\Omega_7) + 3((g^3-g^2)(\mathfrak{I}_1-1)(\mathfrak{I}_2-1)[g^3+\mathfrak{I}_1 g+\mathfrak{I}_2 g^2-3]) \quad (119)$$

$$WW(Y(\mathbb{Z}_{g^3}\times\mathbb{Z}_{\mathfrak{I}_1\mathfrak{I}_2})) = (g^3+g^2+g+g^3(\mathfrak{I}_1+\mathfrak{I}_2)+(\mathfrak{I}_1\mathfrak{I}_2+\mathfrak{I}_1+\mathfrak{I}_2)(g^2+g+1)-14) + 3(\Omega_1+\Omega_2+\Omega_3+\Omega_4+\Omega_5+\Omega_6+\Omega_7) + 6((g^3-g^2)(\mathfrak{I}_1-1)(\mathfrak{I}_2-1)[g^3+\mathfrak{I}_1 g+\mathfrak{I}_2 g^2-3]) \quad (120)$$

Where

$$\Omega_1 = [(\mathfrak{I}_1\mathfrak{I}_2+\mathfrak{I}_1+\mathfrak{I}_2)!+(\mathfrak{I}_2-2)!+(\mathfrak{I}_1-2)! + (g^3-g^2-1)!+((g^3-g^2)(\mathfrak{I}_2-1)-1)! + ((g^3-g^2)(\mathfrak{I}_1-1)-1)!+(g^2-g-1)! + [(g^2-g)(\mathfrak{I}_1\mathfrak{I}_2-\mathfrak{I}_1-\mathfrak{I}_2+1)-1]! + [(g_1{}^2\mathfrak{I}_2-g_1{}^2-g\mathfrak{I}_2+g)-1]!+[(g^2-g)(\mathfrak{I}_1-1)-1]! + (g-2)!+[(g-1)(\mathfrak{I}_1\mathfrak{I}_2-\mathfrak{I}_1-\mathfrak{I}_2+1)-1]! + (g\mathfrak{I}_1-g-\mathfrak{I}_1)!+(g\mathfrak{I}_2-g-\mathfrak{I}_2)!], \quad (121)$$

$$\Omega_2 = (\mathfrak{I}_1\mathfrak{I}_2-\mathfrak{I}_1-\mathfrak{I}_2+1)\{(\mathfrak{I}_1\mathfrak{I}_2-1)(g^2-1)+\mathfrak{I}_1+\mathfrak{I}_2\}, \quad (122)$$

$$\Omega_3 = (\mathfrak{I}_2-1)^2\{(g^3-g^2)+(g^2-g)(\mathfrak{I}_1-1)+(g^2-g) + (g\mathfrak{I}_1-g-\mathfrak{I}_1+1)+(g-1)\}+(\mathfrak{I}_1-1)^2\{(g^3-g^2) + (g^2-g)(\mathfrak{I}_1-1)+(g^2-g)+(g\mathfrak{I}_1-g-\mathfrak{I}_1+1)+(g-1)\}, \quad (123)$$

$$\Omega_4 = (g^3-g^2)\{(\mathfrak{I}_1+\mathfrak{I}_2-2)(g^3-g^2)+\mathfrak{I}_1\mathfrak{I}_2(g^2-g) + g(-1+\mathfrak{I}_1+\mathfrak{I}_2)+1-\mathfrak{I}_1-\mathfrak{I}_2\}, \quad (124)$$

$$\Omega_5 = (g^3-g^2)(\mathfrak{I}_2-1)\{(g^2-g)+(g^2-g)(1+\mathfrak{I}_1\mathfrak{I}_2-\mathfrak{I}_1-\mathfrak{I}_2) + (g^2-g)(\mathfrak{I}_2-1)+(g-1)+(g-1)(\mathfrak{I}_2-1)\} + (g^3-g^2)(\mathfrak{I}_1-1)\{(g^2-g)+(g^2-g)(\mathfrak{I}_1-1) + (g^2-g)(\mathfrak{I}_2-1)+(g-1)+(g-1)(\mathfrak{I}_1-1)\}, \quad (125)$$

$$\Omega_6 = (g^2-g)^2(\mathfrak{I}_1\mathfrak{I}_2-1)+(g^2-g)(\mathfrak{I}_1\mathfrak{I}_2-\mathfrak{I}_1-\mathfrak{I}_2+1) \cdot \{(\mathfrak{I}_1+\mathfrak{I}_2)(g^2-g)+(g-1)(1+\mathfrak{I}_1\mathfrak{I}_2)-2g^2\} + (g^2\mathfrak{I}_2-g^2-g\mathfrak{I}_2+g)\{g^2\mathfrak{I}_1-g^2-2g\mathfrak{I}_1 + g\mathfrak{I}_1\mathfrak{I}_2+g-\mathfrak{I}_1\mathfrak{I}_2+\mathfrak{I}_1\}, \quad (126)$$

$$\Omega_7 = (g^2\mathfrak{I}_1-g^2-g\mathfrak{I}_1+g)\{g\mathfrak{I}_1\mathfrak{I}_2-\mathfrak{I}_1\mathfrak{I}_2-g+1\} + (g\mathfrak{I}_1-g-\mathfrak{I}_1+1)(\mathfrak{I}_2-1)(\mathfrak{I}_1+\mathfrak{I}_2-2\}. \quad (127)$$

4. Conclusion

In this paper, we computed some distance-based polynomial of zero divisor graph $Y(\mathbb{Z}_{g_1{}^2 g_2}\times\mathbb{Z}_{\mathfrak{I}_1\mathfrak{I}_2})$ namely Hosoya polynomial, Harary polynomial, Wiener index, and hyper Wiener index. We have seen that the eccentricity of any vertex of a zero divisor graph is always 2 or 3, which helps for the computation of the above polynomial. These computations can be helpful in computation of eccentricity based topological indices of the considered zero divisor graph.

Data Availability

No data is required to support the study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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