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Research Article

Some Simpson's Riemann–Liouville Fractional Integral Inequalities with Applications to Special Functions

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Based on the Riemann–Liouville fractional integral, a new form of generalized Simpson-type inequalities in terms of the first derivative is discussed. Here, some more inequalities for convexity as well as concavity are established. We expect that present outcomes are the generalization of already obtained results. Applications to beta, q-digamma, and Bessel functions are also provided.

1. Introduction

Integral inequalities have been widely used in various sciences, including mathematical sciences, applied sciences, differential equations, and functional analysis. In most mathematical analysis areas, many types of integral inequalities are used. They are very important in approximation theory and numerical analysis, which estimate the error's approximation [1–3]. Integral inequalities are useful tools in the study of different classes of differential equations and integral equations. Today, they are employed not only in mathematics but also in physics, computer science, and biology.

In several zones of mathematics, convex functions show a vital role. Especially in optimization theory, they are magical due to the number of expedient properties. There is nice connection between the theory of convex functions and mathematical inequalities. There are several important inequalities due to their direct applications in applied sciences (see [4–8]).

Several integral inequalities, such as Hölder's inequality, Simpson's inequality, Newton's inequality, the Hermite–Hadamard inequality, Ostrowski's inequality, Cauchy–Schwarz, and Chebyshev, are well known in classical analysis and have been proven and applied in the setup of q-calculus using classical convexity [9–12]. Much work is being done in the domain of q-analysis, beginning with Euler, in order to achieve proficiency in mathematics that constructs quantum computing. q-Calculus is viewed as a link between mathematics, physics, and statistics. It has numerous applications in various areas of mathematics, including orthogonal polynomials, number theory, hypergeometric functions, and other mechanics, combinatorics, sciences, stochastic theory, quantum theory, and the theory of relativity. This important branch of mathematics appears to have been invented by Euler. In Newton's work with infinite series, he used the q-parameter. Jackson later gave the q-calculus integral without the limit.

During the period 1710–1761, Thomas Simpson developed critical methods for numerical integration and estimation of definite integrals, which became known as the Simpson rule. Nonetheless, Kepler used a similar approximation nearly last decades earlier, so it is also known as the Kepler rule. Simpson rule includes the three-point

Newton–Cotes quadrature rule, so a three-step quadratic kernel estimation is sometimes referred to as a Newton-type result.

An incredible reliance has been found between inequalities and the theory of convex functions. This relationship is the primary mental stability behind the vast data utilizing convex functions. The Simpson-type inequalities have been examined broadly in the course of recent decades. A function $\hbar\colon J \subset \mathfrak{R} \longrightarrow \mathfrak{R}$ is said to be convex on $[\sigma_1, \sigma_2]$, with $\sigma_1 < \sigma_2$ where $\sigma_1, \sigma_2 \in J$:

$$\hbar(\eta\sigma_1 + (1-\eta)\sigma_2) \le \eta\hbar(\sigma_1) + (1-\eta)\hbar(\sigma_2), \quad \eta \in [0,1], \tag{1}$$

an inequality which is notable as Simpson's inequality in [13].

Theorem 1. *Suppose $\hbar\colon [\sigma_1, \sigma_2] \longrightarrow \mathfrak{R}$ is four times continuously differentiable function on (σ_1, σ_2) and $\|\hbar^{(4)}\|_\infty = sup_{\theta\in(\sigma_1,\sigma_2)}|\hbar^{(4)}(\theta)| < \infty$; then, the following inequality holds:*

$$\left|\left[\frac{1}{6}\hbar(\sigma_1) + \frac{2}{3}\hbar\left(\frac{\sigma_1+\sigma_2}{2}\right) + \frac{1}{6}\hbar(\sigma_2)\right] - \frac{1}{\sigma_2-\sigma_1}\int_{\sigma_1}^{\sigma_2}\hbar(\theta)\mathrm{d}\theta\right| \le \frac{(\sigma_2-\sigma_1)^4}{2880}\left\|\hbar^{(4)}\right\|_\infty. \tag{2}$$

Many scholars are interested in the Simpson-type inequality since it has been examined and studied for numerous classes of functions. Due to their efficacy and usefulness in pure and applied mathematics, Simpson-type and Newton-type results have been keenly interpolated for convex functions. The first striking result about the Simpson-type inequality along with its applications to the quadrature formula in numerical integration was given by Dragomir et al. in [14]. Later on, several new Simpson's type inequalities with improved bounds were developed for s-convex functions by Alomari et al. [13] and Sarikaya et al. [15].

Some researchers have demonstrated Simpson's type inequalities and obtained various outcomes. A new generalization and extension of Simpson's type inequalities were presented in [16, 17]. Qaisar et al. in [18, 19] gave Simpson's type inequalities for twice differentiable convex mappings with applications. However, some fractional variants can be observed in [20–22]. There is massive literature regarding improvements and extensions of Simpson's inequality in q-calculus. Recently, Ali et al. in [12] gave new quantum boundaries for quantum Simpson-type and Newton-type inequalities for preinvex functions. They also gave a brief literature review about the development of results connected to quantum Simpson's inequality. The accompanying ongoing improvements for the fractional integral on Simpson's inequality for $\omega > 0$ were demonstrated by Hwang et al. (see [20]).

Theorem 2. *Suppose $\hbar\colon [\sigma_1, \sigma_2] \longrightarrow \mathfrak{R}$ is a differentiable function whose derivative is continuous on (σ_1, σ_2) and $\hbar' \in \mathscr{L}[\sigma_1, \sigma_2]$; then, the following inequality holds:*

$$\begin{aligned}&\left|\frac{\Gamma(\omega+1)}{2(\sigma_2-\sigma_1)^\omega}\left[J^\omega_{\sigma_1^+}\hbar(\sigma_2) + J^\omega_{\sigma_2^-}\hbar(\sigma_1)\right] - \left[\frac{5^\omega-1}{6^\omega}\hbar\left(\frac{\sigma_1+\sigma_2}{2}\right) + \frac{6^\omega-5^\omega+1}{6^\omega}\frac{\hbar(\sigma_1)+\hbar(\sigma_2)}{2}\right]\right|\\ &\quad\le (\sigma_2-\sigma_1)\left[\frac{1}{\omega+1}\left(\frac{2^\omega+1}{2^{\omega+1}} - \frac{5^{\omega+1}+1}{6^{\omega+1}}\right) + \frac{5^\omega-1}{12.6^\omega}\right]\left(\left|\hbar'(\sigma_1)\right| + \left|\hbar'(\sigma_2)\right|\right).\end{aligned} \tag{3}$$

Throughout the text, the class of integrable functions on $[\sigma_1, \sigma_2]$ is denoted by $\mathscr{L}[\sigma_1, \sigma_2]$. Another form utilizing Riemann–Liouville fractional integrals mentioned above for differentiable convex functions was presented by Iqbal et al. (see [22]).

Theorem 3. *Suppose $\hbar\colon [\sigma_1, \sigma_2] \longrightarrow \mathfrak{R}$ is a differentiable function and $\hbar' \in \mathscr{L}[\sigma_1, \sigma_2]$ is integrable on (σ_1, σ_2) with $\sigma_1 < \sigma_2$ and $0 < \omega \le 1$. If $|\hbar'|$ is convex on $[\sigma_1, \sigma_2]$, then we have*

$$\begin{aligned}&\left|\left[\frac{1}{6}\hbar(\sigma_1) + \frac{2}{3}\hbar\left(\frac{\sigma_1+\sigma_2}{2}\right) + \frac{1}{6}\hbar(\sigma_2)\right] - \frac{\Gamma(\omega+1)}{2(\sigma_2-\sigma_1)^\omega}\left[J^\omega_{\sigma_1^+}\hbar(\sigma_2) + J^\omega_{\sigma_2^-}\hbar(\sigma_1)\right]\right| \le \frac{\sigma_2-\sigma_1}{2^{\omega+1}}\times\\ &\left[\frac{2u-\omega+2}{6(\omega+1)} + \left\{\left(\frac{2^\omega-1}{3}+\frac{1}{2}\right)(2\nu-3) - \frac{1}{\omega+1}\left(\frac{5\nu}{3} - \frac{2^{\omega+1}+1}{2}\right)\right\}\right]\left(\left|\hbar'(\sigma_1)\right| + \left|\hbar'(\sigma_2)\right|\right),\end{aligned} \tag{4}$$

where $u = (1/3)^{(1/\omega)}$ *and* $v^{\omega} = ((2(2^{\omega}-1)+3)/3)$.

Proposition 1. *In above Theorem 3, if* $\omega = 1$, *one can obtain*

$$\left|\left[\frac{1}{6}\hbar(\sigma_1)+\frac{2}{3}\hbar\left(\frac{\sigma_1+\sigma_2}{2}\right)+\frac{1}{6}\hbar(\sigma_2)\right]-\frac{1}{\sigma_2-\sigma_1}\int_{\sigma_1}^{\sigma_2}\hbar(\theta)\mathrm{d}\theta\right|\leq\frac{5(\sigma_2-\sigma_1)}{72}\left(\left|\hbar'(\sigma_1)\right|+\left|\hbar'(\sigma_2)\right|\right). \tag{5}$$

Proposition 2. *In above Theorem 3 with* $\hbar(\sigma_1)=\hbar((\sigma_1+\sigma_2)/2)=\hbar(\sigma_2)$, *one can obtain*

$$\left|\frac{1}{\sigma_2-\sigma_1}\int_{\sigma_1}^{\sigma_2}\hbar(\theta)d\theta-\hbar\left(\frac{\sigma_1+\sigma_2}{2}\right)\right|\leq\frac{5(\sigma_2-\sigma_1)}{72}\left(\left|\hbar'(\sigma_1)\right|+\left|\hbar'(\sigma_2)\right|\right). \tag{6}$$

Theorem 4. *Let* $\hbar$ *be defined as in Theorem 3, and if* $|\hbar'|$ *is convex on* $[\sigma_1,\sigma_2]$, *with* $q\geq 1$, *then we have*

$$\begin{aligned}
&\left|\left[\frac{1}{6}\hbar(\sigma_1)+\frac{2}{3}\hbar\left(\frac{\sigma_1+\sigma_2}{2}\right)+\frac{1}{6}\hbar(\sigma_2)\right]-\frac{\Gamma(\omega+1)}{2(\sigma_2-\sigma_1)^{\omega}}\left[J_{\sigma_1^+}^{\omega}\hbar(\sigma_2)+J_{\sigma_2^-}^{\omega}\hbar(\sigma_1)\right]\right|\leq\frac{\sigma_2-\sigma_1}{2^{\omega}}\\
&\left[S_1^{1-(1/q)}\left\{\left(\frac{\Upsilon_5|\hbar'(\sigma_1)|^q+\Upsilon_6|\hbar'(\sigma_2)|^q}{2}\right)^{(1/q)}+\left(\frac{\Upsilon_5|\hbar'(\sigma_1)|^q+\Upsilon_6|\hbar'(\sigma_2)|^q}{2}\right)^{(1/q)}\right\}\right.\\
&\left.+S_2^{1-(1/q)}\left\{\left(\frac{\Upsilon_7|\hbar'(\sigma_1)|^q+\Upsilon_8|\hbar'(\sigma_2)|^q}{2}\right)^{(1/q)}+\left(\frac{\Upsilon_7|\hbar'(\sigma_1)|^q+\Upsilon_8|\hbar'(\sigma_2)|^q}{2}\right)^{(1/q)}\right\}\right],\\
&\Upsilon_1=\int_0^{1-u}\left(\frac{1}{2}(1-\eta)^{\omega}-\frac{1}{6}\right)\mathrm{d}\eta=-\frac{1}{6}(1-u)-\frac{1}{2(\omega+1)}u^{\omega+1}+\frac{1}{2(\omega+1)},\\
&\Upsilon_2=\int_{1-u}^{1}\left(\frac{1}{6}-\frac{1}{2}(1-\eta)^{\omega}\right)\mathrm{d}\eta=\frac{1}{6}-\frac{1}{6}(1-u)-\frac{1}{2(\omega+1)}u^{\omega+1},\\
&\Upsilon_3=\int_0^{v-1}\left(\frac{1}{2(2^{\omega}-1)}-\frac{1}{2(2^{\omega}-1)}(1+\eta)^{\omega}+\frac{1}{3}\right)\mathrm{d}\eta=\left[\frac{1}{3}+\frac{1}{2(2^{\omega}-1)}\right](v-1)-\frac{v^{\omega+1}}{2(2^{\omega}-1)(\omega+1)}+\frac{1}{2(2^{\omega}-1)(\omega+1)},\\
&\Upsilon_4=\int_{v-1}^{1}\left(\frac{1}{2(2^{\omega}-1)}(1+\eta)^{\omega}-\frac{1}{2(2^{\omega}-1)}-\frac{1}{3}\right)\mathrm{d}\eta=\frac{2^{\omega+1}}{2(2^{\omega}-1)(\omega+1)}-\left[\frac{1}{3}+\frac{1}{2(2^{\omega}-1)}\right]\\
&\quad-\frac{v^{\omega+1}}{2(2^{\omega}-1)(\omega+1)}+\left[\frac{1}{3}+\frac{1}{2(2^{\omega}-1)}\right](v-1),\\
&\Upsilon_5=\int_0^1\left|\left(\frac{1}{6}-\frac{1}{2}(1-\eta)^{\omega}\right)\right|(1+\eta)\mathrm{d}\eta=\frac{3(\omega+1)+4\omega(\omega+2)u-\omega(\omega+1)u^2}{12(\omega+1)(\omega+2)}-\frac{1}{8},\\
&\Upsilon_6=\int_0^1\left|\left(\frac{1}{6}-\frac{1}{2}(1-\eta)^{\omega}\right)\right|(1-\eta)\mathrm{d}\eta=\frac{2\omega u^2-\omega+4}{24(\omega+2)},\\
&\Upsilon_7=\int_0^1\left|\frac{1}{2(2^{\omega}-1)}-\frac{1}{2(2^{\omega}-1)}(1+\eta)^{\omega}+\frac{1}{3}\right|(1+\eta)\mathrm{d}\eta\\
&\quad=\frac{1}{2(2^{\omega}-1)}\left[\left(v^2-\frac{5}{2}\right)\left(\frac{(2^{\omega}-1)}{3}+\frac{1}{2}\right)-\frac{1}{(\omega+2)}\left(\frac{5}{3}v^2-\frac{2^{\omega+1}+1}{2}\right)\frac{1}{3}+\frac{1}{2(2^{\omega}-1)}\right],\\
&\Upsilon_8=\int_0^1\left|\frac{1}{2(2^{\omega}-1)}-\frac{1}{2(2^{\omega}-1)}(1+\eta)^{\omega}+\frac{1}{3}\right|(1-\eta)\mathrm{d}\eta\\
&\quad=\frac{1}{2(2^{\omega}-1)}\left[\left(\frac{1}{2}-(2-v)^2\right)\left(\frac{(2^{\omega}-1)}{3}+\frac{1}{2}\right)+\frac{1}{(\omega+1)}\left(\frac{1}{2}-\frac{5v}{3}(2-v)\right)+\frac{1}{(\omega+1)(\omega+2)}\left(\frac{2^{\omega+2}+1}{2}-\frac{5}{3}v^2\right)\right].
\end{aligned} \tag{7}$$

For the execution of differentiation and integration of real or complex number orders, fractional calculus proved as a helpful device which demonstrates its centrality. The theme had pulled in a lot of considerations from many authors who center around investigation of PDEs during the most recent couple of decades. For late outcomes identified with the current study, one can see [23–27]. Among a lot of fractional integrals which are grown up, the Riemann–Liouville fractional integral has been widely considered as a result of uses in numerous fields of sciences.

Definition 1. Suppose $\hbar \in \mathscr{L}[\sigma_1, \sigma_2]$. The left and right Riemann–Liouville fractional integral operator for $\omega > 0$ are

$$\begin{aligned} J^{\omega}_{\sigma_1^+}\hbar(\tau) &= \frac{1}{\Gamma(\omega)}\int_{\sigma_1}^{\tau}(\tau-\eta)^{\omega-1}\hbar(\eta)\mathrm{d}\eta, \quad \sigma_1 < \tau,\\ J^{\omega}_{\sigma_2^-}\hbar(\tau) &= \frac{1}{\Gamma(\omega)}\int_{\tau}^{\sigma_2}(\tau-\eta)^{\omega-1}\hbar(\eta)\mathrm{d}\eta, \quad \tau < \sigma_2. \end{aligned} \tag{8}$$

Gamma function is defined as $\Gamma(\omega) = \int_0^{\infty} e^{-u}u^{\omega-1}\mathrm{d}u$. Note that $J^0_{\sigma_1^+}\hbar(\tau) = J^0_{\sigma_2^-}\hbar(\tau) = \hbar(\tau)$.

If $\omega = 1$, the above integral becomes the classical integral.

2. Inequalities for Simpson's Type

In this section, we give Simpson's inequalities for the Riemann–Liouville integral operator for differentiable functions on (σ_1, σ_2). For this, we give a new Riemann–Liouville integral operator auxiliary identity that will serve to produce subsequent results for improvements.

Lemma 1. *Let $\hbar$: $[\sigma_1, \sigma_2] \longrightarrow \mathfrak{R}$ be a differentiable mapping such that $\hbar' \in \mathscr{L}[\sigma_1, \sigma_2]$ is integrable and $0 < \omega \leq 1$ on (σ_1, σ_2) with $\sigma_1 < \sigma_2$; then, the following identity holds:*

$$\begin{aligned} &\left[\frac{1}{6}\hbar(\sigma_1) + \frac{2}{3}\hbar\left(\frac{\sigma_1+\sigma_2}{2}\right) + \frac{1}{6}\hbar(\sigma_2)\right] - \frac{\Gamma(\omega+1)}{2(\sigma_2-\sigma_1)^{\omega}}\left[J^{\omega}_{\sigma_1^+}\hbar(\sigma_2) + J^{\omega}_{\sigma_2^-}\hbar(\sigma_1)\right]\\ &= \frac{\sigma_2-\sigma_1}{2}\left[A_1 + A_2 + A_3 + A_4\right], \end{aligned} \tag{9}$$

where

$$\begin{aligned} A_1 &= \int_0^{1/2}\left(\eta^{\omega} - \frac{1}{6}\right)\hbar'(\eta\sigma_2 + (1-\eta)\sigma_1)\mathrm{d}\eta,\\ A_2 &= \int_{1/2}^{1}\left(\eta^{\omega} - \frac{5}{6}\right)\hbar'(\eta\sigma_2 + (1-\eta)\sigma_1)\mathrm{d}\eta,\\ A_3 &= \int_0^{1/2}\left(\frac{1}{6} - \eta^{\omega}\right)\hbar'(\eta\sigma_1 + (1-\eta)\sigma_2)\mathrm{d}\eta,\\ A_4 &= \int_{1/2}^{1}\left(\frac{5}{6} - \eta^{\omega}\right)\hbar'(\eta\sigma_1 + (1-\eta)\sigma_2)\mathrm{d}\eta. \end{aligned} \tag{10}$$

Proof. By integration by parts, we obtain

$$
\begin{aligned}
A_1 &= \int_0^{1/2}\left(\eta^{\omega}-\frac{1}{6}\right)\hbar'\left(\eta\sigma_2+(1-\eta)\sigma_1\right)\mathrm{d}\eta \\
&= \frac{1}{\sigma_2-\sigma_1}\left(\frac{1}{2^{\omega}}-\frac{1}{6}\right)\hbar\left(\frac{\sigma_1+\sigma_2}{2}\right)+\frac{1}{6(\sigma_2-\sigma_1)}\hbar(\sigma_1) \\
&\quad -\frac{\omega}{(\sigma_2-\sigma_1)}\int_0^{1/2}\eta^{\omega-1}\hbar'\left(\eta\sigma_2+(1-\eta)\sigma_1\right)\mathrm{d}\eta, \\
A_1(\sigma_2-\sigma_1) &= \left(\frac{1}{2^{\omega}}-\frac{1}{6}\right)\hbar\left(\frac{\sigma_1+\sigma_2}{2}\right)+\frac{1}{6}\hbar(\sigma_1)-\omega\int_0^{1/2}\eta^{\omega-1}\hbar'\left(\eta\sigma_2+(1-\eta)\sigma_1\right)\mathrm{d}\eta, \\
A_2(\sigma_2-\sigma_1) &= \frac{1}{6}\hbar(\sigma_2)-\left(\frac{1}{2^{\omega}}-\frac{5}{6}\right)\hbar\left(\frac{\sigma_1+\sigma_2}{2}\right)-\omega\int_{1/2}^{1}\eta^{\omega-1}\hbar'\left(\eta\sigma_2+(1-\eta)\sigma_1\right)\mathrm{d}\eta \\
\left(\frac{\sigma_2-\sigma_1}{2}\right)(A_1+A_2) &= \left[\frac{4}{12}\hbar\left(\frac{\sigma_1+\sigma_2}{2}\right)+\frac{1}{12}\left(\hbar(\sigma_1)+\hbar(\sigma_2)\right)\right]-\frac{\omega}{2}\int_0^1\eta^{\omega-1}\hbar\left(\eta\sigma_2+(1-\eta)\sigma_1\right)\mathrm{d}\eta \\
&= \left[\frac{4}{12}\hbar\left(\frac{\sigma_1+\sigma_2}{2}\right)+\frac{1}{12}\left(\hbar(\sigma_1)+\hbar(\sigma_2)\right)\right]-\frac{\omega}{2}\int_{\sigma_1}^{\sigma_2}\frac{(\theta-\sigma_1)^{\omega-1}}{(\sigma_2-\sigma_1)^{\omega}}\hbar(\theta)\mathrm{d}\theta \\
&= \left[\frac{4}{12}\hbar\left(\frac{\sigma_1+\sigma_2}{2}\right)+\frac{1}{12}\left(\hbar(\sigma_1)+\hbar(\sigma_2)\right)\right]-\frac{\Gamma(\omega+)}{2(\sigma_2-\sigma_1)^{\omega}}J^{\omega}_{\sigma_2^-}\hbar(\sigma_1).
\end{aligned} \tag{11}
$$

Similarly,

$$
\begin{aligned}
\left(\frac{\sigma_2-\sigma_1}{2}\right)(A_3+A_4) &= \left[\frac{4}{12}\hbar\left(\frac{\sigma_1+\sigma_2}{2}\right)+\frac{1}{12}\left(\hbar(\sigma_1)+\hbar(\sigma_2)\right)\right] \\
&\quad -\frac{\Gamma(\omega+1)}{2(\sigma_2-\sigma_1)^{\omega}}J^{\omega}_{\sigma_1^+}\hbar(\sigma_2),
\end{aligned} \tag{12}
$$

which ends the proof. □

Theorem 5. *Let $\hbar$ be defined as in Lemma 1, and if $|\hbar'|$ is convex on $[\sigma_1,\sigma_2]$, then we have*

$$
\begin{aligned}
&\left|\left[\frac{1}{6}\hbar(\sigma_1)+\frac{2}{3}\hbar\left(\frac{\sigma_1+\sigma_2}{2}\right)+\frac{1}{6}\hbar(\sigma_2)\right]-\frac{\Gamma(\omega+1)}{2(\sigma_2-\sigma_1)^{\omega}}\left[J^{\omega}_{\sigma_1^+}\hbar(\sigma_2)+J^{\omega}_{\sigma_2^-}\hbar(\sigma_1)\right]\right| \\
&\quad \le \frac{(\sigma_2-\sigma_1)}{2}\left[\frac{\beta+5\gamma-4}{3}+\frac{1+2^{\omega}\left(1-2\beta^{\omega+1}-2\gamma^{\omega+1}\right)}{2^{\omega}(\omega+1)}\right]\left(\left|\hbar'(\sigma_1)\right|+\left|\hbar'(\sigma_2)\right|\right).
\end{aligned} \tag{13}
$$

Proof. By using the properties of modulus on Lemma 1, we have

$$
\begin{aligned}
&\left|\left[\frac{1}{6}\hbar(\sigma_1)+\frac{2}{3}\hbar\left(\frac{\sigma_1+\sigma_2}{2}\right)+\frac{1}{6}\hbar(\sigma_2)\right]-\frac{\Gamma(\omega+1)}{2(\sigma_2-\sigma_1)^{\omega}}\left[J^{\omega}_{\sigma_1^+}\hbar(\sigma_2)+J^{\omega}_{\sigma_2^-}\hbar(\sigma_1)\right]\right| \\
&\quad \le \frac{\sigma_2-\sigma_1}{2}\left[A_1+A_2+A_3+A_4\right],
\end{aligned} \tag{14}
$$

where $\beta=(1/6)^{(1/\omega)}$ and $\gamma=(5/6)^{(1/\omega)}$.

By the convexity of $|\hbar'|$, $\omega\in(0,1]$, and $\forall\eta\in[0,1]$, we have

$$\begin{aligned}
&\left|\left[\frac{1}{6}\hbar(\sigma_1)+\frac{2}{3}\hbar\left(\frac{\sigma_1+\sigma_2}{2}\right)+\frac{1}{6}\hbar(\sigma_2)\right]-\frac{\Gamma(\omega+1)}{2(\sigma_2-\sigma_1)^{\omega}}\left[J^{\omega}_{\sigma_1^+}\hbar(\sigma_2)+J^{\omega}_{\sigma_2^-}\hbar(\sigma_1)\right]\right|\\
&\le\frac{(\sigma_2-\sigma_1)}{2}\int_0^{1/2}\left|\eta^{\omega}-\frac{1}{6}\right|\left|\hbar'(\eta\sigma_2+(1-\eta)\sigma_1)\right|d\eta\\
&\quad+\frac{(\sigma_2-\sigma_1)}{2}\int_0^{1/2}\left|\left(\frac{1}{6}-\eta^{\omega}\right)\right|\left|\hbar'(\eta\sigma_1+(1-\eta)\sigma_2)\right|d\eta\\
&\quad+\frac{(\sigma_2-\sigma_1)}{2}\int_{1/2}^{1}\left|\eta^{\omega}-\frac{5}{6}\right|\left|\hbar'(\eta\sigma_2+(1-\eta)\sigma_1)\right|d\eta\\
&\quad+\frac{(\sigma_2-\sigma_1)}{2}\int_{1/2}^{1}\left|\frac{5}{6}-\eta^{\omega}\right|\left|\hbar'(\eta\sigma_1+(1-\eta)\sigma_2)\right|d\eta\\
&=\frac{(\sigma_2-\sigma_1)}{2}\int_0^{1/2}\left|\eta^{\omega}-\frac{1}{6}\right|\left\{\eta\left|\hbar'(\sigma_2)\right|+(1-\eta)\left|\hbar'(\sigma_1)\right|\right\}|d\eta\\
&\quad+\frac{(\sigma_2-\sigma_1)}{2}\int_0^{1/2}\left|\frac{1}{6}-\eta^{\omega}\right|\left\{\eta\left|\hbar'(\sigma_1)\right|+(1-\eta)\left|\hbar'(\sigma_2)\right|\right\}|d\eta\\
&\quad+\frac{(\sigma_2-\sigma_1)}{2}\int_{1/2}^{1}\left|\eta^{\omega}-\frac{5}{6}\right|\left\{\eta\left|\hbar'(\sigma_2)\right|+(1-\eta)\left|\hbar'(\sigma_1)\right|\right\}|d\eta\\
&\quad+\frac{(\sigma_2-\sigma_1)}{2}\int_{1/2}^{1}\left|\frac{5}{6}-\eta^{\omega}\right|\left\{\eta\left|\hbar'(\sigma_1)\right|+(1-\eta)\left|\hbar'(\sigma_2)\right|\right\}|d\eta\\
&\le\frac{(\sigma_2-\sigma_1)}{2}\left[\frac{\beta+5\gamma-4}{3}+\frac{1+2^{\omega}\left(1-2\beta^{\omega+1}-2\gamma^{\omega+1}\right)}{2^{\omega}(\omega+1)}\right]\left(\left|\hbar'(\sigma_1)\right|+\left|\hbar'(\sigma_2)\right|\right).
\end{aligned}\tag{15}$$

Simple calculations yield that

$$\begin{aligned}
S_1&=\int_0^{1/2}\left|\eta^{\omega}-\frac{1}{6}\right|d\eta=\frac{4\beta-1}{12}+\frac{1-\beta^{\omega+1}\cdot 2^{\omega+1}}{2^{\omega+1}(\omega+1)},\\
S_2&=\int_{1/2}^{1}\left|\eta^{\omega}-\frac{5}{6}\right|d\eta=5\left(\frac{4\gamma-3}{12}\right)+\frac{2^{\omega+1}+1-\gamma^{\omega+1}\cdot 2^{\omega+2}}{2^{\omega+1}(\omega+1)},\\
S_3&=\int_0^{1/2}\left|\eta^{\omega}-\frac{1}{6}\right|\eta d\eta=\frac{8\beta^2-1}{48}+\frac{1-\beta^3\cdot 2^{\omega+3}}{2^{\omega+2}(\omega+2)},\\
S_4&=\int_{1/2}^{1}\left|\eta^{\omega}-\frac{5}{6}\right|\eta d\eta=\frac{40\gamma^2-25}{48}+\frac{2^{\omega+2}+1-\gamma^{\omega+2}\cdot 2^{\omega+3}}{2^{\omega+2}(\omega+2)},
\end{aligned}\tag{16}$$

which completes the proof. □

Remark 1. On letting $\omega=1$ in Theorem 5, (13) reduces to inequality (5).

Corollary 1. *By choosing $\hbar(\sigma_1)=\hbar((\sigma_1+\sigma_2)/2)=\hbar(\sigma_2)$ in Theorem 5, inequality (13) becomes*

$$\begin{aligned}
&\left|\left[\hbar\left(\frac{\sigma_1+\sigma_2}{2}\right)-\frac{\Gamma(\omega+1)}{2(\sigma_2-\sigma_1)^{\omega}}\left[J^{\omega}_{\sigma_1^+}\hbar(\sigma_2)+J^{\omega}_{\sigma_2^-}\hbar(\sigma_1)\right]\right]\right|\\
&\le\frac{(\sigma_2-\sigma_1)}{2}\left[\frac{\beta+5\gamma-4}{3}+\frac{1+2^{\omega}\left(1-2\beta^{\omega+1}-2\gamma^{\omega+1}\right)}{2^{\omega}(\omega+1)}\right]\left(\left|\hbar'(\sigma_1)\right|+\left|\hbar'(\sigma_2)\right|\right).
\end{aligned}\tag{17}$$

Remark 2. By letting $\omega = 1$ in the above corollary, we get the following inequality, which looks better than the inequality presented by S. Kirmaci:

$$\left|\hbar\left(\frac{\sigma_1+\sigma_2}{2}\right)-\frac{1}{\sigma_2-\sigma_1}\int_{\sigma_1}^{\sigma_2}\hbar(\theta)\mathrm{d}\theta\right|\leq\frac{\sigma_2-\sigma_1}{2}\left(\frac{5}{36}\right)\left\{\left|\hbar'(\sigma_1)\right|+\left|\hbar'(\sigma_2)\right|\right\}. \tag{18}$$

The corresponding version for powers of the absolute value of the derivative is incorporated as follows.

Theorem 6. *Let $\hbar$ be defined as in Lemma 1, and if $|\hbar'|^q$ is convex on $[\sigma_1,\sigma_2]$, with $q\geq 1$, then we have*

$$\begin{aligned}&\left|\left[\frac{1}{6}\hbar(\sigma_1)+\frac{2}{3}\hbar\left(\frac{\sigma_1+\sigma_2}{2}\right)+\frac{1}{6}\hbar(\sigma_2)\right]-\frac{\Gamma(\omega+1)}{2(\sigma_2-\sigma_1)^\omega}\left[J^\omega_{\sigma_1^+}\hbar(\sigma_2)+J^\omega_{\sigma_2^-}\hbar(\sigma_1)\right]\right|\leq\frac{(\sigma_2-\sigma_1)}{2}\\&\times\left(\frac{1+2^{\omega p+1}}{6^{\omega p+1}(\omega p+1)}\right)^{1/p}\frac{1}{2^{\omega/q-1}}\left[\left(\left|\hbar'(\sigma_1)\right|^q+\left|\hbar'\left(\frac{\sigma_1+\sigma_2}{2}\right)\right|^q\right)^{1/q}+\left(\left|\hbar'\left(\frac{\sigma_1+\sigma_2}{2}\right)\right|^q+\left|\hbar'(\sigma_2)\right|^q\right)^{1/q}\right].\end{aligned} \tag{19}$$

Proof. Using Lemma 1 and convexity, we obtain

$$\begin{aligned}&\left|\left[\frac{1}{6}\hbar(\sigma_1)+\frac{2}{3}\hbar\left(\frac{\sigma_1+\sigma_2}{2}\right)+\frac{1}{6}\hbar(\sigma_2)\right]-\frac{\Gamma(\omega+1)}{2(\sigma_2-\sigma_1)^\omega}\left[J^\omega_{\sigma_1^+}\hbar(\sigma_2)+J^\omega_{\sigma_2^-}\hbar(\sigma_1)\right]\right|\\
&\leq\frac{(\sigma_2-\sigma_1)}{2}\left\{\int_0^{1/2}\left|\eta^\omega-\frac{1}{6}\right|\left|\hbar'(\eta\sigma_2+(1-\eta)\sigma_1)\right|\mathrm{d}\eta+\int_{1/2}^1\left|\eta^\omega-\frac{5}{6}\right|\left|\hbar'(\eta\sigma_2+(1-\eta)\sigma_1)\right|\mathrm{d}\eta+\int_0^{1/2}\left|\frac{1}{6}-\eta^\omega\right|\left|\hbar'(\eta\sigma_1+(1-\eta)\sigma_2)\right|\mathrm{d}\eta\right.\\
&\quad\left.+\int_{1/2}^1\left|\frac{5}{6}-\eta^\omega\right|\left|\hbar'(\eta\sigma_1+(1-\eta)\sigma_2)\right|\mathrm{d}\eta\right\}\\
&\leq\frac{(\sigma_2-\sigma_1)}{2}\left(\int_0^{1/2}\left|\eta^\omega-\frac{1}{6}\right|^p\mathrm{d}\eta\right)^{1/p}\left(\int_0^{1/2}\left|\hbar'(\eta\sigma_2+(1-\eta)\sigma_1)\right|^q\mathrm{d}\eta\right)^{1/q}\\
&\quad+\frac{(\sigma_2-\sigma_1)}{2}\left(\int_{1/2}^1\left|\eta^\omega-\frac{5}{6}\right|^p\mathrm{d}\eta\right)^{1/p}\left(\int_{1/2}^1\left|\hbar'(\eta\sigma_2+(1-\eta)\sigma_1)\right|^q\mathrm{d}\eta\right)^{1/q}\\
&\quad+\frac{(\sigma_2-\sigma_1)}{2}\left(\int_0^{1/2}\left|\eta^\omega-\frac{1}{6}\right|^p\mathrm{d}\eta\right)^{1/p}\left(\int_0^{1/2}\left|\hbar'(\eta\sigma_1+(1-\eta)\sigma_2)\right|^q\mathrm{d}\eta\right)^{1/q}\\
&\quad+\frac{(\sigma_2-\sigma_1)}{2}\left(\int_{1/2}^1\left|\eta^\omega-\frac{5}{6}\right|^p\mathrm{d}\eta\right)^{1/p}\left(\int_{1/2}^1\left|\hbar'(\eta\sigma_1+(1-\eta)\sigma_2)\right|^q\mathrm{d}\eta\right)^{1/q}\\
&=\frac{(\sigma_2-\sigma_1)}{2}\left(\int_0^{1/6}\left|\frac{1}{6}-\eta^\omega\right|^p\mathrm{d}\eta+\int_{1/6}^{1/2}\left|\eta^\omega-\frac{1}{6}\right|^p\mathrm{d}\eta\right)^{1/p}\times\left(\int_0^{1/2}\left|\hbar'(\eta\sigma_1+(1-\eta)\sigma_2)\right|^q\mathrm{d}\eta\right)^{1/q}\\
&\quad+\frac{(\sigma_2-\sigma_1)}{2}\left(\int_{1/2}^{5/6}\left|\frac{5}{6}-\eta^\omega\right|^p\mathrm{d}\eta+\int_{5/6}^1\left|\eta^\omega-\frac{5}{6}\right|^p\mathrm{d}\eta\right)^{1/p}\times\left(\int_{1/2}^1\left|\hbar'(\eta\sigma_2+(1-\eta)\sigma_1)\right|^q\mathrm{d}\eta\right)^{1/q}\\
&\quad+\frac{(\sigma_2-\sigma_1)}{2}\left(\int_0^{1/6}\left|\frac{1}{6}-\eta^\omega\right|^p\mathrm{d}\eta+\int_{1/6}^{1/2}\left|\eta^\omega-\frac{1}{6}\right|^p\mathrm{d}\eta\right)^{1/p}\times\left(\int_0^{1/2}\left|\hbar'(\eta\sigma_1+(1-\eta)\sigma_2)\right|^q\mathrm{d}\eta\right)^{1/q}\\
&\quad+\frac{(\sigma_2-\sigma_1)}{2}\left(\int_{1/2}^{5/6}\left|\frac{5}{6}-\eta^\omega\right|^p\mathrm{d}\eta+\int_{5/6}^1\left|\eta^\omega-\frac{5}{6}\right|^p\mathrm{d}\eta\right)^{1/p}\times\left(\int_{1/2}^1\left|\hbar'(\eta\sigma_1+(1-\eta)\sigma_2)\right|^q\mathrm{d}\eta\right)^{1/q}\\
&\leq\frac{(\sigma_2-\sigma_1)}{2}\left(\frac{1+2^{\omega p+1}}{6^{\omega p+1}(\omega p+1)}\right)^{1/p}\\
&\quad\times\frac{1}{2^{\omega/q-1}}\left[\left(\left|\hbar'(\sigma_1)\right|^q+\left|\hbar'\left(\frac{\sigma_1+\sigma_2}{2}\right)\right|^q\right)^{1/q}+\left(\left|\hbar'\left(\frac{\sigma_1+\sigma_2}{2}\right)\right|^q+\left|\hbar'(\sigma_2)\right|^q\right)^{1/q}\right],\end{aligned} \tag{20}$$

which completes the proof. □

Theorem 7. *Let $\hbar$ be defined as in Lemma 1, and if $|\hbar'|^q$ is convex on $[\sigma_1,\sigma_2]$, with $q\geq 1$, then we have*

$$\left|\left[\frac{1}{6}\hbar(\sigma_1)+\frac{2}{3}\hbar\left(\frac{\sigma_1+\sigma_2}{2}\right)+\frac{1}{6}\hbar(\sigma_2)\right]-\frac{\Gamma(\omega+1)}{2(\sigma_2-\sigma_1)^\omega}\left[J^{\omega}_{\sigma_1^+}\hbar(\sigma_2)+J^{\omega}_{\sigma_2^-}\hbar(\sigma_1)\right]\right|\leq\frac{(\sigma_2-\sigma_1)}{2}$$
$$\times\left[S_1^{1-1/q}\left\{\left(S_3|\hbar'(\sigma_2)|^q+(S_1-S_3)|\hbar'(\sigma_1)|^q\right)^{1/q}+\left(S_3|\hbar'(\sigma_1)|^q+(S_1-S_3)|\hbar'(\sigma_2)|^q\right)^{1/q}\right\}\right.\tag{21}$$
$$\left.+\left[S_2^{1-1/q}\left\{\left(S_4|\hbar'(\sigma_2)|^q+(S_2-S_4)|\hbar'(\sigma_1)|^q\right)^{1/q}+\left(S_4|\hbar'(\sigma_1)|^q+(S_2-S_4)|\hbar'(\sigma_2)|^q\right)^{1/q}\right\}\right]\right].$$

Proof. By the use of the power-mean integral inequality for $q>1$, we obtain

$$\left|\left[\frac{1}{6}\hbar(\sigma_1)+\frac{2}{3}\hbar\left(\frac{\sigma_1+\sigma_2}{2}\right)+\frac{1}{6}\hbar(\sigma_2)\right]-\frac{\Gamma(\omega+1)}{2(\sigma_2-\sigma_1)^\omega}\left[J^{\omega}_{\sigma_1^+}\hbar(\sigma_2)+J^{\omega}_{\sigma_2^-}\hbar(\sigma_1)\right]\right|$$
$$\leq\frac{(\sigma_2-\sigma_1)}{2}\left\{\int_0^{1/2}\left|\eta^\omega-\frac{1}{6}\right||\hbar'(\eta\sigma_2+(1-\eta)\sigma_1)|d\eta+\int_{1/2}^{1}\left|\eta^\omega-\frac{5}{6}\right||\hbar'(\eta\sigma_2+(1-\eta)\sigma_1)|d\eta+\int_0^{1/2}\left|\eta^\omega-\frac{1}{6}\right||\hbar'(\eta\sigma_1+(1-\eta)\sigma_2)|d\eta\right.$$
$$\left.+\int_{1/2}^{1}\left|\eta^\omega-\frac{5}{6}\right||\hbar'(\eta\sigma_1+(1-\eta)\sigma_2)|d\eta\right\}$$
$$\leq\frac{(\sigma_2-\sigma_1)}{2}\left\{\left(\int_0^{1/2}\left|\eta^\omega-\frac{1}{6}\right|d\eta\right)^{1-1/q}\times\left(\int_0^{1/2}|\hbar'(\eta\sigma_2+(1-\eta)\sigma_1)|^q d\eta\right)^{1/q}+\left(\int_{1/2}^{1}\left|\eta^\omega-\frac{5}{6}\right|d\eta\right)^{1-1/q}\right.$$
$$\times\left(\int_{1/2}^{1}|\hbar'(\eta\sigma_2+(1-\eta)\sigma_1)|^q d\eta\right)^{1/q}+\left(\int_0^{1/2}\left|\eta^\omega-\frac{1}{6}\right|d\eta\right)^{1-1/q}\times\left(\int_0^{1/2}|\hbar'(\eta\sigma_1+(1-\eta)\sigma_2)|^q d\eta\right)^{1/q}+\left(\int_{1/2}^{1}\left|\eta^\omega-\frac{5}{6}\right|d\eta\right)^{1-1/q}$$
$$\left.\times\left(\int_{1/2}^{1}|\hbar'(\eta\sigma_1+(1-\eta)\sigma_2)|^q d\eta\right)^{1/q}\right\}.\tag{22}$$

By using the convexity of $|\hbar'|^q$,

$$\int_0^{1/2}\left|\eta^\omega-\frac{1}{6}\right||\hbar'(\eta\sigma_2+(1-\eta)\sigma_1)|^q d\eta$$
$$\leq\int_0^{1/6}\left|\frac{1}{6}-\eta^\omega\right|\left\{\eta|\hbar'(\sigma_1)|^q+(1-\eta)|\hbar'(\sigma_2)|^q\right\}|d\eta$$
$$+\int_{1/6}^{1/2}\left|\eta^\omega-\frac{1}{6}\right|\left\{\eta|\hbar'(\sigma_1)|^q+(1-\eta)|\hbar'(\sigma_2)|^q\right\}|d\eta$$
$$\int_{1/2}^{1}\left|\eta^\omega-\frac{5}{6}\right||\hbar'(\eta\sigma_2+(1-\eta)\sigma_1)|^q d\eta\tag{23}$$
$$\leq\int_{1/2}^{5/6}\left|\frac{5}{6}-\eta^\omega\right|\left\{\eta|\hbar'(\sigma_1)|^q+(1-\eta)|\hbar'(\sigma_2)|^q\right\}|d\eta$$
$$+\int_{5/6}^{1}\left|\eta^\omega-\frac{5}{6}\right|\left\{\eta|\hbar'(\sigma_2)|^q+(1-\eta)|\hbar'(\sigma_1)|^q\right\}|d\eta.$$

By using the calculus tool, we get equation (21), which completes the proof. □

Remark 3. If $\omega = 1$ in Theorem 7, then it reduces to Corollary 2.7 in [19].

In the following theorem, we obtained the estimate of Simpson's inequality (2) for concave functions.

Theorem 8. *Suppose* $\hbar\colon [\sigma_1, \sigma_2] \longrightarrow \mathfrak{R}$ *is a differentiable function on* (σ_1, σ_2) *such that* $\hbar' \in \mathscr{L}[\sigma_1, \sigma_2]$. *If* $|\hbar'|^q$ *is concave on* $[\sigma_1, \sigma_2]$, *for some fixed* $p > 1$ *with* $q = (p/p-1)$, *then we have*

$$\left|\left[\frac{1}{6}\hbar(\sigma_1) + \frac{2}{3}\hbar\left(\frac{\sigma_1+\sigma_2}{2}\right) + \frac{1}{6}\hbar(\sigma_2)\right] - \frac{\Gamma(\omega+1)}{2(\sigma_2-\sigma_1)^{\omega}}\left[J^{\omega}_{\sigma_1^+}\hbar(\sigma_2) + J^{\omega}_{\sigma_2^-}\hbar(\sigma_1)\right]\right| \leq \frac{(\sigma_2-\sigma_1)}{2}\times$$
$$\left[S_1\left\{\left|\hbar'\left(\frac{S_3\sigma_2 + (S_1-S_3)\sigma_1}{S_1}\right)\right| + \left|\hbar'\left(\frac{S_3\sigma_1 + (S_1-S_3)\sigma_2}{S_1}\right)\right|\right\} + S_2\left|\hbar'\left(\frac{S_4\sigma_2 + (S_2-S_4)\sigma_1}{S_2}\right)\right| + \left|\hbar'\left(\frac{S_4\sigma_1 + (S_2-S_4)\sigma_2}{S_2}\right)\right|\right]. \tag{24}$$

Proof. Utilizing concavity of $|\hbar'|^q$ and the power-mean inequality, we obtain

$$\begin{aligned}
\left|\hbar'(\eta\sigma_1 + (1-\eta)\sigma_2)\right|^q &\geq \eta\left|\hbar'(\sigma_1)\right|^q + (1-\eta)\left|\hbar'(\sigma_2)\right|^q \\
&\geq \left(\eta\left|\hbar'(\sigma_1)\right| + (1-\eta)\left|\hbar'(\sigma_2)\right|\right)^q, \\
\left|\hbar'(\eta\sigma_1 + (1-\eta)\sigma_2)\right| &\geq \eta\left|\hbar'(\sigma_1)\right| + (1-\eta)\left|\hbar'(\sigma_2)\right|.
\end{aligned} \tag{25}$$

Therefore, $|\hbar'|$ is also concave. Jensen's integral inequality follows that

$$\begin{aligned}
&\left|\left[\frac{1}{6}\hbar(\sigma_1) + \frac{2}{3}\hbar\left(\frac{\sigma_1+\sigma_2}{2}\right) + \frac{1}{6}\hbar(\sigma_2)\right] - \frac{\Gamma(\omega+1)}{2(\sigma_2-\sigma_1)^{\omega}}\left[J^{\omega}_{\sigma_1^+}\hbar(\sigma_2) + J^{\omega}_{\sigma_2^-}\hbar(\sigma_1)\right]\right| \\
&\leq \frac{(\sigma_2-\sigma_1)}{2}\left(\int_0^{1/2}\left|\eta^{\omega} - \frac{1}{6}\right|\mathrm{d}\eta\right)\left|\hbar'\left(\frac{\int_0^{1/2}\left|\eta^{\omega} - (1/6)\right|(\eta\sigma_2 + (1-\eta)\sigma_1)\mathrm{d}\eta}{\int_0^{1/2}\left|\eta^{\omega} - (1/6)\right|\mathrm{d}\eta}\right)\right|^q \\
&\quad + \frac{(\sigma_2-\sigma_1)}{2}\left(\int_{1/2}^{1}\left|\eta^{\omega} - \frac{5}{6}\right|\mathrm{d}\eta\right)\left|\hbar'\left(\frac{\int_{1/2}^{1}\left|\eta^{\omega} - (5/6)\right|(\eta\sigma_2 + (1-\eta)\sigma_1)\mathrm{d}\eta}{\int_{1/2}^{1}\left|\eta^{\omega} - (5/6)\right|\mathrm{d}\eta}\right)\right|^q \\
&\quad + \frac{(\sigma_2-\sigma_1)}{2}\left(\int_0^{1/2}\left|\frac{1}{6} - \eta^{\omega}\right|\mathrm{d}\eta\right)\left|\hbar'\left(\frac{\int_0^{1/2}\left|(1/6) - \eta^{\omega}\right|(\eta\sigma_1 + (1-\eta)\sigma_2)\mathrm{d}\eta}{\int_0^{1/2}\left|(1/6) - \eta^{\omega}\right|\mathrm{d}\eta}\right)\right|^q \\
&\quad + \frac{(\sigma_2-\sigma_1)}{2}\left(\int_{1/2}^{1}\left|\frac{5}{6} - \eta^{\omega}\right|\mathrm{d}\eta\right)\left|\hbar'\left(\frac{\int_{1/2}^{1}\left|(5/6) - \eta^{\omega}\right|(\eta\sigma_1 + (1-\eta)\sigma_2)\mathrm{d}\eta}{\int_{1/2}^{1}\left|(5/6) - \eta^{\omega}\right|\mathrm{d}\eta}\right)\right|^q \\
&= \frac{(\sigma_2-\sigma_1)}{2}S_1\left|\hbar'\left(\frac{S_3\sigma_2 + (S_1-S_3)\sigma_1}{S_1}\right)\right| + \frac{(\sigma_2-\sigma_1)}{2}S_2\left|\hbar'\left(\frac{S_3\sigma_1 + (S_1-S_3)\sigma_2}{S_1}\right)\right| \\
&\quad + \frac{(\sigma_2-\sigma_1)}{2}S_1\left|\hbar'\left(\frac{S_4\sigma_2 + (S_2-S_4)\sigma_1}{S_2}\right)\right| + \frac{(\sigma_2-\sigma_1)}{2}S_2\left|\hbar'\left(\frac{S_4\sigma_1 + (S_2-S_4)\sigma_2}{S_2}\right)\right|,
\end{aligned} \tag{26}$$

which completes the proof. □

Corollary 2. *By putting* $\omega = 1$ *in Theorem 8, then the following inequality becomes*

$$\left|\left[\frac{1}{6}\hbar(\sigma_1)+\frac{2}{3}\hbar\left(\frac{\sigma_1+\sigma_2}{2}\right)+\frac{1}{6}\hbar(\sigma_2)\right]-\frac{1}{\sigma_2-\sigma_1}\int_{\sigma_1}^{\sigma_2}\hbar(\theta)\mathrm{d}\theta\right| \leq \frac{5(\sigma_2-\sigma_1)}{72}\left[\left|\hbar'\left(\frac{29\sigma_1+61\sigma_2}{90}\right)\right|+\left|\hbar'\left(\frac{61\sigma_1+29\sigma_2}{90}\right)\right|\right]. \quad (27)$$

Remark 4. Inequality (27) is a generalization of the obtained inequality as in Theorem 8 of [13].

3. Applications

3.1. Beta Function. In this section, let $\omega > 0$, $\gamma \geq 3$, $\sigma_1 = 0$, $\sigma_2 = 1$, $\Gamma(\omega)$ be the gamma function, and $\hbar(\theta) = \theta^{\gamma-1}$ ($\theta \in [0,1]$). Then, $|\hbar'|$ is convex on $[0,1]$.

Let us recall that the beta function

$$B(p,q) = \int_0^1 \theta^{p-1}(1-\theta)^{q-1}\mathrm{d}\theta \quad (p,q>0). \quad (28)$$

Remark 5. From Section 2, we have

$$\frac{\Gamma(\omega+1)}{2(\sigma_2-\sigma_1)^{\omega}}J^{\omega}_{\sigma_1^+}\hbar(\sigma_2) = \frac{\omega}{2}\int_0^1 \theta^{\gamma-1}(1-\theta)^{\omega-1}\mathrm{d}\theta = \frac{\omega}{2}B(\gamma,\omega),$$
$$\frac{\Gamma(\omega+1)}{2(\sigma_2-\sigma_1)^{\omega}}J^{\omega}_{\sigma_2^-}\hbar(\sigma_1) = \frac{\omega}{2}\int_0^1 \theta^{\omega+\gamma-2}\mathrm{d}\theta = \frac{\omega}{2(\gamma+\omega-1)}. \quad (29)$$

Proposition 3. *In Theorem 5, the following inequality holds:*

$$\left|\left[\frac{2}{3.2^{\gamma-1}}+\frac{1}{6}\right]-\left[\frac{\omega}{2}B(\gamma,\omega)+\frac{\omega}{2(\gamma+\omega-1)}\right]\right| \leq \frac{(\gamma-1)}{2}\left[\frac{\beta+5\gamma-4}{3}+\frac{1+2^{\omega}\left(1-2\beta^{\omega+1}-2\gamma^{\omega+1}\right)}{2^{\omega}(\omega+1)}\right]. \quad (30)$$

3.2. q-Digamma Function. Suppose $0<\mathbf{q}<1$; the $\mathbf{q}$-digamma function $\varphi_{\mathbf{q}}$ is the $\mathbf{q}$-analogue of the digamma function φ (see [28, 29]) given as

$$\varphi_{\mathbf{q}} = -\ln(1-\mathbf{q})+\ln\mathbf{q}\sum_{k=0}^{\infty}\frac{\mathbf{q}^{k+\zeta}}{1-\mathbf{q}^{k+\zeta}} = -\ln(1-\mathbf{q})+\ln\mathbf{q}\sum_{k=0}^{\infty}\frac{\mathbf{q}^{k\zeta}}{1-\mathbf{q}^{k\zeta}}. \quad (31)$$

For $\mathbf{q}>1$ and $\zeta>0$, $\mathbf{q}$-digamma function $\varphi_{\mathbf{q}}$ can be given as

$$\varphi_{\mathbf{q}} = -\ln(\mathbf{q}-1)+\ln\mathbf{q}\left[\zeta-\frac{1}{2}-\sum_{k=0}^{\infty}\frac{\mathbf{q}^{-(k+\zeta)}}{1-\mathbf{q}^{-(k+\zeta)}}\right] = -\ln(\mathbf{q}-1)+\ln\mathbf{q}\left[\zeta-\frac{1}{2}-\sum_{k=0}^{\infty}\frac{\mathbf{q}^{-k\zeta}}{1-\mathbf{q}^{-k\zeta}}\right]. \quad (32)$$

Proposition 4. *Assume that σ_1, σ_2 are the real numbers such that $0<\sigma_1<\sigma_2$ and $0<\mathbf{q}<1$. Then, the following inequality is valid:*

$$\left|\varphi_{\mathbf{q}}\left(\frac{\sigma_1+\sigma_2}{2}\right)-\frac{1}{\sigma_2-\sigma_1}\int_{\sigma_1}^{\sigma_2}\varphi_{\mathbf{q}}(\varepsilon)\mathrm{d}\varepsilon\right| \leq \left(\frac{\sigma_2-\sigma_1}{2}\right)\left(\frac{5}{36}\right)\left\{\left|\varphi^{(1)}_{\mathbf{q}}(\sigma_1)\right|+\left|\varphi^{(1)}_{\mathbf{q}}(\sigma_2)\right|\right\}. \quad (33)$$

Proof. The assertion can be obtained immediately by using inequality (33) which follows immediately from equation (18), when $\hbar(\varepsilon)=\varphi_{\mathbf{q}}(\varepsilon)$ and $\varepsilon>0$ since $\hbar'(\varepsilon)=\varphi'_{\mathbf{q}}(\varepsilon)$ is convex on $(0,+\infty)$. □

3.3. Modified Bessel Function. We recall the first kind of modified Bessel function $\mathfrak{I}_m$, which has the series representation (see [28], p.77)

$$\mathfrak{I}_m(\zeta) = \sum_{n\geq 0}\frac{(\zeta/2)^{m+2n}}{n!\Gamma(m+n+1)}, \quad (34)$$

where $\zeta \in \mathfrak{R}$ and $m > -1$. Consider the function $\Omega_m(\zeta)$: $\mathfrak{R} \longrightarrow [1,\infty)$ defined by

$$\Omega_m(\zeta) = 2^m\Gamma(m+1)\zeta^{-m}\mathfrak{I}_m(\zeta), \quad (35)$$

where Γ is the gamma function.

The first-order derivative formula of $\Omega_m(\zeta)$ is given by [28]

$$\Omega'_m(\zeta) = \frac{\zeta}{2(m+1)}\Omega_{m+1}(\zeta), \quad (36)$$

and the second derivative can be easily calculated from (36) to be

$$\Omega''_m(\zeta) = \frac{\zeta^2\Omega_{m+2}(\zeta)}{4(m+1)(m+2)}+\frac{\Omega_{m+1}(\zeta)}{2(m+1)}. \quad (37)$$

Proposition 5. *Suppose that $m > -1$ and $0<\sigma_1<\sigma_2$. Then, we have*

$$\left|\frac{\sigma_1+\sigma_2}{4(m+1)}\Omega_{m+1}\left(\frac{\sigma_1+\sigma_2}{2}\right)-\frac{\Omega_m(\sigma_2)-\Omega_m(\sigma_1)}{\sigma_2-\sigma_1}\right| \\ \leq \frac{\sigma_2-\sigma_1}{2}\left(\frac{5}{36}\right)\left\{\left(\frac{\sigma_1^2\Omega_{m+2}(\sigma_1)}{4(m+1)(m+2)}+\frac{\Omega_{m+1}(\sigma_1)}{2(m+1)}\right)+\left(\frac{\sigma_2^2\Omega_{m+2}(\sigma_2)}{4(m+1)(m+2)}+\frac{\Omega_{m+1}(\sigma_2)}{2(m+1)}\right)\right\}. \tag{38}$$

Proof. Apply the inequality in (18) to the mapping $\hbar(\zeta)=\Omega_m'(\zeta)$, $\zeta>0$ (note that all assumptions are satisfied), and identities (36) and (37). □

4. Conclusion

The study dealt with investigating a new method to give fractional Simpson's estimates for differentiable convex functions by taking into account the kernel given by Alomari et al. [13]. Several related estimations by employing Hölder's inequality, bounds by using concave functions, and Jensen's integral inequalities are presented. Finally, a representation of Simpson's inequalities in terms of beta and q-digamma functions is depicted. Some estimation interims of modified Bessel functions are represented. Our technique is also plausible to give extensions for other fractional integral operators, e.g., k-Riemann–Liouville, Katugampola, conformable, and Atangana–Baleanu. Moreover, one can also extend these results in quantum calculus by exhibiting our method of arrangements of kernels for differentiable convex functions. All these results are open to be discussed for generalized convex functions and, in particular, can be given by using the method adopted in this article [12].

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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