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Research Article

Unique Fixed Point Results and Its Applications in Complex-Valued Fuzzy b-Metric Spaces

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The goal of this paper is to extend the concept of complex-valued fuzzy metric space to complex-valued fuzzy b-metric spaces and to discuss various existence results for fixed points to ensure their existence and uniqueness. To demonstrate the viability of the proposed strategies, a nontrivial example is used. Finally, applications to integral equations and initial value problems in mechanical engineering are discussed to demonstrate the superiority of the obtained results.

1. Introduction and Preliminaries

Fixed point theory combines topology, geometry, and analysis in an amazing way. Fixed point theory has emerged as a powerful tool in the study of nonlinear analysis in recent years. In fixed point theory and many other mathematical subjects, multiple separate objects are considered. As a result, mathematics is not only about numbers and shapes but also about prepositions, fluid flows, vector connections, and chemical interactions, among other things. Many researchers investigated the significance of various features of symmetry and demonstrated how they might be applied to many types of mathematical problems [1, 2]. There are several generalizations of the concept of metric spaces in the literature. Azam et al. developed the idea of complex-valued metric space and discovered that the Banach contraction principle may be applied to complex-valued metric spaces [3]. They studied its applications to complex integral equations. After that, fixed point theorems have been studied by many authors in complex-valued metric spaces [4–8].

The concept of b-metric spaces has been introduced by Bakhtin and Czerwik [9, 10]. Later on, many authors studied fixed point theorems for single and multivalued mappings in b-metric spaces for instance [11, 12]. In [13], the author generalized the concept of b-metric spaces by introducing the setting of complex-valued b-metric spaces. Many other researchers worked on complex-valued b-metric, and they extended generalized fixed point theorems in the sense of complex-valued b-metric spaces (see [14, 15] and the references therein).

The concept of fuzzy sets was given by Zadeh [2] and opened the door of new direction in mathematical research. Pao-Ming and Ying-Ming established the notion of fuzzy metric spaces [16]. Afterwards, George and Veeramani improved the settings of fuzzy metric spaces [17]. Heilpern introduced the concept of fuzzy mapping and obtained fixed point results for fuzzy mappings [18]. Heilpern's work was further extended by many authors, for instance, see [19–21]. Shukla et al. worked on the neighborhood structure of fuzzy fixed point [22]. Several other researchers worked on fuzzy metric spaces and obtained the generalizations of related results [23, 24].

George and Veeramani generalized the concept of fuzzy metric to the context of complex-valued fuzzy metric and obtained the complex-valued fuzzy version of Banach contraction mapping result in different forms [17]. Also, they obtain some related fixed point results with valid examples.

In this paper, we introduce the setting of complex-valued fuzzy b-metric spaces to generalize the setting of complex-valued b-metric space and establish the complex-valued fuzzy version of the Banach contraction principle. We also provide examples to back up our findings. The paper concludes with an application to integral and differential equation.

All over the manuscript we have symbolized the set of complex numbers by C. We mark some shortcut representation used in this manuscript, as t_c-norm for a complex-valued continuous triangular norm, CF b-metric for complex-valued fuzzy b-metric, and s.t. for such that.

Let $\mathscr{P} = \{(\xi, \rho): 0 \leq \xi < \infty, 0 \leq \rho < \infty\} \subset C$. The elements $(0,0), (1,1) \in \mathscr{P}$ are denoted by ϑ and ℓ, respectively. The set $\mathscr{P}_\vartheta = \{(\xi, \rho): 0 < \xi < \infty, 0 < \rho < \infty\}$. Clearly for $\varphi, \xi \in C, \xi \preceq \varphi$ iff $\xi - \varphi \in \mathscr{P}_\vartheta$. Let the unit closed complex interval be symbolized by $\mathscr{I} = \{(\xi, \rho): 0 \leq \xi \leq 1, 0 \leq \rho \leq 1\}$ and the open unit complex interval by $\mathscr{I}_0 = \{(\xi, \rho): 0 \leq \xi < 1, 0 \leq \rho < 1\}$.

Definition 1 (see [17]). Define an ordered relation $\preceq$ on C by $\varsigma_1 \preceq \varsigma_2$ if and only if $\varsigma_2 - \varsigma_1 \in \mathscr{P}$. The relations $\varsigma_1 \preceq \varsigma_2$ and $\varsigma_1 \prec \varsigma_2$ indicate that $\operatorname{Re}(\varsigma_1) \leq \operatorname{Re}(\varsigma_2), \operatorname{Im}(\varsigma_1) \leq \operatorname{Im}(\varsigma_2)$ and $\operatorname{Re}(\varsigma_1) < \operatorname{Re}(\varsigma_2), \operatorname{Im}(\varsigma_1) < \operatorname{Im}(\varsigma_2)$, respectively.

Let $B \subset C$. If there exists $\inf B$ such that it i the lower bound of B, that is, $\inf B \preceq a \forall a \in B$ and $v \preceq \inf B$ for every lower bound v of B, then $\inf B$ is called the greatest lower bound of B.

Definition 2 (see [25]). Let X be a nonempty set. A complex fuzzy set M is characterized by a mapping such that domain is X and the range in the closed unit complex interval $\mathscr{I}$.

Definition 3 (see [17]). A binary equation $\star : \mathscr{I} \times \mathscr{I} \longrightarrow \mathscr{I}$ is said to be complex-valued t-norm if the following conditions hold:

(1) $\xi_1 \star \xi_2 = \xi_2 \star \xi_1$

(2) $\xi_1 \star \xi_2 \preceq \xi_3 \star \xi_4$ whenever $\xi_1 \preceq \xi_3, \xi_2 \preceq \xi_4$

(3) $\xi_1 \star (\xi_2 \star \xi_3) = (\xi_1 \star \xi_2) \star \xi_3$

(4) $\xi \star \vartheta = \vartheta, \xi \star \ell = \xi$

for all $\xi, \xi_1, \xi_2, \xi_3, \xi_4 \in \mathscr{I}$.

Some fundamental examples of a t_c-norm are as follows:

(1) $\xi_1 \star_a \xi_2 = \{e_1 e_2, e_3, e_4\}$, for all $\xi_1 = (e_1, e_3), \xi_2 = (e_2, e_4) \in \mathscr{I}$

(2) $\xi_1 \star_b \xi_2 = \{\min\{e_1, e_2\}, \min\{e_3, e_4\}\}$, for all $\xi_1 = (e_1, e_3), \xi_2 = (e_2, e_4) \in \mathscr{I}$

(3) $\xi_1 \star_c \xi_2 = \{\max\{e_1 + e_2 - 1, 0\}, \max\{e_3 + e_4 - 1, 0\}\}$, for all $\xi_1 = (e_1, e_3), \xi_2 = (e_2, e_4) \in \mathscr{I}$

Definition 4 (see [17]). Let $(\mathscr{X}, M, \star)$ be a complex-valued fuzzy metric space. A sequence $\{\varphi_q\}$ in $\mathscr{X}$ is known as a Cauchy sequence if

$$\lim_{q \longrightarrow \infty} \inf_{d > q} M\left(\varphi_q, \varphi_d, t\right) = \ell \forall t \in \mathscr{P}_\vartheta. \tag{1}$$

The complex-valued fuzzy metric space $(\mathscr{X}, M, \star)$ is complete if every Cauchy sequence is convergent in $\mathscr{X}$.

Definition 5 (see [17]). A sequence is monotonic with respect to $\preceq$ if either $\varsigma_b \preceq \varsigma_{b+1}$ or $\varsigma_{b+1} \preceq \varsigma_b \forall b \in N$.

Lemma 6 (see [17]). *Let $(\mathscr{X}, M, \star)$ be a complex-valued fuzzy metric space. If $t, t' \in \mathscr{P}\vartheta$ and $t \preceq t'$, then $M(\varphi, u, t) \preceq M(\varphi, u, t') \forall \varphi, u \in \mathscr{X}$.*

Lemma 7 (see [17]). *Let $(\mathscr{X}, M, \star)$ be complex-valued fuzzy metric space. A sequence $\{\varphi_q\}$ in $\mathscr{X}$ converges to $v \in \mathscr{X}$ iff $\lim_{q \longrightarrow \infty} M(\varphi_q, v, t) = \ell$ holds $\forall t \in \mathscr{P}_\vartheta$.*

Remark 8 (see [17]). Let $\varphi_q \in \mathscr{P} \forall n \in N$ then:

(a) If the sequence $\{\varphi_q\}$ is monotonic with respect to $\preceq$ and there exist $\gamma, \eta \in \mathscr{P}$ with $\gamma^\circ \varphi_q \preceq \eta, \forall q \in \mathrm{N}$, then there exists $\varphi \in \mathscr{P}$ such that $\lim_{q \longrightarrow \infty} \varphi_q = \varphi$

(b) Although the partial ordering $\preceq$ is not a linear order on C, the pair $(C, \preceq)$ is a lattice

(c) If $\mathscr{X} \subset C$ and there exists $\gamma, \eta \in C$ with $\gamma \preceq s \preceq \eta \forall s \in \mathscr{X}$, then $\inf \mathscr{X}$ and $\sup \mathscr{X}$ both exist

Remark 9 (see [17]). Let $\varphi_q, \varphi'_q, \xi \in \mathscr{P}, \forall q \in N$, then

(a) If $\varphi_q \preceq \varphi'_q \preceq \ell \forall q \in N$ and $\lim_{q \longrightarrow \infty} \varphi_q = \ell$, then $\lim_{b \longrightarrow \infty} \varphi'_q = \ell$

(b) If $\varphi_q \preceq \xi \forall q \in N$ and $\lim_{b \longrightarrow \infty} \varphi_q = \varphi$, then $\varphi \preceq \xi$

(c) If $\xi \preceq \varphi_q \forall q \in N$ and $\lim_{q \longrightarrow \infty} \varphi_q = \varphi$, then $\xi \preceq \varphi$

Definition 10 (see [15]). Let $\mathscr{X}$ be a nonempty set and let $b \geq 1$ be a given real number. A function $\mathscr{D} : \mathscr{X} \times \mathscr{X} \longrightarrow C$ is called a complex-valued b-metric on $\mathscr{X}$ if, for all $\xi, \varphi, v \in C$, the following conditions are satisfied:

(i) $D(\xi, \varphi) \succeq 0$

(ii) $D(\xi, \varphi) = 0$ if and only if $\xi = \varphi$

(iii) $D(\xi, \varphi) = D(\varphi, \xi)$

(iv) $b[D(\xi, v) + D(v, \varphi)] \succeq D(\xi, \varphi)$

The pair $(\mathscr{X}, D)$ is called a complex-valued b-metric space.

Example 1 (see [15]). Let $\mathscr{X} = C$. Define the mapping $D : C \times C \longrightarrow C$ by $D(\xi, \varphi) = |\xi - \varphi|^2 + i|\xi - \varphi|^2$ for all $\xi, \varphi, v \in C$. Then, $(C, \mathscr{X})$ is complex-valued b-metric space with $b = 2$.

Definition 11 (see [17]). Let $\mathscr{X}$ be a nonempty set, $\star$ a continuous complex-valued t_C-norm, and M a complex fuzzy set on $\mathscr{X} \times \mathscr{X} \times \mathscr{P}_\theta \longrightarrow \mathscr{I}$ satisfying conditions:

(1) $0 \preceq M(\xi, \varphi, t)$

(2) $M(\xi, \varphi, t) = \ell$ for every $t \in \mathscr{P}_\vartheta$ if and only if $\xi = \varphi$

(3) $M(\xi, \varphi, t) = M(\varphi, \xi, t)$

(4) $M(\xi, \varphi, t) \star M(\varphi, \rho, t') \preceq M(\xi, \rho, t + t')$

(5) $M(\xi, \varphi, \star): \mathscr{P}_\vartheta \longrightarrow \mathscr{I}$ is continuous for all $\xi, \varphi, \rho \in \mathscr{X}$ and $t, t' \in \mathscr{P}_\vartheta$

Then, the triplet $(\mathscr{X}, M, \star)$ is said to be a complex-valued fuzzy metric space, and M is called a complex-valued fuzzy metric on $\mathscr{X}$. The functions $M(\xi, \varphi, t)$ denote the degree of nearness and the degree of nonnearness between ξ and φ with respect to the complex parameter t, respectively.

Example 2 (see [17]). Let $\mathscr{X} = \aleph$. Define $\star$ by $\varsigma' \star \varsigma'' = (s's'', u'u'')$ for all $\varsigma' = (s', u'), \varsigma'' = (s'', u'') \in \mathscr{I}$. Define complex fuzzy set M as

$$M(\xi, \varphi, t) = \begin{cases} \dfrac{\xi}{\varphi} \ell \text{ if } \xi \leq \varphi, \\ \dfrac{\varphi}{\xi} \ell \text{ if } \varphi \leq \xi, \end{cases} \tag{2}$$

for each $\xi, \varphi \in \mathscr{X}, \varsigma \in \mathscr{P}_\theta$. Then, $(\mathscr{X}, M, \star)$ is complex-valued fuzzy metric spaces.

2. Fixed Point Results in Complex-Valued Fuzzy b-Metric Spaces

We start this section with the following definition.

Definition 12. $(\mathscr{X}, M, \star, b)$ is said a complex-valued fuzzy b-metric space if $\mathscr{X}$ is an arbitrary set, $\star$ is a t_C-norm, and M is a fuzzy set on $\mathscr{X} \times \mathscr{X} \longrightarrow \mathscr{P}$ meeting the points below for all $\xi, \varphi \in \mathscr{X}, t, s > \vartheta$ and provided a number $b \pm 1$:

(1) $0 \preceq M(\xi, \varphi, t)$

(2) $M(\xi, \varphi, t) = \ell$ for every $t \in \mathscr{P}_\vartheta$ if and only if $\xi = \varphi$

(3) $M(\xi, \varphi, t) = M(\varphi, \xi, t)$

(4) $M(\xi, \varphi, t/b) \star M(\varphi, \rho, t'/b) \preceq M(\xi, \rho, (t + t'))$

(5) $M(\xi, \varphi, \star): \mathscr{P}_\vartheta \longrightarrow \mathscr{I}$ is continuous for all $\xi, \varphi, \rho \in \mathscr{X}$ and $t, t' \in \mathscr{P}_\vartheta$

Then, the triplet $(\mathscr{X}, M, \star)$ is said to be a complex-valued fuzzy metric space, and M is called a complex-valued fuzzy metric on $\mathscr{X}$.

Example 3. Let $M(\xi, \varphi, t)$ be a complex-valued fuzzy metric defined by $e(-|\varphi - \xi|^r/t)\ell$ such that $t > 1$ be a real number. Then, M is CF b-matric space with $b = 2^{r-1}$.

Proof. (1), (2), (3), and (5) are obvious. Here, we prove (4). For an arbitrary integer b, we have

$$\begin{aligned} |\xi - \rho| &\preceq \frac{b\left(t + t'\right)}{t} |\xi - \varphi| + \frac{b\left(t + t'\right)}{t'} |\varphi - \rho| \frac{|\xi - \rho|}{t + t'} \\ &\preceq \frac{b}{t} |\xi - \varphi| + \frac{b}{t'} |\varphi - \rho| \preceq \frac{|\xi - \varphi|}{t/b} + \frac{|\varphi - \rho|}{t'/b}. \end{aligned} \tag{3}$$

Since e^ξ is an increasing function for ξ, one can write

$$e^{|\xi - \rho|/t + t'} \preceq e^{|\xi - \varphi|/t/b} + e^{|\varphi - \rho|/t'/b}. \tag{4}$$

Thus, we have

$$\begin{aligned} &e^{-|\xi - \rho|/t + t'} \ell \succeq e^{-|\xi - \varphi|/t/b} + e^{-|\varphi - \rho|/t'/b} \ell, \\ &M\left(\xi, \rho, \left(t + t'\right)\right) \succeq M\left(\xi, \varphi, \frac{t}{b}\right) \star M\left(\varphi, \rho, \frac{t'}{b}\right). \end{aligned} \tag{5}$$

□

Remark 13. CF b -metric is the generalization of complex-valued fuzzy metric space. It is obvious from example that is every CF b -metric is complex-valued fuzzy metric for $b = 1$. Similarly, some important results like Lemmas 6 and 7 and definitions of convergence and Cauchy presented in Section 1 can also be defined in the same manner in CF b -metric space as mentioned in complex-valued fuzzy metric space.

Theorem 14. *Let $(\mathscr{X}, M, \star, b)$ be a complete CF b -metric space and let $\varsigma : \mathscr{X} \longrightarrow \mathscr{X}$ be mapping enjoying the following condition:*

$$\frac{\ell}{M(\varsigma\xi, \varsigma\rho, t)} - \ell \preceq q \left[\frac{\ell}{M(\xi, \rho, t)} - \ell \right], \tag{6}$$

for all $\xi, \rho \in \mathscr{X}$ and $q \in [0, 1)$. Then, ς has a unique fixed point $\tau\mathscr{X}$, for all $\tau \in \mathscr{P}_\vartheta$.

Proof. Let $\varphi_0 \in \mathscr{X}$. Define a sequence $\{\varphi_r\}$ in $\mathscr{X}$ by

$$\varphi_r = \varsigma\varphi_{r-1} \text{ for all } r \in N. \tag{7}$$

If $\varphi_0 = \varphi_{r-1}$ for some $r \in N$. Then clearly, ς has a fixed point. Suppose $\varphi_0 = \varphi_{r-1}$ for all $r \in \mathrm{N}$. To show that $\{\varphi_r\}$ is a Cauchy sequence, let define

$$B_r = \left\{ M\left(\varphi_i, \varphi_j, t\right) : j > i \right\} \subset \mathscr{I}. \tag{8}$$

Since $\vartheta \prec M(\varphi_i, \varphi_j, t)$, by Remark 8, the inf $B_r = \beta_r$ exists. For $j, i \in \mathrm{N}$, using (6), we get

$$\begin{aligned} &\frac{\ell}{M\left(\varphi_{i+1}, \varphi_{j+1}, t\right)} - \ell \\ &= \frac{\ell}{M\left(\varsigma\varphi_i, \varsigma\varphi_j, t\right)} - \ell \preceq q\left[\frac{\ell}{M\left(\varphi_i, \varphi_j, t\right)} - \ell\right] \\ &\preceq \frac{\ell}{M\left(\varphi_i, \varphi_j, t\right)} - \ell, \end{aligned} \tag{9}$$

which implies

$$\frac{\ell}{M\left(\varphi_{i+1}, \varphi_{j+1}, t\right)} \preceq \frac{\ell}{M\left(\varphi_i, \varphi_j, t\right)}. \tag{10}$$

Therefore, by definition, we get

$$\ell \preceq \beta_r \preceq \beta_{r+1} \preceq \vartheta, \text{for all } r \in \mathrm{N}. \tag{11}$$

Thus, $\{\varphi_r\}$ is monotonic in $\mathscr{P}$. Using Remark 8 and from (11), there exists $\ell^\star \in \mathscr{P}$, with

$$\lim_{r\infty} \beta_r = \ell^\star. \tag{12}$$

From inequality (9), we have

$$\frac{\ell}{M\left(\varphi_{i+1}, \varphi_{j+1}, t\right)} \preceq \frac{q\ell}{M\left(\varphi_i, \varphi_j, t\right)} + (1-q)\ell, \tag{13}$$

for all i, j and so $\ell/\beta_{i+1} \preceq q\ell/\beta_i + (1-q)\ell$ for every $i \in N$, which yields from (12)

$$(1-q)\ell \preceq (1-q)\ell \star \ell^\star. \tag{14}$$

Since $q \in [0,1)$ and applying Remark 9, we must obtained $\ell = \ell^\star$. Thus,

$$\lim_{r\longrightarrow\infty} \beta_r = \ell. \tag{15}$$

Hence,

$$\lim_{r\longrightarrow\infty} \inf_{j>i} M\left(\varphi_i, \varphi_j, t\right) = \ell, \text{for all } t \in \mathscr{P}_\vartheta. \tag{16}$$

Therefore, from (16), we have that $\{\varphi_r\}$ is a Cauchy sequence. From the completeness of $\mathscr{X}$ and Lemma 7, we get that there exists $\tau \in \mathscr{X}$ such that

$$\lim_{r\longrightarrow\infty} M(\varphi_r, \tau, t) = \ell, \text{for all } t \in \mathscr{P}_\vartheta. \tag{17}$$

Now for $t \in \mathscr{P}_\vartheta$ and $r \in \mathrm{R}$, it yields from (6) that

$$\frac{\ell}{M(\varsigma\varphi_r, \varsigma\tau, t)} - \ell \preceq q\left[\frac{\ell}{M(\varphi_r, \tau, t)} - \ell\right], \tag{18}$$

that is

$$M(\varsigma\varphi_r, \varsigma\tau, t) \succeq \frac{1}{(q/M(\varphi_r, \tau, t)) + (1-q)}. \tag{19}$$

Now, for any $t \in \mathscr{P}_\vartheta$,

$$\begin{aligned} M(\tau, \varsigma\tau, t) &\succeq M\left(\tau, \varphi_{r+1}, \frac{t}{2b}\right) \star M\left(\varphi_{r+1}, \varsigma\varphi_\tau, \frac{t}{2b}\right) \\ &= M\left(\tau, \varphi_{r+1}, \frac{t}{2b}\right) \star M\left(\varsigma\varphi_r, \varsigma\varphi_\tau, \frac{t}{2b}\right). \end{aligned} \tag{20}$$

Taking $r \longrightarrow \infty$ and using (17), (19), and Remark 9, we get that $M(\tau, \varsigma\tau, t) = \ell$ for all $t \in \mathscr{P}_\vartheta$; that is, $\varsigma\tau = \tau$.

Now, we have to show the uniqueness of fixed point τ of ς. On contrary, suppose ν be another fixed point of ς. Then, there exists $t \in \mathscr{P}_\vartheta$ such that $M(\tau, \nu, t) < \ell$, than from (6) we have

$$\frac{\ell}{M(\tau, \nu, t)} - \ell = \frac{\ell}{M(\varsigma\tau, \varsigma\nu, t)} - \ell \preceq q\left[\frac{\ell}{M(\tau, \nu, t)}\ell\right], \tag{21}$$

which is a contradiction. Therefore, we must obtain $M(\tau, \nu, t) = \ell$ for all $t \in \mathscr{P}_\vartheta$. Hence, $\tau = \nu$. □

Corollary 15. *Let $(\mathscr{X}, M, \star, b)$ be a complete CF b -metric space and let $\varsigma : \mathscr{X} \longrightarrow \mathscr{X}$ be mapping enjoying the following condition:*

$$\frac{\ell}{M(\varsigma^r\xi, \varsigma^r\rho, t)} - \ell \preceq q\left[\frac{\ell}{M(\xi, \rho, t)} - \ell\right], \tag{22}$$

for all $\xi, \rho \in \mathscr{X}$ and $q \in [0, 1)$. Then, ς has a unique fixed point $\tau\mathscr{X}$, for all $t \in \mathscr{P}_\vartheta$.

Proof. By the use of Theorem 14, ς^r has a fixed point τ as ς^r observes all conditions. But $\varsigma^r\varsigma\tau = \varsigma\varsigma^r\tau\varsigma\tau$, implies that $\varsigma\tau$ is another fixed point of ς^r. By uniqueness of fixed point, we have $\varsigma\tau = \tau$.As fixed point of ς is also a fixed point ofς. Thus, ς has a unique fixed point. □

Corollary 16. *Let $(\mathscr{X}, M, \star, b)$ be a complete CF b-metric space and let $\varsigma : \mathscr{X} \longrightarrow \mathscr{X}$ be mapping enjoying the following condition:*

$$\frac{\ell}{M(\varsigma^r\xi, \varsigma^r\rho, t)} - \ell \preceq q(t)\left[\frac{\ell}{M(\xi, \rho, t)} - \ell\right], \tag{23}$$

for all $\xi, \rho \in \mathscr{X}$ and $q : \mathscr{P}_\vartheta \longrightarrow [0, 1)$. Then, ς has a unique fixed point $\tau\mathscr{X}$, for all $t \in \mathscr{P}_\vartheta$.

Example 4. Let $\mathscr{X} = [0,\infty)$ and t-norm be defined by $c_1 \star c_2 = c_1 c_2$ for all $c_1 = (a_1, a_2), c_2 = (a_1, a_2) \in \mathscr{I}$. Define M as

$$M(\xi, \rho, t) = \left[\exp^{(\xi-\rho)^2/t}\right]^{-1} \ell \text{ for all } \xi, \rho \in \mathscr{X}, t \in \mathscr{P}_\vartheta. \tag{24}$$

Then, $(\mathscr{X}, M, \star)$ is a CF b-metric space. Define $\varsigma : \mathscr{X} \longrightarrow \mathscr{X}$ as

$$\varsigma(\xi)=\begin{cases} 0, \text{if } \xi = m, \\ \dfrac{\xi}{4}, \text{if } \xi \in (0, m), \\ \dfrac{\xi}{8}, \text{if } \xi \in (m,\infty). \end{cases} \quad (25)$$

Then, we have the following cases.

Case 1. If $\xi, \rho = m$, then $\varsigma\xi, \varsigma\rho = 0$.

Case 2. If $\xi = m$ and $\rho \in (0, m)$, then $\varsigma\xi = 0$ and $\varsigma\rho = \rho/4$.

Case 3. If $\xi = m$ and $\rho \in (m,\infty)$, then $\varsigma\xi = 0$ and $\varsigma\rho = \rho/8$.

Case 4. If $\xi \in [0, m)$ and $\rho \in (m,\infty)$, then $\varsigma\xi = \xi/4$ and $\varsigma\rho = \rho/8$.

Case 5. If $\xi \in [0, m)$ and $\rho \in [0, m)$, then $\varsigma\xi = \xi/4$ and $\varsigma\rho = \rho/4$.

Case 6. If $\xi \in [0, m)$ and $\rho = m$, then $\varsigma\xi = \xi/4$ and $\varsigma\rho = 0$.

Case 7. If $\xi \in (m,\infty)$ and $\rho = m$, then $\varsigma\xi = \xi/8$ and $\varsigma\rho = 0$.

Case 8. If $\xi \in (m,\infty)$ and $\rho \in (m,\infty)$, then $\varsigma\xi = \xi/8$ and $\varsigma\rho = \rho/8$.

The above-mentioned cases observe all conditions of Theorem 14 with $q \in [1/2, 1)$. Thus, the fuzzy contractive mapping ς has a unique fixed point, which is $(0, 0)$.

Theorem 17. *Let $(\mathscr{X}, M, \star, b)$ be a complete CF b -metric space with $t \preceq t \star t$ for $t \in \mathscr{I}_\vartheta$. Let $\varsigma : \mathscr{X} \longrightarrow \mathscr{X}$ be mapping enjoying the following conditions:*

(i) There exists $\varphi_0 \in \mathscr{X}$ and $\varepsilon \in \mathscr{I}_\vartheta$ such that $\ell - \varepsilon \preceq M(\varphi_0, \varsigma\varphi_0, t)$ for all $t \in \mathscr{P}_\vartheta$

(ii) There exists $q \in [0, 1)$ such that for all $\xi, \rho \in \mathscr{B}[\varphi_0, \varepsilon, t]$,

$$\frac{\ell}{M(\varsigma\xi, \varsigma\rho, t)} - \ell \preceq q\left[\frac{\ell}{M(\xi, \rho, t)} - \ell\right]. \quad (26)$$

Then, ς has a unique fixed point in $\mathscr{B}[\varphi_0, \varepsilon, t]$.

Proof. It is enough to proof that $\mathscr{B}[\varphi_0, \varepsilon, t]$ is complete and $\varsigma\varphi \in \mathscr{B}[\varphi_0, \varepsilon, t]$ for all $\varphi \in \mathscr{B}[\varphi_0, \varepsilon, t]$. Let $\{\varphi_r\}$ be a Cauchy sequence in $\mathscr{B}[\varphi_0, \varepsilon, t]$. Since $\mathscr{X}$ is complete thus by the use of Lemma 7, there exists $u \in \mathscr{X}$ such that

$$\lim_{r\longrightarrow\infty} \mathrm{M}(\varphi_r, u, t) = \ell, \quad (27)$$

for all $t \in \mathscr{P}_\vartheta$. Now for all $i, r \in \mathrm{N}$,

$$M\left(\varphi_0, u, t + \frac{t}{i}\right) \succeq M\left(\varphi_0, \varphi_r, \frac{t}{b}\right) \star M\left(\varphi_0, \varphi_r, \frac{t}{ib}\right). \quad (28)$$

Since $\varphi_r \in \mathscr{B}[\varphi_0, \varepsilon, t]$ for every $r \in \mathrm{N}$, also $\lim_{r\longrightarrow\infty} M(\varphi_r, u, t) = \ell$. By using the properties of t-norm and Remark 9, we obtain

$$M\left(\varphi_0, u, t + \frac{t}{i}\right) \succeq (\ell - r) \star \ell = \ell - r, \text{forever } i \in \mathrm{N}. \quad (29)$$

Taking $\lim_{i\longrightarrow\infty}$ and using Remark 9, we get $M(\varphi_0, u, t) \pm \ell - r$. Therefore, $u \in \mathscr{B}[\varphi_0, \varepsilon, t]$.

For every $\varphi \in \mathscr{B}[\varphi_0, \varepsilon, t]$, it yields from (26)

$$\frac{\ell}{M(\varsigma\varphi_0, \varsigma\varphi, t)} - \ell \preceq q\left[\frac{\ell}{M(\varphi_0, \varphi, t)} - \ell\right], \quad (30)$$

that is

$$M(\varsigma\varphi_0, \varsigma\varphi, t) \succeq \frac{1}{(q/M(\varphi_0, \varphi, t)) + (1 - q)}. \quad (31)$$

Thus, for all $i \in \mathrm{N}$, we get

$$\begin{aligned} M\left(\varphi_0, \varsigma\varphi, t + \frac{t}{i}\right) &\pm M\left(\varphi_0, \varsigma\varphi_0, \frac{t}{ib}\right) \\ &\succeq M\left(\varsigma\varphi_0, \varsigma\varphi, \frac{t}{b}\right) \succeq (\ell - \varepsilon) \star \left[\frac{1}{(q/M(\varphi_0, \varphi, t/b)) + (1 - q)}\right] \\ &\succeq (\ell - \varepsilon) \star \left[\frac{1}{(q/(\ell - \varepsilon)) + (1 - q)}\right] \succeq (\ell - \varepsilon) \star (\ell - \varepsilon). \end{aligned} \quad (32)$$

Taking $\lim_{i\longrightarrow\infty}$ and using Remark 9, we have

$$M(\varphi_0, \varsigma\varphi, t) \succeq (\ell - \varepsilon). \quad (33)$$

Therefore, $\varsigma\varphi \in \mathscr{B}[\varphi_0, \varepsilon, t]$. □

Theorem 18. *Let $(\mathscr{X}, M, \star, b)$ be a complete CF b -metric space such that for any sequence $\{t_r\} \in \mathscr{P}_\vartheta$ with $\lim_{r\longrightarrow\infty}\{t_r\} = \infty$, we get $\lim_{r\longrightarrow\infty} \inf_{\rho\in\mathscr{X}} M(\xi, \rho, \{t_r\}) = \ell$, for all $\xi \in \mathscr{X}$. Let $\varsigma : \mathscr{X} \longrightarrow \mathscr{X}$ be a mapping observing that*

$$M(\varsigma\xi, \varsigma\rho, \delta t) \succeq M(\xi, \rho, t), \quad (34)$$

for all $t \in \mathscr{P}_\vartheta$, where $0 < \delta < 1$. Then, ς has a unique fixed point in $\mathscr{X}$.

Proof. Let $\varphi_0 \in \mathscr{X}$. Define a sequence $\{\varphi_r\}$ in $\mathscr{X}$ by

$$\varphi_r = \varsigma\varphi_{r-1} \text{ for all } r \in \mathrm{N}. \quad (35)$$

If $\varphi_0 = \varphi_{r-1}\xi$ for some $r \in \mathrm{N}$. Then clearly, ς has a fixed point. Suppose $\varphi_0 \neq$ for all $r \in \mathrm{N}$. To show that $\{\varphi_r\}$ is a

Cauchy sequence, let define

$$B_r = \{M(\varphi_r, \varphi_s, t): s > r\} \subset \mathcal{I}. \quad (36)$$

Since $\vartheta \prec M(\varphi_r, \varphi_s, t)$, by Remark 8, the inf $B_r = \beta_r$ exists. For $s, r \in \mathrm{N}$, by the use of (??) and Lemma 6, we get

$$\begin{aligned} M(\varphi_{r+1}, \varphi_{s+1}, t) &\succeq M(\varphi_{r+1}, \varphi_{s+1}, \delta t) \succeq M(\varsigma\varphi_r, \varsigma\varphi_s, \delta t) \\ &\succeq M(\varsigma\varphi_r, \varsigma\varphi_s, t), \end{aligned} \quad (37)$$

which yields

$$M(\varsigma\varphi_r, \varsigma\varphi_s, t) \preceq M(\varphi_{r+1}, \varphi_{s+1}, t) \text{ for } s > r. \quad (38)$$

Therefore, by definition, we obtain

$$\vartheta \preceq \beta_r \preceq \beta_{r+1} \preceq \ell, \text{ for all } r \in \mathrm{N}. \quad (39)$$

Hence, $\{\beta_r\}$ is monotonic in $\mathscr{P}$, and by the use of Remark 8 and (39), there exists ℓ^* such that

$$\lim_{r \longrightarrow \infty} \beta_r = \ell^*. \quad (40)$$

For $t \in \mathscr{P}_\vartheta$, once again from (34), we have

$$\begin{aligned} \beta_{r+1} &= \inf_{s>r} M(\varphi_{r+1}, \varphi_{s+1}, t) \succeq \inf_{s>r} M\left(\varphi_r, \varphi_s, \frac{t}{\delta}\right) \\ &= \inf_{s>r} M\left(\varsigma\varphi_r, \varsigma\varphi_s, \frac{t}{\delta}\right) \succeq \inf_{s>r} M\left(\varphi_{r-1}, s_{r-1}, \frac{t}{\delta^2}\right) \\ &= \inf_{s>r} M\left(\varsigma\varphi_{r-2}, \varsigma\varphi_{s-2}, \frac{t}{\delta^2}\right) \succeq \inf_{s>r} M\left(\varsigma\varphi_{r-2}, \varsigma\varphi_{r-2}, \frac{t}{\delta^3}\right) \\ &\succeq \cdots \succeq \inf_{s>r} M\left(\varphi_0, \varphi_{s-r}, \frac{t}{\delta^{r+1}}\right), \end{aligned} \quad (41)$$

for all $r \in \mathrm{N}$ and $t \in \mathscr{P}_\vartheta$, we have

$$\begin{aligned} \beta_{r+1} &= \inf_{s>r} M(\varphi_{r+1}, \varphi_{s+1}, t) \succeq \inf_{s>r} M\left(\varphi_0, \varphi_{s-r}, \frac{t}{\delta^{r+1}}\right) \\ &\succeq \inf_{s>r} \mathrm{M}\left(\varphi_0, \rho, \frac{t}{\delta^{r+1}}\right). \end{aligned} \quad (42)$$

As $\lim_{r \longrightarrow \infty} t/\delta^{r+1} = \infty$, using (40) and assumption, we get

$$\ell^* \pm \lim_{r \longrightarrow \infty} \inf_{\rho \in \mathscr{X}} \mathrm{M}(\varphi_r, \varphi_s, t) \succeq = \ell. \quad (43)$$

From (40) and (43)

$$\lim_{r \longrightarrow \infty} \beta_r = \ell. \quad (44)$$

Thus, $\{\varphi_r\}$ is a Cauchy sequence in $\mathscr{X}$. Since $\mathscr{X}$ is complete, by Lemma 7, there exists $u \in \mathscr{X}$ such that

$$\lim_{r \longrightarrow \infty} M(\varphi_r, u, t) = \ell. \quad (45)$$

For $t \in \mathscr{P}_\vartheta$, (34) yields that

$$\begin{aligned} M(u, \varsigma u, t) &\succeq M\left(u, \varphi_{r+1}, \frac{t}{2b}\right) \star M\left(\varphi_{r+1}, \varsigma u, \frac{t}{2b}\right) \\ &\succeq M\left(u, \varphi_{r+1}, \frac{t}{2b}\right) \star M\left(\varsigma\varphi_r, \varsigma u, \frac{t}{2b}\right) \\ &\succeq M\left(u, \varphi_{r+1}, \frac{t}{2b}\right) \star M\left(\varphi_r, u, \frac{t}{2b\delta}\right). \end{aligned} \quad (46)$$

Taking $\lim_{r \longrightarrow \infty}$ and by (45) and Remark 9, we have $M(u, \varsigma u, t) = \ell$; that is, $\varsigma u = u$.

Now to investigate the uniqueness of fixed point, let on contrary that $v \in \mathscr{X}$ be any other fixed point of ς. So there exist $t \in \mathscr{P}_\vartheta$ with $M(u, v, t) = \ell$; then, (34) yields

$$M(u, v, t) = M(\varsigma u, \varsigma v, t) \succeq M\left(u, v, \frac{t}{\delta}\right). \quad (47)$$

Continuing this way, we obtain

$$M(u, v, t) \succeq M\left(u, v, \frac{t}{\delta^r}\right) \succeq \inf_{\rho \in \mathscr{X}} \mathrm{M}\left(u, v, \frac{t}{\delta^r}\right). \quad (48)$$

Using $\lim_{r \longrightarrow \infty} t/\delta^r = \infty$, it follows that $M(u, v, t) \succeq \ell$, which is contradiction. Thus, $M(u, v, t) = \ell$; that is, $u = v$. □

Example 5. Let $\mathscr{X} = [0, 1]$ and t-norm be defined by $c_1 \star c_2 = c_1 c_2$ for all $c_1 = (a_1, a_2), c_2 = (a_1, a_2) \in \mathscr{I}$. Define M as

$$M(\xi, \rho, t) = \exp^{-|\xi - \rho|/t} \ell \text{ for all } \xi, \rho \in \mathscr{X}, t \in \mathscr{P}_\vartheta. \quad (49)$$

Then, $(\mathscr{X}, \mathrm{M}, \star)$ is a CF b-metric space. Define $\varsigma : \mathscr{X} \longrightarrow \mathscr{X}$ as

$$\varsigma(\xi) = \begin{cases} 0, \text{ if } \xi \in \left[0, \dfrac{1}{2}\right), \\ \dfrac{\xi}{14}, \text{ if } \xi \in \left(\dfrac{1}{2}, 1\right]. \end{cases} \quad (50)$$

For $\lim_{t \longrightarrow \infty} \mathrm{M}(\xi, \rho, t) = \lim_{t \longrightarrow \infty} \exp^{-|\xi - \rho|/t} \ell = \ell$, we obtain that for all values of $\mathscr{X}$ we have $M(\varsigma\xi, \varsigma\rho, \delta t) \pm M(\varsigma\xi, \varsigma\rho, t)$, and for only 0, we have $\lim_{t \longrightarrow \infty} \inf_{\rho \in \mathscr{X}} M(\xi, \rho, t_r) = \exp^0 \ell = \ell$. Thus, all conditions of Theorem 18 are satisfied so, $(0, 0)$ is a unique fixed point of ς.

Example 6. Let $\mathscr{X} = \mathscr{C}([1, 3], R)$, $A > 0$ and for every $\xi, \rho \in \mathscr{X}$ let

$$\mathrm{M}(\xi, \rho, t) = \exp^{-|\xi - \rho|/t} \ell. \quad (51)$$

Let define $\varsigma : \mathscr{X} \longrightarrow \mathscr{X}$ by

$$\varsigma(\xi(\tau)) = 4 + \int_1^\tau (\xi(v) + \rho(v))e^{v-1}dv, t \in [1, 3]. \tag{52}$$

For every $\xi, \rho \in \mathscr{X}$

$$\begin{aligned} M(\varsigma\xi, \varsigma\rho, t) &= \exp^{-|\varsigma\xi(\tau)-\varsigma\rho(\tau)|/t}\ell = \exp^{-\int_1^\tau \max_{\tau\in[1,3]}|\varsigma\xi(\tau)-\varsigma\rho(\tau)|/t}\ell \\ &\geq \exp^{-\int_1^\tau \max_{\tau\in[1,3]}|\varsigma\xi(v)-\varsigma\rho(v)|e^2/t}\ell \geq 2e^2 M(\xi, \rho, t). \end{aligned} \tag{53}$$

Similarly

$$M(\varsigma^r\xi, \varsigma^r\rho, t) \geq \frac{2^r}{r!}e^{2r}M(\xi, \rho, t). \tag{54}$$

Note that

$$e^{2r}\frac{2^r}{r!} = \begin{cases} 537.9 \text{ if } r = 3, \\ 5{,}873.7 \text{ if } r = 5, \\ 1.31 \text{ if } r = 37, \\ 0.202 \text{ if } r = 39. \end{cases} \tag{55}$$

Thus, all conditions of Corollary 15 are satisfied for $q = 0.202$ and $r = 39$, so ς has a fixed point which is a solution of the integral equation

$$\xi(\tau) = 4 + \int_1^\tau (\xi(v) + \rho(v))e^{v-1}dv, t \in [1, 3], \tag{56}$$

or the differential equation

$$\xi'(\tau) = (\xi + \tau^2)e^{\tau-1}\tau \in [1, 3], \xi(1) = 4. \tag{57}$$

3. Application

Integral equations have plenty applications in many scientific fields. It is a ripely rising field in abstract theory. One of its significant approach in the study of integral equations is to apply fixed point results to the function defined by the right-hand side of the equation or to develop homotopy methods, which are highly considered in fixed point theory to find the approximate solution. In this section, firstly, we study application of our main Theorem 14 the existence of unique solution to Fredholm integral equation.

Theorem 19. *Let $\Xi = \mathscr{C}([0, m], R)$ be the spaces of continuous real valued functions defined on interval $[0, m]$, where $m > 0$. The Fredholm integral equation is*

$$z(t) = \int_0^m \mathscr{K}(t, \delta, z(\delta))d\delta. \tag{58}$$

Let $\Xi = \mathscr{C}[0, m, R]$ and $M : \Xi \times \Xi \times \mathscr{I} \longrightarrow \mathscr{I}$ be a CF b -metric defined as follows:

$$M(y, z, c) = \frac{c}{c + |y - z|^2}\ell, y, z \in \mathscr{X}, c > 0. \tag{59}$$

If there exists $q \in (0, 1)$ with

$$\Theta(y, z)(t) \geq \frac{1}{q}\Lambda(y, z)(t), \tag{60}$$

where

$$\begin{aligned} \Theta(y, z)(t) &= \frac{c}{c + \left|\int_0^m \mathscr{K}(t, \delta, y(\delta))d\delta - \int_0^m \mathscr{K}(t, \delta, z(\delta))d\delta\right|^2}\ell, \\ \Lambda(y, z)(t) &= \frac{c}{c + |y(t) - z(t)|^2}\ell, \end{aligned} \tag{61}$$

holds. Then, (58) has a unique solution in $\mathscr{X}$.

Proof. Let $\Gamma : \Xi \longrightarrow \Xi$ define as

$$\Gamma z(t) = \int_0^m \mathscr{K}(t, \delta, z(\delta))d\delta. \tag{62}$$

Then

$$|\Gamma y - \Gamma z|^2 = \left|\int_0^m \mathscr{K}(t, \delta, y(\delta))d\delta - \int_0^m \mathscr{K}(t, \delta, z(\delta))d\delta\right|^2. \tag{63}$$

For all $y, z \in \mathscr{X}$, we have

$$\frac{\ell}{\Theta(y, z)(t)} \preceq \frac{q\ell}{\Lambda(y, z)(t)}, \tag{64}$$

so,

$$\frac{\ell}{\Theta(y, z)(t)} - \ell \preceq \frac{q\ell}{\Lambda(y, z)(t)} - \ell \preceq q\left(\frac{\ell}{\Lambda(y, z)(t)} - \ell\right), \tag{65}$$

which implies that

$$\begin{aligned} &\frac{\ell}{c/c + \left|\int_0^m \mathscr{K}(t, \delta, y(\delta))\delta - \int_0^m \mathscr{K}(t, \delta, z(\delta))d\delta\right|^2} - \ell \\ &\preceq q\left(\frac{\ell}{c/c + |y(t) - z(t)|^2} - \ell\right). \end{aligned} \tag{66}$$

Therefore,

$$\frac{\ell}{M(\Gamma y, \Gamma z, c)} - \ell \preceq q\left(\frac{\ell}{M(y, z, c)} - \ell\right). \tag{67}$$

Since all conditions of Theorem 14 are satisfied, thus (58) has a unique solution in $\mathcal{X}$. □

Next, we study the application of Theorem 18, in mechanical engineering, since the system of auto mobile suspension is an achievable application for the system of spring mass in the field of engineering. We are going to study the motion of an auto mobile spring when its motion is upon a craggy and cleft road, where the forcing term is the craggy road and bumps noticed provide the absorbing. Tension, gravity, and earth quick are the possible external forces acting on the system. We express spring mass by κ and the external force acting on it by Θ. Then the following initial value problem represents the damped motion of the spring mass system under the action of external force Θ.

$$\begin{cases} \kappa \dfrac{d^2\bar{y}}{dt^2} + \pi \dfrac{d\bar{y}}{dt} = \Theta(t, \bar{y}(t)) = 0, \\ \bar{y}(0) = 0, \\ \bar{y}'(0) = 0, \end{cases} \tag{68}$$

where $\pi > 0$ express the damping constant and $\Theta : [0, \phi] \times \bar{R}^+ \longrightarrow \bar{R}$ is a continuous mapping. Clearly, the problem (68) is equivalent to the following integral equation

$$\bar{y}(t) = \int_0^\phi \Lambda(t, \delta)\Theta(\delta, \bar{y}(\delta))d\delta, \text{with } t, \delta \in [0, \phi], \tag{69}$$

where $\Lambda(t, \delta)$ represents the corresponding Green's function and defined as

$$\Lambda(t, \delta) = \begin{cases} \dfrac{1 - e^{\rho(t-\delta)}}{\rho}, \text{for } 0 \le \delta \le t \le \phi, \\ 0 \text{ for } 0 \le t \le \delta, \end{cases} \tag{70}$$

where $\rho = \pi/\kappa$ is a constant ratio. Consider the set of real valued functions $\bar{Y} = \mathscr{C}([0, \phi], \mathrm{R})$. For $b > 1$, consider CF b-mertic space defined by

$$M(y, z, c) = e^{-\sup_{n\in[0,1]} |\bar{y}(t)-\bar{z}(t)|^2/c}, \tag{71}$$

for all $y, z \in \bar{Y}$. WE have to show that problem (68) has a solution iff there exists $\bar{y}^*$ in $\bar{Y}$, a solution of the integral equation (69).

Theorem 20. *Consider problem (68), suppose the following conditions are satisfied:*

(i) $|\Theta(\delta, \bar{y}(\delta)) - \Theta(\delta, \bar{z}(\delta))|^2 \le |\bar{y}(\delta), \bar{z}(\delta)|^2$

(ii) $\int_0^\phi \Lambda(t, \delta) \le 1$

Then, the integral equation (69) has a unique solution in $\bar{Y}$.

Proof. Let define an operator $\Gamma : \bar{Y} \longrightarrow \bar{Y}$

$$\Gamma\bar{y}(t) = \int_0^\phi \Lambda(t, \delta)\Theta(\delta, \bar{y}(\delta))d\delta, \text{with } t, \delta \in [0, \phi]. \tag{72}$$

Now,

$$\begin{aligned} e^{-\sup_{n\in[0,1]} |\Gamma\bar{y}(t)-\Gamma\bar{z}(t)|^2/\lambda c} &\ge e^{-\sup_{n\in[0,1]} \int_0^\phi \Lambda(t,\delta)|\Theta(\delta,\bar{y}(\delta)-\Theta(\delta,\bar{y}(\delta))|^2 d\delta)/\lambda c} \\ &\ge e^{-\sup_{n\in[0,1]} |\Theta(\delta,\bar{y}(\delta)-\Theta(\delta,\bar{y}(\delta))|^2 d\delta)/\lambda c} \\ &\ge e^{-\sup_{n\in[0,1]} |\bar{y}(\delta),\bar{z}(\delta)|^2/\lambda c}, \end{aligned} \tag{73}$$

this yields that

$$e^{-\sup_{n\in[0,1]} |\Gamma\bar{y}(t)-\Gamma\bar{z}(t)|^2/\lambda c} \ell \ge e^{-\sup_{n\in[0,1]} |\bar{y}(\delta),\bar{z}(\delta)|^2/\lambda c} \ell. \tag{74}$$

Consequently, we get

$$M(\Gamma\bar{y}, \Gamma\bar{z}, \lambda c) \succeq M(\bar{y}, \bar{z}, c). \tag{75}$$

Thus, by Theorem 18, we obtained the existence of unique solution to integral equation (69). □

4. Conclusion

In this article, we presented the generalization of CF b -metric space and successfully obtained the generalization of Banach contraction principle to the new established setting herein. In support of our obtained results, we have constructed some examples, and with the help of derived result, we guaranteed the existence of unique solution to integral equation, which makes it possible for more integral equations to be verified in such conditions.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflict of interest.

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