Hindawi
Journal of Function Spaces
Volume 2022, Article ID 2187810, 10 pages
https://doi.org/10.1155/2022/2187810

Research Article

Existence of Two Solutions for a Critical Elliptic Problem with Nonlocal Term in $\mathbb{R}^4$

Khadidja Sabri,[1] Atika Matallah,[2] Safia Benmansour,[2] and Mohammed El Mokhtar Ould El Mokhtar [3]

[1] *University of Oran 2, Laboratory of Analysis and Control of Partial Differential Equations of Sidi Bel Abbes-Algeria, Algeria*
[2] *Higher School of Management-Tlemcen, Laboratory of Analysis and Control of Partial Differential Equations of Sidi Bel Abbes-Algeria, Algeria*
[3] *Qassim University, College of Science, Department of Mathematics, BO 6644, Buraidah: 51452, Saudi Arabia*

Correspondence should be addressed to Mohammed El Mokhtar Ould El Mokhtar; med.mokhtar66@yahoo.fr

Received 18 March 2022; Accepted 4 June 2022; Published 20 June 2022

Academic Editor: Pasquale Vetro

In this paper, we prove the existence of two positive solutions for a critical elliptic problem with nonlocal term and Sobolev exponent in dimension four.

1. Introduction

In this work, we are mainly concerned by the existence and the multiplicity of solutions for the following critical elliptic nonlocal problem:

$$(\mathscr{P})_{a,b}\begin{cases} -\left(a\int_\Omega |\nabla u|^2 dx + b\right)\Delta u = u^3 + f & \text{in } \Omega, \\ u = 0 & \text{on} \partial\Omega, \end{cases} \quad (1)$$

where Ω is a smooth bounded domain of $\mathbb{R}^4$, a and b are positive constants, and f belongs to H^{-1} (H^{-1} is the dual of $H_0^1(\Omega)$) satisfying suitable condition specified afterward.

In our setting, the Laplacian operator is associated to Kirchhoff term $a\int_\Omega |\nabla u|^2 dx + b$, which contains an integral over the entire domain Ω, this implies that the equation in $(\mathscr{P})_{a,b}$ is no longer a pointwise identity and so the problem turns to be nonlocal. This fact brings some mathematical difficulties in the search of the solution, and the solvability of this kind of problems has been under various authors' attention; so, some classical investigations can be seen in the works [1, 2] and the references therein.

Such nonlinear Kirchhoff's equations can be used for describing the dynamic for an axially moving string and was first formulated by Kirchhoff himself [3] in 1883, he take into account the changes in length of the strings produced by transverse vibrations, and his model can be seen as a generalization of the classical D'Alembert wave equation for free vibrations of elastic strings.

Problems which involve nonlocal operator have been widely studied due to their numerous and relevant applications in various fields of sciences. In particular, Kirchhoff type problems proved to be valuable tools for modeling several physical and biological phenomena, and many works have been made to ensure the existence of solutions for such problems; we quote in particular the article of Lions [4]. Since this famous paper, very fruitful development has given rise to many works on this advantageous axis, and in most of them, the used approach relies on topological methods. However, just few improvements were held concerning the multiplicity of solutions. In [5], Maia obtained a multiplicity of solutions for a class of $p(x)$-Choquard equations with a nonlocal and nondegenerate Kirchhoff term by using truncation arguments and Krasnoselskii's genus. In [6], Vetro studied the existence of two different notions of solutions by using Galerkin approximation method, jointly with the theory of pseudomonotone operators.

With this regard, variational approach was solicited instead of topological methods to solve this kind of problems and also to prove the existence of multiple solutions; we refer interested readers to the works [7–10].

We begin by giving an overview about the previous research related to the problem $(\mathscr{P})_{a,b}$ which can be written in the more general form,

$$(\tilde{\mathscr{P}})_{a,b}\begin{cases} -\left(a\int_\Omega |\nabla u|^2 dx + b\right)\Delta u = \varphi(\lambda, x, u) & \text{in}\,\Omega, \\ u = 0 & \text{on}\partial\Omega, \end{cases} \tag{2}$$

where Ω is a smooth bounded domain of $\mathbb{R}^N, N \geq 3$.

Without nonlocal term $(a = 0)$, much interest has grown on problems involving critical exponents, and there are many publications dealing with the existence of solutions, starting from the celebrated paper by Brézis and Nirenberg [11] when $\varphi(\lambda, x, u) = |u|^{2^*-2}u + \lambda u$ and $2^* = 2N/N - 2$ are the critical Sobolev exponent. For convenience of the reader, we give a brief summary of these results: they established existence results in dimension $N = 3$ when Ω is a ball namely, and they ensure the existence of a positive constant λ_0 such that the problem $(\tilde{\mathscr{P}})_{0,1}$ admits a positive solution for $\lambda \in (\lambda_0/4, \lambda_1)$, where λ_1 is the first eigenvalue of the operator $-\Delta$. In higher dimensions $N \geq 4$, they proved the existence of a positive solution for λ sufficiently small, i.e., $\lambda \leq \lambda_0$ and no positive solution for $\lambda > \lambda_0$ and Ω a star-shaped domain.

When $\varphi(\lambda, x, u) = |u|^{2^*-2}u + \lambda u^q, 1 < q < 2^*$, Ambrosetti et al. [12] established a multiplicity result in a bounded domain of $\mathbb{R}^N, N \geq 3$ indeed, they ensured the existence of a positive constant λ_0 such that the problem $(\tilde{\mathscr{P}})_{0,1}$ admits two positive solutions for $\lambda \in (0, \lambda_*)$, a positive solution for $\lambda = \lambda_*$ and no positive solution for $\lambda > \lambda_*$.

For the nonhomogeneous case, namely, when $\varphi(\lambda, x, u) = |u|^{2^*-2}u + f(x)$, Tarantello [13] proved the existence of at least two solutions when f satisfies

$$(f)_1 \int_\Omega f u dx < \frac{4}{N-2}\left(\frac{N-2}{N+2}\right)^{(N+2)/4} \|u\|^{(N+2)/4}, \tag{3}$$

for all $u \in H_0^1(\Omega)$, $\int_\Omega |u|^{2^*} dx = 1$.

We emphases that the extension of the previous results to the nonlocal case, namely, for elliptic problems driven by Kirchhoff type operator are not obvious in high dimensions $N \geq 4$. Therefore, no improvement was hold concerning the multiplicity of solutions in this case.

For the case $a > 0$ and $\varphi(\lambda, x, u) = \mu|u|^{2^*-2}u + \lambda u$, Naimen in [14] treated the problem $(\tilde{\mathscr{P}})_{a,b}$ for $N = 3$ and obtained homologous results than the ones obtained by Brézis and Niremberg [11] in the nonlocal case under a suitable condition on a.

In dimension four, Naimen in [15] used variational methods to explore problem $(\tilde{\mathscr{P}})_{a,b}$ and showed that $(\tilde{\mathscr{P}})_{a,b}$ admits a positive solution when $a > 0, b \geq 0$.

In the same order of ideas and still in the nonlocal case $(a > 0)$, Lei et al. in [16] and Liao et al. in [17] extended the findings of [12] to a more general setting, namely, with the Kirchhoff operator; they established a multiplicity result in dimensions three and four, respectively.

Benmansour and Bouchekif [8] generalized the results obtained by Tarantello [13] to the nonlocal case in dimension three. Indeed, they have shown the existence of two solutions under a sufficient condition on f by introducing the Nehari manifold.

A natural question is to know whether the multiplicity result persists in the case of dimension four.

In the current paper, our main purpose inspired by [8] is to see that the result obtained in [8] can be extended to dimension four. We emphases that our results are new and complement the above works.

In order to study $(\mathscr{P})_{a,b}$, we shall work with the space $H = H_0^1(\Omega)$ endowed with the norm $\|u\| = (\int_\Omega |\nabla u|^2 dx)^{1/2}$, we use also the following notation: $\|u\|_- := \|u\|_{H^{-1}}$, $\|u\|_p := (\int_\Omega |u|^p dx)^{1/p}$ for $1 \leq p < \infty$, C and C_i denote generic positive constants whose exact values are not important, $B_H(0, r) := \{u \in H : \|u\| < r\}$ is the ball of center 0 and radius r, $o_n(1)$ denotes any quantity which tends to zero as n tends to infinity, and S is the best Sobolev constant defined by

$$S := \inf_{u \in H \setminus \{0\}} \frac{\|u\|^2}{\|u\|_4^2}. \tag{4}$$

To state the main results, we define

$$\gamma_{a,b,f} = b^{3/2} \inf_{\substack{u \in H \\ \|u\|_4 = 1}} \left\{ \frac{2}{3^{3/2}} \left(\frac{\|u\|^6}{\|u\|_4^4 - a\|u\|^4} \right)^{1/2} - \int_\Omega \frac{f}{b^{3/2}} u dx \right\}, \tag{5}$$

where $b > 0$, $a < S^{-2}$ a small enough positive number and f belongs to $H^{-1} \setminus \{0\}$.

The main results are concluded as the following theorems, which are news for the case when $a \neq 0$.

Theorem 1. *Assume that $\gamma_{a,b,f} > 0$. Then, the problem $(\mathscr{P})_{a,b}$ admits a local minimal solution u_0 with $I(u_0) < 0$. Furthermore $u_0 \geq 0$ for $f \geq 0$.*

Theorem 2. *Assume that $\gamma_{a,b,f} > 0$. Then, the problem $(\mathscr{P})_{a,b}$ admits another solution u_1 with $I(u_1) > 0$. Furthermore, $u_1 \geq 0$ for $f \geq 0$.*

Notice that $\gamma_{a,b,f} \geq 0$ if

$$\int_\Omega fudx < 2\left(\frac{b}{3}\right)^{3/2}\left(\frac{\|u\|^6}{\|u\|_4 - a\|u\|^4}\right)^{1/2} \text{for all } u \in H. \quad (6)$$

Moreover, the assumption $\gamma_{a,b,f} > 0$ certainly holds if f satisfies certain conditions, for example,

$$(f)_2 \int_\Omega fu < 2\left(\frac{b}{3}\right)^{3/2}\|u\|^3, \quad (7)$$

for all $u \in H$ with $\|u\|_4 = 1$. Indeed, we have $\gamma_{0,b,f}$ is achieved and strictly positive if f satisfies $(f)_2$ (see Lemma 2.2 in [13]). In order as

$$2\left(\frac{b}{3}\right)^{3/2}\left(\frac{\|u\|^6}{\|u\|_4^4}\right)^{1/2} \leq 2\left(\frac{b}{3}\right)^{3/2}\left(\frac{\|u\|^6}{\|u\|_4^4 - a\|u\|^4}\right)^{1/2} \text{for all } u \in H \setminus \{0\}, \quad (8)$$

then, $\gamma_{a,b,f} \geq \gamma_{0,b,f} > 0$.

This paper is structured as follows: in Section 2, we give some basic results useful for what follows. Section 3 is devoted to the proofs of our main results.

2. Some Preliminary Results

We consider the energy functional associated to problem $(\mathscr{P})_{a,b}$ defined for $u \in H$ and given by

$$I(u) = \frac{a}{4}\|u\|^4 + \frac{b}{2}\|u\|^2 - \frac{1}{4}\|u\|_4^4 - \int_\Omega fudx, \quad \text{for all } u \in H. \quad (9)$$

Observe that $I \in C^1(H, \mathbb{R})$, whose derivative at the point $u \in H$ is given by

$$\left\langle I'(u), v\right\rangle = \left(a\|u\|^2 + b\right)\int_\Omega \nabla u \nabla v dx - \int_\Omega u^3 v dx - \int_\Omega f v dx = 0, \text{for all } v \in H. \quad (10)$$

Obviously, if $u \in H$ is a critical point of the functional I; then, u is a weak solution of problem $(\mathscr{P})_{a,b}$.

In general, I is not bounded from below on H, to overcome this and achieve a multiplicity result, the key argument is to use an appropriate manifold called in mathematical literature the Nehari manifold, it is a suitable manifold who has a pertinent property to prove the distinction of two solutions. Indeed, a minimizer in this set may give rise to solution of the corresponding equation. This so called Nehari manifold is defined by

$$\mathcal{N} \ll \left\{u \in H : \left\langle I'(u), u\right\rangle = 0\right\}. \quad (11)$$

Lemma 3. *Assume that $b > 0, a \geq 0$, and $f \in H^{-1} \setminus \{0\}$. Then, the functional I is coercive and bounded from below on $\mathcal{N}$.*

Proof. For $u \in \mathcal{N}$, we have

$$a\|u\|^4 - \|u\|_4^4 = \int_\Omega fudx - b\|u\|^2. \quad (12)$$

Therefore

$$\begin{aligned} I(u) &= \frac{a}{4}\|u\|^4 + \frac{b}{2}\|u\|^2 - \frac{1}{4}\|u\|_4^4 - \int_\Omega fudx \\ &= \frac{b}{4}\|u\|^2 - \frac{3}{4}\int_\Omega fudx \geq \frac{b}{4}\|u\|^2 - \frac{3}{4}\|f\|_{\_}\|u\|. \end{aligned} \quad (13)$$

Thus, I is coercive and bounded from below on $\mathcal{N}$.

Let $h_u(t) = I(tu)$ for $t \in \mathbb{R}^*$ and $u \in H$. These maps are known as fibering maps and were first introduced by Drábek and Pohozaev [18]. The set $\mathcal{N}$ is closely linked to the behavior of $h_u(t)$, for more details, see for example [19] or [20].

It is natural to split $\mathcal{N}$ into three subsets:

$$\mathcal{N}^+ := \left\{u \in \mathcal{N} : h_u''(1) > 0\right\}, \mathcal{N}^0 := \left\{u \in \mathcal{N} : h_u''(1) = 0\right\}, \quad (14)$$

$$\mathcal{N}^- := \left\{u \in \mathcal{N} : h_u''(1) < 0\right\}, \quad (15)$$

where $h_u''(t) = 3t^2 a\|u\|^4 + b\|u\|^2 - 3t^2\|u\|_4^4$. These subsets correspond to local minima, points of inflexion, and local maxima of I, respectively. □

Definition 4. A sequence $\{u_n\} \subset H$ is said to be a Palais Smale sequence at level c ($(PS)_c$ sequence in short) for I if

$$I(u_n) = c + o_n(1) \text{ and } I'(u_n) = o_n(1) \text{ in } H^{-1}. \quad (16)$$

I verifies Palais Smale condition at level c ($(PS)_c$ condition in short) if any $(PS)_c$ sequence has a convergent subsequence in H.

Next, for $u \neq 0, b > 0$, and a, a small enough positive number set

$$\Phi_u(t) = b\|u\|^2 t - \left(\|u\|_4^4 - a\|u\|^4\right)t^3, \quad (17)$$

then $h_u'(t) = \Phi_u(t) - \int_\Omega fudx$. Easy computations show that Φ_u is concave and achieves its maximum at the point $t_{\max}^u$ where

$$t_{\max}^u = \left(\frac{b\|u\|^2}{3\|u\|_4^4 - 3a\|u\|^4}\right)^{1/2}. \quad (18)$$

That is,

$$\max_{t \geq 0}(\Phi_u(t)) = \Phi_u(t_{\max}^u) = 2\left(\frac{b}{3}\right)^{3/2}\left(\frac{\|u\|^6}{\|u\|_4^4 - a\|u\|^4}\right)^{1/2}, \quad (19)$$

$$h'_u(t^u_{\max}) = 2\left(\frac{b}{3}\right)^{3/2}\left(\frac{\|u\|^6}{\|u\|_4^4 - a\|u\|^4}\right)^{1/2} - \int_\Omega fudx. \quad (20)$$

Now, for $u \in H \setminus \{0\}$ set $\Psi(u) = h'_u(t^u_{\max})$, that is

$$\gamma_{a,b,f} = \inf_{\|u\|_4=1} \Psi(u). \quad (21)$$

Fix $t_1 > 0$, then, for $t \geq t_1$, we have $\Psi(tu) = t\Psi(u)$,

$$t_1\,\gamma_{a,b,f} \leq t\,\gamma_{a,b,f} = t \inf_{\|u\|_4=1} \Psi(u) = \inf_{\|tu\|_4=t} \Psi(tu) = \inf_{\|v\|_4=t} \Psi(v), \quad (22)$$

so

$$t_1\,\gamma_{a,b,f} \leq \inf_{\|v\|_4 \geq t_1} \Psi(v). \quad (23)$$

This is crucial for the following.

Lemma 5. *Assume that $\gamma_{a,b,f} > 0$, then, $\mathcal{N}^0 = \{0\}$.*

Proof. Arguing by contradiction we assume that there exists $u \in \mathcal{N}^0 \setminus \{0\}$, i.e., $u \neq 0$ verifies

$$a\|u\|^4 + b\|u\|^2 - \|u\|_4^4 - \int_\Omega fudx = 0, \quad (24)$$

$$3a\|u\|^4 + b\|u\|^2 - 3\|u\|_4^4 = 0. \quad (25)$$

From (24) and (25), we derive that

$$\frac{2}{3}b\|u\|^2 - \int_\Omega fudx = 0, \quad (26)$$

As $0 \leq a < S^{-2}$, we get from (24), (26), and the definition of S

$$3\left(a - S^{-2}\right)\|u\|^4 + b\|u\|^2 \leq 0, \quad (27)$$

that is

$$\|u\|^2\left(3\left(a - S^{-2}\right)\|u\|^2 + b\right) \leq 0. \quad (28)$$

Thus, as $u \neq 0$ and $a < S^{-2}$, we derive that

$$\|u\|^2 \geq \frac{b}{3\left(S^{-2} - a\right)} > 0, \quad (29)$$

therefore, from (25), we obtain $\|u\|_4 \geq t_1$, with

$$t_1 = \left[\frac{ab^2}{9\left(S^{-2} - a\right)^2} + \frac{b^2}{9\left(S^{-2} - a\right)}\right]^{1/4}. \quad (30)$$

Then, from (23), (25), and (26), we get

$$\begin{aligned} 0 < t_1\,\gamma_{a,b,f} &\leq \inf_{\|u\|_4 \geq t_1} \Psi(u) \leq \Psi(u) \\ &= 2\left(\frac{b}{3}\right)^{3/2}\left(\frac{\|u\|^6}{\|u\|_4^4 - a\|u\|^4}\right)^{1/2} - \int_\Omega fudx \\ &= 2\left(\frac{b}{3}\right)^{3/2}\left(\frac{\|u\|^6}{(b/3)\|u\|^2}\right)^{1/2} - \int_\Omega fudx = 2\frac{b}{3}\|u\|^2 - \int_\Omega fudx = 0, \end{aligned} \quad (31)$$

which yields to a contradiction. □

Lemma 6. *Assume that $\gamma_{a,b,f} > 0$, then, for all $u \in H \setminus \{0\}$, there exists unique positive value $t_u^+ = t^+(u)$ such that*

$$t_u^+ > t_u^{\max}, t_u^+ u \in \mathcal{N}^- and I(t_u^+ u) = \max_{t \geq t^u_{\max}} I(tu). \quad (32)$$

Moreover, if $\int_\Omega fudx > 0$, then, there exists unique positive value $t_u^- = t^-(u)$ such that

$$0 < t_u^- < t^u_{\max}, t_u^- u \in \mathcal{N}^+ and I(t_u^- u) = \inf_{0 \leq t \leq t^u_{\max}} I(tu). \quad (33)$$

Proof. We have $h_u(t) = I(tu), h'_u(t) = \Phi_u(t) - \int_\Omega fudx$, and Φ_u is concave and achieves its maximum at the point $t^u_{\max}$. If $\gamma_{a,b,f} > 0$; then, there exists a unique $t_u^+ > 0$, such that $h'_u(t_u^+) = \Phi_u(t_u^+) - \int_\Omega fudx$ and $h_u''(t_u^+) < 0$, which implies that $t_u^+ u \in \mathcal{N}^-$ and $I(t_u^+ u) \geq I(tu)$ for all $t \geq t^u_{\max}$. Moreover, if $\int_\Omega fudx > 0$, then, there exists a unique $t_u^- > 0$, such that $h_u'(t_u^-) = \Phi_u(t_u^-) - \int_\Omega fudx$ and $h_u''(t_u^+) > 0$, which implies that $t_u^- u \in \mathcal{N}^+$ and $I(t_u^+ u) \leq I(tu)$ for all $t \leq t^u_{\max}$.

Set

$$E_1 = \left\{u \in H : u = 0 \text{ or } \|u\| < t^+\left(\frac{u}{\|u\|}\right)\right\}, \quad (34)$$

$$E_2 = \left\{u \in H \setminus \{0\}\colon \|u\| > t^+\left(\frac{u}{\|u\|}\right)\right\}. \quad (35)$$

In the following lemma, we prove that $\mathcal{N}^-$ is closed and disconnects H in exactly two connected components E_1 and E_2. □

Lemma 7. *Assume that $\gamma_{a,b,f} > 0$. Then*

(i) $\mathcal{N}^-$ is closed

(ii) $H \setminus \mathcal{N}^- = E_1 \cup E_2$

(iii) $\mathcal{N}^+ \subset E_1$

Proof. Let $\{u_n\} \subset \mathcal{N}^-$ and $\tilde{u} = \lim_{n\longrightarrow\infty} u_n$, then, $\tilde{u} \in \mathcal{N}$. Assume by contradiction that $\tilde{u} \notin \mathcal{N}^-$, then

$$3a\|u_n\|^4 + b\|u_n\|^2 - 3\|u_n\|_4^4 < 0, \quad (36)$$

$$3a\|\tilde{u}\|^4 + b\|\tilde{u}\|^2 - 3\|\tilde{u}\|_4^4 = 0. \quad (37)$$

So, $\tilde{u} \in \mathcal{N}^0$, this implies that $\tilde{u} = 0$.

From (36) and the definition of S, we get $\|u_n\|^2 \geq bS^2/3$, so $\|\tilde{u}\|^2 \geq bS^2/3$, which yields to a contradiction.

Let $u \in \mathcal{N}^-$ and $v = u/\|u\|$, then, $t^+(u) = 1$, and there exists unique $t^+(v)$ such that $t^+(v)v \in \mathcal{N}^-$. As

$$t^+(v)v = t^+\left(\frac{u}{\|u\|}\right)\frac{1}{\|u\|}u \in \mathcal{N}^-, \quad (38)$$

then

$$t^+\left(\frac{u}{\|u\|}\right)\frac{1}{\|u\|} = t^+(u) = 1. \quad (39)$$

Thus if $u \in H \setminus \{0\}$ and,

$$t^+\left(\frac{u}{\|u\|}\right)\frac{1}{\|u\|} \neq 1, \quad (40)$$

then, $u \notin \mathcal{N}^-$ and

$$H \setminus \mathcal{N}^- = E_1 \cup E_2. \quad (41)$$

Let $u \in \mathcal{N}^+$ then

$$t^-(u) = t^-\left(\frac{u}{\|u\|}\right)\frac{1}{\|u\|} = 1. \quad (42)$$

Since $t^+(u) > t^-(u)$, it follows that

$$t^+(u) = t^+\left(\frac{u}{\|u\|}\right)\frac{1}{\|u\|} > 1. \quad (43)$$

So, $\|u\| < t^+(u/\|u\|)$, and we conclude that $\mathcal{N}^+ \subset E_1$. □

By Lemma 6, we know that $\mathcal{N}$ and $\mathcal{N}^-$ are not empty, so we can define $\theta_0 \leq \theta_1$ with

$$\theta_0 := \inf_{u\in\mathcal{N}} I(u) \text{ and } \theta_1 := \inf_{u\in\mathcal{N}^-} I(u). \quad (44)$$

Lemma 8. *Assume that $\gamma_{a,b,f} > 0$, then, there exists $t_* > 0$ such that*

$$\frac{-9}{16b}\|f\|_-^2 \leq \theta_0 \leq -\frac{b}{4}t_*^2\|f\|_-^2. \quad (45)$$

Proof. Let $u \in \mathcal{N}$, then

$$I(u) = \frac{b}{4}\|u\|^2 - \frac{3}{4}\int_\Omega fudx \geq \frac{b}{4}\|u\|^2 - \frac{3}{4}\|f\|_-\|u\| \geq \frac{-9}{16b}\|f\|_-^2. \quad (46)$$

Thus $\theta_0 \geq -9/16b\|f\|_-^2$.

Set $u_* \in H$ the unique solution of the equation $-\Delta u = f$, it follows

$$\int_\Omega fu_*dx = \|u_*\|^2 = \|f\|_-^2. \quad (47)$$

By Lemma 6, there exists a unique positive value $t_* = t_{u_*}^-$ such that $t_*u_* \in \mathcal{N}^+$. So

$$\begin{aligned} I(t_*u_*) = -\frac{3a}{4}t_*^4\|u_*\|^4 - \frac{b}{2}t_*^2\|u_*\|^2 + \frac{3}{4}t_*^4\|u_*\|_4^4 \leq \frac{b}{4}t_*^2\|u_*\|^2 \\ - \frac{b}{2}t_*^2\|u_*\|^2 = -\frac{b}{4}t_*^2\|f\|_-^2, \end{aligned} \quad (48)$$

consequently

$$\frac{-9}{16b}\|f\|_-^2 \leq \theta_0 \leq -\frac{b}{4}t_*^2\|f\|_-^2. \quad (49)$$

□

The following lemma is needed for prove the existence of Palais Smale sequences.

Lemma 9. *Assume that $\gamma_{a,b,f} > 0$. Then, for any $u \in \mathcal{N} \setminus \{0\}$, there exist $\varepsilon > 0$ and a differentiable function $\zeta : B_H(0,\varepsilon) \longrightarrow \mathbb{R}^+ \setminus \{0\}$ such that*

$$\zeta(0) = 1, \zeta(v)(u - v) \in \mathcal{N}, \forall v \in B_H(0,\varepsilon), \quad (50)$$

$$\left(\zeta'(0), v\right) = \frac{\left(4a\|u\|^2 + 2b\right)\int_\Omega \nabla u\nabla vdx - 4\int_\Omega u^3vdx - \int_\Omega fvdx}{3a\|u\|^4 + b\|u\|^2 - 3\|u\|_4^4}. \quad (51)$$

Proof. Let $u \in \mathcal{N} \setminus \{0\}$ and define: $\mathbb{R} \times H \longrightarrow \mathbb{R}$ as follows

$$(\zeta, v) = a\zeta^3\|u - v\|^4 + b\zeta\|u - v\|^2 - \zeta^3\|u - v\|_4^4 - \int_\Omega f(u - v)dx. \quad (52)$$

Clearly, $(1,0) = 0$. Moreover, from Lemma 5, we derive that

$$\frac{\partial}{\partial\zeta}(1,0) = 3a\|u\|^4 + b\|u\|^2 - 3\|u\|_4^4 \neq 0. \quad (53)$$

Thus, we get our result by a straightforward application of the implicit function theorem to the function at the point $(1,0)$. □

Lemma 10. *Let $\theta \in \{\theta_0, \theta_1\}$. There exist a Palais Smale sequences $\{u_n\} \subset \mathcal{N}$ such that*

$$I(u_n) \longrightarrow \theta, I'(u_n) \longrightarrow 0. \quad (54)$$

Proof. Assume $\theta = \theta_0$, by Lemma 3, I is bounded from below in $\mathcal{N}$, then by applying the Ekeland Variational Principle, we can obtain a minimizing sequence $\{u_n\} \subset \mathcal{N}$ satisfying

$$\theta_0 \leq I(u_n) < \theta_0 + \frac{1}{n}, \quad (55)$$

$$I(w) \geq I(u_n) - \frac{1}{n}\|w - u_n\|, \quad (56)$$

for all $w \in \mathcal{N}$. Thus, $I(u_n) \longrightarrow \theta_0$.

By using Lemma 8, we get for n large enough

$$-\frac{b}{4}t_*^2\|f\|_-^2 \geq \theta_0 + \frac{1}{n} > I(u_n) = \frac{b}{4}\|u_n\|^2 - \frac{3}{4}\int_\Omega f u_n dx, \quad (57)$$

this implies that

$$\int_\Omega f u_n dx \geq \frac{b}{3}\|u_n\|^2 + \frac{b}{3}t_*^2\|f\|_-^2, \quad (58)$$

then

$$\int_\Omega f u_n dx \geq \frac{b}{3}t_*^2\|f\|_-^2 \text{ and} \int_\Omega f u_n dx \geq \frac{b}{3}\|u_n\|^2, \quad (59)$$

and by Holder inequality, we get

$$\|f\|_-\|u_n\| \geq \frac{b}{3}t_*^2\|f\|_-^2 > 0 \text{and} \|f\|_-\|u_n\| \geq \frac{b}{3}\|u_n\|^2. \quad (60)$$

Consequently, $u_n \neq 0$ and

$$\frac{b}{3}t_*^2\|f\|_- \leq \|u_n\| \leq \frac{3}{b}\|f\|_-. \quad (61)$$

Now, we show that $\|I'(u_n)\|$ tend to 0 as n goes to $+\infty$. Arguing by contradiction and fix n with $\|I'(u_n)\| \neq 0$.

Then, by Lemma 9, there exist $\varepsilon > 0$ and a function $\zeta_n : B_H(0, \varepsilon) \longrightarrow \mathbb{R}$ such that $w_n = \zeta_n(v_n)(u_n - v_n) \in \mathcal{N}$ with $v_n = \delta I'(u_n)/\|I'(u_n)\|$ and $0 < \delta < \varepsilon$. By (56) and the Taylor expansion of I, we have

$$\begin{aligned} -\frac{1}{n}\|w_n - u_n\| &\leq I(w_n) - I(u_n) \leq \left\langle I'(u_n), w_n - u_n \right\rangle + \circ(\|w_n - u_n\|) \\ &= (\zeta_n(v_n) - 1)\left\langle I'(u_n), u_n \right\rangle - \delta\zeta_n(v_n)\left\langle I'(u_n), \frac{I'(u_n)}{\|I'(u_n)\|_-} \right\rangle \\ &\quad + \circ(\|w_n - u_n\|). \end{aligned} \quad (62)$$

Then

$$\zeta_n(v_n)\|I'(u_n)\|_- \leq \frac{\zeta_n(v_n) - 1}{\delta}\left\langle I'(u_n), u_n \right\rangle + \frac{\|w_n - u_n\|}{n\delta} + \frac{\circ(\|w_n - u_n\|)}{\delta}. \quad (63)$$

We have

$$\lim_{\delta \longrightarrow 0}\zeta_n(v_n) = 1, \lim_{\delta \longrightarrow 0}\frac{|\zeta_n(v_n) - 1|}{\delta} = \lim_{\delta \longrightarrow 0}\frac{|\zeta_n(v_n) - \zeta_n(0)|}{\delta} \leq \|\zeta_n'(0)\|_-, \quad (64)$$

$$\begin{aligned} \lim_{\delta \longrightarrow 0}\frac{\|w_n - u_n\|}{n\delta} &= \lim_{\delta \longrightarrow 0}\frac{1}{n\delta}\|(\zeta_n(v_n) - 1)u_n - \zeta_n(v_n)v_n\| \leq \lim_{\delta \longrightarrow 0}\frac{1}{n}\left[\left\|\frac{\zeta_n(v_n) - 1}{\delta}u_n\right\| \right. \\ &\quad \left. + \left\|\zeta_n(v_n)\frac{v_n}{\delta}\right\|\right] \leq \frac{1}{n}\left(\|\zeta_n'(0)\|_-\|u_n\| + 1\right). \end{aligned} \quad (65)$$

This, together with (61) implies

$$\|I'(u_n)\|_- \leq \frac{C_3}{n}\left(\|\zeta_n'(0)\|_- + 1\right), \quad (66)$$

for a suitable constant $C_3 > 0$. Now, we must show that $\|\zeta_n'(0)\|_-$ is uniformly bounded in n : indeed, since $\{u_n\}$ is a bounded sequence, we have from Lemma 8

$$|\zeta_n'(0)| \leq \frac{C_4}{\left|3a\|u_n\|^4 + b\|u_n\|^2 - 3\|u_n\|_4^4\right|}, \quad (67)$$

for a suitable constant $C_4 > 0$. Assume by contradiction that for a subsequence still called $\{u_n\}$, we have

$$b\|u_n\|^2 - 3\left(\|u_n\|_4^4 - a\|u_n\|^4\right) = o_n(1). \quad (68)$$

Then, as a is a small enough positive number, we get

$$\left[\frac{b}{3}\|u_n\|^2\right]^{3/2} = \left(\|u_n\|_4^4 - a\|u_n\|^4\right)^{3/2} + o_n(1). \quad (69)$$

So from (61), we derive that

$$\|u_n\| \geq \frac{b}{3}t_*^2\|f\|_-. \quad (70)$$

Also, as $u_n \in \mathcal{N}$, we get from (68)

$$\int_\Omega f u_n dx = 2\left(\|u_n\|_4^4 - a\|u_n\|^4\right) + o_n(1), \quad (71)$$

then

$$\begin{aligned} 0 < t_*^2\|f\|_-\left(\frac{b}{3}\right)^{3/2}\gamma_{a,b,f} &\leq \left[\frac{b}{3}\|u_n\|^2\right]^{1/2}\Psi(u_n) \\ &\leq \left[\|u_n\|_4^4 - a\|u_n\|^4\right]^{1/2}\Psi(u_n) \\ &\leq 2\left(\frac{b}{3}\right)^{3/2}\|u_n\|^3 - 2\left[\|u_n\|_4^4 - a\|u_n\|^4\right]^{3/2} = o_n(1), \end{aligned} \quad (72)$$

which is absurd. At this point, we conclude that $I'(u_n) \longrightarrow 0$ in H^{-1}.

For $\theta = \theta_1$, adopting exactly the same way as in the case where $\theta = \theta_0$.

In the following, we will prove our results. □

3. Proofs of the Main Results

The proof of our main results is divided in two parts.

3.1. Existence of a Solution in $\mathcal{N}^+$. In this subsection, we prove that I has a solution in $\mathcal{N}^+$.

Proposition 11. *Assume that $\gamma_{a,b,f} > 0$. Then, the minimization problem*

$$\theta_0 = \inf_{u\in\mathcal{N}} I(u), \quad (73)$$

attaints its infimum at a point $u_0 \in \mathcal{N}^+$. Moreover, u_0 is a local minimizer for I in H.

Proof. By using Lemma 10, there exists a bounded minimizing sequence $\{u_n\} \subset \mathcal{N}^+$ such that $I(u_n) \longrightarrow \theta_0$ and $I'(u_n) \longrightarrow 0$ in H^{-1}. So, we deduce that $\{u_n\}$ is bounded in H.

Passing to a subsequence if necessary, we have $u_n \rightharpoonup u_0$ weakly in H, then, $\langle I'(u_0), w\rangle = 0$, for all $w \in H$. In addition, from (60), we get $\int_\Omega f u_0 dx > 0$. So, u_0 is a weak solution for $(\mathcal{P})_{a,b}$ and $u_0 \in \mathcal{N}$.

Thus

$$\theta_0 \le I(u_0) = \frac{b}{4}\|u_0\|^2 - \frac{3}{4}\int_\Omega f u_0 dx \le \varliminf_{n\longrightarrow\infty} I(u_n) = \theta_0, \quad (74)$$

then $I(u_0) = \theta_0 = \inf_{u\in\mathcal{N}} I(u)$. It follows that $\{u_n\}$ converges strongly to u_0 in H, then, $u_0 \in \mathcal{N}^+$ and necessarily

$$t^-(u_0) = 1 < t_{\max}^{u_0}. \quad (75)$$

To conclude that u_0 is a local minimum of I, let us recall that we have from Lemma 6

$$I(su) \ge I(u_0) \text{ for every } 0 < s < t_{\max}^{u_0}. \quad (76)$$

Choose $\varepsilon > 0$ sufficiently small to have

$$1 < t_{\max}^{u_0-w} \text{ for } \|w\| < \varepsilon, \quad (77)$$

and $t(w)$ satisfying $t(w)(u_0 - w) \in \mathcal{N}$ for every $\|w\| < \varepsilon$. Since $t(w) \longrightarrow 1$ as $\|w\| \longrightarrow 0$, we can always assume that

$$t(w) < t_{\max}^{u_0-w} \text{ for every } w \text{ such that } \|w\| < \varepsilon, \quad (78)$$

so $t(w)(u_0 - w) \in \mathcal{N}^+$ and for $0 < s < t_{\max}^{u_0-w}$, we have

$$I(s(u_0 - w)) \ge I(t(w)(u_0 - w)) \ge I(u_0), \quad (79)$$

from (75), we can take $s = 1$ and conclude that $I(u_0 - w) \ge I(u_0)$, for all $w \in H$ such that $\|w\| < \varepsilon$. Thus, u_0 is a local minimum of I.

If $f \ge 0$, we have $\int_\Omega f u_0 dx \le \int_\Omega f|u_0|dx$ and clearly $I(|u_0|) \le I(u_0)$, and from (75) necessarily $t^-(|u_0|) \ge 1$. Therefore, as $I(u_0) = \inf_{u\in\mathcal{N}^+} I(u)$, we get

$$I(t^-(|u_0|)|u_0|) \le I(|u_0|) \le I(u_0) \le I(t^-(|u_0|)|u_0|), \quad (80)$$

so, we can always take $u_0 \ge 0$. □

3.2. Existence of a Solution in $\mathcal{N}^-$. The following part is devoted to prove the existence of a second solution u_1 such that $I(u_1) = \theta_1 = \inf_{v\in\mathcal{N}^-} I(v)$. First, we determine the good level for covering the Palais Smale condition.

We have the following important result.

Lemma 12. *Assume that $\gamma_{a,b,f} > 0$. Then, I satisfies the $(PS)_c$ condition for $c < c^*_{a,b}$ with*

$$c^*_{a,b} = \theta_0 + \frac{b^2}{4(S^{-2} - a)}. \quad (81)$$

Proof. Let $\{u_n\}$ be a $(PS)_c$ sequence with $c < c^*_{a,b}$, then

$$\begin{aligned} c + o_n(1) = I(u_n) - \frac{1}{4}\left\langle I'(u_n), u_n\right\rangle &= \frac{b}{4}\|u_n\|^2 \\ - \frac{3}{4}\int_\Omega f u_n dx &\ge \frac{b}{4}\|u_n\|^2 - \frac{3}{4}\|f\|_-\|u_n\|. \end{aligned} \quad (82)$$

Hence, $\{u_n\}$ is a bounded sequence in H. Thus for a subsequence still denoted $\{u_n\}$ and we can find $u_1 \in H$ such that $u_n \rightharpoonup u_1$ in H, $\int_\Omega f u_n dx \longrightarrow \int_\Omega f u_1 dx$ and $u_n \longrightarrow u_1$ a.e in Ω. Therefore, $u_1 \ne 0$, $u_1 \in \mathcal{N}$, and $I(u_1) \ge \theta_1$.

Let $w_n = u_n - u_1$. From Brézis-Lieb Lemma [21], one has

$$\begin{aligned} \|u_n\|^2 = \|w_n\|^2 + \|u_1\|^2 + o_n(1), \|u_n\|^4 = \|w_n\|^4 \\ + 2\|w_n\|^2\|u_1\|^2 + \|u_1\|^4 + o_n(1), \end{aligned} \quad (83)$$

$$\|u_n\|_4^4 = \|w_n\|_4^4 + \|u_1\|_4^4 + o_n(1), \quad (84)$$

this implies that

$$\begin{aligned} o_n(1) = \left\langle I'(u_n), u_n\right\rangle = a\|w_n\|^4 + b\|w_n\|^2 - \|w_n\|_4^4 \\ - \int_\Omega f w_n dx + 2a\|w_n\|^2\|u_1\|^2. \end{aligned} \quad (85)$$

Assume that $\|w_n\| \longrightarrow l$ with $l > 0$, then, by (85) and the Sobolev inequality, we obtain

$$\left(S^{-2} - a\right) l^4 \ge b l^2 + 2a l^2\|u_1\|^2. \quad (86)$$

this implies that

$$\left(S^{-2} - a\right) l^4 - b l^2 \ge 0. \quad (87)$$

Hence

$$l^2 \ge \frac{b}{S^{-2} - a}. \quad (88)$$

On the other hand, we have

$$
\begin{aligned}
c+o_n(1) &= I(u_n)-\frac{1}{4}\left\langle I'(u_n),u_n\right\rangle=\frac{b}{4}\|u_n\|^2-\frac{3}{4}\int_\Omega f u_n dx \\
&= \frac{b}{4}\|w_n\|^2+\frac{b}{4}\|u_1\|^2-\frac{3}{4}\int_\Omega f u_1 dx=\frac{b}{4}\|w_n\|^2 \\
&\quad + I(u_1)-\frac{1}{4}\left\langle I'(u_1),u_1\right\rangle=\frac{b}{4}\|w_n\|^2+I(u_1),
\end{aligned}
\tag{89}
$$

consequently, we obtain

$$
c\geq\theta_0+\frac{b^2}{4\left(S^{-2}-a\right)},
\tag{90}
$$

which is a contradiction. Therefore, $l=0$ and $u_n\longrightarrow u_1$ strongly in H.

Now, it is natural to show that $\theta_1<c^*_{a,b}$. As it is well know, (seeHL), the best Sobolev constant S defined above is attained in $\mathbb{R}^4$ by

$$
U_\varepsilon(x)=\varepsilon\left(\varepsilon^2+|x|^2\right)^{-1}, \varepsilon>0 \text{ and } x\in\mathbb{R}^4.
\tag{91}
$$

For $x_0\in\Omega$ let $V_\varepsilon(x)=U_\varepsilon(x-x_0)\phi\in C_0^\infty(\Omega)$ such that $\phi(x)=1$ for $x\in B^r_{x_0}$, $\phi(x)=0$ for $x\in\mathbb{R}^4\setminus B^{2r}_{x_0}$, and $0\leq\phi\leq 1$, $|\nabla\phi|\leq C$. Now, we shall give some useful estimates of the extremal functions U_ε. Let $V_\varepsilon(x)=\phi(x)(\varepsilon+|x-x_0|^2)^{-2}$ and $u_\varepsilon=V_\varepsilon(x)\|u_\varepsilon\|_4$. The following estimates are obtained in [22] as ε tends to 0

$$
\begin{cases}
\|u_\varepsilon\|_4^2=1,\\
\|u_\varepsilon\|^2=S+O(\varepsilon),\\
C_1\varepsilon^{4-q}\leq\displaystyle\int_\Omega u_\varepsilon^q dx\leq C_2\varepsilon^{4-q} \text{ for } q>2.
\end{cases}
\tag{92}
$$

Let $\Omega'\subset\Omega$ a set of positive measure such that $u_0>0$ on Ω' (if not replace u_0 and f by $-u_0$ and $-f$, respectively). □

Lemma 13. *Let $b>0, 0\leq a<S^{-2}$ a small enough positive number. Assume that $f\in H^{-1}\setminus\{0\}$ satisfies $\gamma_{a,b,f}>0$; then, for every $t>0$ and a.e. $x_0\in\Omega'\subset\Omega$, there exists ε_0 such that $I(u_0+tu_\varepsilon)<c^*_{a,b}$ for every $0<\varepsilon<\varepsilon_0$.*

Proof. For ε small enough, let us consider the functional g defined by

$$
g(t)=\frac{bt^2}{2}\|u_\varepsilon\|^2-\frac{t^4}{4}\|u_\varepsilon\|_4^4.
\tag{93}
$$

We have by (92)

$$
\sup_{t\geq 0} g(t)=\frac{b^2}{4S^{-2}}+O(\varepsilon).
\tag{94}
$$

On the other hand, since $u_0\in\mathcal{N}^+$ is a solution of problem $(\mathcal{P})_{a,b}$, we have $\|u_0\|\leq C$, $I(u_0)=\theta_0$ and $\langle I'(u_0),tu_\varepsilon\rangle=0$.

Then we obtain by (92)

$$
\begin{aligned}
I(u_0+tu_\varepsilon) &= \frac{a}{4}\|u_0+tu_\varepsilon\|^4+\frac{b}{2}\|u_0+tu_\varepsilon\|^2-\frac{1}{4}\|u_0+tu_\varepsilon\|_4^4 \\
&\quad -\int_\Omega f(u_0+tu_\varepsilon)dx=\frac{a}{4}\|u_0\|^4+\frac{b}{2}\|u_0\|^2-\frac{1}{4}\|u_0\|_4^4 \\
&\quad -\int_\Omega f u_0 dx+\frac{at^4}{4}\|u_\varepsilon\|^4+\frac{bt^2}{2}\|u_\varepsilon\|^2-\frac{t^4}{4}\|u_\varepsilon\|_4^4 \\
&\quad +\left(a\|u_0\|^2+b\right)\int_\Omega\nabla u_0\nabla(tu_\varepsilon)dx-\int_\Omega u_0^3(tu_\varepsilon)dx-t\int_\Omega f u_\varepsilon dx+t^2 \\
&\quad \cdot\left[a\left(\int_\Omega\nabla u_0\nabla u_\varepsilon\right)^2 dx+\frac{a}{2}\|u_0\|^2\|u_\varepsilon\|^2-\frac{3}{2}\int_\Omega u_0^2u_\varepsilon^2 dx\right]+t^3 \\
&\quad \cdot\left[a\|u_\varepsilon\|^2\int_\Omega\nabla u_0\nabla u_\varepsilon dx-\int_\Omega u_\varepsilon^3u_0 dx\right]\leq I(u_0)+\left\langle I'(u_0),tu_\varepsilon\right\rangle \\
&\quad +\frac{at^4}{4}\|u_\varepsilon\|^4+\frac{bt^2}{2}\|u_\varepsilon\|^2-\frac{t^4}{4}\|u_\varepsilon\|_4^4+t^2aC_1+t^3[aC_2-C_3\varepsilon]\leq\theta_0+h(t),
\end{aligned}
\tag{95}
$$

where

$$
h(t)=\frac{at^4}{4}\|u_\varepsilon\|^4+\frac{bt^2}{2}\|u_\varepsilon\|^2-\frac{t^4}{4}\|u_\varepsilon\|_4^4+at^2C_1+t^3[aC_2-C_3\varepsilon].
\tag{96}
$$

Since $a<S^{-2}$, we have $h(t)\longrightarrow-\infty$ as t goes to ∞ and $h(t)\longrightarrow 0$ as t goes to 0. This implies that there exist $0<T_1<T_2$ such that

$$
\begin{aligned}
\sup_{t>0} I(u_0+tu_\varepsilon) &\leq \sup_{T_1<t<T_2} I(u_0+tu_\varepsilon)\leq\theta_0+\frac{bt^2}{2}\|u_\varepsilon\|^2-\frac{t^4}{4}\|u_\varepsilon\|_4^4 \\
&\quad +\frac{aT_2^4}{4}\|u_\varepsilon\|^4+aT_2^2C_1+T_1^3[aC_2-C_3\varepsilon]\leq\theta_0+\frac{b^2}{4S^{-2}}+O(\varepsilon) \\
&\quad +a\left(T_2^2C_1+T_1^3C_2\right)-T_1^3C_3\varepsilon.
\end{aligned}
\tag{97}
$$

Then, for $a=\varepsilon^\sigma$ with $\sigma>1$, we conclude

$$
I(u_0+tu_\varepsilon)<\theta_0+\frac{b^2}{4S^{-2}}=c^*_{0,b}\leq c^*_{a,b}.
\tag{98}
$$

Therefore,

$$
\sup_{t>0} I(u_0+tu_\varepsilon)<c^*_{a,b},
\tag{99}
$$

for a small enough positive number. □

Proposition 14. *Let $b>0, 0\leq a<S^{-2}$ a small enough positive number. Assume that $f\in H^{-1}\setminus\{0\}$ satisfies $\gamma_{a,b,f}>0$, then, I has a local minimizer u_1 on $\mathcal{N}^-$ such that $I(u_1)=\theta_1$. Moreover u_1 is a local minimizer for I on H.*

Proof. By Lemma 6, for every $u\in H$ such that $\|u\|=1$, there exists unique $t^+(u)>0$ such that $t^+(u)u\in\mathcal{N}^-$ and $I(t^+u)\geq$

$I(tu)$, for all $t \geq t^u_{\max}$. Then, for suitable constant $T_3 > 0$, we deduce that

$$0 < t^+(u) < T_3, \text{ for all } u \text{ such that } \|u\| = 1. \quad (100)$$

Therefore for $t_0 > 0$ carefully chosen, the estimate (97) holds for ε small enough.

Thus, we derive that

$$\|u_0 + t_0 u_\varepsilon\|^2 > C_1^2 \geq \left[t^+ \left(\frac{u_0 + t_0 u_\varepsilon}{\|u_0 + t_0 u_\varepsilon\|} \right) \right]^2, \text{ for all } \varepsilon > 0 \text{ small enough.} \quad (101)$$

Then, from Lemma 7, we conclude that $u_0 + t_0 u_\varepsilon \in E_2$.

Set

$$\Gamma = \{\xi : [0, 1] \longrightarrow H \text{ continuous}, \xi(0) = u_0, \xi(1) = u_0 + t_0 u_\varepsilon\}. \quad (102)$$

It is obvious that $\xi : [0, 1] \longrightarrow H$ given by $\xi(t) = u_0 + t\, t_0 u_\varepsilon$ belongs to Γ. We conclude from Lemma 13 that

$$c = \inf_{\xi \in \Gamma} \max_{t \in [0,1]} I(\xi(t)) < c^*_{0,b}. \quad (103)$$

As the range of any $\xi \in \Gamma$ intersects $\mathcal{N}^-$, one has

$$c \geq \theta_1 = \inf_{\mathcal{N}^-} I, \quad (104)$$

From Lemma 10, we can obtain a minimizing sequence $\{u_n\} \subset \mathcal{N}^-$ such that

$$I(u_n) \longrightarrow \theta_1 \text{ and } \left\| I'(u_n) \right\| \longrightarrow 0. \quad (105)$$

We also deduce that $\theta_1 < c^*_{0,b}$.

Consequently, we obtain a subsequence still denoted $\{u_n\}$, and we can find $u_1 \in H$ such that

$$u_n \longrightarrow u_1 \text{ strongly in } H. \quad (106)$$

This implies that u_1 is a critical point for I,$u_1 \in \mathcal{N}^-$ and $I(u_1) = \theta_1$.

Finally, for $f \geq 0$, let $t^+(|u_1|) > 0$ satisfying $t^+(|u_1|)|u_1| \in \mathcal{N}^-$. From Lemma 6, we have

$$I(t^+(|u_1|)|u_1|) \geq I(u_1) = \max_{t \geq t_{\max}} I(tu_1) \geq I(t^+(|u_1|)u_1) \geq I(t^+(|u_1|)|u_1|). \quad (107)$$

So we conclude that $u_1 \geq 0$. □

4. Conclusion

In our work, we have searched the critical points as the minimizers of the energy functional associated to the problem on the constraint defined by the Nehari manifold $\mathcal{N}$, which is a solution of our problem. Under some sufficient conditions, we split $\mathcal{N}$ in two disjoint subsets $\mathcal{N}^+$ and $\mathcal{N}^-$. Thus, we consider the minimization problems on $\mathcal{N}^+$ and $\mathcal{N}^-$, respectively. If $\gamma_{a,b,f} > 0$, then, the problem $(\mathcal{P})_{a,b}$ has a local minimal solution u_0 with $I(u_0) < 0$. Furthermore, $u_0 \geq 0$ for $f \geq 0$ and if $\gamma_{a,b,f} > 0$. The problem $(\mathcal{P})_{a,b}$ has another solution u_1 with $I(u_1) > 0$. Furthermore, $u_1 \geq 0$ for $f \geq 0$.

Data Availability

The functions, functionals, and parameters used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

The authors gratefully acknowledge (1) Qassim University, represented by the Deanship of Scientific Research, on the material support for this research under the number (1124) during the academic year 1443AH/2022 AD and (2) Algerian Ministry of Higher Education and Scientific Research on the material support for this research under the number (1124) during the academic year 1443AH/2022 AD.

References

[1] P. D'Ancona and Y. Shibata, "On global solvability of nonlinear viscoelastic equations in the analytic category," *Mathematicsl Methods in the Applied Sciences*, vol. 17, no. 6, pp. 477–486, 1994.

[2] P. D'Ancona and S. Spagnolo, "Global solvability for the degenerate Kirchhoff equation with real analytic data," *Inventiones Mathematicae*, vol. 108, no. 1, pp. 247–262, 1992.

[3] G. Kirchhoff, *Mechanik*, Teubner, Leipzig, 1883.

[4] J. L. Lions, "On some questions in boundary value problems of mathematical physics," *Contemporary Developments in Continuum Mechanics and Partial Differential Equations in: North-Holland Mathematics Studies*, vol. 30, pp. 284–346, 1978.

[5] B. B. V. Maia, "On a class of p(x)-Choquard equations with sign-changing potential and upper critical growth," *Rendiconti del Circolo Matematico di Palermo Series*, vol. 70, no. 3, pp. 1175–1199, 2021.

[6] C. Vetro, "Variable exponent _p_ (_x_)-Kirchhoff type problem with convection," *Journal of Mathematical Analysis and Applications*, vol. 506, no. 2, p. 125721, 2022.

[7] C. O. Alves, F. J. S. A. Correa, and T. F. Ma, "Positive solutions for a quasilinear elliptic equation of Kirchhoff type," *Computers & Mathematcs with Applications*, vol. 49, no. 1, pp. 85–93, 2005.

[8] S. Benmansour and M. Bouchekif, "Nonhomogeneous elliptic problem of Kirchhoff-type involving critical Sobolev exponents," *Journal of Differential Equations*, vol. 2015, pp. 1–11, 2015.

[9] S. H. Liang and S. Y. Shi, "Soliton solutions to Kirchhoff type problems involving the critical growth in," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 81, pp. 31–41, 2013.

[10] T. F. Ma and J. E. Munoz Rivera, “Positive solutions for a nonlinear nonlocal elliptic transmission problem,” *Applied Mathematics Letters*, vol. 16, no. 2, pp. 243–248, 2003.

[11] H. Brézis and L. Nirenberg, “Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents,” *Communications on Pure and Applied Mathematics*, vol. 36, no. 4, pp. 437–477, 1983.

[12] A. Ambrosetti, H. Brezis, and G. Cerami, “Combined effects of concave and convex nonlinearities in some elliptic problems,” *Journal of Functional Analysis*, vol. 122, no. 2, pp. 519–543, 1994.

[13] G. Tarantello, “On nonhomogeneous elliptic equations involving critical Sobolev exponent,” *Annales de l'Institut Henri Poincaré C, Analyse non linéaire*, vol. 9, no. 3, pp. 281–304, 1992.

[14] D. Naimen, “On the Brézis-Niremberg problem with a Kirchhoff type perturbation,” *Advanced Nonlinear Studies*, vol. 15, p. 1168, 2015.

[15] D. Naimen, “The critical problem of Kirchhoff type elliptic equations in dimension four,” *Journal of Differential Equations*, vol. 257, pp. 135–1193, 2014.

[16] C. Y. Lei, G. S. Liu, and L. T. Guo, “Multiple positive solutions for a Kirchhoff type problem with a critical nonlinearity,” *Nonlinear Analysis: Real World Applications*, vol. 31, pp. 343–355, 2016.

[17] J. F. Liao, H. Y. Li, and P. Zhang, “Existence and multiplicity of solutions for a nonlocal problem with critical Sobolev exponent,” *Computers & Mathematics with Applications*, vol. 75, no. 3, pp. 787–797, 2018.

[18] P. Drábek and S. I. Pohozaev, “Positive solutions for thep-Laplacian: application of the fibrering method,” *Proceedings of the Royal Society of Edinburgh*, vol. 127, no. 4, pp. 703–726, 1997.

[19] K. J. Brown and Y. Zhang, “The Nehari manifold for a semilinear elliptic equation with a sign-changing weight function,” *Journal of Differential Equations*, vol. 193, no. 2, pp. 481–499, 2003.

[20] Y. Chen, Y. C. Kuo, and T. F. Wu, “The Nehari manifold for a Kirchhoff type problem involving sign-changing weight functions,” *Journal of Differential Equations*, vol. 250, no. 4, pp. 1876–1908, 2011.

[21] H. Brézis and E. Lieb, “A relation between pointwise convergence of functions and convergence of functionals,” *Proceedings of American Mathematical Society*, vol. 88, no. 3, pp. 486–490, 1983.

[22] T. S. Hsu and H. L. Lin, “Multiple positive solutions for singular elliptic equations with concave-convex nonlinearities and sign-changing weights,” *Boundary Value Problems*, vol. 2009, Article ID 584203, 17 pages, 2009.