Hindawi
Journal of Function Spaces
Volume 2022, Article ID 2195754, 6 pages
https://doi.org/10.1155/2022/2195754

Research Article

Hyers–Ulam Stability Results for a Functional Inequality of (s, t)-Type in Banach Spaces

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Received 20 December 2021; Accepted 9 March 2022; Published 11 April 2022

Academic Editor: Juan Martinez-Moreno

We introduce an additive (s, t)-functional inequality where s and t are nonzero complex numbers with $\sqrt{2}|s| + |t| < 1$. Using the direct method and the fixed point method, we give the Hyers–Ulam stability of such functional inequality in Banach spaces.

1. Introduction and Preliminaries

A problem regarding the stability of homomorphisms was mentioned by Ulam [1] in 1940. The first answer was then found by Hyers in [2] which motivating the study of the stability problems of functional equations. We may roughly say that a given functional equation is stable on a class of functions A when any function in A approximately satisfies such equation. One of the well-known functional equations is the (additive) Cauchy functional equation $f(a+b) = f(a) + f(b)$ which is a useful tool in natural and social sciences. The stability of functional equations has been widely acknowledged as Hyers–Ulam stability. It was notably weakened by Rassias in [3] by making use of a direct method. The result was later extended in [4] which uses a general control function instead of the unbounded Cauchy difference. The concept of stability has been also developed for functional inequalities. Recently, Park introduced additive ρ-functional inequalities (s-type functional inequalities) and investigated the Hyers–Ulam stability in [5, 6]. Over the last decades, stability of functional equations and functional inequalities have been extensively studied, see [7–13], for example.

Not only the direct method, the fixed point method is also one of the most popular methods of proving the stability of functional equations and functional inequalities. Applications of stability of functional equations in a fixed point theory and in nonlinear analysis were introduced in [14]. It was known that Hyers–Ulam stability results can be derived using fixed point theorems while the latter can often be obtained from the former, see [15–20] and there references.

The Hyers–Ulam stability concept is very useful in many applications (i.e., optimization, numerical analysis, biology, and economics), since it can be very difficult to find the exact solutions for those physical problems. It is remarkably used in the field of differential equations. For some recent works, see [21–23] (and references therein) where the Hyers–Ulam stability results concerning (fractional) stochastic functional differential equations were given.

We denote $\mathbb{C}$, $\mathbb{N}$, and $\mathbb{R}^+$ the set of complex numbers, the set of positive integers and the set of positive real numbers, respectively, and let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\mathbb{R}_0^+ = \mathbb{R}^+ \cup \{0\}$.

Now, let $s, t \in \mathbb{C} \setminus \{0\}$ such that $\sqrt{2}|s| + |t| < 1$. Yun and Shin [24] investigated the additive (s, t)-functional inequality:

$$\begin{aligned}\left\|2f\left(\frac{a+b}{2}\right) - f(a) - f(b)\right\| &\leq \|s(f(a+b) + f(a-b) - 2f(a))\| \\ &\quad + \|t(f(a+b) - f(a) - f(b))\|,\end{aligned} \tag{1}$$

while Park [25] proposed the additive (s,t)-functional inequality:

$$\begin{aligned}\|f(a+b)-f(a)-f(b)\| &\le \|s(f(a+b)+f(a-b)-2f(a))\| \\ &+ \left\|t\left(2f\left(\frac{a+b}{2}\right)-f(a)-f(b)\right)\right\|,\end{aligned} \quad (2)$$

and provided the Hyers–Ulam stability results in a Banach space.

In this article, motivated by those (s,t)-type inequalities mentioned above, we introduce the additive (s,t)-functional inequality:

$$\begin{aligned}\left\|2f\left(\frac{a+b}{2}\right)+f(a-b)-2f(a)\right\| &\le \|s(f(a+b)+f(a-b)-2f(a))\| \\ &+ \|t(f(a+b)-f(a)-f(b))\|.\end{aligned} \quad (3)$$

We first investigate the Hyers–Ulam stability of such functional inequality using the direct method in Section 2. Then, in Section 3, we use the fixed point method to prove the Hyers–Ulam stability of such inequality. We also include some example and remarks in the last section. Note that, since (s,t)-type functional inequalities generalize s-type functional inequalities, our results simply extend existing Hyers–Ulam stability results for functional inequalities of s-type in the literature. These results span alongside those regarding other (s,t)-type functional inequalities.

Throughout this article, let X and B be a normed space and a Banach space, respectively, and let $s,t\in\mathbb{C}\setminus\{0\}$ such that $\sqrt{2}|s|+|t|<1$. For convenience, we also require the following classes of mappings:

$$\begin{aligned}\mathscr{F}_0(X,B) &:= \{g : X \longrightarrow B : g(0)=0\}, \\ \mathscr{A}(X,B) &:= \{g : X \longrightarrow B : g \text{ satisfies } (1.1)\}, \\ \mathscr{A}_0(X,B) &:= \mathscr{F}_0(X,B)\cap\mathscr{A}(X,B).\end{aligned} \quad (4)$$

2. Stability Results: Direct Method

In this section, the stability results of the additive (s,t)-functional inequality (3) are proposed by using the direct method. We begin with the lemma showing that any map g in $\mathscr{A}(X,B)$ is additive.

Lemma 1. *If $g\in\mathscr{A}(X,B)$, then g is additive.*

Proof. Taking $a=b=0$ into (3), we obtain that $(1-|t|)\|g(0)\|\le 0$. However, $|t|<1$ implies that $g(0)=0$. Also, if we let $b=0$ in (3), then

$$g(a)=2g\left(\frac{a}{2}\right), \quad (5)$$

for all $a\in X$. From (3) and (5),

$$\begin{aligned}(1-|s|)\|g(a+b)+g(a-b)-2g(a)\| \\ \le |t|\|g(a+b)-g(a)-g(b)\|,\end{aligned} \quad (6)$$

for all $a,b\in X$. Next, taking $c=a+b$ and $d=a-b$ in (3), we have that

$$\begin{aligned}(1-|s|)\left\|g(c)+g(d)-2g\left(\frac{c+d}{2}\right)\right\| \\ \le |t|\left\|g(c)-g\left(\frac{c+d}{2}\right)-g\left(\frac{c-d}{2}\right)\right\|.\end{aligned} \quad (7)$$

Then, from (5),

$$\begin{aligned}(1-|s|)\|g(c+d)-g(c)-f(d)\| \\ \le \frac{|t|}{2}\|g(c+d)+g(c-d)-2g(c)\|,\end{aligned} \quad (8)$$

for all $c,d\in X$. Applying (6) and (8),

$$\begin{aligned}(1-|s|)^2\|g(a+b)-g(a)-g(b)\| \\ \le \frac{|t|^2}{2}\|g(a+b)-g(a)-g(b)\|,\end{aligned} \quad (9)$$

for all $a,b\in X$. Finally, since $\sqrt{2}|s|+|t|<1$, we obtain that g is additive. □

We are now ready to present the main result.

Theorem 2. *Let $\varphi : X\times X \longrightarrow \mathbb{R}_0^+$ be a map such that*

$$\Phi(a,b) := \sum_{j=0}^{\infty} 2^j\varphi\left(2^{-j}a, 2^{-j}b\right) < \infty, \quad (10)$$

for all $a,b\in X$. For any $f\in\mathscr{F}_0(X,B)$ satisfying

$$\begin{aligned}\left\|2f\left(\frac{a+b}{2}\right)+f(a-b)-2f(a)\right\| &\le \|s(f(a+b)+f(a-b)-2f(a))\| \\ &+ \|t(f(a+b)-f(a)-f(b))\|+\varphi(a,b),\end{aligned} \quad (11)$$

for all $a,b\in X$, there exists a unique $F\in\mathscr{A}_0(X,B)$ such that

$$\|f(a)-F(a)\|\le\Phi(a,0), \quad (12)$$

for all $a\in X$.

Proof. We first let $b=0$ in (11). This implies that

$$\left\|2f\left(\frac{a}{2}\right)-f(a)\right\|\le\varphi(a,0), \quad (13)$$

for all $a \in X$. It follows that for any $m, l \in \mathbb{N}_0$ with $m > l$,

$$\begin{aligned} \left\| 2^l f\left(2^{-l}a\right) - 2^m f(2^{-m}a) \right\| &\le \sum_{j=l}^{m-1} \left\| 2^j f(2^{-j}a) - 2^{j+1} f\left(2^{-(j+1)}a\right) \right\| \\ &\le \sum_{j=l}^{m-1} 2^j \varphi(2^{-j}a, 0), \end{aligned} \tag{14}$$

for all $a \in X$. The completeness of B confirms that the Cauchy sequence $\{2^k f(2^{-k}a)\}$ is convergent for any $a \in X$. Define $F : X \longrightarrow B$ by

$$F(a) = \lim_{k \longrightarrow \infty} 2^k f\left(2^{-k}a\right), \tag{15}$$

for all $a \in X$. Clearly, $F \in \mathcal{F}_0(X, B)$. Next, choosing $l = 0$ and letting $m \longrightarrow \infty$ in (14), we have that F satisfies (12). Then, from (10) and (11),

$$\begin{aligned} &\left\| 2F\left(\frac{a+b}{2}\right) + F(a-b) - 2F(a) \right\| \\ &= \lim_{n \longrightarrow \infty} 2^n \left\| 2f\left(2^{-(n+1)}(a+b)\right) + f(2^{-n}(a-b)) - 2f(2^{-n}a) \right\| \\ &\le |s| \lim_{n \longrightarrow \infty} 2^n \| f(2^{-n}(a+b)) + f(2^{-n}(a-b)) - 2f(2^{-n}a) \| \\ &\quad + |t| \lim_{n \longrightarrow \infty} 2^n \| f(2^{-n}(a+b)) - f(2^{-n}a) - f(2^{-n}b) \| \\ &\quad + \lim_{n \longrightarrow \infty} 2^n \varphi(2^{-n}a, 2^{-n}b) = \| s(F(a+b) + F(a-b) - 2F(a)) \| \\ &\quad + \| t(F(a+b) - F(a) - F(b)) \|, \end{aligned} \tag{16}$$

for all $a, b \in X$. By Lemma 1, $F \in \mathcal{A}_0(X, B)$. Finally, let G be another map in $\mathcal{A}_0(X, B)$ satisfying (12). Then, for any $a \in X$,

$$\begin{aligned} \| F(a) - G(a) \| &= \| 2^p F(2^{-p}a) - 2^p G(2^{-p}a) \| \\ &\le \| 2^p F(2^{-p}a) - 2^p f(2^{-p}a) \| \\ &\quad + \| 2^p G(2^{-p}a) - 2^p f(2^{-p}a) \| \\ &\le 2^{p+1} \Phi(2^{-p}a, 0). \end{aligned} \tag{17}$$

Therefore, $\| F(a) - G(a) \| \longrightarrow 0$ as $p \longrightarrow \infty$. The uniqueness of F follows. □

Corollary 3. *For $r, \vartheta \in \mathbb{R}_0^+$ with $r > 1$, if $f \in \mathcal{F}_0(X, B)$ satisfying*

$$\begin{aligned} &\left\| 2f\left(\frac{a+b}{2}\right) + f(a-b) - 2f(a) \right\| \\ &\le \| s(f(a+b) + f(a-b) - 2f(a)) \| \\ &\quad + \| t(f(a+b) - f(a) - f(b)) \| + \vartheta(\|a\|^r + \|b\|^r), \end{aligned} \tag{18}$$

for all $a, b \in X$, then there exists a unique $F \in \mathcal{A}_0(X, B)$ such that

$$\| f(a) - F(a) \| \le \frac{2^r \vartheta}{2^r - 2} \|a\|^r, \tag{19}$$

for all $a \in X$.

Proof. Let $\varphi(a, b) = \vartheta(\|a\|^r + \|b\|^r)$ for all $a, b \in X$ in Theorem 2. The result immediately follows. □

Theorem 4. *Let $\varphi : X \times X \longrightarrow \mathbb{R}_0^+$ be a map satisfying*

$$\Psi(a, b) := \sum_{j=1}^{\infty} 2^{-j} \varphi\left(2^j a, 2^j b\right) < \infty, \tag{20}$$

for all $a, b \in X$, and let $f \in \mathcal{F}_0(X, B)$ satisfy (11). Then, there exists a unique $F \in \mathcal{A}_0(X, B)$ such that

$$\| f(a) - F(a) \| \le \Psi(a, 0), \tag{21}$$

for all $a \in X$.

Proof. It follows from (13) that $\| f(a) - (1/2) f(2a) \| \le (1/2) \varphi(2a, 0)$ for all $a \in X$. Then, for $m, l \in \mathbb{N}_0$ with $m > l$,

$$\begin{aligned} \left\| 2^{-l} f\left(2^l a\right) - 2^{-m} f(2^m a) \right\| &\le \sum_{j=l}^{m-1} \left\| 2^{-j} f(2^j a) - 2^{-(j+1)} f\left(2^{j+1} a\right) \right\| \\ &\le \sum_{j=l+1}^{m} 2^{-j} \varphi\left(2^j a, 0\right), \end{aligned} \tag{22}$$

for all $a \in X$. Now, let $a \in X$. It follows from the completeness of B that $\{2^{-n} f(2^n a)\}$ is convergent in X. Next, define a map $F : X \longrightarrow B$ by

$$F(a) = \lim_{n \longrightarrow \infty} 2^{-n} f(2^n a), \tag{23}$$

for all $a \in X$. Choosing $l = 0$ and taking $m \longrightarrow \infty$ in (22), we have that F satisfies (21). The rest is similar to the Proof of Theorem 2. □

Let $\varphi(a, b) = \vartheta(\|a\|^r + \|b\|^r)$ for all $a, b \in X$. The following result is straightforward.

Corollary 5. *Let $r, \vartheta \in \mathbb{R}_0^+$ with $r < 1$. If $f \in \mathcal{F}_0(X, B)$ satisfies (18), then there exists a unique $F \in \mathcal{A}_0(X, B)$ such that*

$$\| f(a) - F(a) \| \le \frac{2^r \vartheta}{2 - 2^r} \|a\|^r, \tag{24}$$

for all $a \in X$.

3. Stability Results: Fixed Point Method

In this section, we apply the fixed point method to present the Hyers–Ulam stability of the functional inequality (3).

We first state a useful tool in the field of fixed point theory.

Proposition 6. *[26, 27]. Let (X, d) be a complete generalized metric space, and let $\mathscr{L} : X \longrightarrow X$ be a strict Lipschitz contraction with the Lipschitz constant $\alpha < 1$. Then, for $a \in X$, either*

(a) $d(\mathscr{L}^n a, \mathscr{L}^{n+1} a) = \infty$ for all $n \in \mathbb{N}_0$ or

(b) $d(\mathscr{L}^n a, \mathscr{L}^{n+1} a) < \infty$ for all $n \geq n_0$ for some $n_0 \in \mathbb{N}$; $\mathscr{L}^n a \longrightarrow b^*$ where b^* is a unique fixed point of $\mathscr{L}$ in $X_0 := \{b \in X \mid d(\mathscr{L}^{n_0} a, b) < \infty\}$ and $d(b, b^*) \leq (1/(1-\alpha)) d(b, \mathscr{L} b)$ for all $b \in X_0$

Theorem 7. *Let $\varphi : X \times X \longrightarrow \mathbb{R}_0^+$ be a function such that*

$$\varphi\left(\frac{a}{2}, \frac{b}{2}\right) \leq \frac{L}{2} \varphi(a, b), \tag{25}$$

for all $a, b \in X$ for some $L \in \mathbb{R}_0^+$ with $L < 1$. Then, for $f \in \mathscr{F}_0(X, B)$ satisfying (11), there exists a unique $F \in \mathscr{A}_0(X, B)$ such that

$$\|f(a) - F(a)\| \leq \frac{1}{1-L} \varphi(a, 0), \tag{26}$$

for all $a \in X$.

Proof. Firstly, let us equip $\mathscr{F}_0(X, B)$ with the generalized metric d defined by

$$d(g, h) = \inf\{\mu \in \mathbb{R}^+ : \|g(a) - h(a)\| \leq \mu \varphi(a, 0), \text{for all } a \in X\}. \tag{27}$$

□

Then, from [28], $(\mathscr{F}_0(X, B), d)$ is complete. Next, define a map $\mathscr{J} : \mathscr{F}_0(X, B) \longrightarrow \mathscr{F}_0(X, B)$ by

$$\mathscr{J} g(a) = 2g\left(\frac{a}{2}\right), \tag{28}$$

for all $a \in X$. Let $g, h \in \mathscr{F}_0(X, B)$ where $d(g, h) = \varepsilon$. Then,

$$\|g(a) - h(a)\| \leq \varepsilon \varphi(a, 0), \tag{29}$$

for all $a \in X$. Consequently,

$$\begin{aligned} \|\mathscr{J} g(a) - \mathscr{J} h(a)\| &= \left\|2g\left(\frac{a}{2}\right) - 2h\left(\frac{a}{2}\right)\right\| \leq 2\varepsilon \varphi\left(\frac{a}{2}, 0\right) \\ &\leq 2\varepsilon \frac{L}{2} \varphi(a, 0) = L\varepsilon \varphi(a, 0), \end{aligned} \tag{30}$$

for all $a \in X$. Then, $d(\mathscr{J} g, \mathscr{J} h) \leq L\varepsilon$ which means that

$$d(\mathscr{J} g, \mathscr{J} h) \leq L d(g, h), \tag{31}$$

for all $g, h \in \mathscr{F}_0(X, B)$. By (13), we have that $d(f, \mathscr{J} f) \leq 1$.

Now, let $a \in X$. From Proposition 6, there exists $F : X \longrightarrow B$ satisfying the following:

(i) F is a unique fixed point of $\mathscr{J}$, i.e., $F(a) = 2F(a/2)$ for all $a \in X$

(ii) $d(\mathscr{J}^l f, F) \longrightarrow 0$ as $l \longrightarrow \infty$

$$d(f, F) \leq \frac{1}{1-L} d(f, \mathscr{J} f). \tag{32}$$

It follows that

$$\|f(a) - F(a)\| \leq \mu \varphi(a, 0) \tag{33}$$

(a) $\lim_{l \longrightarrow \infty} 2^l f(2^{-l} a) = F(a)$ and

$$\|f(a) - F(a)\| \leq \frac{1}{1-L} \varphi(a, 0) \tag{34}$$

Using the same method as in the proof of Theorem 2, we can prove that $F \in \mathscr{A}_0(X, B)$.

Corollary 8. *Let $r, \vartheta \in \mathbb{R}_0^+$ with $r > 1$. If $f \in \mathscr{F}_0(X, B)$ satisfies (18), then there exists a unique $F \in \mathscr{A}_0(X, B)$ such that*

$$\|f(a) - F(a)\| \leq \frac{2^r \vartheta}{2^r - 2} \|a\|^r, \tag{35}$$

for all $a \in X$.

Proof. By taking $L = 2^{1-r}$ and $\varphi(a, b) = \vartheta(\|a\|^r + \|b\|^r)$ for all $a, b \in X$ in Theorem 7, the result follows. □

Theorem 9. *Let $\varphi : X \times X \longrightarrow \mathbb{R}_0^+$ be a map such that*

$$\varphi(a, b) \leq 2L \varphi\left(\frac{a}{2}, \frac{b}{2}\right), \tag{36}$$

for all $a, b \in X$, for some $L \in \mathbb{R}_0^+$ with $L < 1$. Then, for any $f \in \mathscr{F}_0(X, B)$ satisfying (11), there exists a unique $F \in \mathscr{A}_0(X, B)$ such that

$$\|f(a) - F(a)\| \leq \frac{L}{1-L} \varphi(a, 0), \tag{37}$$

for all $a \in X$.

Proof. We first consider the complete metric space $(\mathscr{F}_0(X, B), d)$ given as in the proof of Theorem 7. Define a mapping $\mathscr{J} : \mathscr{F}_0(X, B) \longrightarrow \mathscr{F}_0(X, B)$ by

$$\mathscr{J} g(a) = \frac{1}{2} g(2a), \tag{38}$$

for all $a \in X$. It follows from (13) and (36) that

$$\left\| f(a) - \frac{1}{2} f(2a) \right\| \le \frac{1}{2} \varphi(2a, 0) \le L\varphi(a, 0), \qquad (39)$$

for all $a \in X$. As in the proof of Theorem 2 and Theorem 7, there exists a unique $F \in \mathscr{A}_0(X, B)$ satisfying (37). □

Corollary 10. *Let $r, \vartheta \in \mathbb{R}_0^+$ with $r < 1$, and let $f \in \mathscr{F}_0(X, B)$ be a map satisfying (18). Then, there exists a unique $F \in \mathscr{A}_0(X, B)$ such that*

$$\| f(a) - F(a) \| \le \frac{2^r \vartheta}{2 - 2^r} \| a \|^r, \qquad (40)$$

for all $a \in X$.

Proof. Taking $L = 2^{r-1}$ and $\varphi(a, b) = \vartheta(\|a\|^r + \|b\|^r)$ for all $a, b \in X$ in Theorem 9, the result follows. □

4. Conclusions and Final Remarks

We have obtained several Hyers–Ulam stability results for the functional inequality (3) using the direct method and the fixed point method. We now discuss some example for Theorem 2 (via Corollary 3). Consider the sequence space l_2 equipped with the 2-norm. Define $f : l_2 \longrightarrow l_2$ by

$$f(a) = (a_1 + a_2, a_1 - a_2, 2a_3, 0, 0, 0, \cdots), \qquad (41)$$

for all $a = (a_1, a_2, a_3, \cdots) \in l_2$. Let $\vartheta = r = 2$. Then, $f \in \mathscr{F}_0(X, B)$ satisfies (18). By Corollary 3, there exists a unique $F \in \mathscr{A}_0(X, B)$ such that $\|f(a) - F(a)\|_2 \le 4\|a\|_2^2$ for all $a \in l_2$. This example is also valid for the other corollaries in the paper.

There could also be other (s, t)-type functional inequalities to be investigated, and thus, of course, their stability results to be examined. Moreover, these functional inequalities can still be possibly generalized in several ways.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

This research was partially supported by the Chiang Mai University and by the Faculty of Science, Chiang Mai University, Thailand.

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