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Research Article

The Sharp Upper Bounds of the Hankel Determinant on Logarithmic Coefficients for Certain Analytic Functions Connected with Eight-Shaped Domains

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The present study's intention is to produce exact estimations of some problems involving logarithmic coefficients for functions belonging to the considered subcollection $\mathscr{BT}_{\sin}$ of the bounded turning class. Furthermore, for the class $\mathscr{BT}_{\sin}$, we look into the accurate bounds of the Zalcman inequality, Fekete-Szegö inequality along with $\mathscr{D}_{2,1}(G_g/2)$ and $\mathscr{D}_{2,2}(G_g/2)$. Importantly, all of these bounds are shown to be sharp.

1. Introduction and Definitions

To properly understand the findings provided in the article, certain important literature on Geometric Function Theory must first be discussed. In this regard, the letters $\mathscr{S}$ and $\mathscr{A}$ stand for the normalized univalent functions class and the normalized holomorphic (or analytic) functions class, respectively. These primary notions are defined in the region $\mathbb{E}_d = \{z \in \mathbb{C} : |z| < 1\}$ by

$$\mathscr{A} = \left\{ g \in \mathscr{H}(\mathbb{E}_d) \colon g(z) = z + \sum_{k=2}^{\infty} b_k z^k (z \in \mathbb{E}_d) \right\}, \qquad (1)$$

where $\mathscr{H}(\mathbb{E}_d)$ symbolizes the holomorphic functions class, and

$$\mathscr{S} = \{ g \in \mathscr{A} : g \text{ is univalent in } \mathbb{E}_d \}. \qquad (2)$$

The following formula defines the logarithmic coefficients β_n of g that belong to $\mathscr{S}$

$$G_g(z) \coloneqq \log \left(\frac{g(z)}{z} \right) = 2 \sum_{n=1}^{\infty} \beta_n z^n \text{ for } z \in \mathbb{E}_d. \qquad (3)$$

In many estimations, these coefficients provide a significant contribution to the concept of univalent functions. In 1985, De Branges [1] proved that

$$\sum_{k=1}^{n} k(n-k+1)|\beta_n|^2 \le \sum_{k=1}^{n} \frac{n-k+1}{k} \forall n \ge 1, \qquad (4)$$

and equality will be achieved if g has the form $z/(1-e^{i\theta}z)^2$ for some $\theta \in \mathbb{R}$. In its most comprehensive version, this inequality offers the famous Bieberbach-Robertson-Milin conjectures regarding Taylor coefficients of $g \in \mathscr{S}$. We refer to [2–4] for further details on the proof of De Branges' finding. By considering the logarithmic coefficients, Kayumov [5] was able to prove Brennan's conjecture for conformal mappings in 2005. For your reference, we mention a few works that have made major contributions to the research of the logarithmic coefficients. Andreev and Duren [6], Alimohammadi et al. [7], Deng [8], Roth [9], Ye [10], Obradović et al. [11], and finally the work of Girela [12] are the major contributions to the study of logarithmic coefficients for different subclasses of holomorphic univalent functions.

As stated in the definition, it is simple to determine that for $g \in \mathcal{S}$, the logarithmic coefficients are computed by

$$\beta_1 = \frac{1}{2} b_2, \tag{5}$$

$$\beta_2 = \frac{1}{2}\left(b_3 - \frac{1}{2} b_2^2\right), \tag{6}$$

$$\beta_3 = \frac{1}{2}\left(b_4 - b_2 b_3 + \frac{1}{3} b_2^3\right), \tag{7}$$

$$\beta_4 = \frac{1}{2}\left(b_5 - b_2 b_4 + b_2^2 b_3 - \frac{1}{2} b_3^2 - \frac{1}{4} b_2^4\right). \tag{8}$$

For given $q, n \in \mathbb{N} = \{1, 2, \cdots\}$, $b_1 = 1$, and $g \in \mathcal{S}$ with the series expansion (1), the Hankel determinant $\mathcal{D}_{q,n}(g)$ is represented by

$$\mathcal{D}_{q,n}(g) = \begin{vmatrix} b_n & b_{n+1} & \cdots & b_{n+q-1} \\ b_{n+1} & b_{n+2} & \cdots & b_{n+q} \\ \vdots & \vdots & \cdots & \vdots \\ b_{n+q-1} & b_{n+q} & \cdots & b_{n+2q-2} \end{vmatrix}. \tag{9}$$

It was defined by Pommerenke [13, 14]. This determinant has indeed been investigated for a number of univalent function subclasses. In specific, the sharp estimate of the functional $|\mathcal{D}_{2,2}(g)| = |b_2 b_4 - b_3^2|$ for the sets $\mathcal{C}$ (convex functions), $\mathcal{S}^*$(starlike functions), and $\mathcal{R}$ (bounded turning functions) has been effectively established in [15, 16]. Later, numerous scholars published their findings on the upper bounds of $|\mathcal{D}_{2,2}(g)|$ for various subcollections of holomorphic functions; see [17–23]. However, for the class of close-to-convex functions, the exact estimation of this determinant is yet unknown [24].

Analogous to the determinant $\mathcal{D}_{q,n}(g)$ mentioned above, Kowalczyk and Lecko [25, 26] considered to examine the following determinant $\mathcal{D}_{q,n}(G_g/2)$ with entries from logarithmic coefficients of g

$$\mathcal{D}_{q,n}\left(\frac{G_g}{2}\right) = \begin{vmatrix} \beta_n & \beta_{n+1} & \cdots & \beta_{n+q-1} \\ \beta_{n+1} & \beta_{n+2} & \cdots & \beta_{n+q} \\ \vdots & \vdots & \cdots & \vdots \\ \beta_{n+q-1} & \beta_{n+q} & \cdots & \beta_{n+2q-2} \end{vmatrix}. \tag{10}$$

It is observed that

$$\begin{aligned} \mathcal{D}_{2,1}\left(\frac{G_g}{2}\right) &= \beta_1\beta_3 - \beta_2^2, \\ \mathcal{D}_{2,2}\left(\frac{G_g}{2}\right) &= \beta_2\beta_4 - \beta_3^2. \end{aligned} \tag{11}$$

For the given functions $G_1, G_2 \in \mathcal{A}$, the subordination between G_1 and G_2 (mathematically written as $G_1 \prec G_2$), if we get a Schwarz function ν with $\nu(0) = 0$ and $|\nu(z)| < 1$ for $z \in \mathbb{E}_d$ in a way such that $G_1(z) = G_2(\nu(z))$ hold true. Additionally, the following relation applies if G_2 in $\mathbb{E}_d$ is univalent:

$$G_1(z) \prec G_2(z), (z \in \mathbb{E}_d), \tag{12}$$

if and only if

$$\begin{aligned} G_1(0) &= G_2(0), \\ G_1(\mathbb{E}_d) &\subset G_2(\mathbb{E}_d). \end{aligned} \tag{13}$$

In 1992, Ma and Minda [27] developed a consolidated version of the collection $\mathcal{S}^*(\pi)$ by using the principle of subordination, and the following is a description of it:

$$\mathcal{S}^*(\pi) := \left\{ g \in \mathcal{S} : \frac{zg'(z)}{g(z)} \prec \pi(z), (z \in \mathbb{E}_d) \right\}, \tag{14}$$

where the univalent function π satisfies

$$\begin{aligned} \pi'(0) &> 0, \\ \Re e\pi &> 0. \end{aligned} \tag{15}$$

The area $\pi(\mathbb{E}_d)$ is also symmetric about x-axis and has a star-shaped form around the point $\pi(0) = 1$. In recent years, a wide variety of the collection $\mathcal{S}$'s subfamilies have been looked into as particular alternatives for the class $\mathcal{S}^*(\pi)$. As an illustration:

(i) $\mathcal{S}\mathcal{S}^*(\xi) \equiv \mathcal{S}^*(\pi(z))$ with $\pi(z) = ((1+z)/(1-z))^\xi$ and $0 < \xi \le 1$ (see [28])

(ii) $\mathcal{S}^*_{\mathcal{L}} \equiv \mathcal{S}^*((1+z)^{1/2})$ (see [29]), and $\mathcal{S}^*_{c\varsigma r} \equiv \mathcal{S}^*(1 + (4/3)z + (2/3)z^2)$ (see [30, 31])

(iii) $\mathcal{S}^*_{\rho} \equiv \mathcal{S}^*(1 + \sinh^{-1} z)$ (see [32]), and $\mathcal{S}^*_{e} \equiv \mathcal{S}^*(e^z)$ (see [33, 34])

(iv) $\mathcal{S}^*_{\cos} \equiv \mathcal{S}^*(\cos z)$ (see [35]), and $\mathcal{S}^*_{\cosh} \equiv \mathcal{S}^*(\cosh z)$ (see [36])

(v) $\mathcal{S}^*_{\tanh} \equiv \mathcal{S}^*(1 + \tanh z)$ (see [37, 38])

In [39], Cho et al. developed a novel subfamily of starlike function described by

$$\mathcal{S}^*_{\sin} := \left\{ g \in \mathcal{A} : \frac{zg'(z)}{g(z)} \prec 1 + \sin z \, (z \in \mathbb{E}_d) \right\}. \tag{16}$$

From the definition of the family $\mathcal{S}^*_{\sin}$, the authors [39] deduced that

$$g \in \mathcal{S}^*_{\sin} \Leftrightarrow g(z) = z \exp\left(\int_0^z \frac{u(t)-1}{t}dt\right), \quad (17)$$

for some $u(z) \prec u_0(z) = 1 + \sin z$. By substituting

$$u(z) = u_0(z) = 1 + \sin z \quad (18)$$

in (17), we acquire the function

$$g_0(z) = z \exp\left(\int_0^z \frac{\sin t}{t}dt\right) = z + z^2 + \frac{1}{2}z^3 + \frac{1}{9}z^4 \cdots, \quad (19)$$

which acts as the extremal function in a variety of $\mathcal{S}^*_{\sin}$ -family problems. In [40], the authors defined the following subfamily $\mathcal{BT}_{\sin}$ of holomorphic functions by using (18):

$$\mathcal{BT}_{\sin} = \left\{g \in \mathcal{S} : g'(z) \prec 1 + \sin z \, (z \in \mathbb{E}_d)\right\}. \quad (20)$$

Our primary objective in the current paper is to compute the problems involving the sharp logarithmic coefficients for the class $\mathcal{BT}_{\sin}$ of bounded turning functions connected to an eight-shaped domain. The sharp bounds of the Zalcman inequality, the Fekete-Szegö type inequality, along with the determinants $\mathcal{D}_{2,1}(G_g/2)$ and $\mathcal{D}_{2,2}(G_g/2)$ for the family $\mathcal{BT}_{\sin}$ are found using logarithmic coefficient entries.

2. Preliminary Lemmas

We must first create the class $\mathcal{P}$ in the below set-builder form in order to declare the Lemmas that are employed in our primary findings:

$$\mathcal{P} = \{p \in \mathcal{H}(\mathbb{E}_d) : p(0) = 1 \,\&\, \mathfrak{Re}\,p > 0, (z \in \mathbb{E}_d)\}. \quad (21)$$

That is, if $p \in \mathcal{P}$, then it has the series representation

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n (z \in \mathbb{E}_d). \quad (22)$$

Lemma 1 (see [41]). *Let $p \in \mathcal{P}$ and has the series form (22). Then for $x, \delta, \rho \in \bar{\mathbb{E}}_d = \mathbb{E}_d \cup 1\{1\}$*

$$2p_2 = p_1^2 + x(4 - p_1^2), \quad (23)$$

$$4p_3 = p_1^3 + 2(4 - p_1^2)p_1 x - p_1(4 - p_1^2)x^2 + 2(4 - p_1^2)(1 - |x|^2)\delta, \quad (24)$$

$$8p_4 = p_1^4 + (4 - p_1^2)x\left[p_1^2(x^2 - 3x + 3) + 4x\right] - 4(4 - p_1^2)(1 - |x|^2), \quad (25)$$

$$\left[p(x-1)\delta + \bar{x}\delta^2 - (1 - |\delta|^2)\rho\right]. \quad (26)$$

Lemma 2. *If $p \in \mathcal{P}$ and has the expansion (22), then*

$$|p_n| \le 2 (n \ge 1), \quad (27)$$

and if $Q \in [0, 1]$ and $Q(2Q - 1) \le R \le Q$, then

$$\left|p_3 - 2Qp_1p_2 + Rp_1^3\right| \le 2. \quad (28)$$

Also,

$$|p_{n+k} - \mu p_n p_k| \le 2 \max\{1, |2\mu - 1|\} = 2\begin{cases} 1, & \text{for } 0 \le \mu \le 1, \\ |2\mu - 1|, & \text{otherwise}. \end{cases} \quad (29)$$

The inequalities (27), (28) and (29) are taken from [42, 43], and [44], respectively.

Lemma 3 (see [45]). *Let τ, ψ, ρ, and ς satisfy the inequalities $0 < \tau < 1, 0 < \varsigma < 1$ and*

$$8\varsigma(1-\varsigma)\left((\tau\psi - 2\rho)^2 + (\tau(\varsigma + \tau) - \psi)^2\right) + \tau(1-\tau)(\psi - 2\varsigma\tau)^2 \le 4\varsigma\tau^2(1-\tau)^2(1-\varsigma). \quad (30)$$

If $p \in \mathcal{P}$ has the form (22), then

$$\left|\rho p_1^4 + \varsigma p_2^2 + 2\tau p_1 p_3 - \frac{3}{2}\psi p_1^2 p_2 - p_4\right| \le 2. \quad (31)$$

3. Coefficient Inequalities for the Class $\mathcal{BT}_{\sin}$

Theorem 4. *If $g \in \mathcal{BT}_{\sin}$ and has the series representation (1), then*

$$|\beta_1| \le \frac{1}{4}, \quad |\beta_2| \le \frac{1}{6}, \quad |\beta_3| \le \frac{1}{8}, \quad |\beta_4| \le \frac{1}{10}. \quad (32)$$

These bounds are sharp and can be obtained from the following extremal functions

$$\begin{aligned} g_0(z) &= \int_0^z (1 + \sin(t))dt = z + \frac{1}{2}z^2 + \cdots, \\ g_1(z) &= \int_0^z \left(1 + \sin\left(t^2\right)\right)dt = z + \frac{1}{3}z^3 + \cdots, \\ g_2(z) &= \int_0^z \left(1 + \sin\left(t^3\right)\right)dt = z + \frac{1}{4}z^4 + \cdots, \\ g_3(z) &= \int_0^z \left(1 + \sin\left(t^4\right)\right)dt = z + \frac{1}{5}z^5 + \cdots. \end{aligned} \quad (33)$$

Proof. Let $g \in \mathcal{BT}_{\sin}$. Consequently, (20) may be expressed using the Schwarz function as

$$g'(z) = 1 + \sin(w(z)), (z \in \mathbb{E}_d). \quad (34)$$

The Schwarz function w may be used to express it if $p \in \mathcal{P}$ as follows

$$p(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + p_1 z + p_2 z^2 + p_3 z^3 + p_4 z^4 + \cdots, \quad (35)$$

equivalently,

$$w(z) = \frac{p(z) - 1}{p(z) + 1} = \frac{p_1 z + p_2 z^2 + p_3 z^3 + \cdots}{2 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots}. \quad (36)$$

From (1), we obtain

$$g'(z) = 1 + 2a_2 z + 3a_3 z^2 + 4a_4 z^3 + \cdots. \quad (37)$$

By simplification and using the series expansion of (36), we get

$$\begin{aligned} 1 + \sin(w(z)) = 1 &+ \frac{1}{2}p_1 z + \left(-\frac{1}{4}p_1^2 + \frac{1}{2}p_2\right)z^2 \\ &+ \left(-\frac{1}{2}p_1 p_2 + \frac{5}{48}p_1^3 + \frac{1}{2}p_3\right)z^3 \\ &+ \left(\frac{1}{2}p_4 - \frac{1}{32}p_1^4 + \frac{5}{16}p_1^2 p_2 - \frac{1}{2}p_1 p_3 - \frac{1}{4}p_2^2\right)z^4 + \cdots. \end{aligned} \quad (38)$$

Comparing (37) and (38), we obtain

$$a_2 = \frac{1}{4}p_1, \quad (39)$$

$$a_3 = -\frac{1}{12}p_1^2 + \frac{1}{6}p_2, \quad (40)$$

$$a_4 = -\frac{1}{8}p_1 p_2 + \frac{5}{192}p_1^3 + \frac{1}{8}p_3, \quad (41)$$

$$a_5 = \frac{1}{10}p_4 - \frac{1}{160}p_1^4 + \frac{5}{80}p_1^2 p_2 - \frac{1}{10}p_1 p_3 - \frac{1}{20}p_2^2. \quad (42)$$

Putting (42) in (5), (6), (7), and (8), we obtain

$$\beta_1 = \frac{1}{8}p_1, \quad (43)$$

$$\beta_2 = -\frac{11}{192}p_1^2 + \frac{1}{12}p_2, \quad (44)$$

$$\beta_3 = -\frac{1}{12}p_1 p_2 + \frac{5}{192}p_1^3 + \frac{1}{16}p_3, \quad (45)$$

$$\beta_4 = \frac{1}{20}p_4 - \frac{1033}{92160}p_1^4 + \frac{17}{288}p_1^2 p_2 - \frac{23}{720}p_2^2 - \frac{21}{320}p_1 p_3. \quad (46)$$

For β_1, using (27), in (43), we obtain

$$|\beta_1| \le \frac{1}{4}. \quad (47)$$

For β_2, putting (29) in (44), we obtain

$$|\beta_2| \le \frac{1}{6}. \quad (48)$$

For β_3, we can rewrite (45) as

$$|\beta_3| = \frac{1}{16}\left|\left(p_3 - \frac{4}{3}p_1 p_2 + \frac{5}{12}p_1^3\right)\right|. \quad (49)$$

Using (28) we get

$$|\beta_3| \le \frac{1}{8}. \quad (50)$$

For β_4, we can rewrite (46) as

$$\beta_4 = -\frac{1}{20}\left(\frac{1033}{4608}p_1^4 + \frac{23}{36}p_2^2 + 2\left(\frac{21}{32}\right)p_1 p_3 - \frac{3}{2}\left(\frac{85}{108}\right)p_1^2 p_2 - p_4\right). \quad (51)$$

Comparing the right side of (51) with

$$\left|\varrho p_1^4 + \varsigma p_2^2 + 2\tau p_1 p_3 - \frac{3}{2}\psi p_1^2 p_2 - p_4\right|, \quad (52)$$

where

$$\begin{aligned} \varrho &= \frac{1033}{4608}, \\ \varsigma &= \frac{23}{36}, \\ \tau &= \frac{21}{32}, \\ \psi &= \frac{85}{108}. \end{aligned} \quad (53)$$

It follows that

$$\begin{aligned} &8\varsigma(1-\varsigma)\left((\tau\psi - 2\rho)^2 + (\tau(\varsigma+\tau) - \psi)^2\right) \\ &\quad + \tau(1-\tau)(\psi - 2\varsigma\tau)^2 = 0.01647, \end{aligned}$$

$$4\varsigma\tau^2(1-\tau)^2(1-\varsigma) = 0.04696. \quad (54)$$

Using (30) we deduce that

$$|\beta_4| \le \frac{1}{10}. \quad (55)$$

□

Theorem 5. *If g has the series form (1) and belongs to $\mathscr{B}\mathscr{T}_{\sin}$, then*

$$\left|\beta_2-\eta\beta_1^2\right|\leq\max\left\{\frac{1}{6},\left|\frac{1}{48}(3+3|\eta|)\right|\right\}. \tag{56}$$

Equality will be attained by using (5), (6), and

$$g_1(z)=\int_0^z\left(1+\sin\left(t^2\right)\right)dt=z+\frac{1}{3}z^3+\cdots. \tag{57}$$

Proof. From (43) to (44), we get

$$\left|\beta_2-\eta\beta_1^2\right|=\left|-\frac{11}{192}p_1^2+\frac{1}{12}p_2-\frac{\eta}{64}p_1^2\right|. \tag{58}$$

Using (29), we have

$$\left|\beta_2-\eta\beta_1^2\right|\leq\frac{1}{12}\max\left\{2,2\left|2\left(\frac{11+3\eta}{16}\right)-1\right|\right\}. \tag{59}$$

After the simplification, we get

$$\left|\beta_2-\eta\beta_1^2\right|\leq\max\left\{\frac{1}{6},\left|\frac{1}{48}(3+3|\eta|)\right|\right\}. \tag{60}$$

□

Theorem 6. *If g has the series expansion (1) and belongs to $\mathscr{B}\mathscr{T}_{\sin}$, then*

$$|\beta_1\beta_2-\beta_3|\leq\frac{1}{8}. \tag{61}$$

Equality can be attained by applying (5), (6), (7), and

$$g_2(z)=\int_0^z\left(1+\sin\left(t^3\right)\right)dt=z+\frac{1}{4}z^4+\cdots. \tag{62}$$

Proof. From (43), (44), and (45), we obtain

$$|\beta_1\beta_2-\beta_3|=\left|-\frac{17}{512}p_1^3+\frac{3}{32}p_1p_2-\frac{1}{16}p_3\right|. \tag{63}$$

After the simplification, we obtain

$$|\beta_1\beta_2-\beta_3|=\frac{1}{16}\left|p_3-\frac{3}{2}p_1p_2+\frac{17}{32}p_1^3\right|. \tag{64}$$

Using (28), we have

$$|\beta_1\beta_2-\beta_3|\leq\frac{1}{8}. \tag{65}$$

□

Theorem 7. *If $g\in\mathscr{B}\mathscr{T}_{\sin}$ has given by (1), then*

$$\left|\beta_4-\beta_2^2\right|\leq\frac{1}{10}. \tag{66}$$

This result is sharp and equality can be achieved by applying (6), (8), and

$$g_3(z)=\int_0^z\left(1+\sin\left(t^4\right)\right)dt=z+\frac{1}{5}z^5+\cdots. \tag{67}$$

Proof. From (44) to (46), we obtain

$$\left|\beta_4-\beta_2^2\right|=\left|-\frac{7}{180}p_2^2+\frac{1}{20}p_4+\frac{79}{1152}p_1^2p_2-\frac{2671}{184320}p_1^4-\frac{21}{320}p_1p_3\right|. \tag{68}$$

After the simplification, we obtain

$$\left|\beta_4-\beta_2^2\right|=-\frac{1}{20}\left|\frac{2671}{9216}p_1^4+\frac{7}{9}p_2^2+2\left(\frac{21}{32}\right)p_1p_3-\frac{3}{2}\left(\frac{395}{432}\right)p_1^2p_2-p_4\right|. \tag{69}$$

Comparing the right side of (69)with

$$\left|\varrho p_1^4+\varsigma p_2^2+2\tau p_1p_3-\frac{3}{2}\psi p_1^2p_2-p_4\right|, \tag{70}$$

where

$$\begin{aligned}\varrho&=\frac{2671}{9216},\\ \varsigma&=\frac{7}{9},\\ \tau&=\frac{21}{32},\\ \psi&=\frac{395}{432}.\end{aligned} \tag{71}$$

It follows that

$$\begin{aligned}&8\varsigma(1-\varsigma)\left((\tau\psi-2\rho)^2+(\tau(\varsigma+\tau)-\psi)^2\right)\\&\quad+\tau(1-\tau)(\psi-2\varsigma\tau)^2=0.004121,\\[6pt]&4\varsigma\tau^2(1-\tau)^2(1-\varsigma)=0.03518.\end{aligned} \tag{72}$$

Using (30) we deduce that

$$\left|\beta_4-\beta_2^2\right|\leq\frac{1}{10}. \tag{73}$$

□

4. Hankel Determinant with Logarithmic Coefficients

Theorem 8. *Let $g \in \mathscr{BT}_{\sin}$ and be of the form (1). Then*

$$\left|\mathscr{D}_{2,1}\left(\frac{G_g}{2}\right)\right| = \left|\beta_1\beta_3 - \beta_2^2\right| \leq \frac{1}{36}. \tag{74}$$

The above stated result is sharp. Equality can be attained with the use of (5), (6), (7), and

$$g_1(z) = \int_0^z \left(1 + \sin\left(t^2\right)\right)dt = z + \frac{1}{3}z^3 + \cdots. \tag{75}$$

Proof. Employing (43), (44), and (45), we obtain

$$\mathscr{D}_{2,1}\left(\frac{G_g}{2}\right) = \frac{1}{128}p_1p_3 - \frac{1}{36864}p_1^4 - \frac{1}{144}p_2^2 - \frac{1}{1152}p_1^2p_2. \tag{76}$$

Using (23) and (24) along with the assumption that $p_1 = p, p \in [0, 2]$, we get

$$\begin{aligned}\left|\mathscr{D}_{2,1}\left(\frac{G_g}{2}\right)\right| = \Bigg| &-\frac{1}{576}\left(4-p^2\right)^2x^2 - \frac{1}{4096}p^4 \\ &- \frac{1}{512}p^2\left(4-p^2\right)x^2 \\ &+ \frac{1}{256}\left(4-p^2\right)\left(1-|x|^2\right)p\delta\Bigg|.\end{aligned} \tag{77}$$

Applying triangle inequality and assuming $|\delta| \leq 1$, $|x| = J, J \leq 1$ and also setting $p \in [0, 2]$, we have

$$\begin{aligned}\left|\mathscr{D}_{2,1}\left(\frac{G_g}{2}\right)\right| \leq &\frac{1}{576}\left(4-p^2\right)^2J^2 + \frac{1}{4096}p^4 \\ &+ \frac{1}{512}p^2\left(4-p^2\right)J^2 \\ &+ \frac{1}{256}\left(4-p^2\right)\left(1-J^2\right)p \coloneqq \phi(p, J).\end{aligned} \tag{78}$$

A little exercise can verify that $\phi'(p, J) \geq 0$ in $[0, 1]$, and this implies $\phi(p, J) \leq \phi(p, 1)$. Thus, by choosing $J = 1$, we achieve

$$\begin{aligned}\left|\mathscr{D}_{2,1}\left(\frac{G_g}{2}\right)\right| \leq &\frac{1}{576}\left(4-p^2\right)^2 + \frac{1}{4096}p^4 + \frac{1}{512}p^2\left(4-p^2\right) \\ &\coloneqq \phi(p, 1).\end{aligned} \tag{79}$$

Now, since $\phi'(p, 1) < 0$, we see that $\phi(p, 1)$ is a decreasing function, and so its maximum value appears at the lowest point $p = 0$, which is

$$\left|\mathscr{D}_{2,1}\left(\frac{G_g}{2}\right)\right| \leq \frac{1}{36}. \tag{80}$$

□

Theorem 9. *If $g \in \mathscr{BT}_{\sin}$ and has the form (1), then*

$$\left|\mathscr{D}_{2,2}\left(\frac{G_g}{2}\right)\right| \leq \frac{1}{64}. \tag{81}$$

The inequality is sharp and can be obtained by using (6), (7), (8), and

$$g_2(z) = \int_0^z \left(1 + \sin\left(t^3\right)\right)dt = z + \frac{1}{4}z^4 + \cdots. \tag{82}$$

Proof. The determinant $\mathscr{D}_{2,2}(G_g/2)$ can be written as

$$\mathscr{D}_{2,2}\left(\frac{G_g}{2}\right) = \beta_2\beta_4 - \beta_3^2. \tag{83}$$

Putting (44), (45), and (46), with $p_1 = p$, we obtain

$$\begin{aligned}\mathscr{D}_{2,2}\left(\frac{G_g}{2}\right) = \frac{1}{17694720}\big(&432p^4p_2 - 3456p^2p_2^2 - 637p^6 \\ &- 50688p^2p_4 + 8928p^3p_3 - 69120p_3^2 \\ &- 47104p_2^3 + 73728p_2p_4 + 87552pp_2p_3\big).\end{aligned} \tag{84}$$

Let $v = 4 - p^2$ in (23), (24), and (25). Now, applying the simplest version of the given lemma, we get

$$432p^4p_2 = 216\left(p^6 + p^4vx\right),$$

$$3456p^2p_2^2 = 1728p^4vx + 864p^2v^2x^2 + 864p^6,$$

$$\begin{aligned}50688p^2p_4 = &6336p^6 - 25344p^3v\left(1-|x|^2\right)\delta x \\ &- 25344p^2v\left(1-|x|^2\right)\bar{x}\delta^2 + 25344vp^2x^2 \\ &+ 25344p^3v\left(1-|x|^2\right)\delta + 19008p^4xv \\ &+ 19008p^4xv + 25344p^2v\left(1-|x|^2\right)\left(1-|\delta|^2\right)\rho \\ &+ 6336p^4vx^3,\end{aligned}$$

$$\begin{aligned}8928p^3p_3 = &4464p^4xv - 2232p^4vx^2 + 2232p^6 \\ &+ 4464p^3v\left(1-|x|^2\right)\delta,\end{aligned}$$

$$\begin{aligned}69120p_3^2 = &-17280x^2v^2\left(1-|x|^2\right)p\delta - 8640p^4vx^2 \\ &+ 4320p^6 + 4320x^4v^2p^2 + 17280p^3v\left(1-|x|^2\right)\delta \\ &+ 17280v^2\left(1-|x|^2\right)^2\delta^2 + 17280p^2x^2v^2 \\ &+ 17280p^4xv + 34560pxv^2\left(1-|x|^2\right)\delta \\ &- 17280x^3v^2p^2,\end{aligned}$$

$$47104p_2^3 = 5888\left(v^3x^3 + p^6\right) + 17664\left(p^4vx + p^2v^2x^2\right),$$

$$\begin{aligned}73728p_2p_4 &= 18432x^3v^2 + 4608p^6 + 18432vp^2x^2 + 4608p^4vx^3 \\ &\quad + 4608x^4v^2p^2 + 18432p^4xv \\ &\quad + 18432pxv^2\left(1-|x|^2\right)\delta - 13824p^4vx^2 \\ &\quad - 13824x^3v^2p^2 - 18432xv^2\bar{x}\left(1-|x|^2\right)\bar{x}\delta^2 \\ &\quad - 18432p^2v\left(1-|x|^2\right)\bar{x}\delta^2 + 13824p^2x^2v^2 \\ &\quad - 18432p^3v\left(1-|x|^2\right)\delta x + 18432p^3v\left(1-|x|^2\right)\delta \\ &\quad + 18432xv^2\left(1-|x|^2\right), \\ &\left(1-|\delta|^2\right)\rho + 18432p^2v\left(1-|x|^2\right)\left(1-|\delta|^2\right)\rho \\ &\quad - 18432x^2v^2\left(1-|x|^2\right)p\delta, \\ 87552pp_2p_3 &= 21888pxv^2\left(1-|x|^2\right)\delta - 10944x^3v^2p^2 \\ &\quad + 32832p^4xv - 10944p^4vx^2 \\ &\quad + 21888p^3v\left(1-|x|^2\right)\delta + 10944p^6 \\ &\quad + 21888p^2x^2v^2. \end{aligned} \tag{85}$$

Putting the above expressions in (84), we get,

$$\begin{aligned}\mathscr{D}_{2,2}\left(\frac{G_g}{2}\right) &= \frac{1}{17694720}\left\{-6912vp^2x^2 - 5888x^3v^3 - 45p^6\right. \\ &\quad + 2160p^3v\left(1-|x|^2\right)\delta + 264p^4xv - 1728p^4vx^3 \\ &\quad + 288x^4v^2p^2 - 96p^2x^2v^2 + 18432x^3v^2 \\ &\quad - 7488x^3v^2p^2 + 6912p^3v\left(1-|x|^2\right)\delta x \\ &\quad + 6912p^2v\left(1-|x|^2\right)\bar{x}\delta^2 + 5760pxv^2\left(1-|x|^2\right)\delta \\ &\quad + 648p^4vx^2 + 18432xv^2\left(1-|x|^2\right)\left(1-|\delta|^2\right)\rho \\ &\quad - 1152x^2v^2\left(1-|x|^2\right)p\delta - 18432xv^2 \\ &\quad \cdot\left(1-|x|^2\right)\bar{x}\delta^2 - 17280v^2\left(1-|x|^2\right)^2\delta^2 \\ &\quad \left.- 6912p^2v\left(1-|x|^2\right)\left(1-|\delta|^2\right)\rho\right\}. \end{aligned} \tag{86}$$

Since $v = 4 - p^2$,

$$\mathscr{D}_{2,2}\left(\frac{G_g}{2}\right) = \frac{1}{17694720}\left(q_1(p,x) + q_2(p,x)\delta + q_3(p,x)\delta^2 + q_4(p,x,\delta)\rho\right), \tag{87}$$

where $\rho, x, \delta \in \bar{\mathbb{U}}_d$, and

$$\begin{aligned} q_1(p,x) &= \left(4-p^2\right)\left[\left(4-p^2\right)\left(288x^4p^2 - 1600x^3p^2 - 5120x^3 - 96x^2p^2\right) - 1728x^3p^4 + 264xp^4 - 6912x^2p^2 + 648x^2p^4\right] - 45p^6, \\ q_2(p,x) &= \left(4-p^2\right)\left(1-|x|^2\right)\left[\left(4-p^2\right)\left(5760xp - 1152x^2p\right) + 2160p^3 + 6912xp^3\right], \\ q_3(p,x) &= \left(4-p^2\right)\left(1-|x|^2\right)\left[\left(4-p^2\right)\left(-17280 - 1152|x|^2\right) + 6912\bar{x}p^2\right], \\ q_4(p,x,\delta) &= \left(4-p^2\right)\left(1-|x|^2\right)\left(1-|\delta|^2\right)\cdot\left[18432x\left(4-p^2\right) - 6912p^2\right]. \end{aligned} \tag{88}$$

Now, by the virtue of $|\delta| = y, |x| = x$, and $|\rho| \le 1$, we get

$$\left|\mathscr{D}_{2,2}\left(\frac{G_g}{2}\right)\right| \le \frac{1}{17694720}\left(|q_1(p,x)| + |q_2(p,x)|y + |q_3(p,x)|y^2 + |q_4(p,x,\delta)|\right) \le \frac{T(p,x,y)}{17694720}, \tag{89}$$

where

$$T(p,x,y) = m_1(p,x) + m_2(p,x)y + m_3(p,x)y^2 + m_4(p,x)\left(1-y^2\right), \tag{90}$$

with

$$\begin{aligned} m_1(p,x) &= \left(4-p^2\right)\left[\left(4-p^2\right)\left(288x^4p^2 + 1600x^3p^2 + 5120x^3 + 96x^2p^2\right) + 1728x^3p^4 + 264xp^4 + 6912x^2p^2 + 648x^2p^4\right] + 45p^6, \\ m_2(p,x) &= \left(4-p^2\right)\left(1-x^2\right)\left[\left(4-p^2\right)\left(5760xp + 1152x^2p\right) + 2160p^3 + 6912xp^3\right], \\ m_3(p,x) &= \left(4-p^2\right)\left(1-x^2\right)\left[\left(4-p^2\right)\left(17280 + 1152x^2\right) + 6912xp^2\right], \\ m_4(p,x) &= \left(4-p^2\right)\left(1-x^2\right)\left[18432x\left(4-p^2\right) + 6912p^2\right]. \end{aligned} \tag{91}$$

To illustrate the sharp bounds of the given problem, we must maximize $T(p,x,y)$ in the closed cuboid $Y : [0,2] \times [0,1] \times [0,1]$.

(1) Interior points of cuboid Y

Let us choose $(p,x,y) \in (0,2) \times (0,1) \times (0,1)$. Then simple calculation yields

$$\frac{\partial T}{\partial y} = 144\left(4-p^2\right)\left(1-x^2\right)\left[16y(x-1)\left(6p^2 + (x-15)\left(4-p^2\right)\right) + 8p\left(x(x+5)\left(4-p^2\right) + p^2\left(6x + \frac{15}{8}\right)\right)\right]. \tag{92}$$

Putting $\partial T/\partial y = 0$, we obtain

$$y = \frac{8p\left(x(x+5)\left(4-p^2\right) + p^2(6x + 15/8)\right)}{16(x-1)(-6p^2 + (15-x)(4-p^2))} = y_0. \tag{93}$$

If $y_0 \in Y$ is a critical point, then $y_0 \in (0,1)$, and it is applicable only if

$$8px(x+5)\left(4-p^2\right)+p^3(48x+15)+16(1-x)(15-x)\cdot\left(4-p^2\right)<96p^2(1-x). \tag{94}$$

$$p^2>\frac{4(15-x)}{21-x}. \tag{95}$$

To check critical points existence, we must find solutions that fulfill both constraints (94) and (95).

Let $k(x)=(4(15-x))/(21-x)$. As $k'(x)<0$ for all $x\in(0,1)$, it is evident that $k(x)$ is decreasing in $(0,1)$. Hence $p^2>14/5$. It is easy to showcase that the inequality (94) does not hold in this scenario for all $x\in[2/5,1)$. As a result, $T(p,x,y)$ does not have a critical point in $(0,2)\times[2/5,1)\times(0,1)$. Assume a critical point $(\tilde{p},\tilde{x},\tilde{y})$ of T exists inside the interior of the cuboid Y, it must unquestionably fulfil that $\tilde{x}<2/5$.

From the arguments above, it is undeniable that $\tilde{p}^2\geq 292/103$ and $\tilde{y}\in(0,1)$. Now let us establish that $T(\tilde{p},\tilde{x},\tilde{y})<276480$. For $(p,x,y)\in((292/103)^{1/2},2)\times(0,2/5)\times(0,1)$, by invoking $x<2/5$ and $1-x^2<1$, it is not hard to observe that

$$\begin{aligned}
m_1(p,x)\leq{}&\left(4-p^2\right)\left[\left(4-p^2\right)\left(288\left(\frac{2}{5}\right)^4p^2+1600\left(\frac{2}{5}\right)^3p^2\right.\right.\\
&+5120\left(\frac{2}{5}\right)^3+96\left(\frac{2}{5}\right)^2p^2+1728\left(\frac{2}{5}\right)^3p^4\\
&\left.\left.+264\left(\frac{2}{5}\right)p^4+6912\left(\frac{2}{5}\right)^2p^2+648\left(\frac{2}{5}\right)^2p^4\right)\right]\\
&+45p^6=\left(4-p^2\right)\left(\frac{799232}{625}p^2+\frac{121712}{625}p^4\right.\\
&\left.+\frac{819200}{625}\right)+45p^6:=\Theta_1(p),\\
m_2(p,x)\leq{}&\left(4-p^2\right)\left[\left(4-p^2\right)\left(5760\left(\frac{2}{5}\right)p+1152\left(\frac{2}{5}\right)^2p\right)\right.\\
&\left.+2160p^3+6912\left(\frac{2}{5}\right)p^3\right]\\
={}&\left(4-p^2\right)\left(\frac{248832}{25}p+\frac{60912}{25}p^3\right):=\Theta_2(p),\\
m_3(p,x)\leq{}&\left(4-p^2\right)\left[\left(4-p^2\right)\left(17280+1152\left(\frac{2}{5}\right)^2\right)\right.\\
&\left.+6912\left(\frac{2}{5}\right)p^2\right]\\
={}&\left(4-p^2\right)\left(\frac{1746432}{25}-\frac{367488}{25}p^2\right):=\Theta_3(p),\\
m_4(p,x)\leq{}&\left(4-p^2\right)\left[18432\left(\frac{2}{5}\right)\left(4-p^2\right)+6912p^2\right]\\
={}&\left(4-p^2\right)\left(\frac{147456}{5}-\frac{2304}{5}p^2\right):=\Theta_4(p).
\end{aligned} \tag{96}$$

Therefore, we have

$$T(p,x,y)\leq\Theta_1(p)+\Theta_4(p)+\Theta_2(p)y+[\Theta_3(p)-\Theta_4(p)]y^2:=\Gamma(p,y). \tag{97}$$

Obviously, it can be seen that

$$\begin{aligned}
\frac{\partial\Gamma}{\partial y}&=\Theta_2(p)+2y[\Theta_3(p)-\Theta_4(p)],\\
\frac{\partial^2\Gamma}{\partial y^2}&=2[\Theta_3(p)-\Theta_4(p)]=2\left(4-p^2\right)\left(\frac{1009152}{25}-\frac{355968}{25}p^2\right).
\end{aligned} \tag{98}$$

Since $\Theta_3(p)-\Theta_4(p)\leq 0$ for $p\in((292/103)^{1/2},2)$, we obtain that $\partial^2\Gamma/\partial y^2\leq 0$ for $y\in(0,1)$, and thus, it follows that

$$\frac{\partial\Gamma}{\partial y}\geq\left.\frac{\partial\Gamma}{\partial y}\right|_{y=1}=\left(4-p^2\right)\left(-\frac{711936}{25}p^2+\frac{60912}{25}p^3+\frac{2018304}{25}+\frac{248832}{25}p\right)\geq 0. \tag{99}$$

Therefore, we have

$$\Gamma(p,y)\leq\Gamma(p,1)=\Theta_1(p)+\Theta_2(p)+\Theta_3(p):=\iota(p). \tag{100}$$

It is easy to be calculated that $\iota(p)$ attains its maximum value 74510.30 at $p\approx 1.68373$. Thus, we have

$$T(p,x,y)<276480,\ (p,x,y)\in\left(\sqrt{\frac{292}{103}},2\right)\times\left(0,\frac{2}{5}\right)\times(0,1). \tag{101}$$

Hence, $T(\tilde{p},\tilde{x},\tilde{y})<276480$. This implies that T is less than 276480 at all the critical points in the interior of Y. Therefore, T has no optimal solution in the interior of Y.

(2) Interior of all the six faces of cuboid Y :

(i) On the face $p=0$, $T(p,x,y)$ yields

$$b_1(x,y)=T(0,x,y)=2048\left(9\left(1-x^2\right)\cdot\left(16x+(x-15)(x-1)y^2\right)+40x^3\right),\ x,y\in(0,1). \tag{102}$$

Differentiating $b_1(x,y)$ with respect to y, we have

$$\frac{\partial b_1}{\partial y}=36864y\left(1-x^2\right)(x-15)(x-1),\ x,y\in(0,1). \tag{103}$$

Thus, $b_1(x,y)$ has no critical point in the interval $(0,1)\times(0,1)$.

(ii) On the face $p=2, T(p,x,y)$ becomes

$$T(2,x,y)=2880. \quad (104)$$

(iii) On the face $x=0, T(p,x,y)$ reduces to

$$b_2(p,y)=T(p,0,y)=\left(4-p^2\right)\left(6912p^2+2160p^3y-24192y^2p^2+69120y^2\right)+45p^6. \quad (105)$$

Differentiating $b_2(p,y)$ partially with respect to y, we have

$$\frac{\partial b_2}{\partial y}=\left(4-p^2\right)\left(2160p^3-48384yp^2+138240y\right). \quad (106)$$

Solving $\partial b_2/\partial y=0$, we obtain

$$y=\frac{5p^3}{16(7p^2-20)}=y_1. \quad (107)$$

For the given range of y, y_1 should belong to $(0,1)$, which is possible only if $p>p_0, p_0\approx 1.7609$. Also derivative of $b_2(p,y)$ partially with respect to p is

$$\frac{\partial b_2}{\partial p}=-4320p^4y-13824p^3+\left(4-p^2\right)\cdot\left(-48384y^2p+6480yp^2+13824p\right)+48384y^2p^3-138240y^2p+270p^5. \quad (108)$$

Putting the value of y in (108), with $\partial b_2/\partial p=0$ and simplifying, we obtain

$$\frac{\partial b_2}{\partial p}=-27\left(49576p^7+35p^9-385072p^5-819200p+983040p^3\right)=0. \quad (109)$$

A calculation gives the solution of (109) in the interval $(0,1)$, that is, $p\approx 1.3851$. Thus, $b_2(p,y)$ has no optimal point in the interval $(0,2)\times(0,1)$.

(iv) On the face $x=1, T(p,x,y)$ becomes

$$b_3(p,y)=T(p,1,y)=45p^6+\left(4-p^2\right)\cdot\left(\left(4-p^2\right)\left(1984p^2+5120\right)+6912p^2+2640p^4\right). \quad (110)$$

Then

$$\frac{\partial b_3}{\partial p}=-3666p^5-28416p^3+36864p. \quad (111)$$

By setting $\partial b_3/\partial p=0$, we get the critical point $p\approx 1.0639$ at which $b_3(p,y)$ attains its maximum value, which is given below

$$T(p,1,y)\le 92795.48842. \quad (112)$$

(v) On the face $y=0, T(p,x,y)$ yields

$$\begin{aligned}b_4(p,x)=T(p,x,0)={}&-128p^6x^3+288p^6x^4-552p^6x^2\\&+19488p^4x-264p^6x-147456p^2x+45p^6\\&+4608p^2x^4+294912x-19200p^4x^3+1536p^2x^2\\&+132096p^2x^3+1824p^4x^2-6912p^4+27648p^2\\&-2304p^4x^4-212992x^3.\end{aligned} \quad (113)$$

A numerical computation shows that the solution for the system of equations

$$\frac{\partial b_4}{\partial p}=0,\quad \frac{\partial b_4}{\partial x}=0 \quad (114)$$

does not exists in the interval $(0,2)\times(0,1)$. Hence, $b_4(p,x)$ has no optimal solution in the interval $(0,2)\times(0,1)$.

(vi) On the face $y=1, T(p,x,y)$ reduces to

$$\begin{aligned}b_5(p,x)=T(p,x,1)={}&45p^6+288p^6x^4+9216p^3x^4\\&+1152p^5x^3-2160p^5-264p^6x-1152p^5x^4\\&+18432p^3x^3+3312p^5x^2+6144p^4x^3-128p^6x^3\\&-43008p^2x^3-3456p^4x^4-552p^6x^2-1152p^5x\\&-138240p^2-21216p^4x^2+276480+13824p^2x^4\\&+81920x^3-5856p^4x-92160px^3-17856p^3x^2\\&-18432px^4-258048x^2+92160px-18432x^4\\&-18432p^3x+8640p^3+18432px^2+17280p^4\\&+27648p^2x+158208p^2x^2.\end{aligned} \quad (115)$$

As in the above case, we conclude the same result for the face $y=0$, that is, system of equations

$$\frac{\partial b_5}{\partial p}=0,\quad \frac{\partial b_5}{\partial x}=0 \quad (116)$$

has no solution in the interval $(0,2)\times(0,1)$.

(3) On the edges of cuboid Y :

(i) On the edge $x=0$ and $y=0, T(p,x,y)$ reduces to

$$T(p,0,0) = -6912p^4 + 45p^6 + 27648p^2 = b_6(p). \quad (117)$$

It follows that

$$b_6'(p) = -27648p^3 + 270p^5 + 55296p. \quad (118)$$

We see that $b_6'(p)=0$ for the critical point $p_0 \approx 1.4285$ at which $b_6(p)$ obtain its maximum value, which is given by

$$T(p,0,0) \le 28018.97. \quad (119)$$

(ii) On the edge $x=0$ and $y=1, T(p,x,y)$ becomes

$$T(p,0,1) = -2160p^5 - 138240p^2 + 17280p^4 + 45p^6 + 8640p^3 + 276480 = b_7(p). \quad (120)$$

Differentiating $b_7(p)$ with respect to p, we have

$$b_7'(p) = -10800p^4 - 276480p + 69120p^3 + 270p^5 + 25920p^2. \quad (121)$$

We know that $b_7'(p) < 0$ in $[0,2]$ follows that $b_7(p)$ is decreasing over $[0,2]$. Therefore, $b_7(p)$ gets its maxima at $p=0$. Hence

$$T(p,0,1) \le 276480. \quad (122)$$

(iii) On the edge $p=0$ and $x=0, T(p,x,y)$ reduces to

$$T(0,0,y) = 276480y^2 = b_8(y). \quad (123)$$

Noting that $b_8'(y) > 0$ in $[0,1]$ shows that $b_8(y)$ is increasing over $[0,1]$. Thus, $b_8(y)$ gets its maxima at $y=1$. Thus, we have

$$T(0,0,y) \le 276480. \quad (124)$$

(iv) On the edges $T(p,1,1)$ and $T(p,1,0)$

Since $T(p,1,y)$ is free of y, therefore

$$T(p,1,1) = T(p,1,0) = -7104p^4 - 611p^6 + 81920 + 18432p^2 = b_9(p). \quad (125)$$

Then

$$b_9'(p) = -28416p^3 - 3666p^5 + 36864p. \quad (126)$$

By putting $b_9'(p)=0$, we obtain the critical point $p_0 \approx 1.0639$ at which $b_9(p)$ attains its maximum value, which is given by

$$T(p,1,1) = T(p,1,0) \le 92795.48. \quad (127)$$

(v) On the edge $p=0$ and $x=1, T(p,x,y)$ becomes

$$T(0,1,y) = 81920. \quad (128)$$

(vi) On the edge $p=2, T(p,x,y)$ reduces to

$$T(2,x,y) = 2880. \quad (129)$$

$T(2,x,y)$ is independent of x and y; therefore

$$T(2,x,0) = T(2,x,1) = T(2,0,y) = T(2,1,y) = 2880. \quad (130)$$

(vii) On the edge $p=0$ and $y=1, T(p,x,y)$ takes the form

$$T(0,x,1) = 81920x^3 - 18432x^4 + 276480 - 258048x^2 = b_{10}(x). \quad (131)$$

It is clear that

$$b_{10}'(x) = 245760x^2 - 73728x^3 - 516096x. \quad (132)$$

We see that $b_{10}'(x) < 0$ in $[0,1]$ shows that $b_{10}(x)$ is decreasing over $[0,1]$. Thus, $b_{10}(x)$ gets its maxima at $x=0$. Hence, we have

$$T(0,x,1) \le 276480. \quad (133)$$

(viii) On the edge $p=0$ and $y=0, T(p,x,y)$ yields

$$T(0,x,0) = 294912x - 212992x^3 = b_{11}(x). \quad (134)$$

It follows that

$$b_{11}'(x) = 294912 - 638976x^2. \quad (135)$$

By taking $b_{11}'(x) = 0$, we obtain the critical point $x_0 \approx 0.6793$ at which $b_{11}(x)$ attains its maximum value, which is given by

$$T(0, x, 0) \leq 133568.833. \tag{136}$$

Hence, from the above cases we deduce that

$$T(p, x, y) \leq 276480 \text{ on } [0, 2] \times [0, 1] \times [0, 1]. \tag{137}$$

From (89)we have

$$\left| \mathcal{D}_{2,2}\left(\frac{G_g}{2}\right) \right| \leq \frac{T(p, x, y)}{17694720} \leq \frac{1}{64}. \tag{138}$$

If $g \in \mathcal{BT}_{\sin}$, then sharp bound for this Hankel determinant is determined by

$$\left| \mathcal{D}_{2,2}\left(\frac{G_g}{2}\right) \right| = \frac{1}{64} \approx 0.0156. \tag{139}$$

Thus, we have completed the proof. □

5. Conclusion

In our current investigation, we have considered a class $\mathcal{B}\mathcal{T}_{\sin}$ of bounded turning functions associated with an eight-shaped domain. For such a class, we studied some interesting problems involving logarithmic coefficients. The Zalcman inequality, the Fekete-Szegö inequality, and the determinants $\mathcal{D}_{2,2}(G_g/2)$ and $\mathcal{D}_{2,1}(G_g/2)$ for the family $\mathcal{B}\mathcal{T}_{\sin}$ have been studied here in this article. All the obtained results are proven to be the best possible.

Data Availability

The numerical data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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