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Retraction

Retracted: Intuitionistic Fuzzy Fixed Point Theorems in Complex-Valued b-Metric Spaces with Applications to Fractional Differential Equations

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This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

(1) Discrepancies in scope

(2) Discrepancies in the description of the research reported

(3) Discrepancies between the availability of data and the research described

(4) Inappropriate citations

(5) Incoherent, meaningless and/or irrelevant content included in the article

(6) Manipulated or compromised peer review

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

References

[1] R. Tabassum, M. S. Shagari, A. Azam, O. K. S. K. Mohamed, and A. A. Bakery, "Intuitionistic Fuzzy Fixed Point Theorems in Complex-Valued b-Metric Spaces with Applications to Fractional Differential Equations," *Journal of Function Spaces*, vol. 2022, Article ID 2261199, 17 pages, 2022.

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Research Article

Intuitionistic Fuzzy Fixed Point Theorems in Complex-Valued *b*-Metric Spaces with Applications to Fractional Differential Equations

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Among various efforts in advancing fuzzy mathematics, a lot of attentions have been paid to examine novel intuitionistic fuzzy analogues of the classical fixed point results. Along this direction, the idea of intuitionistic fuzzy mapping (IFM) is used in this paper to establish some fixed point (FP) results in complex-valued b-metric spaces. Moreover, from application perspective, one of our results is rendered to provide an existence condition for a solution of Caputo-type fractional differential equations. A few nontrivial illustrations are also furnished to authenticate and indicate the usability of the presented results.

1. Introduction

Many FP results of contraction type mappings are specifically beneficial to find the existence and uniqueness of solution to different mathematical problems. In this regard, the Banach contraction principle [1] has become a very powerful source in various fields of applied mathematical analysis. Afterwards, several researchers improved and refined this result using different mappings in various generalized metric spaces (MS). For instance, Nadler refined the Banach FP result for multivalued mappings (MVM). Bakhtin [2] initiated the notion of b-MS as a refinement of classical MS, and Czerwik [3] proved the contraction mapping principle in b-MS. Therefore, a large amount of research has been brought up to obtain FPs of several mappings in b-MS.

It is a familiar fact that fixed point results concerning rational contractions cannot be improved or even meaningless in cone metric spaces. To circumvent this problem, Azam et al. [4] have given a brilliant idea of complex-valued MS and improved the Banach contraction principle for a pair of mappings obeying the rational inequality in the bodywork of complex-valued MS. In sequel, Sintunavarat and Kumam [5] presented some common FP theorems using the control functions in contractive condition instead of constants. Subsequently, Ahmad et al. [6] have reported some new FP results for multivalued mappings obeying the greatest lower bound property in the context of complex-valued MS. Later, there has been much progress in the study of complex-valued MS by many authors [6–9]. In continuation to this, Rao et al. [10] brought in a new view of complex-valued b-MS and proved some common FP results in complex-valued b-MS. Meanwhile, Mukheimer [11] refined the results of Azam et al. [4] and Bhatt et al. [9].

It is well-known that most of the problems in existing nature have several uncertainties, and to handle these uncertainties, there are various theories including fuzzy set [12], the soft set [13], the fuzzy soft set [14], and intuitionistic fuzzy set (IF-set) [15]. However, many physical problems with vague information can be tackled more precisely by means of IF-set approach. Keeping this in view, some FP results for MVM are refined to IFMs given by Shen et al. [16]. However, research on IFMs to establish the fixed point results is relatively recent, and few work has been done for IFMs on various spaces (e.g., see [17–22]).

Stirred by the above-mentioned investigations, we present some FP results for IFMs in the bodywork of $(\mathcal{T}, \mathcal{N}, \tilde{\alpha})$ – level set of an IF-set [23] in complete complex-valued b-MS. Further, some nontrivial examples and an application are given for the reliability of our main results.

2. Preliminaries

We launch here some basic definitions and results which will be useful in what comes hereafter. Let $\mathbb{C}$ be the set of complex numbers and $\mathcal{I}_1, \mathcal{I}_2 \in \mathbb{C}$. We depict a partial order $\prec$ and $\preceq$ on $\mathbb{C}$ as follows:

(i) $\mathcal{I}_1 \prec \mathcal{I}_2$ if and only if $\mathbb{R}e(\mathcal{I}_1) < \mathbb{R}e(\mathcal{I}_2)$ and $\mathbb{I}m(\mathcal{I}_1) < \mathbb{I}m(\mathcal{I}_2)$

(ii) $\mathcal{I}_1 \preceq \mathcal{I}_2$ if and only if $\mathbb{R}e(\mathcal{I}_1) \leq \mathbb{R}e(\mathcal{I}_2)$ and $\mathbb{I}m(\mathcal{I}_1) \leq \mathbb{I}m(\mathcal{I}_2)$

Definition 1 (see [10]). Let Γ be a nonempty set and $Y \geq 1$ be a real number. A function $d_c : \Gamma \times \Gamma \longrightarrow \mathbb{C}$ is termed a complex-valued b-MS, if for all $\mathcal{I}, \ell, \wp \in \Gamma$, the following conditions hold:

(i) $0 \preceq d_c(\mathcal{I}, \ell)$ and $d_c(\mathcal{I}, \ell) = 0$ if and only if $\mathcal{I} = \ell$

(ii) $d_c(\mathcal{I}, \ell) = d_c(\ell, \mathcal{I})$

(iii) $d_c(\mathcal{I}, \ell) \preceq Y[d_c(\mathcal{I}, \wp) + d_c(\wp, \ell)]$

Then, d_c is termed a complex-valued b-metric on Γ and the pair (Γ, d_c) is termed a complex-valued b-MS.

Remark 2. Let (Γ, d_c) be a complex-valued b-MS. If $Y = 1$, then (Γ, d_c) is a complex-valued MS. If $Y = 1$ and $\mathbb{C} = \mathbb{R}$, then (Γ, d_c) is a MS.

Example 1. Let $\Gamma = [0, 1]$ and a mapping $d_c : \Gamma \times \Gamma \longrightarrow \mathbb{C}$ is given by

$$d_c(\mathcal{I}, \ell) = |\mathcal{I} - \ell|^2 + i|\mathcal{I} - \ell|^2, \tag{1}$$

for all $\mathcal{I}, \ell \in \Gamma$. Then, (Γ, d_c) is a complex-valued b-MS with $Y = 2$.

Definition 3 (see [10]). Let (Γ, d_c) be a complex-valued b-MS. A point $\jmath \in \Gamma$ is an interior point of a set $\mathbb{M} \subseteq \Gamma$, whenever we can find $0 \prec r \in \mathbb{C}$:

$$B(\mathcal{I}, r) = \{\ell \in \Gamma : d_c(\mathcal{I}, \ell) \prec r\} \subseteq \mathbb{M}. \tag{2}$$

A point $\jmath \in \Gamma$ is said to be a limit point of a set $\mathbb{M} \subseteq \Gamma$ whenever for every $0 \prec r \in \mathbb{C}$,

$$B(\mathcal{I}, r) \cap (\mathbb{M} \setminus \{\mathcal{I}\}) \neq \phi. \tag{3}$$

$\mathbb{M} \subseteq \Gamma$ is termed an open set if each element of $\mathbb{M}$ is an interior point of $\mathbb{M}$.

Definition 4 (see [10]). Let $\{\mathcal{I}_p\}$ be a sequence in (Γ, d_c) and $\mathcal{I} \in \Gamma$. If for every $c \in \mathbb{C}$ with $0 \prec c$ there is $p_o \in \mathbb{N}$: $d_c(\mathcal{I}_p, \jmath) \prec c$ for all $p > p_o$, then $\{\mathcal{I}_p\}$ is said to be a convergent sequence which converges to $\mathcal{I}$, and we denote this by $\lim_{p\longrightarrow\infty} \jmath_p = \mathcal{I}$. If for every $c \in \mathbb{C}$ with $0 \prec c$ there is $p_o \in \mathbb{N}$: $d_c(\mathcal{I}_p, \mathcal{I}_{p+q}) \prec c$ for all $p, q > p_o$, and $p, q \in \mathbb{N}$, hence $\{\mathcal{I}_p\}$ is said to be a Cauchy sequence in (Γ, d_c). A complex-valued b-MS (Γ, d_c) is termed a complete space if every Cauchy sequence is convergent in (Γ, d_c).

Lemma 5 (see [10]). *Let (Γ, d_c) be a complex-valued b-MS and $\{\mathcal{I}_p\}$ be a sequence in (Γ, d_c). Then, $\{\mathcal{I}_p\}$ converges to $\mathcal{I}$ if and only if $|d_c(\mathcal{I}_p, \mathcal{I})| \longrightarrow 0$ as $p \longrightarrow \infty$.*

Lemma 6 (see [10]). *Let (Γ, d_c) be a complex-valued b-MS and $\{\mathcal{I}_p\}$ be a sequence in (Γ, d_c). Then, $\{\mathcal{I}_p\}$ is a Cauchy sequence if and only if $|d_c(\mathcal{I}_p, \mathcal{I}_{p+q})| \longrightarrow 0$ as $p, q \longrightarrow \infty$.*

Let $\mathbb{CB}(\Gamma)$ be the collection of all nonempty closed and bounded subsets of (Γ, d_c). We denote

$$s(z_1) = \{z_2 \in \mathbb{C} : z_1 \preceq z_2\}, \tag{4}$$

for $z_1 \in \mathbb{C}$ and

$$s(\mathcal{I}, \mathbb{N}) = \bigcup_{n\in\mathbb{N}} s(d_c(\mathcal{I}, n)) = \bigcup_{n\in\mathbb{N}} \{z \in \mathbb{C} : d_c(\mathcal{I}, n) \preceq z\}, \tag{5}$$

for $\mathcal{I} \in \Gamma$ and $\mathbb{N} \in \mathbb{CB}(\Gamma)$. For $\mathbb{M}, \mathbb{N} \in \mathbb{CB}(\Gamma)$, we have

$$s(\mathbb{M}, \mathbb{N}) = \bigcap_{m\in\mathbb{M}} s(m, \mathbb{N}) \cap \bigcap_{n\in\mathbb{N}} s(n, \mathbb{M}). \tag{6}$$

Definition 7. Let (Γ, d_c) be a complex-valued b-MS and $\mathbb{S} : \Gamma \longrightarrow \mathbb{CB}(\Gamma)$ be a MVM. Define

$$W_{\mathcal{I}}(\mathbb{M}) = \{d_c(\mathcal{I}, m): m \in \mathbb{M}\}, \tag{7}$$

for $\mathcal{I} \in \Gamma$ and $\mathbb{M} \in \mathbb{CB}(\Gamma)$. Moreover,

$$W_{\jmath}(\mathbb{S}_\ell) = \{d_c(\mathcal{I}, s): s \in \mathbb{S}_\ell\}, \tag{8}$$

for $\mathcal{I}, \ell \in \Gamma$ and $\mathbb{S}_\ell \in \mathbb{CB}(\Gamma)$.

Definition 8. Let (Γ, d_c) be a complex-valued b-MS. A nonempty subset $\mathbb{M}$ of Γ is said to be bounded from below if we can find some $z \in \mathbb{C}$: $z \preceq m$ for all $m \in \mathbb{M}$.

Definition 9. Let (Γ, d_c) be a complex-valued b-MS. A MVM $\mathbb{T} : \Gamma \longrightarrow 2^C$ is said to be bounded from below if for each $\mathcal{J} \in \Gamma$ we can find $z_J \in \mathbb{C}$:

$$z_{\mathcal{J}} \preceq \wp \text{for all} \wp \in \mathbb{T}_J. \tag{9}$$

Definition 10. Let (Γ, d_c) be a complex-valued b-MS. A MVM $\mathbb{S} : \Gamma \longrightarrow \mathbb{CB}(\Gamma)$ is said to have a lower bound property on (Γ, d_c) if for any $\mathcal{J} \in \Gamma$, the MVM $T_{\mathcal{J}} : \Gamma \longrightarrow 2^C$ given by

$$T_{\mathcal{J}}(\mathbb{S}\ell) = W_{\mathcal{J}}(\mathbb{S}\ell) \tag{10}$$

is termed bounded from below. This means for $\mathcal{J}, \ell \in \Gamma$, there is an element $I_{\mathcal{J}}(\mathbb{S}\ell) \in C$: $I_J(\mathbb{S}\ell) \preceq a$ for all $a \in W_{\mathcal{J}}(\mathbb{S}\ell)$, where $I_{\mathcal{J}}(\mathbb{S}\ell)$ is a lower bound of $\mathbb{S}$ associated with $(\mathcal{J}, \ell)$.

Definition 11. Let (Γ, d_c) be a complex-valued b-MS. A MVM $\mathbb{S} : \Gamma \longrightarrow \mathbb{CB}(\Gamma)$ has the greatest lower bound (g.l.b) property on (Γ, d_c) if the g.l.b of $W_{\mathcal{J}}(\mathbb{S}\ell)$ exists in $\mathbb{C}$ for all $\mathcal{J}, \ell \in \Gamma$. We denote $d_c(\mathcal{J}, \mathbb{S}\ell)$ by the g.l.b of $W_{\mathcal{J}}(\mathbb{S}\ell)$, i.e.,

$$d_c(\mathcal{J}, \mathbb{S}\ell) = \inf \{d_c(\mathcal{J}, s): s \in \mathbb{S}\ell\}. \tag{11}$$

Definition 12 (see [15]). Let $\tilde{W}$ be a universal set. An IF-set $\tilde{F}$ in $\tilde{W}$ is an object of the form $\tilde{F} = \{\langle \wp, \mu_{\tilde{F}}(\wp), \upsilon_{\tilde{F}}(\wp)\rangle : \wp \in \tilde{W}\}$, where $\mu_{\tilde{F}}(\wp)$ and $\upsilon_{\tilde{F}}(\wp)$ denote the membership and nonmembership values of $\wp$ in $\tilde{F}$ obeying $0 \le \mu_{\tilde{F}}(\wp) + \upsilon_{\tilde{F}}(\wp) \le 1$ for every $\wp \in \tilde{W}$.

Definition 13 (see [15]). Let $\tilde{F}$ be an IF-set. Then, the $\tilde{\alpha}$ − level set of $\tilde{F}$ is a crisp set depicted by $[\tilde{F}]_{\tilde{\alpha}}$ and is given by

$$[\tilde{F}]_{\tilde{\alpha}} = \{\wp \in \tilde{W} : \mu_{\tilde{F}}(\wp) \ge \tilde{\alpha} \text{ and } \upsilon_{\tilde{F}}(\wp) \le 1 - \tilde{\alpha}\} \text{if } \tilde{\alpha} \in [0, 1]. \tag{12}$$

Definition 14 (see [24]). A mapping $\mathcal{T} : [0, 1]^2 \longrightarrow [0, 1]$ is termed a triangular norm (t-norm), if the following conditions are obeyed:

(i) $\mathcal{T}(\mathcal{J}, \mathcal{T}(\ell, \wp)) = \mathcal{T}(\mathcal{T}(\mathcal{J}, \ell), \wp)$ for all $\mathcal{J}, \ell, \wp \in \tilde{W}$

(ii) $\mathcal{T}(\mathcal{J}, \ell) = \mathcal{T}(\ell, \mathcal{J})$ for all $\mathcal{J}, \ell \in \tilde{W}$

(iii) If $\mathcal{J}, \ell, \wp \in [0, 1]$ and $\mathcal{J} \le \ell$, then $\mathcal{T}(\mathcal{J}, \wp) \le \mathcal{T}(\ell, \wp)$

(iv) $\mathcal{T}(\mathcal{J}, 1) = \mathcal{J}$ for all $J \in \tilde{W}$

Minimum t-norm depicted by $\mathcal{T}_M$ is given by $\mathcal{T}_M(\mathcal{J}, \ell) = \min (\mathcal{J}, \ell)$ for all $\mathcal{J}, \ell \in [0, 1]$.

Definition 15 (see [24]). Fuzzy negation is a decreasing map $\mathcal{N} : [0, 1] \longrightarrow [0, 1]$ such that $\mathcal{N}(0) = 1$, $\mathcal{N}(1) = 0$. If $\mathcal{N}$ is continuous and strictly nonincreasing, then it is termed strict. Fuzzy negations with $\mathcal{N}(\mathcal{N}(\wp)) = \wp$, for all $\wp \in [0, 1]$, are termed strong fuzzy negations. The example of fuzzy negation is a standard negation given by $\mathcal{N}_S(\wp) = 1 - \wp$, for all $\wp \in \tilde{W}$.

Definition 16 (see [23]). Let $\tilde{F}$ be an IF-set of $\tilde{W}$ and $\mathcal{T}$ and $\mathcal{N}$ be a triangular norm and a fuzzy negation, respectively. Then, $(\mathcal{T}, \mathcal{N}, \tilde{\alpha})$ − level set of $\tilde{F}$ is a crisp set depicted by $[\tilde{F}]_{(\mathcal{T}, \mathcal{N}, \tilde{\alpha})}$ and is given by

$$[\tilde{F}]_{(\mathcal{T}, \mathcal{N}, \tilde{\alpha})} = \{\wp \in \tilde{W} : \mathcal{T}(\mu_{\tilde{F}}(z), \mathcal{N}(\upsilon_{\tilde{F}}(z))) \ge \tilde{\alpha}\} \text{if } \tilde{\alpha} \in [0, 1]. \tag{13}$$

Remark 17. If we take $\mathcal{T} = \mathcal{T}_M$ and $\mathcal{N} = \mathcal{N}_S$, then $(\mathcal{T}, \mathcal{N}, \tilde{\alpha})$ − level set is reduced into original idea of a cut set by Atanassov [15].

Definition 18 (see [16]). Let $\tilde{W}$ be an arbitrary set and Γ be a MS. A mapping $\mathbb{F}$ is termed an IFM if $\mathbb{F}$ is a mapping from $\tilde{W}$ into $(IFS)^\Gamma$ (class of all intuitionistic fuzzy subsets of Γ).

3. Main Results

In this section, first we present few definitions which will be useful in the proof of our main ideas and then establish illustrations to validate their hypotheses.

Definition 19. A point $J^* \in \Gamma$ is said to be an intuitionistic fuzzy FP of an IFM $\mathbb{F} : \Gamma \longrightarrow (IFS)^\Gamma$ if we can find $\tilde{\alpha} \in [0, 1]$: $J^* \in [\mathbb{F}(J^*)]_{(\mathcal{T}, \mathcal{N}, \tilde{\alpha})}$.

Definition 20. Let (Γ, d_c) be a complex-valued b-MS and $\mathbb{F} : \Gamma \longrightarrow (IFS)^\Gamma$ be an intuitionistic fuzzy map. Suppose that for each $\ell \in \Gamma$, we can find $\tilde{\alpha}_{\mathbb{F}}(\ell) \in [0, 1]$: $[\mathbb{F}(\ell)]_{(\mathcal{T}, \mathcal{N}, \tilde{\alpha}_{\mathbb{F}(\ell)})} \in \mathbb{CB}(\Gamma)$. We can depict

$$U_J\left([\mathbb{F}(\ell)]_{(\mathcal{T}, \mathcal{N}, \tilde{\alpha}_{\mathbb{F}(\ell)})}\right) = \left\{d_c(\mathcal{J}, \wp): \wp \in [\mathbb{F}(\mathcal{J})]_{(\mathcal{T}, \mathcal{N}, \tilde{\alpha}_{\mathbb{F}(\ell)})}\right\}, \tag{14}$$

for $J, \ell \in \Gamma$.

Definition 21. Let (Γ, d_c) be a complex-valued b-metric space and $\mathbb{F} : \Gamma \longrightarrow (IFS)^\Gamma$ be an intuitionistic fuzzy map. Suppose that for each $\ell \in \Gamma$, we can find $\tilde{\alpha}_{\mathbb{F}}(\ell) \in [0, 1]$: $[\mathbb{F}(\ell)]_{(\mathcal{T}, \mathcal{N}, \tilde{\alpha}_{\mathbb{F}(\ell)})} \in \mathbb{CB}(\Gamma)$. An IFM $\mathbb{F}$ is said to have the greatest lower bound (g.l.b) property on (Γ, d_c), if the g.l.b of $U_J([\mathbb{F}(\ell)]_{(\mathcal{T}, \mathcal{N}, \tilde{\alpha}_{\mathbb{F}(\ell)})})$ exists in $\mathbb{C}$ for all $\mathcal{J}, \ell \in \Gamma$. We denote the g.l.b of $U_{\mathcal{J}}([\mathbb{F}(\ell)]_{(\mathcal{T}, \mathcal{N}, \tilde{\alpha}_{\mathbb{F}(\ell)})})$ by $d_c(\mathcal{J}, [\mathbb{F}(\ell)]_{(\mathcal{T}, \mathcal{N}, \tilde{\alpha}_{\mathbb{F}(\ell)})})$ and is given by

$$d_c\left(\mathcal{J}, [\mathbb{F}(\ell)]_{(\mathcal{T}, \mathcal{N}, \tilde{\alpha}_{\mathbb{F}(\ell)})}\right) = \inf \left\{d_c(\mathcal{J}, \ell_1): b \in [\mathbb{F}(\ell)]_{(\mathcal{T}, \mathcal{N}, \tilde{\alpha}_{\mathbb{F}(\ell)})}\right\}. \tag{15}$$

Theorem 22. *Let (Γ, d_c) be a complete complex-valued b-MS and $\mathbb{F}, \mathbb{G} : \Gamma \longrightarrow (IFS)^\Gamma$ be a pair of IFMs obeying the g.l.b property. Assume that for each $\mathcal{J} \in \Gamma$, we can* *find* $\tilde{\alpha}_{\mathbb{F}}(\mathcal{I}),\ \tilde{\alpha}_{\mathbb{G}}(\mathcal{I}) \in [0, 1]$*:* $[\mathbb{F}(\mathcal{I})]_{(\mathcal{T},\mathcal{N},\tilde{\alpha}_{F(\mathcal{I})})}, [\mathbb{G}(\mathcal{I})]_{(\mathcal{T},\mathcal{N},\tilde{\alpha}_{G(\mathcal{I})})} \in \mathbb{CB}(\Gamma)$*. If for all* $\mathcal{I}, \ell \in \Gamma$*,*

$$B^*(\mathcal{I}, \ell, \mathbb{F}, \mathbb{G}, \tilde{\alpha}) \in s\left([\mathbb{F}(\mathcal{I})]_{(\mathcal{T},\mathcal{N},\tilde{\alpha}_{\mathbb{F}}(\mathcal{I}))}, [\mathbb{G}(\ell)]_{(\mathcal{T},\mathcal{N},\tilde{\alpha}_{\mathbb{G}}(\ell))}\right), \tag{16}$$

where

$$\begin{aligned} B^*(\mathcal{I}, \ell, \mathbb{F}, \mathbb{G}, \tilde{\alpha}) &= \beta d_c(\mathcal{I}, \ell) \\ &\quad + \frac{\mu d_c\left(\mathcal{I}, [\mathbb{F}(\mathcal{I})]_{(\mathcal{T},\mathcal{N},\tilde{\alpha}_{\mathbb{F}}(\mathcal{I}))}\right) d_c\left(\ell, [\mathbb{G}(\ell)]_{(\mathcal{T},\mathcal{N},\tilde{\alpha}_{\mathbb{G}}(\ell))}\right)}{1 + d_c(\mathcal{I}, \ell)} \\ &\quad + \frac{\lambda d_c\left(\ell, [\mathbb{F}(\mathcal{I})]_{(\mathcal{T},\mathcal{N},\tilde{\alpha}_{\mathbb{F}}(\mathcal{I}))}\right) d_c\left(\mathcal{I}, [\mathbb{G}(\ell)]_{(\mathcal{T},\mathcal{N},\tilde{\alpha}_{\mathbb{G}}(\ell))}\right)}{1 + d_c(\mathcal{I}, \ell)}, \end{aligned} \tag{17}$$

and β, μ, λ *are nonnegative real numbers with* $Y\beta + \mu + \lambda < 1$*, where* $Y \geq 1$*. Then,* $\mathbb{F}$ *and* $\mathbb{G}$ *have a common FP in* Γ*.*

Proof. Suppose that $\mathcal{I}_0$ is an arbitrary and fixed element of Γ; then by assumption $[\mathbb{F}(\mathcal{I}_0)]_{(\mathcal{T},\mathcal{N},\tilde{\alpha}_{\mathbb{F}}(\mathcal{I}_0))} \in \mathbb{CB}(\Gamma)$, so we can take $\jmath_1 \in [\mathbb{F}(\mathcal{I}_0)]_{(\mathcal{T},\mathcal{N},\tilde{\alpha}_{\mathbb{F}}(\mathcal{I}_0))}$, and therefore from (16),

$$\begin{aligned} &B^*(\mathcal{I}_0, \mathcal{I}_1, \mathbb{F}, \mathbb{G}, \tilde{\alpha}) \\ &\quad \in s\left([\mathbb{F}(\mathcal{I}_0)]_{(\mathcal{T},\mathcal{N},\tilde{\alpha}_{\mathbb{F}}(\mathcal{I}_0))}, [\mathbb{G}(\mathcal{I}_1)]_{(\mathcal{T},\mathcal{N},\tilde{\alpha}_{\mathbb{G}}(\mathcal{I}_1))}\right). \end{aligned} \tag{18}$$

It follows that

$$\begin{aligned} &B^*(\mathcal{I}_0, \mathcal{I}_1, \mathbb{F}, \mathbb{G}, \tilde{\alpha}) \\ &\quad \in \bigcap_{a' \in [\mathbb{F}(\mathcal{I}_0)]_{(\mathcal{T},\mathcal{N},\tilde{\alpha}_{\mathbb{F}}(\mathcal{I}_0))}} s\left(a', [\mathbb{G}(\mathcal{I}_1)]_{(\mathcal{T},\mathcal{N},\tilde{\alpha}_{\mathbb{G}}(\mathcal{I}_1))}\right), \\ &B^*(\mathcal{I}_0, \mathcal{I}_1, \mathbb{F}, \mathbb{G}, \tilde{\alpha}) \\ &\quad \in s\left(a', [\mathbb{G}(\mathcal{I}_1)]_{(\mathcal{T},\mathcal{N},\tilde{\alpha}_{\mathbb{G}}(\mathcal{I}_1))}\right) \quad \forall a' \in [\mathbb{F}(\mathcal{I}_0)]_{(\mathcal{T},\mathcal{N},\tilde{\alpha}_{\mathbb{F}}(\mathcal{I}_0))}. \end{aligned} \tag{19}$$

As $\jmath_1 \in [\mathbb{F}(\mathcal{I}_0)]_{(\mathcal{T},\mathcal{N},\tilde{\alpha}_{\mathbb{F}}(\mathcal{I}_0))}$, then we obtain

$$B^*(\mathcal{I}_0, \mathcal{I}_1, \mathbb{F}, \mathbb{G}, \tilde{\alpha}) \in s\left(\mathcal{I}_1, [\mathbb{G}(\mathcal{I}_1)]_{(\mathcal{T},\mathcal{N},\tilde{\alpha}_{\mathbb{G}}(\mathcal{I}_1))}\right). \tag{20}$$

Therefore,

$$B^*(\mathcal{I}_0, \mathcal{I}_1, \mathbb{F}, \mathbb{G}, \tilde{\alpha}) \in \bigcup_{b' \in [\mathbb{G}(\mathcal{I}_1)]_{(\mathcal{T},\mathcal{N},\tilde{\alpha}_{\mathbb{G}}(\mathcal{I}_1))}} s\left(d_c\left(\mathcal{I}_1, b'\right)\right). \tag{21}$$

Thus, we can find some $\mathcal{I}_2 \in [\mathbb{G}(\mathcal{I}_1)]_{(\mathcal{T},\mathcal{N},\tilde{\alpha}_{\mathbb{G}}(\mathcal{I}_2))}$:

$$B^*(\mathcal{I}_0, \mathcal{I}_1, \mathbb{F}, \mathbb{G}, \tilde{\alpha}) \in s(d_c(\mathcal{I}_1, \mathcal{I}_2)). \tag{22}$$

It yields

$$d_c(\mathcal{I}_1, \mathcal{I}_2) \preceq B^*(\mathcal{I}_0, \mathcal{I}_1, \mathbb{F}, \mathbb{G}, \tilde{\alpha}). \tag{23}$$

Using the g.l.b property of $\mathbb{F}$ and $\mathbb{G}$, we get

$$\begin{aligned} d_c(\mathcal{I}_1, \mathcal{I}_2) &\preceq \beta d_c(\mathcal{I}_0, \mathcal{I}_1) \\ &\quad + \frac{\mu d_c(\mathcal{I}_0, \mathcal{I}_1) d_c(\mathcal{I}_1, \mathcal{I}_2) + \lambda d_c(\mathcal{I}_1, \mathcal{I}_1) d_c(\mathcal{I}_0, \mathcal{I}_2)}{1 + d_c(\mathcal{I}_0, \mathcal{I}_1)} \\ &\preceq \beta d_c(\mathcal{I}_0, \mathcal{I}_1) \\ &\quad + \frac{\mu d_c(\mathcal{I}_0, \mathcal{I}_1) d_c(\mathcal{I}_1, \mathcal{I}_2) + \lambda d_c(\mathcal{I}_1, \mathcal{I}_1) d_c(\mathcal{I}_0, \mathcal{I}_2)}{1 + d_c(\mathcal{I}_0, \mathcal{I}_1)}. \end{aligned} \tag{24}$$

This implies

$$\begin{aligned} |d_c(\mathcal{I}_1, \mathcal{I}_2)| &\leq \beta |d_c(\mathcal{I}_0, \mathcal{I}_1)| + \frac{\mu |d_c(\mathcal{I}_0, \mathcal{I}_1)| |d_c(\mathcal{I}_1, \mathcal{I}_2)|}{1 + |d_c(\mathcal{I}_0, \mathcal{I}_1)|} \\ &= \beta |d_c(\mathcal{I}_0, \mathcal{I}_1)| + \mu |d_c(\mathcal{I}_1, \mathcal{I}_2)| \\ &\leq \frac{\beta}{(1-\mu)} |d_c(\mathcal{I}_0, \mathcal{I}_1)|. \end{aligned} \tag{25}$$

Inductively, we can develop a sequence $\{\jmath_p\}$ in Γ: for $p = 0, 1, 2 \cdots$,

$$\left|d_c\left(\mathcal{I}_p, \mathcal{I}_{p+1}\right)\right| \leq \kappa^p |d_c(\mathcal{I}_0, \mathcal{I}_1)|, \tag{26}$$

where $\kappa = (\beta/1-\mu) < 1$.

Now for $q \in \mathbb{N}$ and using the triangular inequality of (Γ, d_c),

$$\begin{aligned} d_c\left(\mathcal{I}_p, \mathcal{I}_{p+q}\right) &\preceq Y\left[d_c\left(\mathcal{I}_p, \mathcal{I}_{p+1}\right) + d_c\left(\mathcal{I}_{p+1}, \mathcal{I}_{p+q}\right)\right] \\ &\preceq Y d_c\left(\mathcal{I}_p, \mathcal{I}_{p+1}\right) + Y^2 d_c\left(\mathcal{I}_{p+1}, \mathcal{I}_{p+2}\right) \\ &\quad + Y^2 d_c\left(\mathcal{I}_{p+2}, \mathcal{I}_{p+q}\right) \\ &\preceq Y d_c\left(\mathcal{I}_p, \mathcal{I}_{p+1}\right) + Y^2 d_c\left(\mathcal{I}_{p+1}, \mathcal{I}_{p+2}\right) \\ &\quad + Y^3 d_c\left(\mathcal{I}_{p+2}, \mathcal{I}_{p+3}\right) + \cdots + Y^q d_c\left(\mathcal{I}_{p+q-1}, \mathcal{I}_{p+q}\right) \\ &\preceq Y\kappa^p d_c(\mathcal{I}_o, \mathcal{I}_1) + Y^2 \kappa^{p+1} d_c(\mathcal{I}_o, \mathcal{I}_1) \\ &\quad + Y^3 \kappa^{p+2} d_c(\mathcal{I}_o, \mathcal{I}_1) + \cdots + Y^q \kappa^{p+q-1} d_c(\mathcal{I}_o, \mathcal{I}_1) \\ &\preceq Y\kappa^p d_c(\mathcal{I}_o, \mathcal{I}_1)\left[1 + Y\kappa + (Y\kappa)^2 + \cdots + (Y\kappa)^{q-1}\right]. \end{aligned} \tag{27}$$

Therefore,

$$\begin{aligned}\left|d_c\left(\mathcal{J}_p, \mathcal{J}_{p+q}\right)\right| &\leq \left|Y\kappa^p d_c(\mathcal{J}_o, \mathcal{J}_1)\left[1 + Y\kappa + (Y\kappa)^2 + \cdots + (Y\kappa)^{q-1}\right]\right| \\ &\leq Y\kappa^p\left|d_c(\mathcal{J}_o, \mathcal{J}_1)\right|\left[1 + Y\kappa + (Y\kappa)^2 + \cdots + (Y\kappa)^{q-1}\right] \\ &\leq \frac{Y\kappa^p}{1 - Y\kappa}\left|d_c(\mathcal{J}_o, \mathcal{J}_1)\right|,\end{aligned} \tag{28}$$

since $Y\beta + \mu + \lambda < 1 \Rightarrow (Y\beta/1 - \mu) < 1$ and $|d_c(\mathcal{J}_p, \mathcal{J}_{p+q})| \longrightarrow 0$ as $p, q \longrightarrow \infty$.

By Lemma 6, $\{\xi_p\}$ is a Cauchy sequence in Γ, which is complete; so we can find some $\omega \in \Gamma$,: $\lim_{p\longrightarrow\infty} \mathcal{J}_p = \omega$.

However, from (16), we have

$$\begin{aligned}&B^*\left(\mathcal{J}_{2p}, \omega, \mathbb{F}, \mathbb{G}, \tilde{\alpha}\right) \\ &\quad\in s\left(\left[\mathbb{F}\left(\mathcal{J}_{2p}\right)\right]_{\left(\mathcal{T}, \mathcal{N}, \tilde{\alpha}_{\mathbb{F}}\left(\mathcal{J}_{2p}\right)\right)}, \left[\mathbb{G}(\omega)\right]_{\left(\mathcal{T}, \mathcal{N}, \tilde{\alpha}_{\mathbb{G}}(\omega)\right)}\right).\end{aligned} \tag{29}$$

This implies that

$$\begin{aligned}&B^*\left(\mathcal{J}_{2p}, \omega, \mathbb{F}, \mathbb{G}, \tilde{\alpha}\right) \\ &\quad\in \bigcap_{a' \in \left[\mathbb{F}\left(\mathcal{J}_{2p}\right)\right]_{\left(\mathcal{T}, \mathcal{N}, \tilde{\alpha}_{\mathbb{F}}\left(\mathcal{J}_{2p}\right)\right)}} s\left(a', \left[\mathbb{G}(\omega)\right]_{\left(\mathcal{T}, \mathcal{N}, \tilde{\alpha}_{\mathbb{G}}(\omega)\right)}\right).\end{aligned} \tag{30}$$

Since $\mathcal{J}_{2p+1} \in \left[\mathbb{F}\left(\mathcal{J}_{2p}\right)\right]_{\left(\mathcal{T}, \mathcal{N}, \tilde{\alpha}_{\mathbb{F}}\left(\mathcal{J}_{2p}\right)\right)}$,

$$B^*\left(\mathcal{J}_{2p}, \omega, \mathbb{F}, \mathbb{G}, \tilde{\alpha}\right) \in s\left(\mathcal{J}_{2p+1}, \left[\mathbb{G}(\omega)\right]_{\left(\mathcal{T}, \mathcal{N}, \tilde{\alpha}_{\mathbb{G}}(\omega)\right)}\right). \tag{31}$$

Therefore,

$$B^*\left(\mathcal{J}_{2p}, \omega, \mathbb{F}, \mathbb{G}, \tilde{\alpha}\right) \in \bigcup_{b' \in \left[\mathbb{G}(\omega)\right]_{\left(\mathcal{T}, \mathcal{N}, \tilde{\alpha}_{\mathbb{G}}(\omega)\right)}} s\left(d_c\left(\mathcal{J}_{2p+1}, b'\right)\right). \tag{32}$$

This implies that we can find some $\omega_p \in \left[\mathbb{G}(\omega)\right]_{\left(\mathcal{T}, \mathcal{N}, \tilde{\alpha}_{\mathbb{G}}(\omega)\right)}$:

$$B^*\left(\mathcal{J}_{2p}, \omega, \mathbb{F}, \mathbb{G}, \tilde{\alpha}\right) \in s\left(d_c\left(\mathcal{J}_{2p+1}, \omega_p\right)\right). \tag{33}$$

It yields

$$d_c\left(\mathcal{J}_{2p+1}, \omega_p\right) \preceq B^*\left(\mathcal{J}_{2p}, \omega, \mathbb{F}, \mathbb{G}, \tilde{\alpha}\right). \tag{34}$$

Next,

$$\begin{aligned}d_c\left(\omega, \omega_p\right) &\preceq Y\left[d_c\left(\omega, \mathcal{J}_{2p+1}\right) + d_c\left(\mathcal{J}_{2p+1}, \omega_p\right)\right] \\ &\preceq Y d_c\left(\omega, \mathcal{J}_{2p+1}\right) + Y B^*\left(\mathcal{J}_{2p}, \omega, \mathbb{F}, \mathbb{G}, \tilde{\alpha}\right).\end{aligned} \tag{35}$$

However,

$$\begin{aligned}&\left|d_c\left(\omega, \omega_p\right)\right| \\ &\quad\leq Y\left|d_c\left(\omega, \mathcal{J}_{2p+1}\right)\right| + Y\beta\left|d_c\left(\mathcal{J}_{2p}, \omega\right)\right| \\ &\qquad+ \frac{Y\mu\left|d_c\left(\mathcal{J}_{2p}, \mathcal{J}_{2p+1}\right)\right|\left|d_c\left(\omega, \omega_p\right)\right| + Y\lambda\left|d_c\left(\omega, \mathcal{J}_{2p+1}\right)\right|\left|d_c\left(\mathcal{J}_{2p}, \omega_p\right)\right|}{1 + \left|d_c\left(\mathcal{J}_{2p}, \omega\right)\right|},\end{aligned} \tag{36}$$

since $|d_c(\omega, \omega_p)| \longrightarrow 0$ as $p \longrightarrow \infty$.

By Lemma 5, we have $\omega_p \longrightarrow \omega$. Since $\left[\mathbb{G}(\omega)\right]_{\left(\mathcal{T}, \mathcal{N}, \tilde{\alpha}_{\mathbb{G}}(\omega)\right)}$ is closed, so $\omega \in \left[\mathbb{G}(\omega)\right]_{\left(\mathcal{T}, \mathcal{N}, \tilde{\alpha}_{\mathbb{G}}(\omega)\right)}$. Similarly, it follows that $\omega \in \left[\mathbb{F}(\omega)\right]_{\left(\mathcal{T}, \mathcal{N}, \tilde{\alpha}_{\mathbb{F}}(\omega)\right)}$. Thus, $\mathbb{F}$ and $\mathbb{G}$ have a common FP in Γ.

By setting $\lambda = 0$ in Theorem 22, we have the following result. □

Corollary 23. *Let (Γ, d_c) be a complete complex-valued b-MS and $\mathbb{F}, \mathbb{G} : \Gamma \longrightarrow (IFS)^\Gamma$ be a pair of IFMs obeying the g.l.b property. Assume that for each $\jmath \in \Gamma$, we can find $\tilde{\alpha}_{\mathbb{F}}(\mathcal{J})$, $\tilde{\alpha}_{\mathbb{G}}(\mathcal{J}) \in [0, 1]$: $\left[\mathbb{F}(\mathcal{J})\right]_{\left(\mathcal{T}, \mathcal{N}, \tilde{\alpha}_{\mathbb{F}(\mathcal{J})}\right)}, \left[\mathbb{G}(\mathcal{J})\right]_{\left(\mathcal{T}, \mathcal{N}, \tilde{\alpha}_{G(\mathcal{J})}\right)} \in \mathbb{CB}(\Gamma)$. If for all $\mathcal{J}, \ell \in \Gamma$,*

$$B^o(\mathcal{J}, \ell, \mathbb{F}, \tilde{\alpha}) \in s\left(\left[\mathbb{F}(\mathcal{J})\right]_{\left(\mathcal{T}, \mathcal{N}, \tilde{\alpha}_{\mathbb{F}}(\mathcal{J})\right)}, \left[\mathbb{G}(\ell)\right]_{\left(\mathcal{T}, \mathcal{N}, \tilde{\alpha}_{\mathbb{G}}(\ell)\right)}\right), \tag{37}$$

where

$$\begin{aligned}&B^o(\mathcal{J}, \ell, \mathbb{F}, \mathbb{G}, \tilde{\alpha}) \\ &\quad= \beta d_c(\mathcal{J}, \ell) \\ &\qquad+ \frac{\mu d_c\left(\mathcal{J}, \left[\mathbb{F}(\mathcal{J})\right]_{\left(\mathcal{T}, \mathcal{N}, \tilde{\alpha}_{\mathbb{F}}(\mathcal{J})\right)}\right) d_c\left(\ell, \left[\mathbb{G}(\ell)\right]_{\left(\mathcal{T}, \mathcal{N}, \tilde{\alpha}_{\mathbb{G}}(\ell)\right)}\right)}{1 + d_c(\mathcal{J}, \ell)},\end{aligned} \tag{38}$$

and β, μ are nonnegative real numbers with $Y\beta + \mu < 1$, where $Y \geq 1$. Then, $\mathbb{F}$ and $\mathbb{G}$ have a common FP in Γ.

By letting $\mathbb{F} = \mathbb{G}$ in Theorem 22, we have the following corollary.

Corollary 24. *Let (Γ, d_c) be a complete complex-valued b-MS and $\mathbb{F} : \Gamma \longrightarrow (IFS)^\Gamma$ be an IFM obeying the g.l.b property. Assume that for each $\mathcal{J} \in \Gamma$, we can find $\tilde{\alpha}_{\mathbb{F}}(\mathcal{J}) \in [0, 1]$: $\left[\mathbb{F}(\mathcal{J})\right]_{\left(\mathcal{T}, \mathcal{N}, \tilde{\alpha}_{F(\mathcal{J})}\right)} \in \mathbb{CB}(\Gamma)$. If for all $\mathcal{J}, \ell \in \Gamma$,*

$$B^*(\mathcal{J}, \ell, \mathbb{F}, \tilde{\alpha}) \in s\left(\left[\mathbb{F}(\mathcal{J})\right]_{\left(\mathcal{T}, \mathcal{N}, \tilde{\alpha}_{\mathbb{F}}(\jmath)\right)}, \left[\mathbb{F}(\ell)\right]_{\left(\mathcal{T}, \mathcal{N}, \tilde{\alpha}_{\mathbb{F}}(\ell)\right)}\right), \tag{39}$$

where

$$\begin{aligned}&B^*(\mathcal{J}, \ell, \mathbb{F}, \tilde{\alpha}) \\ &\quad= \beta d_c(\mathcal{J}, \ell) \\ &\qquad+ \frac{\mu d_c\left(\mathcal{J}, \left[\mathbb{F}(\mathcal{J})\right]_{\left(\mathcal{T}, \mathcal{N}, \tilde{\alpha}_{\mathbb{F}}(\mathcal{J})\right)}\right) d_c\left(\ell, \left[\mathbb{F}(\ell)\right]_{\left(\mathcal{T}, \mathcal{N}, \tilde{\alpha}_{\mathbb{F}}(\ell)\right)}\right)}{1 + d_c(\mathcal{J}, \ell)} \\ &\qquad+ \frac{\lambda d_c\left(\ell, \left[\mathbb{F}(\mathcal{J})\right]_{\left(\mathcal{T}, \mathcal{N}, \tilde{\alpha}_{\mathbb{F}}(\mathcal{J})\right)}\right) d_c\left(\mathcal{J}, \left[\mathbb{F}(\ell)\right]_{\left(\mathcal{T}, \mathcal{N}, \tilde{\alpha}_{\mathbb{F}}(\ell)\right)}\right)}{1 + d_c(\mathcal{J}, \ell)},\end{aligned} \tag{40}$$

and β, μ, λ are nonnegative real numbers with $Y\beta + \mu + \lambda < 1$, where $Y \geq 1$. Then, $\mathbb{F}$ has a FP in Γ.

Corollary 25. *Let (Γ, d_c) be a complete complex-valued b-MS and $\mathbb{F}, \mathbb{G} : \Gamma \longrightarrow \mathbb{CB}(\Gamma)$ be a pair of MVM obeying the g.l.b property*

$$\beta d_c(\mathcal{J}, \ell) + \frac{\mu d_c(\mathcal{J}, \mathbb{F}(\mathcal{J}))d_c(\ell, \mathbb{G}(\ell))}{1 + d_c(\mathcal{J}, \ell)} + \frac{\lambda d_c(\ell, \mathbb{F}(\jmath))d_c(\mathcal{J}, \mathbb{G}(\ell))}{1 + d_c(\mathcal{J}, \ell)} \in s(\mathbb{F}(\mathcal{J}), \mathbb{G}(\ell)), \tag{41}$$

for all $\mathcal{J}, \ell \in \Gamma$, and β, μ, λ are nonnegative real numbers with $Y\beta + \mu + \lambda < 1$, where $Y \geq 1$. Then, $\mathbb{F}$ and $\mathbb{G}$ have a common FP in Γ.

Example 2. Let $\Gamma = [0, 1]$, $d_c(\mathcal{J}, \ell) = |\mathcal{J} - \ell|^2 e^{i\psi}$, where $\psi = tan^{-1}(\ell/\mathcal{J})$, for $\mathcal{J}, \ell \in \Gamma$ and $\zeta_1, \zeta_2, \sigma_1, \sigma_2 \in [0, 1]$.

Then, (Γ, d_c) is a complete complex-valued b-MS with $Y = 2$. Assume that $\zeta_1, \zeta_2, \sigma_1, \sigma_2 \in [0, 1]$ and a pair of IFMs $\mathbb{F} = \langle \mu_{\mathbb{F}}, \upsilon_{\mathbb{F}} \rangle$, $\mathbb{G} = \langle \mu_{\mathbb{G}}, \upsilon_{\mathbb{G}} \rangle$: $\Gamma \longrightarrow (IFS)^{\Gamma}$ are given by

$$\mu_{\mathbb{F}(\mathcal{J})}(t) = \begin{cases} \zeta_1, & \text{if } 0 \leq t \leq \dfrac{\mathcal{J}}{15}, \\ \dfrac{\zeta_1}{5}, & \text{if } \dfrac{\mathcal{J}}{15} < t \leq \dfrac{\mathcal{J}}{13}, \\ \dfrac{\zeta_1}{8}, & \text{if } \dfrac{\mathcal{J}}{11} < t < \mathcal{J}, \\ 0, & \text{if } \mathcal{J} \leq t < \infty, \end{cases}$$

$$\upsilon_{\mathbb{F}(\mathcal{J})}(t) = \begin{cases} 0, & \text{if } 0 \leq t \leq \dfrac{\mathcal{J}}{15}, \\ \dfrac{\zeta_2}{6}, & \text{if } \dfrac{\mathcal{J}}{15} < t \leq \dfrac{\mathcal{J}}{12}, \\ \dfrac{\zeta_2}{3}, & \text{if } \dfrac{\mathcal{J}}{10} < t < \mathcal{J}, \\ \zeta_2, & \text{if } \mathcal{J} \leq t < \infty, \end{cases} \tag{42}$$

$$\mu_{\mathbb{G}(\mathcal{J})}(t) = \begin{cases} \sigma_1, & \text{if } 0 \leq t \leq \dfrac{\mathcal{J}}{20}, \\ \dfrac{\sigma_1}{4}, & \text{if } \dfrac{\mathcal{J}}{20} < t \leq \dfrac{\mathcal{J}}{18}, \\ \dfrac{\sigma_1}{9}, & \text{if } \dfrac{\mathcal{J}}{16} < t < \mathcal{J}, \\ 0, & \text{if } \mathcal{J} \leq t < \infty, \end{cases}$$

$$\upsilon_{\mathbb{G}(\mathcal{J})}(t) = \begin{cases} 0, & \text{if } 0 \leq t \leq \dfrac{\mathcal{J}}{20}, \\ \dfrac{\sigma_2}{7}, & \text{if } \dfrac{\mathcal{J}}{20} < t \leq \dfrac{\mathcal{J}}{17}, \\ \dfrac{\sigma_2}{2}, & \text{if } \dfrac{\mathcal{J}}{14} < t < \mathcal{J}, \\ \sigma_2, & \text{if } \mathcal{J} \leq t < \infty. \end{cases}$$

If $\tilde{\alpha}_{\mathbb{F}(\mathcal{J})} = \zeta_1$ and $\tilde{\alpha}_{\mathbb{G}(\mathcal{J})} = \sigma_1$, then we have

$$\begin{aligned} [\mathbb{F}(\mathcal{J})]_{(\mathcal{T}, \mathcal{N}, \zeta_1)} &= \left\{ t \in \Gamma : \mathcal{T}\left(\mu_{\mathbb{F}(\mathcal{J})}(t), \mathcal{N}\left(\upsilon_{\mathbb{F}(\mathcal{J})}(t)\right)\right) = \zeta_1 \right\} \\ &= \left[0, \frac{\mathcal{J}}{15}\right], \\ [\mathbb{G}(\mathcal{J})]_{(\mathcal{T}, \mathcal{N}, \sigma_1)} &= \left\{ t \in \Gamma : \mathcal{T}\left(\mu_{\mathbb{G}(\mathcal{J})}(t), \mathcal{N}\left(\upsilon_{\mathbb{G}(\mathcal{J})}(t)\right)\right) = \sigma_1 \right\} \\ &= \left[0, \frac{\mathcal{J}}{20}\right]. \end{aligned} \tag{43}$$

Thus, the contractive condition of Theorem 22 becomes trivial when $\jmath = \ell = 0$.

Assume that without loss of generality, $\mathcal{J}, \ell \neq 0$, and $\mathcal{J} < \ell$, and then, we have

$$\begin{aligned} d_c(\mathcal{J}, \ell) &= |\mathcal{J} - \ell|^2 e^{i\psi}, \\ d_c\left(\mathcal{J}, [\mathbb{F}(\mathcal{J})]_{(\mathcal{T}, \mathcal{N}, \tilde{\alpha}_{\mathbb{F}(\jmath)})}\right) &= \left|\mathcal{J} - \frac{\mathcal{J}}{15}\right|^2 e^{i\psi}, \\ d_c\left(\ell, [\mathbb{G}(\ell)]_{(\mathcal{T}, \mathcal{N}, \tilde{\alpha}_{\mathbb{G}(\ell)})}\right) &= \left|\ell - \frac{\ell}{20}\right|^2 e^{i\psi}, \\ d_c\left(\ell, [\mathbb{F}(\mathcal{J})]_{(\mathcal{T}, \mathcal{N}, \tilde{\alpha}_{\mathbb{F}(\mathcal{J})})}\right) &= \left|\ell - \frac{\mathcal{J}}{15}\right|^2 e^{i\psi}, \\ d_c\left(\mathcal{J}, [\mathbb{G}(\ell)]_{(\mathcal{T}, \mathcal{N}, \tilde{\alpha}_{\mathbb{G}(\ell)})}\right) &= \left|\mathcal{J} - \frac{\ell}{20}\right|^2 e^{i\psi}, \\ s\left(d_c\left([\mathbb{F}(\mathcal{J})]_{(\mathcal{T}, \mathcal{N}, \tilde{\alpha}_{\mathbb{F}(\mathcal{J})})}, [\mathbb{G}(\ell)]_{(\mathcal{T}, \mathcal{N}, \tilde{\alpha}_{\mathbb{G}(\ell)})}\right)\right) &= s\left(\left|\frac{\mathcal{J}}{15} - \frac{\ell}{20}\right|^2\right) e^{i\psi}. \end{aligned} \tag{44}$$

Thus, clearly for $\tilde{\alpha} = 1/225$ and any value of μ and λ, we have

$$\begin{aligned} &\left|\frac{\mathcal{J}}{15} - \frac{\ell}{20}\right|^2 \\ &\quad \leq \frac{1}{225}|\mathcal{J} - \ell|^2 \\ &\qquad + \frac{\mu|\mathcal{J} - (\mathcal{J}/15)|^2|\ell - (\ell/20)|^2 + \lambda|\ell - (\mathcal{J}/15)|^2|\mathcal{J} - (\ell/20)|^2}{|1 + d_c(\mathcal{J}, \ell)|}. \end{aligned} \tag{45}$$

Hence, all the conditions of Theorem 22 are obeyed to obtain a common FP of $\mathbb{F}$ and $\mathbb{G}$.

Theorem 26. *Let (Γ, d_c) be a complete complex-valued b-MS and $\mathbb{F}, \mathbb{G} : \Gamma \longrightarrow (IFS)^{\Gamma}$ be a pair of IFMs obeying the g.l.b property. Assume that for each $\mathcal{J} \in \Gamma$, we can find $\tilde{\alpha}_{\mathbb{F}}(\mathcal{J})$,*

$\tilde{\alpha}_{\mathbb{G}}(\mathcal{J}) \in [0,1]$: $[\mathbb{F}(\mathcal{J})]_{(\mathcal{T},\mathcal{N},\tilde{\alpha}_{\mathbb{F}}(\mathcal{J}))}, [\mathbb{G}(\mathcal{J})]_{(\mathcal{T},\mathcal{N},\tilde{\alpha}_{\mathbb{G}}(\mathcal{J}))} \in \mathbb{CB}(\Gamma)$. *If for all* $\mathcal{J}, \ell \in \bar{B}(\jmath_0, r)$,

$$B^*(\mathcal{J}, \ell, \mathbb{F}, \mathbb{G}, \tilde{\alpha}) \in s\left([\mathbb{F}(\mathcal{J})]_{(\mathcal{T},\mathcal{N},\tilde{\alpha}_{\mathbb{F}}(\jmath))}, [\mathbb{G}(\ell)]_{(\mathcal{T},\mathcal{N},\tilde{\alpha}_{\mathbb{G}}(\ell))}\right), \tag{46}$$

$$(1-\epsilon)r \in s\left(\mathcal{J}_0, [\mathbb{F}(\jmath_0)]_{(\mathcal{T},\mathcal{N},\tilde{\alpha}_{\mathbb{F}}(\mathcal{J}_0))}\right), \tag{47}$$

where

$$\begin{aligned} B^*(\mathcal{J}, \ell, \mathbb{F}, \mathbb{G}, \tilde{\alpha}) &= \beta d_c(\mathcal{J}, \ell) \\ &+ \frac{\mu d_c\left(\mathcal{J}, [\mathbb{F}(\mathcal{J})]_{(\mathcal{T},\mathcal{N},\tilde{\alpha}_{\mathbb{F}}(\mathcal{J}))}\right) d_c\left(\ell, [\mathbb{G}(\ell)]_{(\mathcal{T},\mathcal{N},\tilde{\alpha}_{\mathbb{G}}(\ell))}\right)}{1 + d_c(\mathcal{J}, \ell)} \\ &+ \frac{\lambda d_c\left(\ell, [\mathbb{F}(\mathcal{J})]_{(\mathcal{T},\mathcal{N},\tilde{\alpha}_{\mathbb{F}}(\mathcal{J}))}\right) d_c\left(\mathcal{J}, [\mathbb{G}(\ell)]_{(\mathcal{T},\mathcal{N},\tilde{\alpha}_{\mathbb{G}}(\ell))}\right)}{1 + d_c(\mathcal{J}, \ell)}, \end{aligned} \tag{48}$$

and β, μ, λ *are nonnegative real numbers with* $Y\beta + \mu + \lambda < 1$ *and* $\epsilon = \tau\beta/1 - \mu < 1$ *for any* $Y \geq 1$. *Thus, we can find* ω *in* $\bar{B}(\mathcal{J}_0, r)$:

$$\omega \in [\mathbb{F}(\omega)]_{(\mathcal{T},\mathcal{N},\tilde{\alpha}_{\mathbb{F}}(\omega))} \cap [\mathbb{G}(\omega)]_{(\mathcal{T},\mathcal{N},\tilde{\alpha}_{\mathbb{G}}(\omega))}. \tag{49}$$

Proof. Let $\jmath_0$ be an arbitrary but fixed element in Γ. Then by hypothesis, $[\mathbb{F}(\jmath_0)]_{(\mathcal{T},\mathcal{N},\tilde{\alpha}_{\mathbb{F}}(\jmath_0))} \in \mathbb{CB}(\Gamma)$. So we can find $\mathcal{J}_1 \in [\mathbb{F}(\mathcal{J}_0)]_{(\mathcal{T},\mathcal{N},\tilde{\alpha}_{\mathbb{F}}(\jmath_0))}$: $(1-\epsilon)r \in s(d_c(\mathcal{J}_0, \mathcal{J}_1))$. From (47), it is easy to see that

$$d_c(\mathcal{J}_0, \mathcal{J}_1) \preceq (1-\epsilon)r. \tag{50}$$

This implies

$$|d_c(\mathcal{J}_0, \mathcal{J}_1)| \leq (1-\epsilon)|r|. \tag{51}$$

Therefore, $\jmath_1 \in \bar{B}(\jmath_0, r)$. From (46), we have

$$\begin{aligned} &B^*(\mathcal{J}_0, \mathcal{J}_1, \mathbb{F}, \mathbb{G}, \tilde{\alpha}) \\ &\in s\left([\mathbb{F}(\mathcal{J}_0)]_{(\mathcal{T},\mathcal{N},\tilde{\alpha}_{\mathbb{F}}(\mathcal{J}_0))}, [\mathbb{G}(\mathcal{J}_1)]_{(\mathcal{T},\mathcal{N},\tilde{\alpha}_{\mathbb{G}}(\mathcal{J}_1))}\right). \end{aligned} \tag{52}$$

From here, by following the remaining steps in the proof of Theorem 26, we obtain

$$|d_c(\mathcal{J}_1, \mathcal{J}_2)| \leq \epsilon |d_c(\mathcal{J}_0, \mathcal{J}_1)|. \tag{53}$$

From (51), we have

$$|d_c(\mathcal{J}_1, \mathcal{J}_2)| \leq \epsilon |d_c(\mathcal{J}_0, \mathcal{J}_1)| \leq \epsilon(1-\epsilon)|r|. \tag{54}$$

Using the triangular inequality of (Γ, d_c),

$$\begin{aligned} |d_c(\mathcal{J}_0, \mathcal{J}_2)| &\leq Y[|d_c(\mathcal{J}_0, \mathcal{J}_1)| + |d_c(\mathcal{J}_1, \mathcal{J}_2)|] \\ &\leq Y[(1-\epsilon)|r| + \epsilon(1-\epsilon)|r|] \\ &\leq Y\left(1-\epsilon^2\right)|r|. \end{aligned} \tag{55}$$

It follows that $\mathcal{J}_2 \in \bar{B}(\mathcal{J}_0, r)$. From (46), we have

$$\begin{aligned} &B^*(\mathcal{J}_1, \mathcal{J}_2, \mathbb{G}, \mathbb{F}, \tilde{\alpha}) \\ &\in s\left([\mathbb{G}(\mathcal{J}_1)]_{(\mathcal{T},\mathcal{N},\tilde{\alpha}_{\mathbb{G}}(\mathcal{J}_1))}, [\mathbb{F}(\jmath_2)]_{(\mathcal{T},\mathcal{N},\tilde{\alpha}_{\mathbb{F}}(\mathcal{J}_2))}\right). \end{aligned} \tag{56}$$

By repetition of the above steps and using the fact that (Γ, d_c) is a complex-valued b-MS, we can generate a sequence $\{\mathcal{J}_n\}_{n\in\mathbb{N}}$ in $\bar{B}(\jmath_0, r)$:

$$\begin{aligned} |d_c(\mathcal{J}_{2n}, \mathcal{J}_{2n+1})| &\leq \epsilon^{2n}|d_c(\mathcal{J}_0, \mathcal{J}_1)|, \\ |d_c(\mathcal{J}_{2n+1}, \mathcal{J}_{2n+2})| &\leq \epsilon^{2n+1}|d_c(\mathcal{J}_0, \mathcal{J}_1)|, \\ \mathcal{J}_{2n+1} &\in [\mathbb{F}(\mathcal{J}_{2n})]_{(\mathcal{T},\mathcal{N},\tilde{\alpha}_{\mathbb{F}}(\mathcal{J}_{2n}))}, \\ \mathcal{J}_{2n+2} &\in [\mathbb{G}(\mathcal{J}_{2n+1})]_{(\mathcal{T},\mathcal{N},\tilde{\alpha}_{\mathbb{G}}(\mathcal{J}_{2n+1}))}. \end{aligned} \tag{57}$$

Inductively, we can construct a sequence $\{\jmath_n\}_{n\in\mathbb{N}}$ in Γ:

$$|d_c(\mathcal{J}_n, \mathcal{J}_{n+1})| \leq \epsilon^n |d_c(\mathcal{J}_0, \mathcal{J}_1)|. \tag{58}$$

Now, for $m, n \in \mathbb{N}$ with $n < m$, using triangle inequality and the iterative scheme (58), we have

$$\begin{aligned} |d_c(\mathcal{J}_n, \mathcal{J}_m)| &\leq Y[|d_c(\mathcal{J}_n, \mathcal{J}_{n+1})| \\ &\quad + |d_c(\mathcal{J}_{n+1}, \mathcal{J}_{n+2})| + \cdots + |d_c(\mathcal{J}_{m-1}, \mathcal{J}_m)|] \\ &\leq Y\left|\epsilon^n + \epsilon^{n+1} + \cdots + \epsilon^{m-1}\right| |d_c(\mathcal{J}_0, \mathcal{J}_1)| \\ &\leq \frac{Y\epsilon^n}{1-\epsilon}|d_c(\mathcal{J}_0, \mathcal{J}_1)|. \end{aligned} \tag{59}$$

Consequently,

$$|d_c(\mathcal{J}_n, \mathcal{J}_m)| \leq \frac{Y\epsilon^n}{1-\epsilon}|d_c(\mathcal{J}_0, \mathcal{J}_1)| \longrightarrow 0 \cdots \text{as} \quad n, m \longrightarrow \infty. \tag{60}$$

Hence, $\{\jmath_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\bar{B}(\mathcal{J}_0, r)$. Since $\bar{B}(\jmath_0, r)$ is a closed subspace of a complete MS Γ, therefore, we can find $\omega \in \bar{B}(\mathcal{J}_0, r)$: $\mathcal{J}_n \longrightarrow \omega$ as $n \longrightarrow \infty$. Now, to show that $\omega \in [\mathbb{F}(\omega)]_{(\mathcal{T},\mathcal{N},\tilde{\alpha}_{\mathbb{F}}(\omega))} \cap [\mathbb{G}(\omega)]_{(\mathcal{T},\mathcal{N},\tilde{\alpha}_{\mathbb{G}}(\omega))}$, from (46), we have

$$B^*(\mathcal{J}_{2n}, \omega, \mathbb{F}, \mathbb{G}, \tilde{\alpha}) \in s\left([\mathbb{F}(\mathcal{J}_{2n})]_{(\mathcal{T},\mathcal{N},\tilde{\alpha}_{\mathbb{F}}(\mathcal{J}_{2n}))}, [\mathbb{G}(\omega)]_{(\mathcal{T},\mathcal{N},\tilde{\alpha}_{\mathbb{G}}(\omega))}\right). \tag{61}$$

This implies

$$B^*(\mathscr{I}_{2n}, \omega, \mathbb{F}, \mathbb{G}, \tilde{\alpha}) \in s\left(\mathscr{I}_{2n+1}, [\mathbb{G}(\omega)]_{(\mathscr{T},\mathscr{N},\tilde{\alpha}_{\mathbb{G}}(\omega))}\right). \quad (62)$$

By definition, we can find some $\omega_n \in [\mathbb{G}(\omega)]_{(\mathscr{T},\mathscr{N},\tilde{\alpha}_{\mathbb{G}}(\omega))}$:

$$B^*(\mathscr{I}_{2n}, \omega, \mathbb{F}, \mathbb{G}, \tilde{\alpha}) \in s(d_c(\mathscr{I}_{2n+1}, \omega_n)). \quad (63)$$

Using the g.l.b property of $\mathbb{F}$ and $\mathbb{G}$, we get

$$\begin{aligned} d_c(\mathscr{I}_{2n+1}, \omega_n) &\preceq \beta d_c(\mathscr{I}_{2n}, \omega) \\ &+ \frac{\mu d_c(\mathscr{I}_{2n}, \mathscr{I}_{2n+1}) d_c(\omega, \omega_n) + \lambda d_c(\omega, \mathscr{I}_{2n+1}) d_c(\mathscr{I}_{2n}, \omega_n)}{1 + d_c(\mathscr{I}_{2n}, \omega)}. \end{aligned} \quad (64)$$

From triangle inequality, it follows that

$$\begin{aligned} d_c(\omega, \omega_n) &\preceq Y[d_c(\omega, \mathscr{I}_{2n+1}) + d_c(\mathscr{I}_{2n+1}, \omega_n)] \\ &\preceq Y d_c(\omega, \mathscr{I}_{2n+1}) + Y\beta d_c(\mathscr{I}_{2n}, \omega_n) \\ &+ Y \frac{\mu d_c(\mathscr{I}_{2n}, \mathscr{I}_{2n+1}) d_c(\omega, \omega_n)}{1 + d_c(\mathscr{I}_{2n}, \omega)} \\ &+ Y \frac{\lambda |d_c(\omega, \mathscr{I}_{2n+1})||d_c(\mathscr{I}_{2n}, \omega_n)|}{|1 + d_c(\mathscr{I}_{2n}, \omega)|}. \end{aligned} \quad (65)$$

Therefore,

$$\begin{aligned} |d_c(\omega, \omega_n)| &\leq Y|d_c(\omega, \mathscr{I}_{2n+1})| + Y\beta|d_c(\mathscr{I}_{2n}, \omega)| \\ &+ Y \frac{\mu |d_c(\mathscr{I}_{2n}, \mathscr{I}_{2n+1})||d_c(\omega, \omega_n)|}{|1 + d_c(\mathscr{I}_{2n}, \omega)|} \\ &+ Y \frac{\lambda |d_c(\omega, \mathscr{I}_{2n+1})||d_c(\mathscr{I}_{2n}, \omega_n)|}{|1 + d_c(\mathscr{I}_{2n}, \omega)|}. \end{aligned} \quad (66)$$

Applying $n \longrightarrow \infty$ in (66), we have $|d_c(\omega, \omega_n)| \longrightarrow 0$. This implies $\omega_n \longrightarrow \omega$ as $n \longrightarrow \infty$. Since $[\mathbb{G}(\omega)]_{(\mathscr{T},\mathscr{N},\tilde{\alpha}_{\mathbb{G}}(\omega))}$ is closed, therefore $\omega \in [\mathbb{G}(\omega)]_{(\mathscr{T},\mathscr{N},\tilde{\alpha}_{\mathbb{G}}(\omega))}$. Similarly, one can show that $\omega \in [\mathbb{F}(\omega)]_{(\mathscr{T},\mathscr{N},\tilde{\alpha}_{\mathbb{F}}(\omega))}$.

Hence,

$$\omega \in [\mathbb{F}(\omega)]_{(\mathscr{T},\mathscr{N},\tilde{\alpha}_{\mathbb{F}}(\omega))} \cap [\mathbb{G}(\omega)]_{(\mathscr{T},\mathscr{N},\tilde{\alpha}_{\mathbb{G}}(\omega))}. \quad (67)$$

By setting $\mathbb{F} = \mathbb{G}$ in Theorem 26, we obtain the following result as a corollary. □

Corollary 27. *Let (Γ, d_c) be a complete complex-valued b-MS and $\mathbb{F} : \Gamma \longrightarrow (IFS)^{\Gamma}$ be an IFM obeying the g.l.b property. Assume that for each $\jmath \in \Gamma$, we can find $\tilde{\alpha}_{\mathbb{F}}(\jmath) \in [0, 1]$: $[\mathbb{F}(\jmath)]_{(\mathscr{T},\mathscr{N},\tilde{\alpha}_{\mathbb{F}}(\jmath))} \in \mathbb{CB}(\Gamma)$. If for all $\mathscr{I}, \ell \in \bar{B}(\jmath_0, r)$,*

$$\begin{aligned} &B^*(\mathscr{I}, \ell, \mathbb{F}, \tilde{\alpha}) \in s\left([\mathbb{F}(\mathscr{I})]_{(\mathscr{T},\mathscr{N},\tilde{\alpha}_{\mathbb{F}}(\mathscr{I}))}, [\mathbb{F}(\ell)]_{(\mathscr{T},\mathscr{N},\tilde{\alpha}_{\mathbb{F}}(\ell))}\right), \\ &(1 - \epsilon) r \in s\left(\mathscr{I}_0, [\mathbb{F}(\mathscr{I}_0)]_{(\mathscr{T},\mathscr{N},\tilde{\alpha}_{\mathbb{F}}(\mathscr{I}_0))}\right), \end{aligned} \quad (68)$$

where

$$\begin{aligned} &B^*(\mathscr{I}, \ell, \mathbb{F}, \mathbb{G}, \tilde{\alpha}) \\ &= \beta d_c(\mathscr{I}, \ell) \\ &+ \frac{\mu d_c\left(\mathscr{I}, [\mathbb{F}(\mathscr{I})]_{(\mathscr{T},\mathscr{N},\tilde{\alpha}_{\mathbb{F}}(\mathscr{I}))}\right) d_c\left(\ell, [\mathbb{G}(\ell)]_{(\mathscr{T},\mathscr{N},\tilde{\alpha}_{\mathbb{G}}(\ell))}\right)}{1 + d_c(\mathscr{I}, \ell)} \\ &+ \frac{\lambda d_c\left(\ell, [\mathbb{F}(\mathscr{I})]_{(\mathscr{T},\mathscr{N},\tilde{\alpha}_{\mathbb{F}}(\mathscr{I}))}\right) d_c\left(\mathscr{I}, [\mathbb{G}(\ell)]_{(\mathscr{T},\mathscr{N},\tilde{\alpha}_{\mathbb{G}}(\ell))}\right)}{1 + d_c(\mathscr{I}, \ell)}, \end{aligned} \quad (69)$$

and β, μ, λ are nonnegative real numbers with $Y\beta + \mu + \lambda < 1$ and $\epsilon = \tau\beta/1 - \mu < 1$ for any $Y \geq 1$. Thus, we can find ω in $\bar{B}(\jmath_0, r)$:

$$\omega \in [\mathbb{F}(\omega)]_{(\mathscr{T},\mathscr{N},\tilde{\alpha}_{\mathbb{F}}(\omega))}. \quad (70)$$

Corollary 28. *Let (Γ, d_c) be a complete complex-valued b-MS and $\mathbb{F}, \mathbb{G} : \Gamma \longrightarrow CB(\Gamma)$ be a pair of MVM obeying the g.l.b property*

$$\begin{aligned} &\beta d_c(\mathscr{I}, \ell) + \frac{\mu d_c(\mathscr{I}, \mathbb{F}(\mathscr{I})) d_c(\ell, \mathbb{G}(\ell))}{1 + d_c(\mathscr{I}, \ell)} \\ &+ \frac{\lambda d_c(\ell, \mathbb{F}(\mathscr{I})) d_c(\mathscr{I}, \mathbb{G}(\ell))}{1 + d_c(\mathscr{I}, \ell)} \in s(\mathbb{F}(\mathscr{I}), \mathbb{G}(\ell)), \\ &(1 - \epsilon) r \in s(\mathscr{I}_0, \mathbb{F}(\mathscr{I}_0)), \end{aligned} \quad (71)$$

for all $\mathscr{I}, \ell \in \bar{B}(\mathscr{I}_0, r)$, and β, μ, λ are nonnegative real numbers with $Y\beta + \mu + \lambda < 1$, and $\epsilon = \tau\beta/1 - \mu < 1$ for any $Y \geq 1$. Thus, we can find ω in $\bar{B}(\mathscr{I}_0, r)$:

$$\omega \in \mathbb{F}(\omega) \cap G(\omega). \quad (72)$$

Theorem 29. *Let (Γ, d_c) be a complete complex-valued b-MS and $\mathbb{F} : \Gamma \longrightarrow (IFS)^{\Gamma}$ be an IFM obeying the g.l.b property. Assume that for each $\jmath \in \Gamma$, we can find $\tilde{\alpha}_{\mathbb{F}}(\mathscr{I}) \in [0, 1]$: $[\mathbb{F}(\mathscr{I})]_{(\mathscr{T},\mathscr{N},\tilde{\alpha}_{\mathbb{F}}(\mathscr{I}))} \in \mathbb{CB}(\Gamma)$. If for all $\mathscr{I}, \ell \in \Gamma$,*

$$\begin{aligned} &\lambda_1 d_c(\mathscr{I}_0, \mathscr{I}_1) + \lambda_2 K^o(\mathscr{I}_0, \mathscr{I}_1, (\mathscr{T}, \mathscr{N}, \tilde{\alpha})) \\ &+ \lambda_3 C^o(\mathscr{I}_0, \mathscr{I}_1, (\mathscr{T}, \mathscr{N}, \tilde{\alpha})) + \lambda_4 D^o(\mathscr{I}_0, \mathscr{I}_1, (\mathscr{T}, \mathscr{N}, \tilde{\alpha})) \\ &+ \lambda_5 E^o(\mathscr{I}_0, \mathscr{I}_1, (\mathscr{T}, \mathscr{N}, \tilde{\alpha})) \\ &\in s\left(\mathbb{F}(\mathscr{I}_0)_{(\mathscr{T},\mathscr{N},\tilde{\alpha}_{\mathbb{F}}(\mathscr{I}_0))}, [\mathbb{F}(\mathscr{I}_1)]_{(\mathscr{T},\mathscr{N},\tilde{\alpha}_{\mathbb{F}}(\mathscr{I}_1))}\right), \end{aligned} \quad (73)$$

where

$$\begin{aligned}
&K^o(\mathcal{J},\ell,(\mathcal{T},\mathcal{N},\tilde{\alpha})) \\
&\quad = \frac{d_c\left(\mathcal{J},[\mathbb{F}(\mathcal{J})]_{(\mathcal{T},\mathcal{N},\tilde{\alpha}_{\mathbb{F}}(\mathcal{J}))}\right) d_c\left(\ell,[\mathbb{F}(\ell)]_{(\mathcal{T},\mathcal{N},\tilde{\alpha}_{\mathbb{F}}(\ell))}\right)}{1+d_c(\mathcal{J},\ell)}, \\
&C^o(\mathcal{J},\ell,(\mathcal{T},\mathcal{N},\tilde{\alpha})) \\
&\quad = \frac{d_c\left(\ell,[\mathbb{F}(\mathcal{J})]_{(\mathcal{T},\mathcal{N},\tilde{\alpha}_{\mathbb{F}}(\mathcal{J}))}\right) d_c\left(\mathcal{J},[\mathbb{F}(\ell)]_{(\mathcal{T},\mathcal{N},\tilde{\alpha}_{\mathbb{F}}(\ell))}\right)}{1+d_c(\mathcal{J},\ell)}, \\
&D^o(\mathcal{J},\ell,(\mathcal{T},\mathcal{N},\tilde{\alpha})) \\
&\quad = \frac{d_c\left(\mathcal{J},[\mathbb{F}(\mathcal{J})]_{(\mathcal{T},\mathcal{N},\tilde{\alpha}_{\mathbb{F}}(\mathcal{J}))}\right) d_c\left(\mathcal{J},[\mathbb{F}(\ell)]_{(\mathcal{T},\mathcal{N},\tilde{\alpha}_{\mathbb{F}}(\ell))}\right)}{1+d_c(\mathcal{J},\ell)}, \\
&E^o(\mathcal{J},\ell,(\mathcal{T},\mathcal{N},\tilde{\alpha})) \\
&\quad = \frac{d_c\left(\ell,[\mathbb{F}(\mathcal{J})]_{(\mathcal{T},\mathcal{N},\tilde{\alpha}_{\mathbb{F}}(\mathcal{J}))}\right) d_c\left(\ell,[\mathbb{F}(\ell)]_{(\mathcal{T},\mathcal{N},\tilde{\alpha}_{\mathbb{F}}(\ell))}\right)}{1+d_c(\mathcal{J},\ell)},
\end{aligned} \tag{74}$$

and $\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5$ *are nonnegative real numbers with* $\lambda_1+\lambda_2+\lambda_3+2Y\lambda_4+\lambda_5<1$. *Then,* $\mathbb{F}$ *has a FP in* Γ.

Proof. Suppose that $\mathcal{J}_0$ is an arbitrary and fixed element of Γ, then by assumption, $[\mathbb{F}(\mathcal{J}_0)]_{(\mathcal{T},\mathcal{N},\tilde{\alpha}_{\mathbb{F}}(\mathcal{J}_0))} \in \mathbb{CB}(\Gamma)$, so we can take $j_1 \in [\mathbb{F}(\mathcal{J}_0)]_{(\mathcal{T},\mathcal{N},\tilde{\alpha}_{\mathbb{F}}(\mathcal{J}_0))}$; therefore from Theorem 26,

$$\begin{aligned}
&\lambda_1 d_c(\mathcal{J}_0,\mathcal{J}_1) + \lambda_2 K^o(\mathcal{J}_0,\mathcal{J}_1,(\mathcal{T},\mathcal{N},\tilde{\alpha})) \\
&\quad + \lambda_3 C^o(\mathcal{J}_0,\mathcal{J}_1,(\mathcal{T},\mathcal{N},\tilde{\alpha})) + \lambda_4 D^o(\mathcal{J}_0,\mathcal{J}_1,(\mathcal{T},\mathcal{N},\tilde{\alpha})) \\
&\quad + \lambda_5 E^o(\mathcal{J}_0,\mathcal{J}_1,(\mathcal{T},\mathcal{N},\tilde{\alpha})) \\
&\in s\left([\mathbb{F}(\mathcal{J}_0)]_{(\mathcal{T},\mathcal{N},\tilde{\alpha}_{\mathbb{F}}(\mathcal{J}_0))}, [\mathbb{F}(\mathcal{J}_1)]_{(\mathcal{T},\mathcal{N},\tilde{\alpha}_{\mathbb{F}}(\mathcal{J}_1))}\right).
\end{aligned} \tag{75}$$

This implies

$$\begin{aligned}
&\lambda_1 d_c(\mathcal{J}_0,\mathcal{J}_1) + \lambda_2 K^o(\mathcal{J}_0,\mathcal{J}_1,(\mathcal{T},\mathcal{N},\tilde{\alpha})) \\
&\quad + \lambda_3 C^o(\mathcal{J}_0,\mathcal{J}_1,(\mathcal{T},\mathcal{N},\tilde{\alpha})) \\
&\quad + \lambda_4 D^o(\mathcal{J}_0,\mathcal{J}_1,(\mathcal{T},\mathcal{N},\tilde{\alpha})) + \lambda_5 E^o(\mathcal{J}_0,\mathcal{J}_1,(\mathcal{T},\mathcal{N},\tilde{\alpha})) \\
&\in \bigcap_{a\in[\mathbb{F}(\mathcal{J}_0)]_{(\mathcal{T},\mathcal{N},\tilde{\alpha}_{\mathbb{F}}(\mathcal{J}_0))}} s\left(a, [\mathbb{F}(\mathcal{J}_1)]_{(\mathcal{T},\mathcal{N},\tilde{\alpha}_{\mathbb{F}}(\mathcal{J}_1))}\right).
\end{aligned} \tag{76}$$

Thus,

$$\begin{aligned}
&\lambda_1 d_c(\mathcal{J}_0,\mathcal{J}_1) + \lambda_2 K^o(\mathcal{J}_0,\mathcal{J}_1,(\mathcal{T},\mathcal{N},\tilde{\alpha})) \\
&\quad + \lambda_3 C^o(\mathcal{J}_0,\mathcal{J}_1,(\mathcal{T},\mathcal{N},\tilde{\alpha})) \\
&\quad + \lambda_4 D^o(\mathcal{J}_0,\mathcal{J}_1,(\mathcal{T},\mathcal{N},\tilde{\alpha})) + \lambda_5 E^o(\mathcal{J}_0,\mathcal{J}_1,(\mathcal{T},\mathcal{N},\tilde{\alpha})) \\
&\in s\left(a, [\mathbb{F}(\mathcal{J}_1)]_{(\mathcal{T},\mathcal{N},\tilde{\alpha}_{\mathbb{F}}(\mathcal{J}_1))}\right),
\end{aligned} \tag{77}$$

for all $a \in [\mathbb{F}(\mathcal{J}_0)]_{(\mathcal{T},\mathcal{N},\tilde{\alpha}_{\mathbb{F}}(\mathcal{J}_0))}$.

Since $\mathcal{J}_1 \in [\mathbb{F}(\mathcal{J}_0)]_{(\mathcal{T},\mathcal{N},\tilde{\alpha}_{\mathbb{F}}(\mathcal{J}_0))}$, then we obtain

$$\begin{aligned}
&\lambda_1 d_c(\mathcal{J}_0,\mathcal{J}_1) + \lambda_2 K^o(\mathcal{J}_0,\mathcal{J}_1,(\mathcal{T},\mathcal{N},\tilde{\alpha})) \\
&\quad + \lambda_3 C^o(\mathcal{J}_0,\mathcal{J}_1,(\mathcal{T},\mathcal{N},\tilde{\alpha})) + \lambda_4 D^o(\mathcal{J}_0,\mathcal{J}_1,(\mathcal{T},\mathcal{N},\tilde{\alpha})) \\
&\quad + \lambda_5 E^o(\mathcal{J}_0,\mathcal{J}_1,(\mathcal{T},\mathcal{N},\tilde{\alpha})) \\
&\in s\left(\mathcal{J}_1, [\mathbb{F}(\mathcal{J}_1)]_{(\mathcal{T},\mathcal{N},\tilde{\alpha}_{\mathbb{F}}(\mathcal{J}_1))}\right).
\end{aligned} \tag{78}$$

It yields

$$\begin{aligned}
&\lambda_1 d_c(\mathcal{J}_0,\mathcal{J}_1) + \lambda_2 K^o(\mathcal{J}_0,\mathcal{J}_1,(\mathcal{T},\mathcal{N},\tilde{\alpha})) \\
&\quad + \lambda_3 C^o(\mathcal{J}_0,\mathcal{J}_1,(\mathcal{T},\mathcal{N},\tilde{\alpha})) + \lambda_4 D^o(\mathcal{J}_0,\mathcal{J}_1,(\mathcal{T},\mathcal{N},\tilde{\alpha})) \\
&\quad + \lambda_5 E^o(\mathcal{J}_0,\mathcal{J}_1,(\mathcal{T},\mathcal{N},\tilde{\alpha})) \\
&\in \bigcup_{b\in[\mathbb{F}(\mathcal{J}_1)]_{(\mathcal{T},\mathcal{N},\tilde{\alpha}_{\mathbb{F}}(\mathcal{J}_1))}} s(d_c(\mathcal{J}_1,b)).
\end{aligned} \tag{79}$$

Therefore, we can find some $\mathcal{J}_2 \in [\mathbb{F}(j_1)]_{(\mathcal{T},\mathcal{N},\tilde{\alpha}_{\mathbb{F}}(\mathcal{J}_1))}$:

$$\begin{aligned}
&\lambda_1 d_c(\mathcal{J}_0,\mathcal{J}_1) + \lambda_2 K^o(\mathcal{J}_0,\mathcal{J}_1,(\mathcal{T},\mathcal{N},\tilde{\alpha})) \\
&\quad + \lambda_3 C^o(\mathcal{J}_0,\mathcal{J}_1,(\mathcal{T},\mathcal{N},\tilde{\alpha})) + \lambda_4 D^o(\mathcal{J}_0,\mathcal{J}_1,(\mathcal{T},\mathcal{N},\tilde{\alpha})) \\
&\quad + \lambda_5 E^o(\mathcal{J}_0,\mathcal{J}_1,(\mathcal{T},\mathcal{N},\tilde{\alpha})) \\
&\in s(d_c(\mathcal{J}_1,\mathcal{J}_2)).
\end{aligned} \tag{80}$$

This implies

$$\begin{aligned}
&d_c(\mathcal{J}_1,\mathcal{J}_2)^\circ \lambda_1 d_c(\mathcal{J}_0,\mathcal{J}_1) + \lambda_2 K^o(\mathcal{J}_0,\mathcal{J}_1,(\mathcal{T},\mathcal{N},\tilde{\alpha})) \\
&\quad + \lambda_3 C^o(\mathcal{J}_0,\mathcal{J}_1,(\mathcal{T},\mathcal{N},\tilde{\alpha})) + \lambda_4 D^o(\mathcal{J}_0,\mathcal{J}_1,(\mathcal{T},\mathcal{N},\tilde{\alpha})) \\
&\quad + \lambda_5 E^o(\mathcal{J}_0,\mathcal{J}_1,(\mathcal{T},\mathcal{N},\tilde{\alpha})).
\end{aligned} \tag{81}$$

Using the g.l.b property of $\mathbb{F}$, we have

$$\begin{aligned}
d_c(\mathcal{J}_1,\mathcal{J}_2) \preceq{} & \lambda_1 d_c(\mathcal{J}_0,\mathcal{J}_1) + \lambda_2 \frac{d_c(\mathcal{J}_0,\mathcal{J}_1) d_c(\mathcal{J}_1,\mathcal{J}_2)}{1+d(\mathcal{J}_0,\mathcal{J}_1)} \\
& + \lambda_3 \frac{d_c(\mathcal{J}_1,\mathcal{J}_1) d(\mathcal{J}_0,\mathcal{J}_1)}{1+d_c(\mathcal{J}_0,\mathcal{J}_1)} \\
& + \lambda_4 \frac{d_c(\mathcal{J}_0,\mathcal{J}_1) d_c(\mathcal{J}_0,\mathcal{J}_1)}{1+d(\mathcal{J}_0,\mathcal{J}_1)} \\
& + \lambda_5 \frac{d_c(\mathcal{J}_1,\mathcal{J}_1) d(\mathcal{J}_1,\mathcal{J}_2)}{1+d(\mathcal{J}_0,\mathcal{J}_1)}.
\end{aligned} \tag{82}$$

However,

$$\begin{aligned}|d_c(\mathcal{J}_1, \mathcal{J}_2)| &\le \lambda_1 |d_c(\mathcal{J}_0, \mathcal{J}_1)| + \lambda_2 |d_c(\mathcal{J}_1, \mathcal{J}_2)| \\ &\quad + \lambda_4 |d_c(\mathcal{J}_0, \mathcal{J}_2)|, \\ (1 - \lambda_2 - Y\lambda_4)|d_c(\mathcal{J}_1, \mathcal{J}_2)| &\le (\lambda_1 + Y\lambda_4)|d_c(\mathcal{J}_0, \mathcal{J}_1)|.\end{aligned} \tag{83}$$

This implies

$$|d_c(\mathcal{J}_1, \mathcal{J}_2)| \le \Omega |d_c(\mathcal{J}_0, \mathcal{J}_1)|, \tag{84}$$

where $\Omega = ((\lambda_1 + Y\lambda_4)/(1 - \lambda_2 - Y\lambda_4)) < 1$.

Inductively, we develop a sequence $\{j_p\}$ in Γ:

$$|d_c(\mathcal{J}_1, \mathcal{J}_2)| \le \Omega^p |d_c(\mathcal{J}_0, \mathcal{J}_1)|. \tag{85}$$

Now for $q \in \mathbb{N}$ and using the triangle inequality of (Γ, d_c), we have

$$\begin{aligned} d_c(\mathcal{J}_p, \mathcal{J}_{p+q}) &\le Y\left[d_c(\mathcal{J}_p, \mathcal{J}_{p+1}) + d_c(\mathcal{J}_{p+1}, \mathcal{J}_{p+q})\right] \\ &\le Y d_c(\mathcal{J}_p, \mathcal{J}_{p+1}) + Y^2 d_c(\mathcal{J}_{p+1}, \mathcal{J}_{p+2}) \\ &\quad + Y^2 d_c(\mathcal{J}_{p+2}, \mathcal{J}_{p+q}) \\ &\le Y d_c(\mathcal{J}_p, \mathcal{J}_{p+1}) + Y^2 d_c(\mathcal{J}_{p+1}, \mathcal{J}_{p+2}) \\ &\quad + Y^3 d_c(\mathcal{J}_{p+2}, \mathcal{J}_{p+3}) + \cdots + Y^q d_c(\mathcal{J}_{p+q-1}, \mathcal{J}_{p+q}) \\ &\le Y\Omega^p d_c(\mathcal{J}_0, \mathcal{J}_1) + Y^2 \Omega^{p+1} d_c(\mathcal{J}_0, \mathcal{J}_1) \\ &\quad + Y^3 \Omega^{p+2} d_c(\mathcal{J}_0, \mathcal{J}_1) + \cdots + Y^q \Omega^{p+q-1} d_c(\mathcal{J}_0, \mathcal{J}_1) \\ &\le Y\Omega^p d_c(\mathcal{J}_0, \mathcal{J}_1)\left[1 + Y\Omega + (Y\Omega)^2 + \cdots + (Y\Omega)^{q-1}\right]. \end{aligned} \tag{86}$$

However,

$$\begin{aligned} \left|d_c(\mathcal{J}_p, \mathcal{J}_{p+q})\right| &\le \left|Y\Omega^p d_c(\mathcal{J}_0, \mathcal{J}_1)\left[1 + Y\Omega + (Y\Omega)^2 + \cdots + (Y\Omega)^{q-1}\right]\right| \\ &\le Y\Omega^p |d_c(\mathcal{J}_0, \mathcal{J}_1)|\left[1 + Y\Omega + (Y\Omega)^2 + \cdots + (Y\Omega)^{q-1}\right] \\ &\le \frac{Y\Omega^p}{1 - Y\Omega}|d_c(\mathcal{J}_0, \mathcal{J}_1)|. \end{aligned} \tag{87}$$

Letting $p, q \longrightarrow \infty$, therefore, $|d_c(\mathcal{J}_p, \mathcal{J}_{p+q})| \longrightarrow 0$.

By Lemma 6, $\{j_p\}$ is a Cauchy sequence in Γ, which is complete, so we can find some $\omega \in \Gamma$,: $\lim_{p \longrightarrow \infty} j_p = \omega$. Next, we show that $\omega \in [\mathbb{F}(\omega)]_{(\mathcal{T}, \mathcal{N}, \tilde{\alpha}_{\mathbb{F}}(\omega))}$. From Theorem 26, we have

$$\begin{aligned} &\lambda_1 d_c(\mathcal{J}_{2p}, \omega) + \lambda_2 K^o(\mathcal{J}_{2p}, \omega, (\mathcal{T}, \mathcal{N}, \tilde{\alpha})) \\ &\quad + \lambda_3 C^o(\mathcal{J}_{2p}, \omega, (\mathcal{T}, \mathcal{N}, \tilde{\alpha})) + \lambda_4 D^o(\mathcal{J}_{2p}, \omega, (\mathcal{T}, \mathcal{N}, \tilde{\alpha})) \\ &\quad + \lambda_5 E^o(\mathcal{J}_{2p}, \omega, (\mathcal{T}, \mathcal{N}, \tilde{\alpha})) \\ &\in s\left(\left[\mathbb{F}(\mathcal{J}_{2p})\right]_{(\mathcal{T}, \mathcal{N}, \tilde{\alpha}_{\mathbb{F}}(\mathcal{J}_{2p}))}, [\mathbb{F}(\omega)]_{(\mathcal{T}, \mathcal{N}, \tilde{\alpha}_{\mathbb{F}}(\omega))}\right). \end{aligned} \tag{88}$$

This implies

$$\begin{aligned} &\lambda_1 d_c(\mathcal{J}_{2p}, \omega) + \lambda_2 K^o(\mathcal{J}_{2p}, \omega, (\mathcal{T}, \mathcal{N}, \tilde{\alpha})) \\ &\quad + \lambda_3 C^o(\mathcal{J}_{2p}, \omega, (\mathcal{T}, \mathcal{N}, \tilde{\alpha})) + \lambda_4 D^o(\mathcal{J}_{2p}, \omega, (\mathcal{T}, \mathcal{N}, \tilde{\alpha})) \\ &\quad + \lambda_5 E^o(\mathcal{J}_{2p}, \omega, (\mathcal{T}, \mathcal{N}, \tilde{\alpha})) \\ &\in \bigcap_{a \in \left[\mathbb{F}(\mathcal{J}_{2p})\right]_{(\mathcal{T}, \mathcal{N}, \tilde{\alpha}_{\mathbb{F}}(\mathcal{J}_{2p}))}} s\left(a, [\mathbb{F}(\omega)]_{(\mathcal{T}, \mathcal{N}, \tilde{\alpha}_{\mathbb{F}}(\omega))}\right). \end{aligned} \tag{89}$$

Since $\mathcal{J}_{2p+1} \in \left[\mathbb{F}(\mathcal{J}_{2p})\right]_{(\mathcal{T}, \mathcal{N}, \tilde{\alpha}_{\mathbb{F}}(\mathcal{J}_{2p}))}$, then we obtain

$$\begin{aligned} &\lambda_1 d_c(\mathcal{J}_{2p}, \omega) + \lambda_2 K^o(\mathcal{J}_{2p}, \omega, (\mathcal{T}, \mathcal{N}, \tilde{\alpha})) \\ &\quad + \lambda_3 C^o(\mathcal{J}_{2p}, \omega, (\mathcal{T}, \mathcal{N}, \tilde{\alpha})) + \lambda_4 D^o(\mathcal{J}_{2p}, \omega, (\mathcal{T}, \mathcal{N}, \tilde{\alpha})) \\ &\quad + \lambda_5 E^o(\mathcal{J}_{2p}, \omega, (\mathcal{T}, \mathcal{N}, \tilde{\alpha})) \\ &\in s\left(\mathcal{J}_{2p+1}, [\mathbb{F}(\omega)]_{(\mathcal{T}, \mathcal{N}, \tilde{\alpha}_{\mathbb{F}}(\omega))}\right). \end{aligned} \tag{90}$$

It yields

$$\begin{aligned} &\lambda_1 d_c(\mathcal{J}_{2p}, \omega) + \lambda_2 K^o(\mathcal{J}_{2p}, \omega, (\mathcal{T}, \mathcal{N}, \tilde{\alpha})) \\ &\quad + \lambda_3 C^o(\mathcal{J}_{2p}, \omega, (\mathcal{T}, \mathcal{N}, \tilde{\alpha})) + \lambda_4 D^o(\mathcal{J}_{2p}, \omega, (\mathcal{T}, \mathcal{N}, \tilde{\alpha})) \\ &\quad + \lambda_5 E^o(\mathcal{J}_{2p}, \omega, (\mathcal{T}, \mathcal{N}, \tilde{\alpha})) \\ &\in \bigcup_{b \in [\mathbb{F}(\omega)]_{(\mathcal{T}, \mathcal{N}, \tilde{\alpha}_{\mathbb{F}}(\omega))}} s(d_c(\mathcal{J}_{2p+1}, b)). \end{aligned} \tag{91}$$

So, we can find $\Omega_p \in [\mathbb{F}(\Omega)]_{(\mathcal{T}, \mathcal{N}, \tilde{\alpha}_{\mathbb{F}}(\Omega))}$:

$$\begin{aligned} &\lambda_1 d_c(\mathcal{J}_{2p}, \omega) + \lambda_2 K^o(\mathcal{J}_{2p}, \omega, (\mathcal{T}, \mathcal{N}, \tilde{\alpha})) \\ &\quad + \lambda_3 C^o(\mathcal{J}_{2p}, \omega, (\mathcal{T}, \mathcal{N}, \tilde{\alpha})) + \lambda_4 D^o(\mathcal{J}_{2p}, \omega, (\mathcal{T}, \mathcal{N}, \tilde{\alpha})) \\ &\quad + \lambda_5 E^o(\mathcal{J}_{2p}, \omega, (\mathcal{T}, \mathcal{N}, \tilde{\alpha})) \\ &\in s(d_c(\mathcal{J}_{2p+1}, \omega_p)). \end{aligned} \tag{92}$$

Therefore,

$$\begin{aligned} d_c(\mathcal{J}_{2p+1}, \omega_p) &\le \lambda_1 d_c(\mathcal{J}_{2p}, \omega) + \lambda_2 K^o(\mathcal{J}_{2p}, \omega, (\mathcal{T}, \mathcal{N}, \tilde{\alpha})) \\ &\quad + \lambda_3 C^o(\mathcal{J}_{2p}, \omega, (\mathcal{T}, \mathcal{N}, \tilde{\alpha})) \\ &\quad + \lambda_4 D^o(\mathcal{J}_{2p}, \omega, (\mathcal{T}, \mathcal{N}, \tilde{\alpha})) \\ &\quad + \lambda_5 E^o(\mathcal{J}_{2p}, \omega, (\mathcal{T}, \mathcal{N}, \tilde{\alpha})). \end{aligned} \tag{93}$$

Using the g.l.b property, we have

$$
\begin{aligned}
d_c\left(\mathscr{J}_{2\mathrm{p}+1}, \omega_p\right) \leq \lambda_1 d_c\left(\mathscr{J}_{2\mathrm{p}}, \omega\right) &+ \lambda_2 \frac{d_c\left(\mathscr{J}_{2\mathrm{p}}, \mathscr{J}_{2\mathrm{p}+1}\right) d_c\left(\omega, \omega_p\right)}{1 + d\left(\mathscr{J}_{2\mathrm{p}}, \omega\right)} \\
&+ \lambda_3 \frac{d_c\left(\omega, \mathscr{J}_{2\mathrm{p}+1}\right) d\left(\mathscr{J}_{2\mathrm{p}}, \omega_p\right)}{1 + d_c\left(\mathscr{J}_{2\mathrm{p}}, \omega\right)} \\
&+ \lambda_4 \frac{d_c\left(\mathscr{J}_{2\mathrm{p}}, \mathscr{J}_{2\mathrm{p}+1}\right) d_c\left(\mathscr{J}_{2\mathrm{p}}, \omega_p\right)}{1 + d_c\left(\mathscr{J}_{2\mathrm{p}}, \omega\right)} \\
&+ \lambda_5 \frac{d_c\left(\omega, \mathscr{J}_{2\mathrm{p}+1}\right) d_c\left(\omega, \omega_p\right)}{1 + d_c\left(\mathscr{J}_{2\mathrm{p}}, \omega\right)}.
\end{aligned}
\tag{94}
$$

By triangle inequality,

$$
d_c\left(\omega, \omega_p\right) \leq Y\left[d_c\left(\omega, \mathscr{J}_{2\mathrm{p}+1}\right) + d_c\left(\mathscr{J}_{2\mathrm{p}+1}, \omega_p\right)\right]. \tag{95}
$$

Thus,

$$
\begin{aligned}
d_c\left(\mathscr{I}_{2\mathrm{p}+1}, \omega_p\right) \leq Y d_c\left(\omega, \mathscr{J}_{2\mathrm{p}+1}\right) &+ Y\lambda_1 d_c\left(\mathscr{J}_{2\mathrm{p}}, \omega\right) \\
&+ Y\lambda_2 \frac{d_c\left(\mathscr{J}_{2\mathrm{p}}, \mathscr{J}_{2\mathrm{p}+1}\right) d_c\left(\omega, \omega_p\right)}{1 + d\left(\mathscr{J}_{2\mathrm{p}}, \omega\right)} \\
&+ Y\lambda_3 \frac{d_c\left(\omega, \mathscr{J}_{2\mathrm{p}+1}\right) d_c\left(\mathscr{J}_{2\mathrm{p}}, \omega_p\right)}{1 + d_c\left(\mathscr{J}_{2\mathrm{p}}, \omega\right)} \\
&+ Y\lambda_4 \frac{d_c\left(\mathscr{J}_{2\mathrm{p}}, \mathscr{J}_{2\mathrm{p}+1}\right) d_c\left(\mathscr{J}_{2\mathrm{p}}, \omega_p\right)}{1 + d_c\left(\mathscr{J}_{2\mathrm{p}}, \omega\right)} \\
&+ Y\lambda_5 \frac{d_c\left(\omega, \mathscr{J}_{2\mathrm{p}+1}\right) d_c\left(\omega, \omega_p\right)}{1 + d_c\left(\mathscr{J}_{2\mathrm{p}}, \omega\right)}.
\end{aligned}
\tag{96}
$$

This implies

$$
\begin{aligned}
\left|d_c\left(\mathscr{I}_{2\mathrm{p}+1}, \omega_p\right)\right| \leq Y\left|d_c\left(\omega, \mathscr{J}_{2\mathrm{p}+1}\right)\right| &+ Y\lambda_1\left|d_c\left(\mathscr{I}_{2\mathrm{p}}, \omega\right)\right| \\
&+ Y\lambda_2 \left|\frac{d_c\left(\mathscr{J}_{2\mathrm{p}}, \mathscr{I}_{2\mathrm{p}+1}\right) d_c\left(\omega, \omega_p\right)}{1 + d\left(\mathscr{I}_{2\mathrm{p}}, \omega\right)}\right| \\
&+ Y\lambda_3 \left|\frac{d_c\left(\omega, \mathscr{I}_{2\mathrm{p}+1}\right) d_c\left(\mathscr{I}_{2\mathrm{p}}, \omega_p\right)}{1 + d_c\left(\mathscr{J}_{2\mathrm{p}}, \omega\right)}\right| \\
&+ Y\lambda_4 \left|\frac{d_c\left(\mathscr{J}_{2\mathrm{p}}, \mathscr{I}_{2\mathrm{p}+1}\right) d_c\left(\mathscr{J}_{2\mathrm{p}}, \omega_p\right)}{1 + d_c\left(\mathscr{J}_{2\mathrm{p}}, \omega\right)}\right| \\
&+ \left|Y\lambda_5 \frac{d_c\left(\omega, \mathscr{I}_{2\mathrm{p}+1}\right) d_c\left(\omega, \omega_p\right)}{1 + d_c\left(\mathscr{I}_{2\mathrm{p}}, \omega\right)}\right|.
\end{aligned}
\tag{97}
$$

As $p \longrightarrow \infty$, we get

$$
\left|d_c\left(\mathscr{I}_{2\mathrm{p}+1}, \omega_p\right)\right| \longrightarrow 0. \tag{98}
$$

By Lemma 5, we have $\omega_p \longrightarrow \omega$ as $p \longrightarrow \infty$. Since $[\mathbb{F}(\omega)]_{(\mathscr{T}, \mathscr{N}, \tilde{\alpha}_{\mathbb{F}}(\omega))}$ is closed, therefore, $\mathbb{F}$ has a FP. □

Corollary 30. *Let (Γ, d_c) be a complete complex-valued b-MS and $\mathbb{F} : \Gamma \longrightarrow \mathbb{CB}(\Gamma)$ be a MVM obeying the g.l.b property such that*

$$
\begin{aligned}
\lambda_1 d_c(\mathscr{J}, \ell) &+ \lambda_2 \frac{d_c(\mathscr{I}, \mathbb{F}(\mathscr{J})) d_c(\ell, \mathbb{F}(\ell))}{1 + d_c(\mathscr{J}, \ell)} \\
&+ \lambda_3 \frac{d_c(\ell, \mathbb{F}(\mathscr{I})) d_c(\mathscr{I}, \mathbb{F}(\ell))}{1 + d_c(\mathscr{J}, \ell)} \\
&+ \lambda_4 \frac{d_c(\mathscr{J}, \mathbb{F}(\mathscr{J})) d_c(\mathscr{J}, \mathbb{F}(\ell))}{1 + d_c(\mathscr{J}, \ell)} \\
&+ \lambda_5 \frac{d_c(\ell, \mathbb{F}(\mathscr{I})) d_c(\ell, \mathbb{F}(\ell))}{1 + d_c(\mathscr{J}, \ell)} \\
&\in s(\mathbb{F}(\mathscr{I}), \mathbb{F}(\ell)),
\end{aligned}
\tag{99}
$$

for all $j, \ell \in \Gamma$, and $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$ are nonnegative real numbers with $\lambda_1 + \lambda_2 + \lambda_3 + 2Y\lambda_4 + \lambda_5 < 1$. Then, $\mathbb{F}$ has a FP in Γ.

4. Applications to Fractional Mixed Volterra-Fredholm Integrodifferential Equations with Integral Boundary Conditions

In this section, first some definitions and results are given from the existing literature and then Corollary 24 of Theorem 22 is applied to study the existence of a solution of fractional mixed Volterra-Fredholm integrodifferential equation with integral boundary conditions.

Definition 31 [25, 26]. Let g be a function given on $[a, b]$; then, the Caputo fractional order derivative of g is given by

$$
{}_a D_t^{\tilde{\alpha}} g(t) = \frac{1}{\Gamma(n - \tilde{\alpha})} \int_a^t (t - s)^{n - \tilde{\alpha} - 1} g^{(n)}(s) ds, \tag{100}
$$

where $n = [\tilde{\alpha}] + 1$ and $[\tilde{\alpha}]$ denotes the integer part of $\tilde{\alpha}$.

Definition 32 [25]. Let g be a function given almost everywhere on $[a, b]$, for $\tilde{\alpha} > 0$; we depict

$$
{}_a D_b^{-\tilde{\alpha}} g = \frac{1}{\Gamma(\tilde{\alpha})} \int_a^b (b - t)^{\tilde{\alpha} - 1} g(t) dt, \tag{101}
$$

provided that the Lebesgue integral exists.

Lemma 33 [27]. *Let $g : [0, \delta] \times X \longrightarrow X$ be a continuous function, $D^{\tilde{\alpha}}$ be the Caputo fractional derivative, and $1 < \tilde{\alpha} \leq 2$; then, the solution of fractional mixed Volterra-Fredholm integrodifferential equation with boundary conditions,*

$$
D^{\tilde{\alpha}} \ell(t) = g\left(t, \ell(t), \int_0^t k(t, s, \ell(s)) ds, \int_0^\delta r_1(t, s, \ell(s)) ds\right), \tag{102}
$$

$$\ell(0) - \ell'(0) = \int_0^\delta f(\ell(s))ds, \ell(\delta) - \ell'(\delta) = \int_0^\delta r(\ell(s))ds, \tag{103}$$

is given by

$$\begin{aligned}\ell(t) = &\left(\frac{1+t}{\delta}\right)\int_0^\delta r(\ell(s))ds + \left(1 - \frac{(1+t)}{\delta}\right)\int_0^\delta f(\ell(s))ds \\ &- \left(\frac{1+t}{\delta}\right)\int_0^\delta \frac{(\delta - s)^{\tilde{\alpha}-1}}{\Gamma(\tilde{\alpha})} \\ &\cdot g\left(s, \ell(s), \int_0^s k(s, \tau, \ell(\tau))d_c\tau, \int_0^\delta r_1(s, \tau, \ell(\tau))d_c\tau\right)ds \\ &+ \left(\frac{1+t}{\delta}\right)\int_0^\delta \frac{(\delta - s)^{\tilde{\alpha}-2}}{\Gamma(\tilde{\alpha}-1)} \\ &\cdot g\left(s, \ell(s), \int_0^s k(s, \tau, \ell(\tau))d_c\tau, \int_0^\delta r_1(s, \tau, \ell(\tau))d_c\tau\right)ds \\ &+ \int_0^t \frac{(t-s)^{\tilde{\alpha}-1}}{\Gamma(\tilde{\alpha})} \\ &\cdot g\left(s, \ell(s), \int_0^s k(s, \tau, \ell(\tau))d_c\tau, \int_0^\delta r_1(s, \tau, \ell(\tau))d_c\tau\right)ds.\end{aligned} \tag{104}$$

Theorem 34. *Let Γ be a complete complex-valued b-metric space endowed with the metric $d_c : \Gamma \times \Gamma \longrightarrow \mathbb{C}$ given by $d_c(\mathcal{J}, \ell) = |\mathcal{J} - \ell|^2 e^{i\theta}$ and $C = ([0, \delta], \Gamma)$ be the space of all continuous functions from $[0, \delta]$ into Γ. Suppose that the function $g : [0, \delta] \times \Gamma \times \Gamma \times \Gamma \longrightarrow \Gamma$, $k, r, r_1 : [0, \delta] \times [0, \delta] \times \Gamma \longrightarrow \Gamma$ and $f, r : \Gamma \longrightarrow \Gamma$ be chosen: the following conditions are obeyed:*

(C1) For each $\ell \in \Gamma$ and $t \in [0, \delta]$, we have $\Lambda_\ell \in \Gamma$, where

$$\begin{aligned}\Lambda_\ell = &\left(\frac{1+t}{\delta}\right)\int_0^\delta r(\ell(s))ds + \left(1 - \frac{(1+t)}{\delta}\right)\int_0^\delta f(\ell(s))ds \\ &- \left(\frac{1+t}{\delta}\right)\int_0^\delta \frac{(\delta - s)^{\tilde{\alpha}-1}}{\Gamma(\tilde{\alpha})} \\ &\cdot g\left(s, \ell(s), \int_0^s k(s, \tau, \ell(\tau))d_c\tau, \int_0^\delta r_1(s, \tau, \ell(\tau))d_c\tau\right)ds \\ &+ \left(\frac{1+t}{\delta}\right)\int_0^\delta \frac{(\delta - s)^{\tilde{\alpha}-2}}{\Gamma(\tilde{\alpha}-1)} \\ &\cdot g\left(s, \ell(s), \int_0^s k(s, \tau, \ell(\tau))d_c\tau, \int_0^\delta r_1(s, \tau, \ell(\tau))d_c\tau\right)ds \\ &+ \int_0^t \frac{(t-s)^{\tilde{\alpha}-1}}{\Gamma(\tilde{\alpha})} \\ &\cdot g\left(s, \ell(s), \int_0^s k(s, \tau, \ell(\tau))d_c\tau, \int_0^\delta r_1(s, \tau, \ell(\tau))d_c\tau\right)ds\end{aligned} \tag{105}$$

(C2) We can find a constant $\beta \in (0, 1)$: for all $\mathcal{J}, \ell \in \Gamma$,

$$|\mathcal{J} - \ell|^2 \leq \beta|\mathcal{J} - \ell|^2 + \left\|\Lambda_{\mathcal{J}} - \Lambda_\ell\right\| \tag{106}$$

(C3) We can find some constants a_1, a_2: for all $\mathcal{J}, \ell \in \Gamma$,

$$\|r(\mathcal{J}) - r(\ell)\| \leq a_1\|\mathcal{J} - \ell\| \text{ and } \|f(\mathcal{J}) - f(\ell)\| \leq a_2\|\mathcal{J} - \ell\| \tag{107}$$

(C4) We can find continuous functions $\Theta_1, \Theta_2 : [0, \delta] \longrightarrow [0,\infty)$:

$$\begin{aligned}\left\|\int_0^t (k(t, s, \mathcal{J}) - k(t, s, \ell))ds\right\| &\leq \Theta_1\|\mathcal{J} - \ell\|, \\ \left\|\int_0^\delta (r_1(t, s, \mathcal{J}) - r_1(t, s, \ell))ds\right\| &\leq \Theta_2\|\mathcal{J} - \ell\|\end{aligned} \tag{108}$$

(C5) We can find a continuous function $K : [0, \delta] \longrightarrow [0,\infty)$ and a continuous nondecreasing function $L : [0,\infty) \longrightarrow (0,\infty)$:

$$\begin{aligned}&\|g(t, \mathcal{J}_1, \ell_1, z_1) - g(t, \mathcal{J}_2, \ell_2, z_2)\| \\ &\quad \leq K(t)L(\|\mathcal{J}_1 - \mathcal{J}_2\| + \|\ell_1 - \ell_2\| + \|z_1 - z_2\|),\end{aligned} \tag{109}$$

where the function L satisfies $L(\xi(t)\mathcal{J}) \leq \xi(t)L(\mathcal{J})$, provided $\xi : [0, \delta] \longrightarrow [0,\infty)$ is continuous

(C6) We can find a multivalued function $M : \Gamma \longrightarrow \mathbb{CB}(\Gamma)$ and some constants η, β, Y, μ with $\eta \in (0, 1)$ and $Y \geq 1$:

$$\eta\|\mathcal{J} - \ell\| \leq \mu\mathcal{J}(\mathcal{J}, \ell)(t) + \lambda E(\mathcal{J}, \ell)(t), \tag{110}$$

where

$$\begin{aligned}J(\mathcal{J}, \ell) &= \frac{|j - Mu|^2|\ell - Mv|^2 e^{i\theta}}{1 + |\mathcal{J} - \ell|^2 e^{i\theta}}, \\ E(\mathcal{J}, \ell) &= \frac{|\ell - Mu|^2|\mathcal{J} - Mv|^2 e^{i\theta}}{1 + |\mathcal{J} - \ell|^2 e^{i\theta}}, \\ \theta &= \tan^{-1}\left|\frac{\ell}{\mathcal{J}}\right|,\end{aligned} \tag{111}$$

$$Y\beta + \mu + \lambda < 1$$

Then, the Caputo integrodifferential equations (102) and (103) have a solution.

Proof. By Lemma 33, the equivalent integral reformulation of problem (100)–(101) is given by

$$\begin{aligned}\ell(t) &= \left(\frac{1+t}{\delta}\right)\int_0^\delta r(\ell(s))ds + \left(1-\frac{(1+t)}{\delta}\right)\int_0^\delta f(\ell(s))ds \\ &\quad - \left(\frac{1+t}{\delta}\right)\int_0^\delta \frac{(\delta-s)^{\tilde\alpha-1}}{\Gamma(\tilde\alpha)} \\ &\quad \cdot g\left(s,\ell(s),\int_0^s k(s,\tau,\ell(\tau))d_c\tau,\int_0^\delta r_1(s,\tau,\ell(\tau))d_c\tau\right)ds \\ &\quad + \left(\frac{1+t}{\delta}\right)\int_0^\delta \frac{(\delta-s)^{\tilde\alpha-2}}{\Gamma(\tilde\alpha-1)} \\ &\quad \cdot g\left(s,\ell(s),\int_0^s k(s,\tau,\ell(\tau))d_c\tau,\int_0^\delta r_1(s,\tau,\ell(\tau))d_c\tau\right)ds \\ &\quad + \int_0^t \frac{(t-s)^{\tilde\alpha-1}}{\Gamma(\tilde\alpha)} \\ &\quad \cdot g\left(s,\ell(s),\int_0^s k(s,\tau,\ell(\tau))d_c\tau,\int_0^\delta r_1(s,\tau,\ell(\tau))d_c\tau\right)ds.\end{aligned} \tag{112}$$

By (C1), for each $\ell \in \Gamma$, we have $\Lambda_\ell \in \Gamma$, where

$$\begin{aligned}\Lambda_\ell &= \left(\frac{1+t}{\delta}\right)\int_0^\delta r(\ell(s))ds + \left(1-\frac{(1+t)}{\delta}\right)\int_0^\delta f(\ell(s))ds \\ &\quad - \left(\frac{1+t}{\delta}\right)\int_0^\delta \frac{(\delta-s)^{\tilde\alpha-1}}{\Gamma(\tilde\alpha)} \\ &\quad \cdot g\left(s,\ell(s),\int_0^s k(s,\tau,\ell(\tau))d_c\tau,\int_0^\delta r_1(s,\tau,\ell(\tau))d_c\tau\right)ds \\ &\quad + \left(\frac{1+t}{\delta}\right)\int_0^\delta \frac{(\delta-s)^{\tilde\alpha-2}}{\Gamma(\tilde\alpha-1)} \\ &\quad \cdot g\left(s,\ell(s),\int_0^s k(s,\tau,\ell(\tau))d_c\tau,\int_0^\delta r_1(s,\tau,\ell(\tau))d_c\tau\right)ds \\ &\quad + \int_0^t \frac{(t-s)^{\tilde\alpha-1}}{\Gamma(\tilde\alpha)} \\ &\quad \cdot g\left(s,\ell(s),\int_0^s k(s,\tau,\ell(\tau))d_c\tau,\int_0^\delta r_1(s,\tau,\ell(\tau))d_c\tau\right)ds.\end{aligned} \tag{113}$$

By (C2), for all $\mathcal{J}, \ell \in \Gamma$ and each $t \in [0,\delta]$, we have

$$\begin{aligned}|\mathcal{J}-\ell|^2 &\le \beta|\mathcal{J}-\ell|^2 + \|\Lambda_{\mathcal{J}}-\Lambda_\ell\| \\ &\le \beta|\mathcal{J}-\ell|^2 + \left(\frac{1+t}{\delta}\right)\int_0^\delta \|r(\mathcal{J})-r(\ell)\|ds \\ &\quad + \left(1-\frac{(1+t)}{\delta}\right)\int_0^\delta \|f(\mathcal{J})-f(\ell)\|ds \\ &\quad + \left(\frac{1+t}{\delta}\right)\int_0^\delta \frac{(\delta-s)^{\tilde\alpha-1}}{\Gamma(\tilde\alpha)} \\ &\quad \cdot g\left(s,\mathcal{J}(s),\int_0^s k(s,\tau,\mathcal{J}(\tau))d_c\tau,\int_0^\delta r_1(s,\tau,\mathcal{J}(\tau))d_c\tau\right) \\ &\quad - g\left(s,\ell(s),\int_0^s k(s,\tau,\ell(\tau))d_c\tau,\int_0^\delta r_1(s,\tau,\ell(\tau))d_c\tau\right)ds \\ &\quad + \left(\frac{1+t}{\delta}\right)\int_0^\delta \frac{(\delta-s)^{\tilde\alpha-2}}{\Gamma(\tilde\alpha-1)} \\ &\quad \cdot \left\|g\left(s,\mathcal{J}(s),\int_0^s k(s,\tau,\mathcal{J}(\tau))d_c\tau,\int_0^\delta r_1(s,\tau,\mathcal{J}(\tau))d_c\tau\right)\right. \\ &\quad \left.- g\left(s,\ell(s),\int_0^s k(s,\tau,\ell(\tau))d_c\tau,\int_0^\delta r_1(s,\tau,\ell(\tau))d_c\tau\right)\right\|ds \\ &\quad + \int_0^t \frac{(t-s)^{\tilde\alpha-1}}{\Gamma(\tilde\alpha)} \\ &\quad \cdot \left\|g\left(s,\mathcal{J}(s),\int_0^s k(s,\tau,\mathcal{J}(\tau))d_c\tau,\int_0^\delta r_1(s,\tau,\mathcal{J}(\tau))d_c\tau\right)\right. \\ &\quad \left.- g\left(s,\ell(s),\int_0^s k(s,\tau,\ell(\tau))d_c\tau,\int_0^\delta r_1(s,\tau,\ell(\tau))d_c\tau\right)\right\|ds.\end{aligned} \tag{114}$$

By (C2)-(C5), we have

$$\begin{aligned}|\mathcal{J}-\ell|^2 &\le \beta|\mathcal{J}-\ell|^2 + a_1\left(\frac{1+t}{\delta}\right)\int_0^\delta \|\mathcal{J}-\ell\|ds \\ &\quad + a_2\left(1-\frac{(1+t)}{\delta}\right)\int_0^\delta \|\mathcal{J}-\ell\|ds + \left(\frac{1+t}{\delta}\right) \\ &\quad \cdot \int_0^\delta \frac{(\delta-s)^{\tilde\alpha-2}}{\Gamma(\tilde\alpha-1)} K(S)L\left(\|\mathcal{J}(s)-\ell(s)\|\right. \\ &\quad + \left\|\int_0^s (k(s,\tau,\mathcal{J}(\tau))-k(s,\tau,\ell(\tau)))d_c\tau\right\| \\ &\quad \left.+ \left\|\int_0^\delta (r_1(s,\tau,\mathcal{J}(\tau))-r_1(s,\tau,\ell(\tau)))d_c\tau\right\|\right) \\ &\quad + \left(\frac{1+t}{\delta}\right)\int_0^\delta \frac{(\delta-s)^{\tilde\alpha-1}}{\Gamma(\tilde\alpha)} K(s) \\ &\quad \times L\left(\|\mathcal{J}(s)-\ell(s)\|\right. \\ &\quad + \left\|\int_0^s (k(s,\tau,\mathcal{J}(\tau))-k(s,\tau,\ell(\tau)))d_c\tau\right\| \\ &\quad \left.+ \left\|\int_0^\delta (r_1(s,\tau,\mathcal{J}(\tau))-r_1(s,\tau,\ell(\tau)))d_c\tau\right\|\right) \\ &\quad + \int_0^t \frac{(t-s)^{\tilde\alpha-1}}{\Gamma(\tilde\alpha)} K(s)\times L\left(\|\mathcal{J}(s)-\ell(s)\|\right. \\ &\quad + \|(k(s,\tau,\mathcal{J}(\tau))-k(s,\tau,\ell(\tau)))d_c\tau\| \\ &\quad \left.+ \left\|\int_0^\delta (r_1(s,\tau,\mathcal{J}(\tau))-r_1(s,\tau,\ell(\tau)))d_c\tau\right\|\right) \\ &\le \beta|\mathcal{J}-\ell|^2 + a_1\left(\frac{1+t}{\delta}\right)\int_0^\delta \|\mathcal{J}-\ell\|ds\end{aligned}$$

$$
\begin{aligned}
&+ a_2\left(1-\frac{(1+t)}{\delta}\right)\int_0^\delta \|\mathcal{J}-\ell\| ds \\
&+ \left(\frac{1+t}{\delta}\right)\int_0^\delta \frac{(\delta-s)^{\tilde{\alpha}-2}}{\Gamma(\tilde{\alpha}-1)} \\
&\times K(s)L(\|\mathcal{J}-\ell\| + \Theta_1(s)\|\mathcal{J}-\ell\| + \Theta_2\|\mathcal{J}-\ell\|)ds \\
&+ \left(\frac{1+t}{\delta}\right)\int_0^\delta \frac{(\delta-s)^{\tilde{\alpha}-1}}{\Gamma(\tilde{\alpha})}K(s) \\
&\times L(\|\mathcal{J}-\ell\| + \Theta_1(s)\|\mathcal{J}-\ell\| + \Theta_2(s)\|\mathcal{J}-\ell\|) \\
&+ \int_0^t \frac{(t-s)^{\tilde{\alpha}-1}}{\Gamma(\tilde{\alpha})} \\
&\cdot K(s)L(\|\mathcal{J}-\ell\| + \Theta_1(s)\|\mathcal{J}-\ell\| + \Theta_2(s)\|\mathcal{J}-\ell\|) \\
\leq &\beta|\mathcal{J}-\ell|^2 + a_1\left(\frac{1+t}{\delta}\right)\int_0^\delta \|\mathcal{J}-\ell\| ds \\
&+ a_2\left(1-\frac{(1+t)}{\delta}\right)\int_0^\delta \|\mathcal{J}-\ell\| ds \\
&+ \left(\frac{1+t}{\delta}\right)\int_0^\delta \frac{(\delta-s)^{\tilde{\alpha}-2}}{\Gamma(\tilde{\alpha}-1)} \\
&\cdot K(s)(1+\Theta_1(s)+\Theta_2(s))L\|\mathcal{J}-\ell\| ds \\
&+ \left(\frac{1+t}{\delta}\right)\int_0^\delta \frac{(\delta-s)^{\tilde{\alpha}-1}}{\Gamma(\tilde{\alpha})} \\
&\times K(s)(1+\Theta_1(s)+\Theta_2(s))L\|\mathcal{J}-\ell\| ds \\
&+ \int_0^t \frac{(t-s)^{\tilde{\alpha}-1}}{\Gamma(\tilde{\alpha})}K(s)(1+\Theta_1(s)+\Theta_2(s))L\|\mathcal{J}-\ell\| ds.
\end{aligned} \tag{115}
$$

Let $\Delta(t) = K(t)(1+\Theta_1(t)+\Theta_2(t))$ and $\Delta^* = \sup\{\Delta(t): t \in [0,\delta]\}$. Then,

$$
\begin{aligned}
|\mathcal{J}-\ell|^2 \leq &\beta|\mathcal{J}-\ell|^2 + a_1\left(\frac{1+t}{\delta}\right)\int_0^\delta \|\mathcal{J}-\ell\| ds \\
&+ a_2\left(1-\frac{(1+t)}{\delta}\right)\int_0^\delta \|\mathcal{J}-\ell\| ds \\
&+ L\Delta^*\left(\frac{1+t}{\delta}\right)\int_0^\delta \frac{(\delta-s)^{\tilde{\alpha}-2}}{\Gamma(\tilde{\alpha}-1)}\|\mathcal{J}-\ell\| ds \\
&+ \frac{L\Delta^*(1+t)}{\delta}\int_0^\delta \frac{(\delta-s)^{\tilde{\alpha}-1}}{\Gamma(\tilde{\alpha})}\|\mathcal{J}-\ell\| ds \\
&+ L\Delta\int_0^t \frac{(t-s)^{\tilde{\alpha}-1}}{\Gamma(\tilde{\alpha})}\|\mathcal{J}-\ell\| ds \\
\leq &\beta|\mathcal{J}-\ell|^2 + \left(a_1(1+\delta) + a_2(\delta-1) + \frac{L\Delta^* \mathrm{y}}{\Gamma(\tilde{\alpha}+1)\delta^{2-\tilde{\alpha}}}\right) \\
&\cdot \|\mathcal{J}-\ell\|,
\end{aligned} \tag{116}
$$

where $\mathrm{y} = 2\delta^2 + \delta + \tilde{\alpha}(1+\delta)$.

Choose a_1, a_2: $a_1(1+\delta) + a_2(\delta-1) + (L\Delta^* \mathrm{y}/\Gamma(\tilde{\alpha}+1)\delta^{2-\tilde{\alpha}}) = \eta < 1$. Then by (C6), we have

$$
\begin{aligned}
|\mathcal{J}-\ell|^2 \leq &\beta|\mathcal{J}-\ell|^2 + \eta\|\mathcal{J}-\ell\| \\
\leq &\beta|\mathcal{J}-\ell|^2 + \frac{\mu|\mathcal{J}-Mu|^2|\ell-Mv|^2e^{i\theta}}{1+|\mathcal{J}-\ell|e^{i\theta}} \\
&+ \frac{\lambda|\ell-Mu|^2|\mathcal{J}-Mv|^2e^{i\theta}}{1+|\mathcal{J}-\ell|^2e^{i\theta}} \\
\leq &\beta|\mathcal{J}-\ell|^2 \\
&+ \frac{\mu|\mathcal{J}-Mu|^2|\ell-Mv|^2e^{i\theta} + \lambda|\ell-Mu|^2|\mathcal{J}-Mv|^2e^{i\theta}}{1+|\mathcal{J}-\ell|e^{i\theta}}.
\end{aligned} \tag{117}
$$

Let $P, Q : \Gamma \longrightarrow (0,1]$ be any two arbitrary mappings and $M : \Gamma \longrightarrow \mathbb{CB}(\Gamma)$ be the given multivalued mapping. Consider an intuitionistic fuzzy mapping $G : \Gamma \longrightarrow (IFS)^\Gamma$ as follows:

$$
\begin{aligned}
\mu_{G(\mathcal{J})}(\omega) &= \begin{cases} P(\mathcal{J}), & \omega \in Mu, \\ 0, & \text{otherwise}, \end{cases} \\
\nu_{G(\mathcal{J})}(\omega) &= \begin{cases} 0, & \omega \in Mu, \\ Q(\mathcal{J}), & \text{otherwise}. \end{cases}
\end{aligned} \tag{118}
$$

By letting $\tilde{\alpha}_{\mathbb{F}(\mathcal{J})} = P(j)$ and $\nu_{\mathbb{F}(\mathcal{J})} = 0$, we obtain

$$
\begin{aligned}
&[\mathbb{F}(\mathcal{J})]_{(\mathcal{T},\mathcal{N},\tilde{\alpha}_{\mathbb{F}(\mathcal{J})})} \\
&= \left\{\omega \in \Gamma : \mathcal{T}\left(\mu_{\mathbb{F}(\mathcal{J})}(\omega), \mathcal{N}\left(\nu_{\mathbb{F}(\mathcal{J})}(\omega)\right)\right) = P(\mathcal{J})\right\} \\
&= Mu.
\end{aligned} \tag{119}
$$

Therefore, the inequality (117) can be written as

$$
\begin{aligned}
|\mathcal{J}-\ell|^2 \leq &\beta|\mathcal{J}-\ell|^2 \\
&+ \frac{\mu\left|j-[\mathbb{F}(\mathcal{J})]_{(\mathcal{T},\mathcal{N},\tilde{\alpha}_{\mathbb{F}(\mathcal{J})})}\right|^2\left|\ell-[\mathbb{F}(\ell)]_{(\mathcal{T},\mathcal{N},\tilde{\alpha}_{\mathbb{F}(\ell)})}\right|^2e^{i\theta}}{1+|\mathcal{J}-\ell|^2e^{i\theta}} \\
&+ \frac{\lambda\left|\ell-[\mathbb{F}(\mathcal{J})]_{(\mathcal{T},\mathcal{N},\tilde{\alpha}_{G(\mathcal{J})})}\right|^2\left|\mathcal{J}-[\mathbb{F}(\ell)]_{(\mathcal{T},\mathcal{N},\tilde{\alpha}_{G(\ell)})}\right|^2e^{i\theta}}{1+|\mathcal{J}-\ell|^2e^{i\theta}}.
\end{aligned} \tag{120}
$$

Multiplying inequality (120) by $e^{i\theta}$, we get

$$
\begin{aligned}
d_c(\mathcal{J},\ell) \leq &\beta d_c(\mathcal{J},\ell) \\
&+ \frac{\mu d_c\left(\mathcal{J}, [\mathbb{F}(\mathcal{J})]_{(\mathcal{T},\mathcal{N},\tilde{\alpha}_{\mathbb{F}(\mathcal{J})})}\right)d_c\left(\ell, [\mathbb{F}(\ell)]_{(\mathcal{T},\mathcal{N},\tilde{\alpha}_{\mathbb{F}(\ell)})}\right)}{1+d_c(\mathcal{J},\ell)} \\
&+ \frac{\lambda d_c\left(\ell, [\mathbb{F}(\mathcal{J})]_{(\mathcal{T},\mathcal{N},\tilde{\alpha}_{\mathbb{F}(\mathcal{J})})}\right)d_c\left(\mathcal{J}, [\mathbb{F}(\ell)]_{(\mathcal{T},\mathcal{N},\tilde{\alpha}_{\mathbb{F}(\ell)})}\right)}{1+d_c(\mathcal{J},\ell)}.
\end{aligned} \tag{121}
$$

By definition, inequality (121) implies

$$\begin{aligned}
&\beta d_c(\mathcal{I},\ell)+\frac{\mu d_c\left(\mathcal{I},[\mathbb{F}(\mathcal{I})]_{(\mathcal{T},\mathcal{N},\tilde{\alpha}_{\mathbb{F}(\mathcal{I})})}\right)d_c\left(\ell,[\mathbb{F}(\ell)]_{(\mathcal{T},\mathcal{N},\tilde{\alpha}_{\mathbb{F}(\ell)})}\right)}{1+d_c(\mathcal{I},\ell)}\\
&\quad+\frac{\lambda d_c\left(\ell,[\mathbb{F}(\mathcal{I})]_{(\mathcal{T},\mathcal{N},\tilde{\alpha}_{\mathbb{F}(\mathcal{I})})}\right)d_c\left(\mathcal{I},[\mathbb{F}(\ell)]_{(\mathcal{T},\mathcal{N},\tilde{\alpha}_{\mathbb{F}(\ell)})}\right)}{1+d_c(\mathcal{I},\ell)}\\
&=B^*(\mathcal{I},\ell,\mathbb{F},\tilde{\alpha})\in s\left([\mathbb{F}(\mathcal{I})]_{(\mathcal{T},\mathcal{N},\tilde{\alpha}_{\mathbb{F}(\mathcal{I})})},[\mathbb{F}(\ell)]_{(\mathcal{T},\mathcal{N},\tilde{\alpha}_{\mathbb{F}(\ell)})}\right).
\end{aligned}\tag{122}$$

Hence, the conclusion of Theorem 34 follows the application of Corollary 24. □

Example 3. Consider the following fractional integrodifferential equation:

$$\begin{aligned}
\ell^{\left(\frac{9}{5}\right)}(t)&=\frac{1}{12}+\frac{1}{12+|\ell(t)|}+\int_0^t e^{-\left(\frac{1}{12}\right)\ell(s)}ds\\
&\quad+\int_0^1\frac{|\ell(s)|e^{-s}}{12+|\ell(s)|^2}ds,\ t\in[0,1],
\end{aligned}\tag{123}$$

with integral boundary conditions

$$\begin{aligned}
\ell(0)-\ell'(0)&=\int_0^1\frac{1}{12+|\ell(s)|}ds,\ \ell(1)-\ell'(1)\\
&=\int_0^1\frac{1}{12+e^{-|\ell(s)|}}ds.
\end{aligned}\tag{124}$$

Here,

$$\begin{aligned}
\|r(\mathcal{I})-r(\ell)\|&=\left\|\frac{1}{12+e^{-|\mathcal{I}(t)|}}-\frac{1}{12+e^{-|\ell(t)|}}\right\|\\
&=\left\|\frac{e^{-|\ell(t)|}-e^{-|\mathcal{I}(t)|}}{\left(12+e^{-|\mathcal{I}(t)|}\right)\left(12+e^{-|\ell(t)|}\right)}\right\|\\
&\le\frac{1}{144}\|\mathcal{I}-\ell\|,\\
\|f(\mathcal{I})-f(\ell)\|&=\left\|\frac{1}{12+|\mathcal{I}(t)|}-\frac{1}{12+|\ell(t)|}\right\|\\
&=\left\|\frac{|\ell(t)|-|\mathcal{I}(t)|}{(12+|\mathcal{I}(t)|)(12+|\ell(t)|)}\right\|\\
&\le\frac{1}{144}\|\mathcal{I}-\ell\|,\\
\left\|\int_0^t(k(t,s,\mathcal{I})-k(t,s,\ell))ds\right\|&=\left\|\int_0^t\left(e^{-\left(\frac{1}{12}\right)\mathcal{I}(s)}-e^{-\left(\frac{1}{12}\ell(s)\right)}\right)ds\right\|\\
&\le\frac{1}{12}\|\mathcal{I}-\ell\|,\\
\left\|\int_0^1(r_1(t,s,\mathcal{I})-r_1(t,s,\ell))ds\right\|&=\left\|\int_0^1\left(\frac{|\mathcal{I}(s)|e^{-s}}{12+|\ell(s)|^2}-\frac{|\ell(s)|e^{s}}{12+|\ell(s)|^2}\right)\right\|\\
&\le\frac{1}{144}\|\mathcal{I}-\ell\|.
\end{aligned}\tag{125}$$

However,

$$\begin{aligned}
&\|g(t,\mathcal{I}_1,\ell_1,z_1)-g(t\mathcal{I}_2,\ell_2,z_2)\|\\
&\quad\le\frac{1}{12+t}(\|\mathcal{I}_1-\mathcal{I}_2\|+\|\ell_1-\ell_2\|+\|z_1-z_2\|).
\end{aligned}\tag{126}$$

Notice that by Lemma 33, Condition (C2) of Theorem 34 can be verified by direct calculation.

Let $\Gamma=[0,1]$ and $d_c:\Gamma\times\Gamma\longrightarrow\mathbb{C}$ be given by

$$d_c(\mathcal{I},\ell)=|\mathcal{I}-\ell|^2e^{i\theta},\theta=\tan^{-1|\ell/\mathcal{I}|}.\tag{127}$$

Then, (Γ,d_c) is a complete complex-valued b-MS. Suppose that $M:\Gamma\longrightarrow\mathbb{CB}(\Gamma)$ is given by

$$Mu=\left[0,\frac{\mathcal{I}}{7}\right].\tag{128}$$

Assume without loss of generality that for $\mathcal{I},\ell\in\Gamma$, $\mathcal{I}\ne\ell$, and $\mathcal{I}<\ell$. Thus, we have

$$\begin{aligned}
d_c(\mathcal{I},Mu)&=\left|\mathcal{I}-\frac{\mathcal{I}}{7}\right|^2e^{i\theta},d_c(\ell,Mv)=\left|\ell-\frac{\ell}{7}\right|^2e^{i\theta},\\
d_c(\ell,Mu)&=\left|\ell-\frac{\mathcal{I}}{7}\right|^2e^{i\theta},d_c(\mathcal{I},Mv)=\left|\mathcal{I}-\frac{\ell}{7}\right|^2e^{i\theta}.
\end{aligned}\tag{129}$$

Clearly, for any value of μ and λ and $\eta=1/7$, we have

$$\begin{aligned}
\left|\frac{\mathcal{I}}{7}-\frac{\ell}{7}\right|&\le\frac{1}{7}|\mathcal{I}-\ell|+\frac{\mu|\mathcal{I}-(\mathcal{I}/7)|^2|\ell-(\ell/7)|^2e^{i\theta}}{1+|\mathcal{I}-\ell|^2e^{i\theta}}\\
&\quad+\frac{\lambda|\ell-(\mathcal{I}/7)|^2e^{i\theta}}{1+|\mathcal{I}-\ell|^2e^{i\theta}}.
\end{aligned}\tag{130}$$

Hence, all the conditions of Theorem 34 are obeyed. We conclude that the fractional mixed Volterra-Fredholm integrodifferential equation has a solution in Γ.

5. Conclusion

A number of practical and theoretic problems in economics, engineering, management sciences, and medical science and a substantial number of other fields involve vagueness and the complexity of modeling data possessing nonstatistical uncertainties. Prototypal mathematical techniques are not usually successful because the imprecisions in these domains may be of various kinds. Several innovative models such as fuzzy set theory, rough set theory, intuitionistic fuzzy set theory, and other related mathematical tools have been established to manipulate data with incomplete information. On the other hand, fractional differential equations (FDEs) have instigated, in recent years, considerable interests due to their enormous applications in both physical sciences, chemical process in engineering, and social sciences. Different methods for solving FDEs and investigating the existence of their solutions were proposed by many authors; some of these techniques are series methods, the iteration method, the operational calculus, and so on. In this direction, one may see the textbooks [25, 26, 28] for both introductory and detail analysis.

Motivated by the above developments, in this article, we have used the ideas of complex-valued b-MS to establish sufficient conditions for the existence of common FPs of a pair of IFMs obeying some contractive conditions involving rational inequalities. In Theorem 22, we presented a Banach type results involving rational expression, while the Banach type locally contractive condition is discussed in Corollary 24. On the other hand, Theorem 26 is a slight improvement of Definition 19. Moreover, in continuation of the role of the classical Banach contraction theorem in the study of existence of solutions of nonlinear integrodifferential equations, we included the application section. In this regard, Corollary 24 is applied to provide some conditions for the existence of a solution of mixed Volterra-Fredholm integrodifferential equation with integral boundary conditions. In each case of the above-mentioned key results, examples are supplied to support or authenticate their usability. While the current paper is theoretical, at least a note justifying its study is in order. To start with, in several MS, e.g., the cone MS, FP results involving rational expressions cannot be refined, whereas, in complex-valued MS, this is feasible. Thus, by incorporating the notion of complex-valued IFMs into the existing concepts, our remarkable results will be beneficial and significant in the modeling and solution of optimization problems in mathematical analysis.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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