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Research Article

Approximation Properties and q-Statistical Convergence of Stancu-Type Generalized Baskakov-Szász Operators

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In this article, we introduce Stancu-type modification of generalized Baskakov-Szász operators. We obtain recurrence relations to calculate moments for these new operators. We study several approximation properties and q-statistical approximation for these operators.

1. Introduction

In 1912, Bernstein [1] proposed the famous polynomial known as the Bernstein polynomial to give a simple, short, and most elegant proof of the Weierstrass approximation theorem. Since then, several papers have appeared to study approximation properties in different settings and spaces. Many new operators were constructed, e.g., Szász [2], Mirakjan [3], Kantorovic [4], Durrmeyer [5], Stancu [6], and many more [7–9]. These operators provide the improvement of approximating functions of different classes and give better and better estimates. For example, the Baskakov operators were given in [10]:

$$V_p(\mathfrak{h};\mathfrak{v}) == \sum_{l=0}^{\infty}\binom{p+l-1}{l}\frac{\mathfrak{v}^l}{(1+\mathfrak{v})^{p+l}}\mathfrak{h}\left(\frac{l}{p}\right). \quad (1)$$

For $\mathfrak{h}\in C[0,\infty)$, the space of all continuous functions on $[0,\infty)$ normed with standard sup-norm $\|\cdot\|_{\infty}$.

Devore and Lorentz [11] introduced a generalization of operators (1) dependent on a constant $a>0$ and independent of p as follows:

$$B_p(\mathfrak{h};u) = \sum_{j=0}^{\infty} W_{p,j}^a(u)\mathfrak{h}\left(\frac{j}{p}\right), \quad (2)$$

where

$$W_{p,j}^a(u) = e^{-au/(1+u)}\frac{G_j(p,a)}{j!}\frac{u^j}{(1+u)^{p+j}}, \sum_{j=0}^{\infty} W_{p,j}^a(u) = 1,$$
$$G_j(p,a) = \sum_{i=0}^{j}\binom{j}{i}(p)_i a^{j-i}, \quad (3)$$

and $(p)_i = p(p+1)\cdots(p+i-1), (p)_0 = 1$.

Recently, Agrawal et al. [12] studied the following operators (2):

$$L_{p,a}^*(\mathfrak{h};u) = p\sum_{j=0}^{\infty} W_{p,j}^a(u)\int_0^{\infty} s_{p,j}(t)\mathfrak{h}(t)dt, \quad (4)$$

for $\mathfrak{h}\in C_\gamma[0,\infty) := \{\mathfrak{h}\in C[0,\infty): |\mathfrak{h}(t)|\le M_{\mathfrak{h}}e^{\gamma t}$, for some $\gamma>0, M_{\mathfrak{h}}>0\}$, where $s_{p,j}(t) = e^{-pt}((pt)^j/j!)$.

Inspired by Stancu's work [6], we have studied recently the Stancu-type generalization in [13]. Now, we propose the Stancu-type generalization of operators (4) as follows:

$$\mathfrak{L}_{p,a}^{(\lambda,\mu)}(\mathfrak{h};\mathfrak{v}) = p\sum_{k=0}^{\infty} W_{p,k}^{a}(\mathfrak{v})\int_0^{\infty} s_{p,k}(t)\mathfrak{h}\left(\frac{pt+\lambda}{p+\mu}\right)dt, \quad (5)$$

for any bounded and integrable function $\mathfrak{h}$ defined on $[0,\infty)$, where $0 \le \lambda \le \mu$. For $\lambda=\mu=0$, the operators (5) reduce to operators (4).

We establish recurrence relations to find moments and central moments. We study some approximation properties and the Voronovskaja-type asymptotic formula. We also study weighted approximation.

2. Auxiliary Results

Our first result is the recurrence formula for moments.

Theorem 1. *The m^{th} order moment for (5) is defined by*

$$\mathfrak{T}_{p,a,m}^{(\lambda,\mu)}(\mathfrak{v}) \coloneqq \mathfrak{L}_{p,a}^{(\lambda,\mu)}(t^m;\mathfrak{v}) = p\sum_{k=0}^{\infty} W_{p,k}^{a}(\mathfrak{v})\int_0^{\infty} s_{p,k}(t)\mathfrak{h} \cdot \left(\frac{pt+\lambda}{p+\mu}\right)^m dt. \quad (6)$$

$m=0,1,2,\cdots$. Then, $\mathfrak{T}_{p,a,0}^{(\lambda,\mu)}(\mathfrak{v})=1$, and

$$\begin{aligned}(m+1)(1+\mathfrak{v})\mathfrak{T}_{p,a,m+1}^{(\lambda,\mu)}(\mathfrak{v}) &= \mathfrak{v}(1+\mathfrak{v})^2\left[\mathfrak{T}_{p,a,m}^{(\lambda,\mu)}(\mathfrak{v})\right]' + \{(1+\mathfrak{v})(\lambda+p\mathfrak{v}+m\\ &\quad +1)+a\mathfrak{v}\}\mathfrak{T}_{p,a,m}^{(\lambda,\mu)}(\mathfrak{v}) - \frac{\lambda m}{p+\mu}(1+\mathfrak{v})\mathfrak{T}_{p,a,m-1}^{(\lambda,\mu)}(\mathfrak{v}).\end{aligned} \quad (7)$$

Proof. We use the identity

$$\mathfrak{v}(1+\mathfrak{v})^2\left(W_{p,k}^{a}(\mathfrak{v})\right)' = [(k-p\mathfrak{v})(1+\mathfrak{v})-a\mathfrak{v}]W_{p,k}^{a}(\mathfrak{v}). \quad (8)$$

Then,

$$\begin{aligned}\mathfrak{v}(1+\mathfrak{v})^2\left[\mathfrak{T}_{p,a,m}^{(\lambda,\mu)}(\mathfrak{v})\right]' &= p\sum_{k=0}^{\infty}\mathfrak{v}(1+\mathfrak{v})^2\left(W_{p,k}^{a}(\mathfrak{v})\right)'\int_0^{\infty}s_{p,k}(t)\mathfrak{h}\left(\frac{pt+\lambda}{p+\mu}\right)^m dt\\ &= p\sum_{k=0}^{\infty}[(k-p\mathfrak{v})(1+\mathfrak{v})-a\mathfrak{v}]W_{p,k}^{a}(\mathfrak{v})\int_0^{\infty}s_{p,k}(t)\left(\frac{pt+\lambda}{p+\mu}\right)^m dt\\ &= p(1+\mathfrak{v})\sum_{k=0}^{\infty}(k-p\mathfrak{v})W_{p,k}^{a}(\mathfrak{v})\int_0^{\infty}s_{p,k}(t)\left(\frac{pt+\lambda}{p+\mu}\right)^m dt\\ &\quad - a\mathfrak{v}.p\sum_{k=0}^{\infty}W_{p,k}^{a}(\mathfrak{v})\int_0^{\infty}s_{p,k}(t)\left(\frac{pt+\lambda}{p+\mu}\right)^m dt,\end{aligned} \quad (9)$$

$$\mathfrak{v}(1+\mathfrak{v})^2\left[\mathfrak{T}_{p,a,m}^{(\lambda,\mu)}(\mathfrak{v})\right]' = I - ax\mathfrak{T}_{p,a,m}^{(\lambda,\mu)}(\mathfrak{v}), \quad (10)$$

where

$$\begin{aligned}I &= p(1+\mathfrak{v})\sum_{k=0}^{\infty}(k-p\mathfrak{v})W_{p,k}^{a}(\mathfrak{v})\int_0^{\infty}s_{p,k}(t)\left(\frac{pt+\lambda}{p+\mu}\right)^m dt\\ &= p(1+\mathfrak{v})\sum_{k=0}^{\infty}[k-(pt+\lambda)+(pt+\lambda)-p\mathfrak{v}] \cdot W_{p,k}^{a}(\mathfrak{v})\int_0^{\infty}s_{p,k}(t)\left(\frac{pt+\lambda}{p+\mu}\right)^m dt\\ &= p(1+\mathfrak{v})\sum_{k=0}^{\infty}[(k-pt)+(pt+\lambda)-(\lambda+p\mathfrak{v})] \cdot W_{p,k}^{a}(\mathfrak{v})\int_0^{\infty}s_{p,k}(t)\left(\frac{pt+\lambda}{p+\mu}\right)^m dt\\ &= p(1+\mathfrak{v})\sum_{k=0}^{\infty}W_{p,k}^{a}(\mathfrak{v})\int_0^{\infty}(k-pt)s_{p,k}(t)\left(\frac{pt+\lambda}{p+\mu}\right)^m dt\\ &\quad + p(1+\mathfrak{v})(p+\mu)\sum_{k=0}^{\infty}W_{p,k}^{a}(\mathfrak{v})\int_0^{\infty}(k-pt)s_{p,k}(t)\left(\frac{pt+\lambda}{p+\mu}\right)^{m+1}\\ &\quad \cdot dt - p(1+\mathfrak{v})(\lambda+p\mathfrak{v})\sum_{k=0}^{\infty}W_{p,k}^{a}(\mathfrak{v})\int_0^{\infty}s_{p,k}(t)\left(\frac{pt+\lambda}{p+\mu}\right)^m dt,\end{aligned} \quad (11)$$

$$I = \sum_1 + \sum_2 + \sum_3, \text{ say} \quad (12)$$

where

$$\sum_2 = (1+\mathfrak{v})(p+\mu)\mathfrak{T}_{p,a,m+1}^{(\lambda,\mu)}(\mathfrak{v}), \quad (13)$$

$$\sum_3 = -(1+\mathfrak{v})(\lambda+p\mathfrak{v})\mathfrak{T}_{p,a,m}^{(\lambda,\mu)}(\mathfrak{v}), \quad (14)$$

$$\begin{aligned}\sum_1 &= p(1+\mathfrak{v})\sum_{k=0}^{\infty}W_{p,k}^{a}(\mathfrak{v})\int_0^{\infty}(k-pt)s_{n,k}(t)\left(\frac{pt+\lambda}{p+\mu}\right)^m dt\\ &= (1+\mathfrak{v})\sum_{k=0}^{\infty}W_{p,k}^{a}(\mathfrak{v})\int_0^{\infty}pt\left(s_{p,k}(t)\right)' \cdot \left(\frac{pt+\lambda}{p+\mu}\right)^m dt, \left(\text{using} t\left(s_{p,k}(t)\right)' = \int_0^{\infty}(k-pt)s_{p,k}(t)\right),\\ &= (p+\mu)(1+\mathfrak{v})\sum_{k=0}^{\infty}W_{p,k}^{a}(\mathfrak{v})\int_0^{\infty}\left(\frac{pt+\lambda}{p+\mu}-\frac{\lambda}{p+\mu}\right) \cdot \left(s_{p,k}(t)\right)'\left(\frac{pt+\lambda}{p+\mu}\right)^m dt = (p+\mu)(1+\mathfrak{v})\\ &\quad \cdot \sum_{k=0}^{\infty}W_{p,k}^{a}(\mathfrak{v})\int_0^{\infty}\left(s_{p,k}(t)\right)'\left(\frac{pt+\lambda}{p+\mu}\right)^{m+1} dt\\ &\quad - \lambda(1+\mathfrak{v})\sum_{k=0}^{\infty}W_{p,k}^{a}(\mathfrak{v})\int_0^{\infty}\left(s_{p,k}(t)\right)'\left(\frac{pt+\lambda}{p+\mu}\right)^m dt\\ &= \mathcal{J}_1 + \mathcal{J}_2, \text{ say}\end{aligned} \quad (15)$$

where

$$\begin{aligned}\mathcal{J}_1 &= (p+\mu)(1+\mathfrak{v})W_{p,k}^a(\mathfrak{v})\int_0^\infty (s_{p,k}(t))'\left(\frac{pt+\lambda}{p+\mu}\right)^{m+1}dt\\ &= -(1+\mathfrak{v})(m+1)p\sum_{k=0}^\infty W_{p,k}^a(\mathfrak{v})\int_0^\infty s_{p,k}(t)\left(\frac{pt+\lambda}{p+\mu}\right)^{m}dt\\ &\quad-(1+\mathfrak{v})(m+1)\mathfrak{T}_{p,a,m}^{(\lambda,\mu)}(\mathfrak{v}),\\ \mathcal{J}_2 &= -\lambda(1+\mathfrak{v})\sum_{k=0}^\infty W_{p,k}^a(\mathfrak{v})\int_0^\infty (s_{p,k}(t))'\left(\frac{pt+\lambda}{p+\mu}\right)^{m}dt\\ &= -\lambda(1+\mathfrak{v})\left(\frac{-mn}{p+\mu}\right)\sum_{k=0}^\infty W_{p,k}^a(\mathfrak{v})\int_0^\infty s_{p,k}(t)\left(\frac{pt+\lambda}{p+\mu}\right)^{m-1}dt\\ &= \left(\frac{-m\lambda}{p+\mu}\right)(1+\mathfrak{v})\mathfrak{T}_{p,a,m-1}^{(\lambda,\mu)}(\mathfrak{v}).\end{aligned} \tag{16}$$

Therefore,

$$\sum_1 = \left(\frac{-m\lambda}{p+\mu}\right)(1+\mathfrak{v})\mathfrak{T}_{p,a,m-1}^{(\lambda,\mu)}(\mathfrak{v})-(1+\mathfrak{v})(m+1)\mathfrak{T}_{p,a,m}^{(\lambda,\mu)}(\mathfrak{v}). \tag{17}$$

Substituting (17), (14), and (13) in (12), we get

$$\begin{aligned}I &= \frac{-m\lambda}{p+\mu}(1+\mathfrak{v})\mathfrak{T}_{p,a,m-1}^{(\lambda,\mu)}(\mathfrak{v})-(1+\mathfrak{v})(p\mathfrak{v}+\lambda\\ &\quad+m+1)\mathfrak{T}_{p,a,m}^{(\lambda,\mu)}(\mathfrak{v})+(1+\mathfrak{v})(p+\mu)\mathfrak{T}_{p,a,m+1}^{(\lambda,\mu)}(\mathfrak{v}).\end{aligned} \tag{18}$$

Further, substituting (18) in (10), we get the result. □

Corollary 2. *From the above theorem, we get*

(i) $\mathfrak{L}_{p,a}^{(\lambda,\mu)}(1;\mathfrak{v})=1$

(ii) $\mathfrak{L}_{p,a}^{(\lambda,\mu)}(t;\mathfrak{v})=1/(p+\mu)(p\mathfrak{v}+(a\mathfrak{v}/(1+\mathfrak{v}))+1+\lambda)$

(iii) $\mathfrak{L}_{p,a}^{(\lambda,\mu)}(t^2;\mathfrak{v}) = 1/(p+\mu)^2[\{p^2+p+(a^2/(1+\mathfrak{v})^2)+(2ap/(1+\mathfrak{v}))\}\mathfrak{v}^2+\{4p+(4a/(1+\mathfrak{v}))+2\lambda p+(2a\lambda/(1+\mathfrak{v}))\}\mathfrak{v}+\{\lambda^2+2\lambda+2\}]$

Theorem 3. *The m^{th} order central moment is defined by*

$$\begin{aligned}\mathfrak{M}_{p,a,m}^{(\lambda,\mu)}(\mathfrak{v}) &:= \mathfrak{L}_{p,a}^{(\lambda,\mu)}((t-\mathfrak{v})^m;\mathfrak{v})\\ &= p\sum_{k=0}^\infty W_{p,k}^a(\mathfrak{v})\int_0^\infty s_{p,k}(t)\mathfrak{h}\left(\frac{pt+\lambda}{p+\mu}-u\right)^m dt.\end{aligned} \tag{19}$$

$m=0,1,2,\cdots$. The following recurrence relation holds:

$$\begin{aligned}&\mathfrak{v}(1+\mathfrak{v})^2\left(\mathfrak{M}_{p,a,m+1}^{(\lambda,\mu)}(\mathfrak{v})\right)'\\ &\quad=\left(\frac{1+\mathfrak{v}}{p+\mu}\right)(\lambda-p\mathfrak{v}-\mu\mathfrak{v}-mp\mathfrak{v}-m\mu\mathfrak{v}-mp\mathfrak{v}^2\\ &\qquad-m\mu\mathfrak{v}^2)\mathfrak{M}_{p,a,m-1}^{(\lambda,\mu)}(\mathfrak{v})-(m+m\mathfrak{v}+1+\mathfrak{v}\\ &\qquad-\mu\mathfrak{v}+\lambda-\mu\mathfrak{v}^2+\lambda\mathfrak{v}+a\mathfrak{v})\mathfrak{M}_{p,a,m}^{(\lambda,\mu)}(\mathfrak{v})\\ &\qquad+(1+\mathfrak{v})(p+\mu)\mathfrak{M}_{p,a,m+1}^{(\lambda,\mu)}(\mathfrak{v}).\end{aligned} \tag{20}$$

Corollary 4. *From the above theorem, we get*

(a) $\mathfrak{L}_{p,a}^{(\lambda,\mu)}((t-\mathfrak{v});\mathfrak{v})=1/(p+\mu)(-\mu\mathfrak{v}+(a\mathfrak{v}/(1+\mathfrak{v}))+\lambda+1)$

(b) $\mathfrak{L}_{p,a}^{(\lambda,\mu)}((t-\mathfrak{v})^2;\mathfrak{v})=1/(p+\mu)^2(p+\mu^2+(a^2/(1+\mathfrak{v})^2)-(2a\mu/(1+\mathfrak{v})))\mathfrak{v}^2+(2/(p+\mu)^2)(p-\mu-\lambda\mu+((2+\lambda)a/(1+\mathfrak{v})))\mathfrak{v}+(1/(p+\mu)^2)(\lambda^2+2\lambda+2)$

Corollary 5. *We further get*

(a) $\lim_{p\longrightarrow\infty}p\mathfrak{L}_{p,a}^{(\lambda,\mu)}((t-\mathfrak{v});\mathfrak{v})=-\mu u+(a\mathfrak{v}/1+\mathfrak{v})+\lambda+1$

(b) $\lim_{p\longrightarrow\infty}p\mathfrak{L}_{p,a}^{(\lambda,\mu)}((t-\mathfrak{v})^2;\mathfrak{v})=\mathfrak{v}(\mathfrak{v}+2)$

3. Main Results

Peetre's K-functional is defined as

$$K_2(\mathfrak{h},\delta) := \inf\left\{\|\mathfrak{h}-\mathfrak{g}\|+\delta\|\mathfrak{g}''\| : \mathfrak{g}\in C_B^2[0,\infty)\right\}, \tag{21}$$

for $\mathfrak{h}\in C_B[0,\infty)$, $\delta>0$, where $C_B[0,\infty):=\{\mathfrak{h}\in C_B[0,\infty)$: $\mathfrak{h}$ is bounded on $[0,\infty)\}$ and $C_B^2[0,\infty):=\{\mathfrak{g}\in C_B[0,\infty)$: $\mathfrak{g}',\mathfrak{g}''\in C_B[0,\infty)\}$. Note that

$$K_2(\mathfrak{h};\delta)\le M\omega_2\left(\mathfrak{h};\sqrt{\delta}\right), M>0, \tag{22}$$

where $\omega_2(\mathfrak{h};\delta)$ is the second-order modulus of continuity [11].

$$\omega_2(\mathfrak{h},\delta)=\sup_{0<l\le\delta}\sup_{\mathfrak{v}\in 0,\infty)}|\mathfrak{h}(\mathfrak{v}+2l)-2\mathfrak{h}(\mathfrak{v}+l)+\mathfrak{h}(\mathfrak{v})|, \quad \delta>0. \tag{23}$$

The usual modulus of continuity of $\mathfrak{h}\in C_B[0,\infty)$ is defined as

$$\omega_1(\mathfrak{h},\delta)=\sup_{0<l\le\delta}\sup_{\mathfrak{v}\in 0,\infty)}|\mathfrak{h}(\mathfrak{v}+l)-\mathfrak{h}(\mathfrak{v})|. \tag{24}$$

Theorem 6. *For* $\mathfrak{h} \in C_B[0,\infty)$,

$$\left|\mathfrak{L}_{p,a}^{(\lambda,\mu)}(\mathfrak{h};\mathfrak{v})-\mathfrak{h}(\mathfrak{v})\right| \le M\omega_2\left(\mathfrak{h};\sqrt{\phi_{p,a}^{(\lambda,\mu)}}\right)+\omega_1\left(\mathfrak{h};\frac{1}{p+\mu}\cdot\left(1+\lambda-\mu\mathfrak{v}+\frac{a\mathfrak{v}}{1+\mathfrak{v}}\right)\right), \tag{25}$$

where $M>0$ *and*

$$\phi_{p,a}^{(\lambda,\mu)}=\frac{1}{(p+\mu)^2}\left\{\left(p^2+\mu^2+\frac{a^2}{(1+\mathfrak{v})^2}-\frac{2a\mu}{(1+\mathfrak{v})}\right)\mathfrak{v}^2+\left(2p-2\mu-2\lambda\mu+\frac{2a(2+\lambda)}{(1+\mathfrak{v})}\right)\mathfrak{v}\right\}+\frac{1}{(p+\mu)^2}\left\{\left(1+\lambda-\mu\mathfrak{v}+\frac{a\mathfrak{v}}{1+\mathfrak{v}}\right)^2+\lambda^2+2\lambda+2\right\}. \tag{26}$$

Proof. Put

$$\tilde{\mathfrak{L}}_{p,a}^{(\lambda,\mu)}(\mathfrak{h};\mathfrak{v})=\mathfrak{L}_{p,a}^{(\lambda,\mu)}(\mathfrak{h};\mathfrak{v})+\mathfrak{h}(\mathfrak{v})-\mathfrak{h}\left(\frac{1+\lambda}{p+\mu}+\frac{p\mathfrak{v}}{p+\mu}+\frac{1}{p+\mu}\frac{a\mathfrak{v}}{1+\mathfrak{v}}\right). \tag{27}$$

Note that $\tilde{\mathfrak{L}}_{p,a}^{(\lambda,\mu)}(1;\mathfrak{v})=1$ and $\tilde{\mathfrak{L}}_{p,a}^{(\lambda,\mu)}(t;\mathfrak{v})=\mathfrak{v}$. Let $\mathfrak{E}\in C_B^2[0,\infty)$. Then, by using Taylor's theorem, we may write

$$\mathfrak{E}(t)=\mathfrak{E}(\mathfrak{v})+(t-\mathfrak{v})\mathfrak{E}'(x)\int_{\mathfrak{v}}^t(t-y)\mathfrak{E}''(y)dy, \tag{28}$$

which gives

$$\begin{aligned}\tilde{\mathfrak{L}}_{p,a}^{(\lambda,\mu)}(\mathfrak{E};\mathfrak{v})-\mathfrak{E}(\mathfrak{v}) &=\mathfrak{E}'(\mathfrak{v})\tilde{\mathfrak{L}}_{p,a}^{(\lambda,\mu)}((t-\mathfrak{v});\mathfrak{v})+\tilde{\mathfrak{L}}_{p,a}^{(\lambda,\mu)}\cdot\left(\int_{\mathfrak{v}}^t(t-y)\mathfrak{E}''(y)dy;\mathfrak{v}\right)\\ &=\tilde{\mathfrak{L}}_{p,a}^{(\lambda,\mu)}\left(\int_{\mathfrak{v}}^t(t-y)\mathfrak{E}''(y)dy;\mathfrak{v}\right)\\ &=\mathfrak{L}_{p,a}^{(\lambda,\mu)}\left(\int_{\mathfrak{v}}^t(t-y)\mathfrak{E}''(y)dy;\mathfrak{v}\right)-\int_{\mathfrak{v}}^{1/(p+\mu)(1+\lambda+p\mathfrak{v}+(a\mathfrak{v}/1+\mathfrak{v}))}\left(\frac{1+\lambda}{p+\mu}+\frac{p\mathfrak{v}}{p+\mu}+\frac{1}{p+\mu}\frac{a\mathfrak{v}}{1+\mathfrak{v}}-\mathfrak{v}\right)\mathfrak{E}''(y)dy.\end{aligned} \tag{29}$$

Hence,

$$\begin{aligned}\left|\tilde{\mathfrak{L}}_{p,a}^{(\lambda,\mu)}(\mathfrak{E};\mathfrak{v})-\mathfrak{E}(\mathfrak{v})\right| &\le \mathfrak{L}_{p,a}^{(\lambda,\mu)}\left(\left|\int_{\mathfrak{v}}^t(t-y)\mathfrak{E}''(y)dy\right|;\mathfrak{v}\right)\\ &+\left|\int_{\mathfrak{v}}^{1/(p+\mu)(1+\lambda+p\mathfrak{v}+(a\mathfrak{v}/(1+\mathfrak{v})))}\left(\frac{1+\lambda}{p+\mu}+\frac{p\mathfrak{v}}{p+\mu}+\frac{1}{p+\mu}\frac{a\mathfrak{v}}{1+\mathfrak{v}}-y\right)\mathfrak{E}''(y)dy\right|.\end{aligned} \tag{30}$$

Since $\left|\int_{\mathfrak{v}}^t(t-y)\mathfrak{E}''(y)dy\right|\le(t-\mathfrak{v})^2\|\mathfrak{E}''\|$ and

$$\left|\int_{\mathfrak{v}}^{1/(p+\mu)(1+\lambda+p\mathfrak{v}+(a\mathfrak{v}/(1+\mathfrak{v})))}\left(\frac{1+\lambda}{p+\mu}+\frac{p\mathfrak{v}}{p+\mu}+\frac{1}{p+\mu}\frac{a\mathfrak{v}}{1+\mathfrak{v}}-y\right)\mathfrak{E}''(y)dy\right| \le\left(\frac{1+\lambda}{p+\mu}-\frac{\mu\mathfrak{v}}{p+\mu}+\frac{1}{p+\mu}\frac{a\mathfrak{v}}{1+\mathfrak{v}}\right)^2\|\mathfrak{E}''\|, \tag{31}$$

we have

$$\left|\tilde{\mathfrak{L}}_{p,a}^{(\lambda,\mu)}(\mathfrak{E};\mathfrak{v})-\mathfrak{E}(\mathfrak{v})\right|\le\mathfrak{L}_{p,a}^{(\lambda,\mu)}\left((t-\mathfrak{v})^2;\mathfrak{v}\right)+\left(\frac{1+\lambda}{p+\mu}-\frac{\mu\mathfrak{v}}{p+\mu}+\frac{1}{p+\mu}\frac{a\mathfrak{v}}{1+\mathfrak{v}}\right)^2\|\mathfrak{E}''\|. \tag{32}$$

Now, by Corollary 4 (b), we get

$$\left|\tilde{\mathfrak{L}}_{p,a}^{(\lambda,\mu)}(\mathfrak{E};\mathfrak{v})-\mathfrak{E}(\mathfrak{v})\right|\le\phi_{p,a}^{(\lambda,\mu)}(\mathfrak{v})\|\mathfrak{E}''\|. \tag{33}$$

By (27), we get

$$\begin{aligned}\left|\mathfrak{L}_{p,a}^{(\lambda,\mu)}(\mathfrak{h};\mathfrak{v})-\mathfrak{h}(\mathfrak{v})\right| &\le\left|\tilde{\mathfrak{L}}_{p,a}^{(\lambda,\mu)}(\mathfrak{h}-\mathfrak{E};\mathfrak{v})\right|+\left|(\mathfrak{h}-\mathfrak{E})(\mathfrak{v})\right|+\left|\tilde{\mathfrak{L}}_{p,a}^{(\lambda,\mu)}(\mathfrak{E};\mathfrak{v})-\mathfrak{E}(\mathfrak{v})\right|\\ &+\left|\mathfrak{h}\left(\frac{1+\lambda}{p+\mu}+\frac{p\mathfrak{v}}{p+\mu}+\frac{1}{p+\mu}\frac{a\mathfrak{v}}{1+\mathfrak{v}}\right)-\mathfrak{h}(\mathfrak{v})\right|.\end{aligned} \tag{34}$$

Since $\left|\tilde{\mathfrak{L}}_{p,a}^{(\lambda,\mu)}(\mathfrak{h};\mathfrak{v})\right|\le 3\|\mathfrak{h}\|$, we get

$$\begin{aligned}\left|\mathfrak{L}_{p,a}^{(\lambda,\mu)}(\mathfrak{h};\mathfrak{v})-\mathfrak{h}(\mathfrak{v})\right| &\le 4\|\mathfrak{h}-\mathfrak{E}\|+\left|\tilde{\mathfrak{L}}_{p,a}^{(\lambda,\mu)}(\mathfrak{E};\mathfrak{v})-\mathfrak{E}(\mathfrak{v})\right|\\ &+\left|\mathfrak{h}\left(\frac{1+\lambda}{p+\mu}+\frac{p\mathfrak{v}}{p+\mu}+\frac{1}{p+\mu}\frac{a\mathfrak{v}}{1+\mathfrak{v}}\right)-\mathfrak{h}(\mathfrak{v})\right|.\end{aligned} \tag{35}$$

From (33), we get

$$|\mathfrak{L}_{p,a}^{(\lambda,\mu)}(\mathfrak{h};\mathfrak{v})-\mathfrak{h}(\mathfrak{v})|\leq 4\|\mathfrak{h}-\mathfrak{E}\|+\phi_{p,a}^{(\lambda,\mu)}(\mathfrak{v})\|\mathfrak{E}''\|+\omega_1\left(\mathfrak{h}:\frac{1+\lambda}{p+\mu}-\frac{\mu\mathfrak{v}}{p+\mu}+\frac{1}{p+\mu}\frac{a\mathfrak{v}}{1+\mathfrak{v}}\right). \quad (36)$$

Now, taking the infimum over all $\mathfrak{E}\in C_B^2[0,\infty)$, we obtain

$$|\mathfrak{L}_{p,a}^{(\lambda,\mu)}(\mathfrak{h};\mathfrak{v})-\mathfrak{h}(\mathfrak{v})|\leq 4K_2\left(\mathfrak{h};\phi_{p,a}^{(\lambda,\mu)}(\mathfrak{v})\right)+\omega_1\left(f:\frac{1+\lambda}{p+\mu}-\frac{\mu\mathfrak{v}}{p+\mu}+\frac{1}{p+\mu}\frac{a\mathfrak{v}}{1+\mathfrak{v}}\right). \quad (37)$$

Hence, by using (22), we get the result. □

For our next result, we consider the functions belonging to the Lipschitz class:

$$\operatorname{lip}_{\mathscr{M}}(\gamma)=\left\{\mathfrak{h}\in C_B[0,\infty):|\mathfrak{h}(t)-\mathfrak{h}(\mathfrak{v})|\leq\mathscr{M}\frac{|t-\mathfrak{v}|^\gamma}{(t+\mathfrak{v})^{\gamma/2}}\right\}, \quad (38)$$

where $\mathscr{M}>0$ and $0<\gamma\leq 1$; $\mathfrak{v},t\in 0,\infty)$.

Theorem 7. *For $\mathfrak{h}\in lip_{\mathscr{M}}(\gamma)$, we have*

$$|\mathfrak{L}_{p,a}^{(\lambda,\mu)}(\mathfrak{h};\mathfrak{v})-\mathfrak{h}(\mathfrak{v})|\leq\mathscr{M}\left(\frac{\varphi_p^{(\lambda,\mu)}(\mathfrak{v})}{\mathfrak{v}}\right)^{\gamma/2}, \quad (39)$$

where $\varphi_{p,a}^{(\lambda,\mu)}(\mathfrak{v})=\mathfrak{L}_{p,a}^{(\lambda,\mu)}((e_1-\mathfrak{v})^2;\mathfrak{v})$.

Proof. First, we prove for $\gamma=1$. For $\mathfrak{h}\in\operatorname{lip}_{\mathscr{M}}(\gamma)$, we get

$$\begin{aligned}&\left|\mathfrak{L}_{p,a}^{(\lambda,\mu)}(\mathfrak{h};\mathfrak{v})-\mathfrak{h}(\mathfrak{v})\right|\\ &\leq p\sum_{k=0}^{\infty}W_{p,k}^a(\mathfrak{v})\int_0^\infty s_{p,k}(t)\left|\mathfrak{h}\left(\frac{pt+\lambda}{p+\mu}\right)-\mathfrak{h}(\mathfrak{v})\right|dt\\ &\leq\mathscr{M}p\sum_{k=0}^{\infty}W_{p,k}^a(\mathfrak{v})\int_0^\infty s_{p,k}(t)\frac{|((pt+\lambda)/(p+\mu))-\mathfrak{v}|}{\sqrt{((pt+\lambda)/(p+\mu))+\mathfrak{v}}}dt.\end{aligned} \quad (40)$$

Since $\sqrt{\mathfrak{v}}<\sqrt{((pt+\lambda)/(p+\mu))+\mathfrak{v}}$, we get by the Cauchy-Schwarz inequality:

$$\begin{aligned}&\left|\mathfrak{L}_{p,a}^{(\lambda,\mu)}(\mathfrak{h};\mathfrak{v})-\mathfrak{h}(\mathfrak{v})\right|\\ &\leq\frac{\mathscr{M}}{\sqrt{\mathfrak{v}}}p\sum_{k=0}^{\infty}W_{p,k}^a(\mathfrak{v})\int_0^\infty s_{n,k}(t)\left|\frac{pt+\lambda}{p+\mu}-u\right|dt\\ &=\frac{\mathscr{M}}{\sqrt{\mathfrak{v}}}\mathfrak{L}_{p,a}^{(\lambda,\mu)}\left((e_1-\mathfrak{v})^2;\mathfrak{v}\right)\leq\mathscr{M}\sqrt{\frac{\varphi_{p,a}^{(\lambda,\mu)}(\mathfrak{v})}{\mathfrak{v}}}.\end{aligned} \quad (41)$$

For $0<\gamma<1$, applying Hölder's inequality, we get

$$\begin{aligned}&\left|\mathfrak{L}_{p,a}^{(\lambda,\mu)}(\mathfrak{h};\mathfrak{v})-\mathfrak{h}(\mathfrak{v})\right|\\ &\leq p\sum_{k=0}^{\infty}W_{p,k}^a(\mathfrak{v})\int_0^\infty s_{p,k}(t)\left|\mathfrak{h}\left(\frac{pt+\lambda}{p+\mu}\right)-\mathfrak{h}(\mathfrak{v})\right|dt\\ &\leq p\sum_{k=0}^{\infty}\left\{W_{p,k}^a(\mathfrak{v})\left(\int_0^\infty s_{p,k}(t)\left|\mathfrak{h}\left(\frac{pt+\lambda}{p+\mu}\right)-\mathfrak{h}(\mathfrak{v})\right|dt\right)^{1/\gamma}\right\}^\gamma\\ &\leq p\sum_{k=0}^{\infty}\left\{W_{p,k}^a(\mathfrak{v})\int_0^\infty s_{p,k}(t)\left|\mathfrak{h}\left(\frac{pt+\lambda}{p+\mu}\right)-\mathfrak{h}(\mathfrak{v})\right|^{1/\gamma}dt\right\}^\gamma.\end{aligned} \quad (42)$$

Since $\mathfrak{h}\in\operatorname{lip}_M(\gamma)$, we have

$$\begin{aligned}&|\mathfrak{L}_{p,a}^{(\lambda,\mu)}(\mathfrak{h};\mathfrak{v})-\mathfrak{h}(\mathfrak{v})\\ &\leq\frac{\mathscr{M}}{\mathfrak{v}^{\gamma/2}}\left\{p\sum_{k=0}^{\infty}W_{p,k}^a(\mathfrak{v})\int_0^\infty s_{p,k}(t)\left|\frac{pt+\lambda}{p+\mu}-\mathfrak{v}\right|dt\right\}^\gamma\\ &=\frac{\mathscr{M}}{\mathfrak{v}^{\gamma/2}}\mathfrak{L}_{p,a}^{(\lambda,\mu)}(|e_1-\mathfrak{v}|;\mathfrak{v})^\gamma=\frac{\mathscr{M}}{\mathfrak{v}^{\gamma/2}}\mathfrak{L}_{p,a}^{(\lambda,\mu)}\left((e_1-\mathfrak{v})^2;\mathfrak{v}\right)^\gamma\\ &\leq\mathscr{M}\left(\sqrt{\frac{\varphi_{p,a}^{(\lambda,\mu)}(\mathfrak{v})}{\mathfrak{v}}}\right)^\gamma.\end{aligned} \quad (43)$$

Therefore, we get (39). □

Next, we obtain a Voronovskaja-type asymptotic formula.

Theorem 8. *If $\mathfrak{h}''$ exists at a point $\mathfrak{v}\in 0,\infty)$ for $\mathfrak{h}\in C_\gamma[0,\infty)$, then*

$$\begin{aligned}&\lim_{p\longrightarrow\infty}p\left(\mathfrak{L}_{p,a}^{(\lambda,\mu)}(\mathfrak{h}\simeq\mathfrak{v})-\mathfrak{h}(\mathfrak{v})\right)\\ &=\left(1+\lambda-\mu\mathfrak{v}+\frac{a\mathfrak{v}}{1+\mathfrak{v}}\right)\mathfrak{h}'(\mathfrak{v})+\frac{\mathfrak{v}}{2}(2+\mathfrak{v})\mathfrak{h}''(\mathfrak{v}).\end{aligned} \quad (44)$$

Proof. From Taylor's expansion of $\mathfrak{v}$, we may write

$$\mathfrak{h}(t)=\mathfrak{h}(\mathfrak{v})+(t-\mathfrak{v})\mathfrak{h}'(\mathfrak{v})+\frac{1}{2}(t-\mathfrak{v})^2\mathfrak{h}''(\mathfrak{v})+R(t,\mathfrak{v})(t-\mathfrak{v})^2, \quad (45)$$

where $R(t, \mathfrak{v}) \longrightarrow 0(t \longrightarrow \mathfrak{v})$. By operating $\mathfrak{L}_{p,a}^{(\lambda,\mu)}$, we obtain

$$\begin{aligned}\mathfrak{L}_{p,a}^{(\lambda,\mu)}(\mathfrak{h} \simeq \mathfrak{v}) - \mathfrak{h}(\mathfrak{v}) &= \mathfrak{L}_{p,a}^{(\lambda,\mu)}((t-\mathfrak{v});\mathfrak{v})\mathfrak{h}'(\mathfrak{v}) \\ &+ \mathfrak{L}_{p,a}^{(\lambda,\mu)}\left((t-\mathfrak{v})^2;\mathfrak{v}\right)\frac{\mathfrak{h}''(\mathfrak{v})}{2} \\ &+ \mathfrak{L}_{p,a}^{(\lambda,\mu)}\left(R(t,\mathfrak{v})(t-\mathfrak{v})^2;\mathfrak{v}\right).\end{aligned} \quad (46)$$

By the Cauchy-Schwarz inequality, we get

$$\begin{aligned}&\mathfrak{L}_{p,a}^{(\lambda,\mu)}\left(R(t,\mathfrak{v})(t-\mathfrak{v})^2;\mathfrak{v}\right) \\ &\leq \left(\mathfrak{L}_{p,a}^{(\lambda,\mu)}\left(R^2(t,\mathfrak{v});\mathfrak{v}\right)\right)^{1/2}\left(\mathfrak{L}_{p,a}^{(\lambda,\mu)}\left((t-\mathfrak{v})^4;\mathfrak{v}\right)\right)^{1/2}.\end{aligned} \quad (47)$$

Since $\mathfrak{L}_{p,a}^{(\lambda,\mu)}(\mathfrak{h} \simeq \mathfrak{v}) \longrightarrow \mathfrak{h}(\mathfrak{v})$, we get

$$\begin{aligned}&\lim_{p\longrightarrow\infty}\mathfrak{L}_{p,a}^{(\lambda,\mu)}\left(R^2(t,\mathfrak{v});\mathfrak{v}\right) = R^2(\mathfrak{v},\mathfrak{v}) = 0, \\ &\lim_{p\longrightarrow\infty} p\mathfrak{L}_{p,a}^{(\lambda,\mu)}\left(R(t,\mathfrak{v})(t-\mathfrak{v})^2;\mathfrak{v}\right) = 0.\end{aligned} \quad (48)$$

Now, combining the above equations and using Corollary 5, we get

$$\begin{aligned}&\lim_{p\longrightarrow\infty} p\left(\mathfrak{L}_{p,a}^{(\lambda,\mu)}(\mathfrak{h} \simeq \mathfrak{v}) - \mathfrak{h}(\mathfrak{v})\right) \\ &= \lim_{p\longrightarrow\infty} p\left(\mathfrak{L}_{p,a}^{(\lambda,\mu)}((t-\mathfrak{v});\mathfrak{v})\right)\mathfrak{h}'(\mathfrak{v}) \\ &+ \lim_{p\longrightarrow\infty} p\left(\mathfrak{L}_{p,a}^{(\lambda,\mu)}\left((t-\mathfrak{v})^2;\mathfrak{v}\right)\right)\frac{\mathfrak{h}''(\mathfrak{v})}{2} \\ &+ \lim_{p\longrightarrow\infty} p\left(\mathfrak{L}_{p,a}^{(\lambda,\mu)}\left(R(t,\mathfrak{v})(t-\mathfrak{v})^2;\mathfrak{v}\right)\right) \\ &= \left(1+\lambda-\mu\mathfrak{v}+\frac{a\mathfrak{v}}{1+\mathfrak{v}}\right)\mathfrak{h}'(\mathfrak{v}) + \frac{\mathfrak{v}}{2}(2+\mathfrak{v})\mathfrak{h}''(\mathfrak{v}).\end{aligned} \quad (49)$$

□

Let $B_\sigma[0,\infty) = \{\mathfrak{h} : [0,\infty) \longrightarrow \mathbb{R} | |\mathfrak{h}(\mathfrak{v})| \leq \mathscr{K}_{\mathfrak{h}}\sigma(\mathfrak{v}), \mathfrak{v} \geq 0\}$, where $\mathscr{K}_{\mathfrak{h}}$ is a constant which depends only on $\mathfrak{h}$, and

$$\|\mathfrak{h}\|_\sigma = \sup_{\mathfrak{v}\in 0,\infty)} \frac{|\mathfrak{h}(\mathfrak{v})|}{\sigma(\mathfrak{v})}. \quad (50)$$

Also, let $C_\sigma[0,\infty) = \{\mathfrak{h} \in B_\sigma[0,\infty)$: $\mathfrak{h}$ be continuous on $[0,\infty)\}$, and

$$C_\sigma^0[0,\infty) = \left\{\mathfrak{h} \in C_\sigma[0,\infty) : \lim_{\mathfrak{v}\longrightarrow\infty}\frac{|\mathfrak{h}(\mathfrak{v})|}{\sigma(\mathfrak{v})}\text{exists}\right\}, \quad (51)$$

where $\sigma(\mathfrak{v}) = 1 + \mathfrak{v}^2$.

The weighted modulus of continuity [14] is defined by

$$\Omega(l,\delta) := \sup_{0<l\leq\delta}\sup_{\mathfrak{v}\in 0,\infty)} \frac{|\mathfrak{h}(\mathfrak{v}+l) - \mathfrak{h}(\mathfrak{v})|}{1+(\mathfrak{v}+l)^2}. \quad (52)$$

Lemma 9 (see [14]). *Let $\mathfrak{h} \in C_\sigma^0[0,\infty)$. Then,*

(i) $\Omega(l,\delta)$ is a monotone increasing function of δ

(ii) $\Omega(l,\delta) \longrightarrow 0$ as $\delta \longrightarrow 0$

(iii) $\Omega(l,k\delta) \leq k\Omega(l,\delta)$ for each $k \in \mathbb{N}$

(iv) $\Omega(l,\alpha\delta) \leq (1+\alpha)\Omega(l,\delta)$ for each $\alpha \in \mathbb{R}^+$

Theorem 10. *For $\mathfrak{h} \in C_\sigma^0[0,\infty)$, we have*

$$\sup_{\mathfrak{v}\in 0,\infty)} \frac{|\mathfrak{L}_{p,a}^{(\lambda,\mu)}(\mathfrak{h};\mathfrak{v}) - \mathfrak{h}(\mathfrak{v})|}{(1+\mathfrak{v}^2)^{5/2}} \leq M\Omega\left(l,\frac{1}{p}\right), M > 0. \quad (53)$$

Proof. By Lemma 9, we have

$$\begin{aligned}|\mathfrak{h}(t) - \mathfrak{h}(\mathfrak{v})| &\leq (1+(\mathfrak{v}+|t-\mathfrak{v}|)^2\Omega(l,|t-\mathfrak{v}|) \\ &\leq 2\left(1+\mathfrak{v}^2\right)\left(1+(t-\mathfrak{v})^2\right)\left(1+\frac{|t-\mathfrak{v}|}{\delta}\right)\Omega(l,\delta).\end{aligned} \quad (54)$$

Operating $\mathfrak{L}_{p,a}^{(\lambda,\mu)}$, we get

$$\begin{aligned}&\left|\mathfrak{L}_{p,a}^{(\lambda,\mu)}(\mathfrak{h};\mathfrak{v}) - \mathfrak{h}(\mathfrak{v})\right| \\ &\leq 2\left(1+\mathfrak{v}^2\right)\Omega(l,\delta)\left\{1+\mathfrak{L}_{p,a}^{(\lambda,\mu)}\left((t-\mathfrak{v})^2;\mathfrak{v}\right)\right. \\ &\left.+\mathfrak{L}_{p,a}^{(\lambda,\mu)}\left(1+(t-\mathfrak{v})^2\frac{|\mathfrak{v}-t|}{\delta};\mathfrak{v}\right)\right\}.\end{aligned} \quad (55)$$

Using a second-order central moment, we get

$$\mathfrak{L}_{p,a}^{(\lambda,\mu)}\left((t-\mathfrak{v})^2;\mathfrak{v}\right) \leq M_1\frac{\left(1+\mathfrak{v}^2\right)}{(p+\mu)} \leq M_1\frac{\left(1+\mathfrak{v}^2\right)}{p}, M_1 > 0. \quad (56)$$

Applying the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}&\mathfrak{L}_{p,a}^{(\lambda,\mu)}\left(1+(t-\mathfrak{v})^2\frac{|\mathfrak{v}-t|}{\delta};u\right) \\ &\leq \frac{1}{\delta}\left(\mathfrak{L}_{p,a}^{(\lambda,\mu)}\left((t-\mathfrak{v})^2;\mathfrak{v}\right)\right)^{1/2} \\ &+\frac{1}{\delta}\left(\mathfrak{L}_{p,a}^{(\lambda,\mu)}\left((t-\mathfrak{v})^4;\mathfrak{v}\right)\right)^{1/2}\left(\mathfrak{L}_{p,a}^{(\lambda,\mu)}\left((t-\mathfrak{v})^2;\mathfrak{v}\right)\right)^{1/2}.\end{aligned} \quad (57)$$

Again, using the central moment of order 4, we get

$$\left(\mathfrak{L}_{p,a}^{(\lambda,\mu)}\left((t-\mathfrak{v})^4;\mathfrak{v}\right)\right)^{1/2} \leq M_2\frac{\left(1+\mathfrak{v}^2\right)}{(p+\mu)} \leq M_2\frac{\left(1+\mathfrak{v}^2\right)}{p}, M_2 > 0. \quad (58)$$

Combining the estimates (55)–(58) and choosing $M = 2(1+M_1+\sqrt{M_1}+M_2\sqrt{M_1})$, $\delta = 1/\sqrt{p}$, we get the required result. □

4. q-Statistical Convergence

Defining a q-analog of the Cesàro matrix C_1 is not unique (see [15, 16]). Here, we consider the q-Cesàro matrix, $C_1(q) = (c^1_{nk}(q^k))^{\infty}_{n,k=0}$, defined by

$$c^1_{nk}\left(q^k\right) = \begin{cases} \dfrac{q^k}{[n+1]_q}, & \text{if } k \le n, \\ 0, & \text{otherwise,} \end{cases} \quad (59)$$

which is regular for $q \ge 1$.

Let $\mathscr{K} \subseteq \mathbb{N}$ (the set of natural numbers). Then, $\delta(\mathscr{K}) = \lim_r (1/r)\#\{k \le r : k \in \mathscr{K}\}$ is called the asymptotic density of $\mathscr{K}$, where # denotes the cardinality of the enclosed set. A sequence $\eta = (\eta_k)$ is called statistically convergent to the number $\mathfrak{s}$ if $\delta(\mathscr{K}_\varepsilon) = 0$ for each $\varepsilon > 0$, where $\mathscr{K}_\varepsilon = \{k \le r : |\eta_k - \mathfrak{s}| > \varepsilon\}$ (see [17]).

Recently, Aktuğlu and Bekar [16] defined q-density and q-statistical convergence. The q-density is defined by

$$\begin{aligned} \delta_q(\mathscr{K}) &= \delta_{C_1^q}(\mathscr{K}) = \lim_{n \longrightarrow \infty} \inf \left(C_1^q \chi_{\mathscr{K}}\right)_n \\ &= \lim_{n \longrightarrow \infty} \inf \sum_{k \in K} \frac{q^{k-1}}{[n]}, q \ge 1. \end{aligned} \quad (60)$$

A sequence $\eta = (\eta_k)$ is said to be q-statistically convergent to the number $\mathscr{L}$ if $\delta_q(\mathscr{K}_\varepsilon) = 0$, where $\mathscr{K}_\varepsilon = \{k \le n : |\eta_k - \mathscr{L}| \ge \varepsilon\}$ for every $\varepsilon > 0$. That is, for each $\varepsilon > 0$,

$$\lim_n \frac{1}{[n]} \#\left\{k \le n : q^{k-1} \mid \eta_k - \mathscr{L}| \ge \varepsilon\right\} = 0, \quad (61)$$

and we write $St_q - \lim \eta_k = \mathscr{L}$.

If $\delta(\mathscr{K}) = 0$ for an infinite set $\mathscr{K}$, then $\delta_q(\mathscr{K}) = 0$; hence, statistical convergence implies q-statistical convergence but not conversely (c.f. [16, Example 15]). Recently in [18], authors proved Korovkin's type theorem via q-statistical convergence. Using the same technique we prove the following theorem.

Theorem 11. *For all* $\mathfrak{h} \in C^0_\rho$, *we have*

$$St_q - \lim_p \left\| \mathfrak{L}^{(\lambda,\mu)}_{p,a}(\mathfrak{h}; \mathfrak{v}) - \mathfrak{h}(\mathfrak{v}) \right\|_\sigma = 0, \mathfrak{v} \in [0, \infty). \quad (62)$$

Proof. It is sufficient to show that $St_q - \lim_p \|\mathfrak{L}^{(\lambda,\mu)}_{p,a}(e_i; \mathfrak{v}) - e_i\|_\sigma = 0$, for $i = 0, 1, 2$, where $e_i(\mathfrak{v}) = \mathfrak{v}^i$. It is clear that

$$St_q - \lim_p \left\| \mathfrak{L}^{(\lambda,\mu)}_{p,a}(e_0; \mathfrak{v}) - e_0 \right\|_\sigma = 0. \quad (63)$$

By Corollary 2 (ii), we have

$$\begin{aligned} \left\| \mathfrak{L}^{(\lambda,\mu)}_{p,a}(t; \mathfrak{v}) - \mathfrak{v} \right\|_\sigma &= \sup_{u \in 0,\infty)} \frac{1}{1+\mathfrak{v}^2} \left| \frac{1}{p+\mu}\left(p\mathfrak{v} + \frac{a\mathfrak{v}}{1+\mathfrak{v}} \right.\right. \\ &\quad \left.\left. + 1 + \lambda\right) - \mathfrak{v} \right| \le \frac{1}{p+\mu} |-\mu + 1 + \lambda + a|. \end{aligned} \quad (64)$$

For $\varepsilon > 0$, define the sets:

$$\begin{aligned} \mathscr{E}_1 &:= \left\{p \in \mathbb{N} : \|\mathfrak{L}^{(\lambda,\mu)}_{p,a}(e_1; \mathfrak{v}) - e_1\|_\sigma \ge \varepsilon\right\}, \\ \mathscr{E}_2 &:= \left\{p \in \mathbb{N} : \left|\frac{1+\lambda-\mu+a}{(p+\mu)}\right| \ge \varepsilon\right\}. \end{aligned} \quad (65)$$

Then,

$$\delta_q(\mathscr{E}_2) = \lim_{p \longrightarrow \infty} \inf \left(C_1^q \chi_{\mathscr{E}_2}\right)_p = \lim_{p \longrightarrow \infty} \inf \sum_{k \in \mathscr{E}_2} \frac{q^{k-1}}{[p]} = 0. \quad (66)$$

Since $\mathscr{E}_1 \subseteq \mathscr{E}_2$, we have $\delta_q(\mathscr{E}_1) \le \delta_q(\mathscr{E}_2)$. Hence,

$$St_q - \lim_p \left\| \mathfrak{L}^{(\lambda,\mu)}_{p,a}(e_1; \mathfrak{v}) - e_1 \right\|_\sigma = 0. \quad (67)$$

Again, by Corollary 2 (iii), we obtain

$$\begin{aligned} &\left\| \mathfrak{L}^{(\lambda,\mu)}_{p,a}(t^2; \mathfrak{v}) - \mathfrak{v}^2 \right\|_\sigma \\ &\le \sup_{\mathfrak{v} \in 0,\infty)} \frac{1}{1+\mathfrak{v}^2} \left| \frac{1}{(p+\mu)^2} \left\{p + \mu^2 + \frac{a^2}{(1+\mathfrak{v})^2} - \frac{2a\mu}{(1+\mathfrak{v})}\right\} \mathfrak{v}^2 \right| \\ &\quad + \sup_{\mathfrak{v} \in 0,\infty)} \frac{1}{1+\mathfrak{v}^2} \left| \frac{2}{(p+\mu)^2} \left(p - \mu - \lambda\mu + \frac{(2+\lambda)a}{(1+\mathfrak{v})}\right) \mathfrak{v} \right. \\ &\quad \left. + \frac{1}{(p+\mu)^2}\left(\lambda^2 + 2\lambda + 2\right) \right| \le \frac{1}{(p+\mu)^2}\left\{p + \mu^2 + a^2 - 2a\mu\right\} \\ &\quad + \frac{2}{(p+\mu)^2}(p - \mu - \lambda\mu + (2+\lambda)a) + \frac{1}{(p+\mu)^2}\left(\lambda^2 + 2\lambda + 2\right). \end{aligned} \quad (68)$$

For $\varepsilon > 0$, define the sets:

$$\begin{aligned} \mathfrak{D}_1 &:= \left\{p \in \mathbb{N} : \|\mathfrak{L}^{(\lambda,\mu)}_{p,a}(e_2; \mathfrak{v}) - e_2\|_\sigma \ge \varepsilon\right\}, \\ \mathfrak{D}_2 &:= \left\{p \in \mathbb{N} : \left(\frac{1}{(p+\mu)^2}\left\{p + \mu^2 + a^2 - 2a\mu\right\}\right) \ge \frac{\varepsilon}{3}\right\}, \\ \mathfrak{D}_3 &:= \left\{p \in \mathbb{N} : \frac{2}{(p+\mu)^2}(p - \mu - \lambda\mu + (2+\lambda)a) \ge \frac{\varepsilon}{3}\right\}, \\ \mathfrak{D}_4 &:= \left\{p \in \mathbb{N} : \frac{1}{(p+\mu)^2}\left(\lambda^2 + 2\lambda + 2\right) \ge \frac{\varepsilon}{3}\right\}. \end{aligned} \quad (69)$$

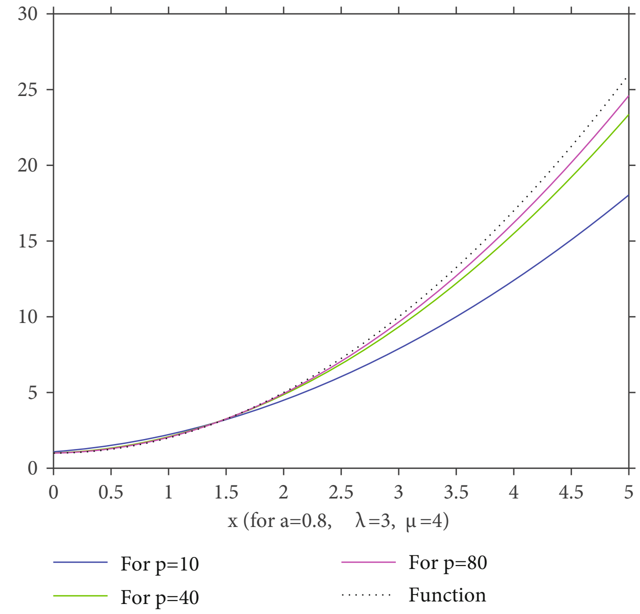

FIGURE 1: Convergence of the operator towards the function $f(x) = x^2 + 1$.

Then, $\delta_q(\mathfrak{D}_2) = 0 = \delta_q(\mathfrak{D}_3) = \delta_q(\mathfrak{D}_4)$. Since $\mathfrak{D}_1 \subseteq \mathfrak{D}_2 \cup \mathfrak{D}_3 \cup \mathfrak{D}_4$ which implies that $\delta_q(\mathfrak{D}_1) \leq \delta_q(\mathfrak{D}_2) + \delta_q(\mathfrak{D}_3) + \delta_q(\mathfrak{D}_4)$,

$$St_q - \lim_p \left\| \mathfrak{L}_{p,a}^{(\lambda,\mu)}(e_2\,;\mathfrak{v}) - e_2 \right\|_\rho = 0. \tag{70}$$

Hence, the proof is completed. □

Example 12. Let $\eta = (\eta_p)$ be defined by

$$\eta_p = \begin{pmatrix} 1\left(2^{2n}\text{times}\right) \\ 0\left(2^{2n-1}\text{times}\right) \end{pmatrix} n = 0, 1, 2, \cdots. \tag{71}$$

That is, 1 occurs 2^{2n} times and 0 occurs 2^{2n-1} times ($n = 0, 1, 2, \cdots$), respectively. Let $\mathcal{K} = \{k \in \mathbb{N} : \eta_k = 1\}$. Then, $\lim_{n\longrightarrow\infty}(C_1^q \chi_{\mathcal{K}})_{2^{2n}-1} = 0$, i.e., $St_q - \lim \eta_k = 0$, but $\delta(\mathcal{K})$ does not exist, so η is not statistically convergent.

Define $\mathcal{A}_p^{(\lambda,\mu)} = (1 + \eta_p)\mathfrak{L}_{p,a}^{(\lambda,\mu)}$, where it is defined by (71). Then, obviously $st - \lim_p \|\mathcal{A}_p^{(\lambda,\mu)}(e_i\,;\mathfrak{v}) - e_i\|_\sigma = 0 (i = 0, 1, 2)$. Applying the above theorem, we have

$$St_q - \lim_p \left\| \mathcal{A}_p^{(\lambda,\mu)}(\mathfrak{h}\,;\mathfrak{v}) - \mathfrak{h} \right\|_\sigma = 0 \text{ for all } \mathfrak{h} \in C_\rho^0. \tag{72}$$

On the other hand, since $\eta = (\eta_p)$ is q-statistically convergent but neither convergent nor statistically convergent, the sequence $(\mathcal{A}_p^{(\lambda,\mu)})$ can not be convergent, while it is q-statistically convergent.

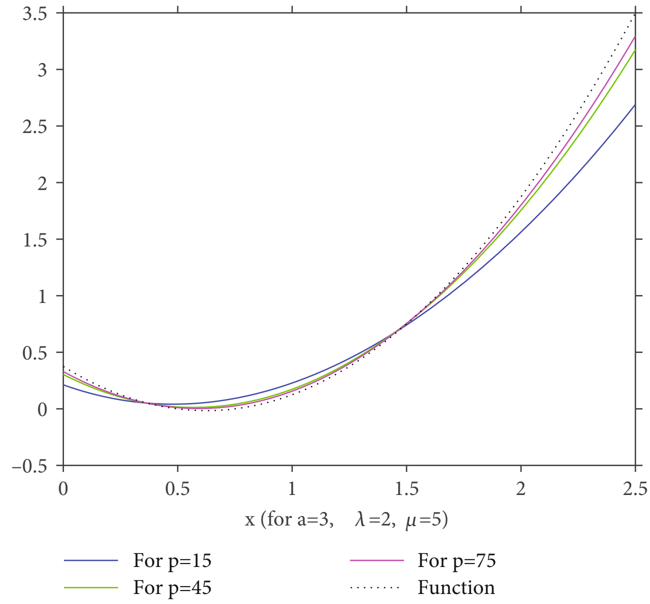

FIGURE 2: Convergence of the operator towards the function $f(x) = (x - (1/2))(x - (3/4))$.

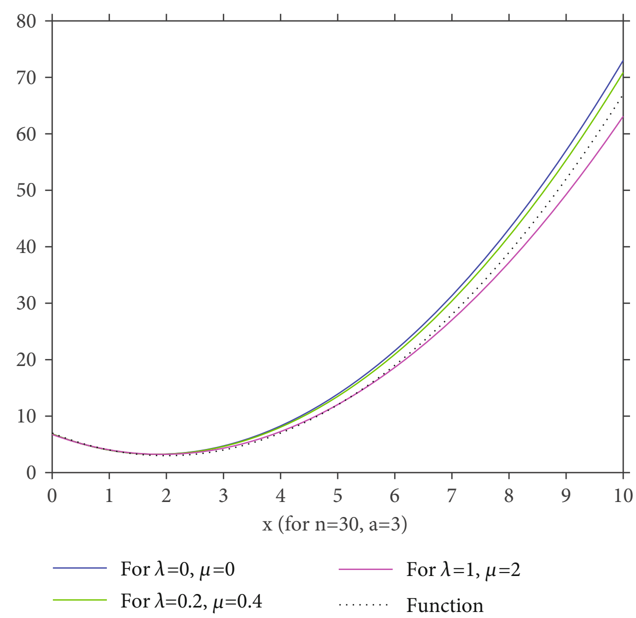

FIGURE 3: Comparison of convergence of the operator.

5. Graphical Analysis

In this section, we will give some numerical examples with illustrative graphics with the help of MATLAB.

Example 13. Let $f(x) = x^2 + 1$, $\lambda = 3$, $\mu = 4$, $a = 0.8$, and $p \in \{10,40,80\}$. The convergence of the operator towards the function $f(x)$ is shown in Figure 1.

Example 14. Let $f(x) = (x - (1/2))(x - (3/4))$, $\lambda = 2$, $\mu = 5$, $a = 3$, and $p \in \{15,45,75\}$. The convergence of the operator towards the function $f(x)$ is shown in Figure 2.

From these examples, we observe that the approximation of function by the operators becomes better when we take larger values of n.

Notice that for $\lambda = \mu = 0$, the operators (5) reduce to operators (4).

Example 15. Let $f(x) = x^2 - 4x + 7$. For $a = 3, p = 30$, comparison of convergence of the constructed operator (5) (green and pink) with the previously defined operator (4) (blue) is shown in Figure 3. From this figure, it is clear that the constructed operator gives a better approximation to $f(x)$ than the previously defined operator.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally to writing this paper. All authors read and approved the manuscript.

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