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Research Article

The Boundedness of Doob's Maximal and Fractional Integral Operators for Generalized Grand Morrey-Martingale Spaces

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In this paper, we introduce the generalized grand Morrey spaces in the framework of probability space setting in the spirit of the martingale theory and grand Morrey spaces. The Doob maximal inequalities on the generalized grand Morrey spaces are provided. Moreover, we present the boundedness of fractional integral operators for regular martingales in this new framework.

1. Introduction

A real-valued function f is said to belong to the Morrey space $\mathscr{L}_{p,\lambda}(\mathbb{R}^N)$ on the N-dimensional Euclidean space $\mathbb{R}^N$ provided the following norm is finite:

$$\|f\|_{\mathscr{L}_{p,\lambda}(\mathbb{R}^N)} = \left(\sup_{(x,r)\in\left(\mathbb{R}^N\times\mathbb{R}_+\right)} r^{\lambda-N}\int_{B(x,r)} |f(y)|^p dy \right)^{1/p}. \quad (1)$$

Here $1 \le p < \infty$, $0 \le \lambda \le N$, $\mathbb{R}_+ = (0,\infty)$, and $B(x,r)$ are a ball in $\mathbb{R}^N$ centered at x of radius r. This class of functions was first introduced by Morrey [1] in order to study regularity problem arising in Calculus of Variations, describe local regularity more precisely than Lebesgue spaces. In the past, Morrey spaces have been studied heavily, such as the maximal operators, fractional integral operators, and singular operators. The results are extensively applied not only in partial differential equations but also in harmonic analysis. We refer the readers to [2, 3] and the references therein.

The Morrey spaces on Euclidean spaces have been developed to the generalization versions, for example, the generalized Morrey spaces [4, 5], the Orlicz-Morrey spaces [6, 7], the Triebel-Lizorkin-Morrey spaces [8], and the variable exponent Morrey spaces [9]. Especially, Meskhi [10] introduced the grand Morrey spaces and established the boundedness of the Hardy-Littlewood maximal, Calderón-Zygmund, and potential operators in these spaces. The generalized grand Morrey spaces in a general setting of the quasi-metric measure spaces are studied by Kokilashvili et al. [11, 12].

Moreover, in probability theory, Nakai and Sadasue [13] introduced Morrey spaces of martingales as the following:

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space and $\{\mathscr{F}_n\}_{n\ge0}$ be a nondecreasing sequence of sub-σ-algebras of $\mathscr{F}$ such that $\mathscr{F} = \sigma(\cup_{n\ge0} \mathscr{F}_n)$.

We assume that every σ-algebra $\mathscr{F}_n$ is generated by countable atoms, where $B \in \mathscr{F}_n$ is called an atom, if any $A \subset B$ with $A \in \mathscr{F}_n$ satisfies $\mathbb{P}(A) = 0$ or $\mathbb{P}(A) = \mathbb{P}(B)$. Denote by $A(\mathscr{F}_n)$ the set of all atoms in $\mathscr{F}_n$. For $p \in [1,\infty)$ and $\mu \in (-\infty, \infty)$, martingale Morrey space $L_{p,\mu}(\Omega)$ consists of all $f \in L_1(\Omega)$ having the finite norm

$$\|f\|_{L_{p,\mu}(\Omega)} = \sup_{n\ge0} \sup_{B\in A(\mathscr{F}_n)} \frac{1}{\mathbb{P}(B)^\mu} \left(\frac{1}{\mathbb{P}(B)} \int_B |f|^p d\mathbb{P} \right)^{1/p}. \quad (2)$$

They introduced some basic properties of the martingale Morrey spaces. Furthermore, the Doob maximal inequality was established, and the mapping properties for the fractional integral operators were investigated on these spaces. Two generalized versions of them introduced in [14, 15]. Ho [16] presented atomic decompositions of martingale Hardy-Morrey spaces. Later on, he [17] introduced a version of martingale Morrey spaces equipping with Banach function spaces. Jiao et al. [18] studied the maximal operator, atom decompositions, and fractional integral operators on martingale Morrey spaces with variable exponents.

Recently, Deng and Li [19] studied the Doob maximal operator and fractional integral operator in the framework of grand Morrey-martingale spaces associated with an almost decreasing function. Moreover, compared with classical martingale spaces, the grand martingale spaces have not of absolutely continuous norm based on [20]. Consequently, we need a further research about grand martingale spaces. Motivated by the works of this and [11], the paper is to investigate the generalized grand Morrey space theory for the martingale setting. More precisely, we first introduce the generalized grand Morrey-martingale spaces and then establish the Doob maximal inequality in this new framework. As an application, we discuss the boundedness of fractional integral operators for regular martingales in the generalized grand Morrey-martingale spaces.

2. Preliminaries

Now we recall some standard notations from martingale theory. Refer to [21, 22] for more information on martingale theory. The expectation is denoted by E with respect to $(\Omega, \mathscr{F}, \mathbb{P})$. Recall that the conditional expectation operator relative to $\mathscr{F}_n$ is denoted by E_n, i.e., $E(f|\mathscr{F}_n)=E_n(f)$. A sequence of measurable functions $f=(f_n)_{n\geq 0}\subset L_1(\Omega)$ is called a martingale with respect to $(\mathscr{F}_n)_{n\geq 0}$ if $E_n(f_{n+1})=f_n$ for every $n\geq 0$. Let $\mathscr{M}$ be the set of all martingale $f=(f_n)_{n\geq 0}$ relative to $(\mathscr{F}_n)_{n\geq 0}$ such that $f_0=0$. For $f\in\mathscr{M}$, denote its martingale difference by $d_nf=f_n-f_{n-1}$ $(n\geq 0$, with convention $d_0f=0)$.

The maximal function of $f\in\mathscr{M}$ is defined by

$$M_mf=\sup_{n\leq m}|f_n|, Mf=\sup_{n\geq 0}|f_n|. \tag{3}$$

For $p>1$ and $f\in L_p(\mathscr{M})$, we have

$$\|Mf\|_{L_p}\leq\frac{p}{p-1}\|f\|_{L_p}, \tag{4}$$

which is well known in the literature as the Doob maximal inequality (see [22]).

Hence, it follows from the above inequality that if $p\in(1,\infty)$, then L_p-bounded martingale converges in L_p. Moreover, if $p\in[1,\infty)$, then, for any $f\in L_p$, its corresponding martingale $(f_n)_{n\geq 0}$ with $f_n=E_nf$ is an L_p-bounded martingale and converges to f in L_p (see [21]). For this reason, a function $f\in L_1$ and the corresponding martingale $(f_n)_{n\geq 0}$ will be denoted by the same symbol f.

It is convenient for us to state the generalized grand Morrey-martingale spaces, we first need to recall the definition of martingale Morrey spaces $\mathscr{L}_{p,\lambda}=\mathscr{L}_{p,\lambda}(\Omega)$ as follows.

Definition 1. For $p\in[1,\infty)$ and $\lambda\in(-\infty,\infty)$, let

$$\mathscr{L}_{p,\lambda}=\left\{f\in\mathscr{M}:\|f\|_{\mathscr{L}_{p,\lambda}}<\infty\right\}, \tag{5}$$

where

$$\|f\|_{\mathscr{L}_{p,\lambda}}=\sup_{n\geq 0}\sup_{B\in A(\mathscr{F}_n)}\left(\frac{1}{\mathbb{P}(B)^{\lambda}}\int_B|f|^p d\mathbb{P}\right)^{1/p}. \tag{6}$$

Remark 2. If $\lambda=pu+1$, the above definition of $\|\cdot\|_{\mathscr{L}_{p,\lambda}}$ is equivalent to $\|\cdot\|_{L_{p,\mu}}$ (see (2)), which introduced by Nakai and Sadasue [13].

If $\lambda=0$ and $\mathscr{F}_0=\{\varnothing,\Omega\}$, then the Morrey-martingale space $\mathscr{L}_{p,\lambda}$ is L_p by the above definition.

Now we introduce a new type Morrey-martingale spaces as follows.

Definition 3. Let $1<p<\infty$, $0\leq\lambda<1$, φ be a nondecreasing real-valued nonnegative function defined on $(0,p-1]$ with $\lim_{x\longrightarrow 0^+}\varphi(x)=0$, and δ be a positive number. The generalized grand Morrey-martingale space $\mathscr{L}^{\delta,\varphi}_{p),\lambda)}(\Omega)$ consists of $f\in\mathscr{M}$ such that

$$\|f\|_{\mathscr{L}^{\delta,\varphi}_{p),\lambda)}}=\sup_{\substack{0<\varepsilon\leq s\\ s=\min\{p-1,\alpha\}}}\varepsilon^{\delta/(p-\varepsilon)}\sup_{n\geq 0}\sup_{B\in A(\mathscr{F}_n)}\cdot\left(\frac{1}{\mathbb{P}(B)^{\lambda-\varphi(\varepsilon)}}\int_B|f|^{p-\varepsilon}d\mathbb{P}\right)^{1/(p-\varepsilon)} \tag{7}$$

is finite, where $\alpha=\sup\{x>0:\varphi(x)\leq\lambda\}$.

Notice that, in the above condition, $\|\cdot\|_{\mathscr{L}^{\delta,\varphi}_{p)\lambda)}}$ is a norm and can be expressed as

$$\|f\|_{\mathscr{L}^{\delta,\varphi}_{p),\lambda)}}=\sup_{\substack{0<\varepsilon\leq s\\ s=\min\{p-1,\alpha\}}}\varepsilon^{\delta/(p-\varepsilon)}\|f\|_{\mathscr{L}_{p-\varepsilon,\lambda-\varphi(\varepsilon)}}. \tag{8}$$

Remark 4. If $\lambda>0$ and $\varphi\equiv 0$, then $\mathscr{L}^{\delta,\varphi}_{p),\lambda)}(\Omega)$ is called grand Morrey-martingale space, which was introduced in [19]. If $\lambda=0$, $\delta=1$, and $\varphi\equiv 0$, we recover the grand Lebesgue spaces for martingales introduced in [23]. In this case, if consider $\Omega=[0,1)$, we have grand Lebesgue spaces introduced in [24]. We mention that there exists a martingale $f=(f_n)_{n\geq 0}$ such that it does not converge in $L_{p)}([0,1))$. Indeed, $L_{p)}([0,1))$ is a rearrangement-invariant Banach function space, $L_{p)}([0,1))\neq L_1([0,1))$, but is not of absolutely continuous norm from [20]. According to Theorem 3.3 in [25], there exists a martingale $f=(f_n)_{n\geq 0}$ such that it does not converge in $L_{p)}([0,1))$.

The stochastic basis $\{\mathscr{F}_n\}_{n\geq 0}$ is said to be regular, if there exists a constant $R \geq 2$ such that

$$f_n \leq R f_{n-1} \tag{9}$$

holds for all nonnegative martingale $(f_n)_{n\geq 0}$ adapted to $\{\mathscr{F}_n\}_{n\geq 0}$.

For regular stochastic basis, there has the following property, proved in [13].

Lemma 5. *Let $\{\mathscr{F}_n\}_{n\geq 0}$ be regular. Then, for every sequence*

$$B_n \subset B_{n-1} \subset \cdots \subset B_k \subset \cdots \subset B_0, B_k \in A(\mathscr{F}_k), \tag{10}$$

we have

$$B_k = B_{k-1} \text{ or } \left(1+\frac{1}{R}\right)\mathbb{P}(B_k) \leq \mathbb{P}(B_{k-1}) \leq R\mathbb{P}(B_k)(1 \leq k \leq n), \tag{11}$$

where R is the positive constant in (9).

3. The Doob Maximal Operator

In this section, we present the boundedness of Doob's maximal operator on generalized grand Morrey-martingale spaces.

Theorem 6. *Let $1 < p < \infty$, $\delta > 0$, and $0 \leq \lambda < 1$. Then,*

$$\|Mf\|_{\mathscr{L}^{\delta,\varphi}_{p),\lambda)}} \leq C\|f\|_{\mathscr{L}^{\delta,\varphi}_{p),\lambda)}}, \tag{12}$$

where the constant C satisfies

$$C = \inf_{0<\theta<s} s^{\delta/(p-\varepsilon)}\theta^{-\delta/(p-\theta)}\left(\frac{p-\theta}{p-\theta-1}+1\right), \tag{13}$$

which only depends on the parameters $p, \lambda, \delta, \varphi$ for $s = \min\{p-1, \alpha\}$ and $\alpha = \sup\{x > 0 : \varphi(x) \leq \lambda\}$.

In order to prove Theorem 6, we need the following useful lemma:

Lemma 7. *Let $f = (f_n)_{n\geq 0} \in L_1$, $1 < p < \infty$, $0 \leq \lambda < 1$. Then,*

$$\|Mf\|_{\mathscr{L}_{p,\lambda}} \leq \left(\frac{p}{p-1}+1\right)\|f\|_{\mathscr{L}_{p,\lambda}}. \tag{14}$$

Proof. For any $B \in A(\mathscr{F}_m)$ and $m \geq 0$, suppose that $f = g + h$ and $g = f\chi_B$.

Then, according to the well-known Doob's maximal inequality, that is,

$$\|Mf\|_{L_p} \leq \frac{p}{p-1}\|f\|_{L_p}, \tag{15}$$

we have

$$\int_B |Mg|^p d\mathbb{P} \leq \int_\Omega |Mg|^p d\mathbb{P} \leq \left(\frac{p}{p-1}\right)^p \int_\Omega |g|^p d\mathbb{P} = \left(\frac{p}{p-1}\right)^p \int_B |f|^p d\mathbb{P}. \tag{16}$$

Hence,

$$\left(\frac{1}{\mathbb{P}(B)^\lambda}\int_B |Mg|^p d\mathbb{P}\right)^{1/p} \leq \frac{p}{p-1}\|f\|_{\mathscr{L}_{p,\lambda}}. \tag{17}$$

Next, take $B_n \in A(\mathscr{F}_n)$, $n = 0, 1, \cdots, m$, such that $B = B_m \subset B_{m-1} \subset \cdots \subset B_0$. Then, for a.e. $\omega \in B$,

$$E_n h(\omega) = \begin{cases} 0, & \text{if } n \geq m, \\ \dfrac{1}{\mathbb{P}(B_n)}\displaystyle\int_{B_n} h d\mathbb{P}, & \text{if } n < m. \end{cases} \tag{18}$$

If $n < m$, according to Jensen's inequality, then

$$|E_n h(\omega)| \leq \left(\frac{1}{\mathbb{P}(B_n)}\int_{B_n} |h|^p d\mathbb{P}\right)^{1/p} \leq \mathbb{P}(B_n)^{(\lambda-1)/p}\|f\|_{\mathscr{L}_{p,\lambda}} \leq \mathbb{P}(B)^{(\lambda-1)/p}\|f\|_{\mathscr{L}_{p,\lambda}}, \tag{19}$$

where the last inequality dues to $0 \leq \lambda < 1$ and $\mathbb{P}(B) \leq \mathbb{P}(B_n)$. This means

$$(Mh)(\omega) \leq \mathbb{P}(B)^{(\lambda-1)/p}\|f\|_{\mathscr{L}_{p,\lambda}} \text{ for any } \omega \in B. \tag{20}$$

Then, we obtain

$$\left(\frac{1}{\mathbb{P}(B)^\lambda}\int_B |Mh|^p d\mathbb{P}\right)^{1/p} \leq \|f\|_{\mathscr{L}_{p,\lambda}}. \tag{21}$$

Combining inequalities (17) and (21) and $Mf \leq Mg + Mh$, we can get

$$\left(\frac{1}{\mathbb{P}(B)^\lambda}\int_B (Mf)^p d\mathbb{P}\right)^{1/p} \leq \left(\frac{p}{p-1}+1\right)\|f\|_{\mathscr{L}_{p,\lambda}}. \tag{22}$$

The proof is complete. □

Note that Nakai and Sadasue [13] proved that, for $1 < p < \infty$ and $-1/p \leq \mu < 0$,

$$\|Mf\|_{L_{p,\mu}} \leq C_p\|f\|_{L_{p,\mu}}. \tag{23}$$

The proof of Lemma 7 is devoted to determination of the constant C_p. Now we prove Theorem 6:

Proof. Let $0<\theta<s$, and we have

$$\|Mf\|_{\mathscr{L}^{\delta,\varphi}_{p),\lambda)}} = \sup_{\substack{0<\varepsilon\le s\\ s=\min\{p-1,\alpha\}}} \varepsilon^{\delta/(p-\varepsilon)}\|Mf\|_{\mathscr{L}_{p-\varepsilon,\lambda-\varphi(\varepsilon)}} = \max\left\{\sup_{0<\varepsilon<\theta}\varepsilon^{\delta/(p-\varepsilon)}\|Mf\|_{\mathscr{L}_{p-\varepsilon,\lambda-\varphi(\varepsilon)}}, \sup_{\substack{\theta\le\varepsilon<s\\ s=\min\{p-1,\alpha\}}}\varepsilon^{\delta/(p-\varepsilon)}\|Mf\|_{\mathscr{L}_{p-\varepsilon,\lambda-\varphi(\varepsilon)}}\right\}, \tag{24}$$

where $\alpha=\sup\{x>0:\varphi(x)\le\lambda\}$. Let

$$I=\sup_{\substack{\theta\le\varepsilon<s\\ s=\min\{p-1,\alpha\}}}\varepsilon^{\delta/(p-\varepsilon)}\|Mf\|_{\mathscr{L}_{p-\varepsilon,\lambda-\varphi(\varepsilon)}}. \tag{25}$$

Note that the function $h(\varepsilon)\coloneqq\varepsilon^{\delta/(p-\varepsilon)}$ is increasing in $0<\varepsilon<p$, which means

$$\begin{aligned}
I &\le s^{\delta/(p-s)}\sup_{\substack{\theta\le\varepsilon<s\\ s=\min\{p-1,\alpha\}}}\sup_{n\ge0}\sup_{B\in A(\mathscr{F}_n)}\cdot\left(\frac{1}{\mathbb{P}(B)^{\lambda-\varphi(\varepsilon)}}\int_B|Mf|^{p-\varepsilon}d\mathbb{P}\right)^{1/(p-\varepsilon)}\\
&\le s^{\delta/(p-s)}\sup_{\substack{\theta\le\varepsilon<s\\ s=\min\{p-1,\alpha\}}}\sup_{n\ge0}\sup_{B\in A(\mathscr{F}_n)}\cdot\frac{1}{\mathbb{P}(B)^{(\lambda-\varphi(\varepsilon)-1)/(p-\varepsilon)}}\left(\frac{1}{\mathbb{P}(B)}\int_B|Mf|^{p-\varepsilon}d\mathbb{P}\right)^{1/(p-\varepsilon)}\\
&\le s^{\delta/(p-s)}\sup_{\substack{\theta\le\varepsilon<s\\ s=\min\{p-1,\alpha\}}}\sup_{n\ge0}\sup_{B\in A(\mathscr{F}_n)}\cdot\frac{\theta^{\delta/(p-\theta)}\theta^{-\delta/(p-\theta)}}{\mathbb{P}(B)^{\Delta(\varepsilon,\theta)}}\left(\frac{1}{\mathbb{P}(B)^{\lambda-\varphi(\theta)}}\int_B|Mf|^{p-\theta}d\mathbb{P}\right)^{1/(p-\theta)},
\end{aligned} \tag{26}$$

where

$$\Delta(\varepsilon,\theta)\coloneqq\frac{\lambda-\varphi(\varepsilon)-1}{p-\varepsilon}-\frac{\lambda-\varphi(\theta)-1}{p-\theta}. \tag{27}$$

Note that for $\theta\le\varepsilon$,

$$\begin{aligned}
\Delta(\varepsilon,\theta) &= \frac{(\varepsilon-\theta)(\lambda-1)+\varphi(\theta)(p-\varepsilon)-\varphi(\varepsilon)(p-\theta)}{(p-\varepsilon)(p-\theta)}\\
&\le \frac{(\varepsilon-\theta)(\lambda-1)+\varphi(\varepsilon)(p-\varepsilon)-\varphi(\varepsilon)(p-\theta)}{(p-\varepsilon)(p-\theta)}\\
&\le \frac{(\varepsilon-\theta)(\lambda-1)+\varphi(\varepsilon)(\theta-\varepsilon)}{(p-\varepsilon)(p-\theta)}\le 0.
\end{aligned} \tag{28}$$

Then, $0<1/\mathbb{P}(B)^{\Delta(\varepsilon,\theta)}\le1$, and we obtain $I\le s^{\delta/(p-s)}\theta^{-\delta/(p-\theta)}(\theta^{\delta/(p-\theta)}\|Mf\|_{\mathscr{L}_{p-\theta,\lambda-\varphi(\theta)}})$. Obviously, $s^{\delta/(p-s)}\theta^{-\delta/(p-\theta)}>1$ as $0<\theta<s$. Thus, according to Lemma 7, we deduce that

$$\begin{aligned}
\|Mf\|_{\mathscr{L}^{\delta,\varphi}_{p),\lambda)}} &\le s^{\delta/(p-s)}\theta^{-\delta/(p-\theta)}\sup_{0<\varepsilon\le\theta}\varepsilon^{\delta/(p-\varepsilon)}\|Mf\|_{\mathscr{L}_{p-\varepsilon,\lambda-\varphi(\varepsilon)}}\\
&\le s^{\delta/(p-s)}\theta^{-\delta/(p-\theta)}\sup_{0<\varepsilon\le\theta}\varepsilon^{\delta/(p-\varepsilon)}\left(\frac{p-\varepsilon}{p-\varepsilon-1}+1\right)\|f\|_{\mathscr{L}_{p-\varepsilon,\lambda-\varphi(\varepsilon)}}\\
&\le s^{\delta/(p-s)}\theta^{-\delta/(p-\theta)}\left(\frac{p-\theta}{p-\theta-1}+1\right)\|f\|_{\mathscr{L}^{\delta,\varphi}_{p),\lambda)}}.
\end{aligned} \tag{29}$$

Taking the infimum over all θ, we obtain that

$$\|Mf\|_{\mathscr{L}^{\delta,\varphi}_{p),\lambda)}}\le C\|f\|_{\mathscr{L}^{\delta,\varphi}_{p),\lambda)}}, \tag{30}$$

where

$$C=\inf_{\substack{0<\theta<s\\ s=\min\{p-1,\alpha\}\\ \alpha=\sup\{x>0:\varphi(x)\le\lambda\}}} s^{\delta/(p-s)}\theta^{-\delta/(p-\theta)}\left(\frac{p-\theta}{p-\theta-1}+1\right)\in(1,\infty). \tag{31}$$

□

4. The Fractional Integral Operator

In this section, we present the boundedness of the fractional integral operator in the new type grand Morrey-martingale spaces. In martingale theory, Chao and Ombe [26] introduced the fractional integrals for dyadic martingales. The fractional integrals in this section are defined for more general martingale setting as in [13, 14] (see also [15, 27–32]).

Definition 8. Let $f=(f_n)_{n\ge0}\in\mathscr{M}$ and $\iota>0$, and the fractional integral $I^\iota f=((I^\iota f)_n)_{n\ge0}$ of martingale f is defined by

$$(I^\iota f)_n=\sum_{k=0}^{n}b_{k-1}^{\iota}d_kf, \tag{32}$$

where b_k is an $\mathscr{F}_k$-measurable function such that

$$b_k(\omega)=\sum_{B\in A(\mathscr{F}_k)}\mathbb{P}(B)\chi_B(\omega),\quad \omega\in\Omega. \tag{33}$$

Remark 9. Obviously, b_k is bounded in above definition; there $I^\iota f=((I^\iota f)_n)_{n\ge0}$ is a martingale transform of f.

The following lemma was shown in [13]. Here we focus on more accurate upper boundedness.

Lemma 10. *Suppose that $\{\mathscr{F}_n\}_{n\ge0}$ is regular. Let $1<p<\infty$, $1\le q\le(\nu/u)p$, $-1/p\le\nu<0$, and $u=\nu+\iota<0$. Then, for $f\in L_1$,*

$$\|M(I^\iota f)\|_{L_{q,u}}\le C_{q,u,p,\nu}\|f\|_{L_{p,\nu}}, \tag{34}$$

where $C_{q,u,p,\nu} = [(1 + (1/R)^{\nu})/(1 - (1 + 1/R)^{u}) + 2] (p/(p-1) + 1)^{p/q}$ *and R is the constant in formula (9).*

Proof. Since $\|f\|_{L_{q_1,\lambda}} \le \|f\|_{L_{q_2,\lambda}}$ for $q_1 \le q_2$ by Hölder's inequality, it is enough to prove it in the case where $q = (\nu/u)p$. Without loss of generality, we let $\|f\|_{L_{p,\nu}} \neq 0$.

First, we prove the following inequality holds for any $n \ge 1$, and any $B_n \in A(\mathscr{F}_n)$,

$$\left|(I^{\iota} f)_n(\omega)\right| \le \left(\frac{1 + (1/R)^{\nu}}{1 - (1 + 1/R)^{u}} + 2\right)(Mf(\omega))^{u/\nu}\|f\|_{L_{p,\nu}}^{-\iota/\nu}, \quad \omega \in B_n. \tag{35}$$

Choose $B_k \in A(\mathscr{F}_k)$ and $0 \le k < n$, such that $B_n \subset B_{n-1} \subset \cdots \subset B_0$, and let

$$K = \{k : 0 < k \le n, B_k \neq B_{k-1}\} = \{k_1, k_2, \cdots, k_h\}, \tag{36}$$

where $0 = k_0 < k_1 < k_2 < \cdots < k_h$.

Since $\{\mathscr{F}_n\}_{n\ge 0}$ is regular, according to Lemma 5, we have

$$\left(1 + \frac{1}{R}\right) b_{k_j} \le b_{k_{j-1}} \le R\, b_{k_j} \text{ on } B_n. \tag{37}$$

So, for $k \notin K$, we have $b_k = b_{k-1}$ and $d_k f = 0$. Hence, we obtain

$$(I^{\iota} f)_n = \sum_{0<k_j\le n} b_{k_{j-1}}^{\iota} d_{k_j} f = \sum_{j=1}^{h} b_{k_{j-1}}^{\iota} d_{k_j} f \text{ on } B_n. \tag{38}$$

For $\omega \in B_n$,

$$\begin{aligned}
\left|d_{k_j} f(\omega)\right| &= \left|f_{k_j}(\omega) - f_{k_{j-1}}(\omega)\right| \le \left|f_{k_j}(\omega)\right| + \left|f_{k_{j-1}}(\omega)\right| \\
&= \left|\frac{1}{\mathbb{P}\left(B_{k_j}\right)}\int_{B_{k_j}} f\, d\mathbb{P}\right| + \left|\frac{1}{\mathbb{P}\left(B_{k_{j-1}}\right)}\int_{B_{k_{j-1}}} f\, d\mathbb{P}\right| \\
&\le \left(\frac{1}{\mathbb{P}\left(B_{k_j}\right)}\int_{B_{k_j}} |f|^p d\mathbb{P}\right)^{1/p} + \left(\frac{1}{\mathbb{P}\left(B_{k_{j-1}}\right)}\int_{B_{k_{j-1}}} |f|^p d\mathbb{P}\right)^{1/p} \\
&\le \left(\mathbb{P}\left(B_{k_j}\right)^{\nu} + \mathbb{P}\left(B_{k_{j-1}}\right)^{\nu}\right)\|f\|_{L_{p,\nu}} \\
&\le \left(1 + \left(\frac{1}{R}\right)^{\nu}\right) b_{k_{j-1}}(\omega)^{\nu}\|f\|_{L_{p,\nu}}.
\end{aligned} \tag{39}$$

Then, for $\omega \in B_n$ and when $0 < k \le m$ where $m \le n$,

$$\begin{aligned}
\left|\sum_{k=0}^{m} b_{k-1}(\omega)^{\iota} d_k f(\omega)\right| &\le \left(1 + \left(\frac{1}{R}\right)^{\nu}\right)\sum_{0<k_j\le m} b_{k_{j-1}}(\omega)^{\nu+\iota}\|f\|_{L_{p,\nu}} \\
&= \left(1 + \left(\frac{1}{R}\right)^{\nu}\right)\sum_{0<k_j\le m} b_{k_{j-1}}(\omega)^{u}\|f\|_{L_{p,\nu}} \\
&\le \frac{1 + (1/R)^{\nu}}{1 - (1 + 1/R)^{u}} b_m(\omega)^{u}\|f\|_{L_{p,\nu}}.
\end{aligned} \tag{40}$$

For $\omega \in B_n$ and when $m + 1 \le k < n$, let $j(k) = \min\{j : k < k_j\}$, and we have

$$\begin{aligned}
&\left|\sum_{k=m+1}^{n} b_{k-1}(\omega)^{\iota} d_k f(\omega)\right| \\
&= \left|\sum_{j=j(k)}^{h} b_{k_{j-1}}(\omega)^{\iota} d_{k_j} f(\omega)\right| \\
&= \left|\sum_{j=j(k)}^{h} b_{k_{j-1}}(\omega)^{\iota} f_{k_j}(\omega) - \sum_{j=j(k)}^{h} b_{k_{j-1}}(\omega)^{\iota} f_{k_{j-1}}(\omega)\right| \\
&= \left|b_{k_{h-1}}(\omega)^{\iota} f_{k_h}(\omega) + \sum_{j=j(k)}^{h-1}\left(b_{k_{j-1}}(\omega)^{\iota} - b_{k_j}(\omega)^{\iota}\right) f_{k_j}(\omega) - b_{k_{j(k)-1}}(\omega)^{\iota} f_{k_{j(k)-1}}(\omega)\right| \\
&\le b_{k_{h-1}}(\omega)^{\iota} Mf(\omega) + \sum_{j=j(k)}^{h-1}\left|b_{k_{j-1}}(\omega)^{\iota} - b_{k_j}(\omega)^{\iota}\right| Mf(\omega) + b_{k_{j(k)-1}}(\omega)^{\iota} Mf(\omega) \\
&\le 2 b_{k_{j(k)-1}}(\omega)^{\iota} Mf(\omega) = 2 b_m(\omega)^{\iota} Mf(\omega).
\end{aligned} \tag{41}$$

Now let

$$\Lambda_1 = \left\{\omega \in \Omega : \left(\frac{Mf(\omega)}{\|f\|_{L_{p,\nu}}}\right)^{1/\nu} \le b_0(\omega)\right\} \text{ and } \Lambda_2 = \Omega \setminus \Lambda_1. \tag{42}$$

Next we estimate $(I^{\iota} f)_n$ from the following cases. For the first case, if $\omega \in \Lambda_1 \cap B_n$ and

$$\left(\frac{Mf(\omega)}{\|f\|_{L_{p,\nu}}}\right)^{1/\nu} \le b_n(\omega), \tag{43}$$

then, by formula (40) and $u = \nu + \iota < 0$, we have

$$\begin{aligned}
\left|(I^{\iota} f)_n(\omega)\right| &\le \frac{1 + (1/R)^{\nu}}{1 - (1 + 1/R)^{u}} b_n(\omega)^{u}\|f\|_{L_{p,\nu}} \\
&\le \frac{1 + (1/R)^{\nu}}{1 - (1 + 1/R)^{u}}\left(\frac{Mf(\omega)}{\|f\|_{L_{p,\nu}}}\right)^{u/\nu}\|f\|_{L_{p,\nu}} \\
&= \frac{1 + (1/R)^{\nu}}{1 - (1 + 1/R)^{u}}(Mf(\omega))^{u/\nu}\|f\|_{L_{p,\nu}}^{-\iota/\nu}.
\end{aligned} \tag{44}$$

For the second case, if $\omega \in \Lambda_1 \cap B_n$ and

$$b_n(\omega) < \left(\frac{Mf(\omega)}{\|f\|_{L_{p,\nu}}}\right)^{1/\nu}, \tag{45}$$

then there exists m such that

$$\frac{1}{R} b_m(\omega) < \left(\frac{Mf(\omega)}{\|f\|_{L_{p,\nu}}}\right)^{1/\nu} \le b_m(\omega). \tag{46}$$

Combining (46) with formulas (40) and (41), we have

$$\begin{aligned}\left|(I^\iota f)_n(\omega)\right| &\le \frac{1+(1/R)^\nu}{1-(1+1/R)^u} b_m(\omega)^u \|f\|_{L_{p,\nu}} + 2b_m(\omega)^\iota Mf(\omega) \\ &\le \frac{1+(1/R)^\nu}{1-(1+1/R)^u}\left(\frac{Mf(\omega)}{\|f\|_{L_{p,\nu}}}\right)^{u/\nu}\|f\|_{L_{p,\nu}} + 2\left(\frac{Mf(\omega)}{\|f\|_{L_{p,\nu}}}\right)^{\iota/\nu} Mf(\omega) \\ &\le \left(\frac{1+(1/R)^\nu}{1-(1+1/R)^u}+2\right)(Mf(\omega))^{u/\nu}\|f\|_{L_{p,\nu}}^{-\iota/\nu}.\end{aligned} \tag{47}$$

For the third case, if $\omega \in \Lambda_2 \cap B_n$, then by (41), we have

$$\begin{aligned}\left|(I^\iota f)_n(\omega)\right| &\le 2b_0(\omega)^\iota Mf(\omega) \\ &\le 2\left(\frac{Mf(\omega)}{\|f\|_{L_{p,\nu}}}\right)^{\iota/\nu} Mf(\omega) = 2(Mf(\omega))^{u/\nu}\|f\|_{L_{p,\nu}}^{-\iota/\nu}.\end{aligned} \tag{48}$$

Formulas (44), (47), and (48) give that

$$\left|(I^\iota f)_n(\omega)\right| \le \left(\frac{1+(1/R)^\nu}{1-(1+1/R)^u}+2\right)(Mf(\omega))^{u/\nu}\|f\|_{L_{p,\nu}}^{-\iota/\nu}, \omega \in B_n, \tag{49}$$

which implies that

$$\begin{aligned}&\left(\frac{1}{\mathbb{P}(B)}\int_B |M(I^\iota f(\omega))|^q d\mathbb{P}\right)^{1/q} \\ &\quad\le \left(\frac{1+(1/R)^\nu}{1-(1+1/R)^u}+2\right)\left(\frac{1}{\mathbb{P}(B)}\int_B (Mf(\omega))^{qu/\nu} d\mathbb{P}\right)^{1/q}\|f\|_{L_{p,\nu}}^{-\iota/\nu} \\ &\quad= \left(\frac{1+(1/R)^\nu}{1-(1+1/R)^u}+2\right)\left(\frac{1}{\mathbb{P}(B)}\int_B (Mf(\omega))^p d\mathbb{P}\right)^{(1/p)(p/q)}\|f\|_{L_{p,\nu}}^{1-p/q}.\end{aligned} \tag{50}$$

Moreover, by Lemma 7, we have

$$\begin{aligned}\left(\frac{1}{\mathbb{P}(B)}\int_B (Mf(\omega))^p d\mathbb{P}\right)^{(1/p)(p/q)} &\le \left(\mathbb{P}(B)^\nu \|Mf\|_{L_{p,\nu}}\right)^{p/q} \\ &\le \left(\frac{p}{p-1}+1\right)^{p/q}\mathbb{P}(B)^u\|f\|_{L_{p,\nu}}^{p/q}.\end{aligned} \tag{51}$$

It follows from the above inequality and (50) that

$$\begin{aligned}&\left(\frac{1}{\mathbb{P}(B)}\int_B |M(I^\iota f(\omega))|^q d\mathbb{P}\right)^{1/q} \\ &\quad\le \left(\frac{1+(1/R)^\nu}{1-(1+1/R)^u}+2\right)\left(\frac{p}{p-1}+1\right)^{p/q}\mathbb{P}(B)^u\|f\|_{L_{p,\nu}},\end{aligned} \tag{52}$$

that is to say,

$$\|M(I^\iota f)\|_{L_{q,u}} \le \left(\frac{1+(1/R)^\nu}{1-(1+1/R)^u}+2\right)\left(\frac{p}{p-1}+1\right)^{p/q}\|f\|_{L_{p,\nu}}. \tag{53}$$

The proof is complete. □

Theorem 11. *Let $1<q<\infty$, $0\le\lambda<1$, $0<\iota<(1-\lambda)/q$, $1/q-1/p=\iota/(1-\lambda)$, $\delta_2>0$, and $\delta_1\ge\delta_2(1+\iota p/(1-\lambda))$. Suppose that φ_1 and φ_2 are continuous nonnegative and nondecreasing real-valued functions on $(0,p-1]$ and $(0,q-1]$, respectively, satisfying*

- *(i) $\varphi_1 \in C^1(0,\kappa]$ for some positive $\kappa>0$*
- *(ii) $\lim_{x\longrightarrow 0^+}\varphi_1(x)=0$*
- *(iii) $0\le \lim_{x\longrightarrow 0^+} d\varphi_1(x)/dx < (1-\lambda)^2/(\iota p^2)$*
- *(iv) $\varphi_2(\eta)=\varphi_1(\phi^{-1}(\eta))$, where ϕ^{-1} is the inverse of ϕ on $(0,\kappa]$ for $\kappa>0$, and $\phi(x)=q-(p-x)(1-\lambda+\varphi_1(x))/[1-\lambda+\varphi_1(x)+\iota(p-x)]$. Then, for $f\in\mathscr{L}^{\delta_2,\varphi_2}_{q),\lambda)}$,*

$$\|M(I^\iota f)\|_{\mathscr{L}^{\delta_1,\varphi_1}_{p),\lambda)}} \le C(p,\delta_1,\delta_2,\varphi_1,\lambda)\|f\|_{\mathscr{L}^{\delta_2,\varphi_2}_{q),\lambda)}}, \tag{54}$$

where $C(p,\delta_1,\delta_2,\varphi_1,\lambda)$ only depends on p, δ_1, δ_2, φ_1, and λ.

Proof. The equation $1/q-1/p=\iota/(1-\lambda)$ and $\lim_{x\longrightarrow 0^+}\varphi_1(x)=0$ give that $\lim_{x\longrightarrow 0^+}\phi(x)=0$. The condition (iii) ensures that $\lim_{x\longrightarrow 0^+} d\phi(x)/dx>0$. Then, there exists small positive number $\epsilon<\min\{1,\kappa\}$ such that ϕ is increasing in $(0,\epsilon]$ and $\phi(\epsilon)<(q-1)/2$. Now fix

$$\theta \in (0, \min\{s,\epsilon\}), \tag{55}$$

where $s=\min\{p-1,\alpha\}$ and $\alpha=\sup\{x>0:\varphi_1(x)\le\lambda\}$.

Firstly, we consider the case of $\varepsilon\in(\theta,s)$. In this situation, let

$$I(\varepsilon) := \varepsilon^{\delta_1/(p-\varepsilon)}\left(\frac{1}{\mathbb{P}(B)^{\lambda-\varphi_1(\varepsilon)}}\int_B |M(I^\iota f)|^{p-\varepsilon}d\mathbb{P}\right)^{1/(p-\varepsilon)}. \tag{56}$$

Since $p-\varepsilon<p-\theta$, then it follows from Jensen's inequality that

$$I(\varepsilon)=\varepsilon^{\delta_1/(p-\varepsilon)}\mathbb{P}(B)^{(\varphi_1(\varepsilon)+1-\lambda)/(p-\varepsilon)}\left(\frac{1}{\mathbb{P}(B)}\int_B|M(I^\iota f)|^{p-\varepsilon}d\mathbb{P}\right)^{1/(p-\varepsilon)} \leq \varepsilon^{\delta_1/(p-\varepsilon)}\mathbb{P}(B)^{(\varphi_1(\varepsilon)+1-\lambda)/(p-\varepsilon)}\left(\frac{1}{\mathbb{P}(B)}\int_B|M(I^\iota f)|^{p-\theta}d\mathbb{P}\right)^{1/(p-\theta)}. \tag{57}$$

Note that $[\varphi_1(x)+1-\lambda]/(p-x)$ is a nonnegative and nondecreasing function on $(\theta,s]$; hence,

$$I(\varepsilon)\leq \varepsilon^{\delta_1/(p-\varepsilon)}\mathbb{P}(B)^{(\varphi_1(\theta)+1-\lambda)/(p-\theta)}\left(\frac{1}{\mathbb{P}(B)}\int_B|M(I^\iota f)|^{p-\theta}d\mathbb{P}\right)^{1/(p-\theta)} \leq s^{\delta_1/(p-s)}\theta^{-\delta_1/(p-\theta)}\sup_{0<t\leq\theta}t^{\delta_1/(p-t)}\sup_{n\geq 0}\sup_{B\in A(\mathcal{F}_n)} \cdot\left(\frac{1}{\mathbb{P}(B)^{\lambda-\varphi_1(t)}}\int_B|M(I^\iota f)|^{p-t}d\mathbb{P}\right)^{1/(p-t)}. \tag{58}$$

Since $s^{\delta_1/(p-s)}\theta^{-\delta_1/(p-\theta)}>1$ for $\theta<s$, then the following inequality holds

$$\|M(I^\iota f)\|_{\mathscr{L}^{\delta_1,\varphi_1}_{p),\lambda)}}\leq s^{\delta_1/(p-s)}\theta^{-\delta_1/(p-\theta)}\sup_{0<t\leq\theta}t^{\delta_1/(p-t)}\sup_{n\geq 0}\sup_{B\in A(\mathcal{F}_n)} \cdot\left(\frac{1}{\mathbb{P}(B)^{\lambda-\varphi_1(t)}}\int_B|M(I^\iota f)|^{p-t}d\mathbb{P}\right)^{1/(p-t)}. \tag{59}$$

Next, we consider $t\in(0,\theta]$ in the following discussion. Since $1/q-1/p=\iota/(1-\lambda)$, we can choose η and t satisfying

$$\frac{1}{q-\eta}-\frac{1}{p-t}=\frac{\iota}{1-\lambda+\varphi_1(t)}. \tag{60}$$

Obviously we know that $t\longrightarrow 0$ if and only if $\eta\longrightarrow 0$, and we obtain η with respect to t as follows:

$$\eta=q-\frac{(p-t)(1-\lambda+\varphi_1(t))}{1-\lambda+\varphi_1(t)+\iota(p-t)}=\phi(t). \tag{61}$$

Let

$$\tilde{p}=q-\eta,\ \tilde{q}=p-t,\ \tilde{v}=\frac{\lambda-\varphi_1(t)-1}{q-\eta},\quad \tilde{u}=\frac{\lambda-\varphi_1(t)-1}{p-t}. \tag{62}$$

It is not hard to see that $1\leq\tilde{q}=(\tilde{v}/\tilde{u})\tilde{p}$, $-1/\tilde{p}\leq\tilde{v}<0$, $\tilde{v}+\iota=\tilde{u}\leq-(1-\lambda)/(p-t)<0$, and

$$q-\eta=q-\phi(t)\geq q-\phi(\epsilon)\geq\frac{q+1}{2}>1. \tag{63}$$

Moreover,

$$C_{\tilde{q},\tilde{u},\tilde{p},\tilde{v}}=\left(\frac{1+(1/R)^{\tilde{v}}}{1-(1+1/R)^{\tilde{u}}}+2\right)\left(\frac{\tilde{p}}{\tilde{p}-1}+1\right)^{\tilde{p}/\tilde{q}} \leq\left(\frac{1+R^{1+\varphi_1(\theta)}}{1-(R/(R+1))^{(1-\lambda)/p}}+2\right)\left(\frac{q-\phi(\varepsilon)}{q-\phi(\varepsilon)-1}+1\right)^{q} \leq\left(\frac{1+R^{1+\varphi_1(\theta)}}{1-(R/(R+1))^{(1-\lambda)/p}}+2\right)\left(\frac{q-\phi(\theta)}{q-\phi(\theta)-1}+1\right)^{q} =: C(\theta). \tag{64}$$

Notice that $\varphi_1(\theta)\leq\varphi_1(s)$ and $q-\phi(\theta)-1\geq q-\phi(\epsilon)-1\geq(q-1)/2$. This implies that $C(\theta)<\infty$.

Thus, according to the inequalities (59) and (64) and Lemma 10, we have

$$\begin{aligned}\|M(I^\iota f)\|_{\mathscr{L}^{\delta_1,\varphi_1}_{p),\lambda)}}&\leq s^{\delta_1/(p-s)}\theta^{-\delta_1/(p-\theta)}\sup_{0<t\leq\theta}t^{\delta_1/(p-t)}\sup_{n\geq 0}\sup_{B\in A(\mathcal{F}_n)} \cdot\frac{1}{\mathbb{P}(B)^{(\lambda-\varphi_1(t)-1)/(p-t)}}\left(\frac{1}{\mathbb{P}(B)}\int_B|M(I^\iota f)|^{p-t}d\mathbb{P}\right)^{1/(p-t)}\\ &=s^{\delta_1/(p-s)}\theta^{-\delta_1/(p-\theta)}\sup_{0<t\leq\theta}t^{\delta_1/(p-t)}\|M(I^\iota f)\|_{L_{\tilde{q},\tilde{u}}}\\ &\leq C(\theta)s^{\delta_1/(p-s)}\theta^{-\delta_1/(p-\theta)}\sup_{0<t\leq\theta}t^{\delta_1/(p-t)}\|f\|_{L_{\tilde{p},\tilde{v}}}\\ &=C(\theta)s^{\delta_1/(p-s)}\theta^{-\delta_1/(p-\theta)}\sup_{0<t\leq\theta}t^{\delta_1/(p-t)}\sup_{n\geq 0}\sup_{B\in A(\mathcal{F}_n)} \cdot\frac{1}{\mathbb{P}(B)^{\lambda-\varphi_1(t)-1/q-\eta}}\left(\frac{1}{\mathbb{P}(B)}\int_B|f|^{q-\eta}d\mathbb{P}\right)^{1/(q-\eta)}\\ &=C(\theta)s^{\delta_1/(p-s)}\theta^{-\delta_1/(p-\theta)}\sup_{0<t\leq\theta}t^{\delta_1/(p-t)}\sup_{n\geq 0}\sup_{B\in A(\mathcal{F}_n)} \cdot\left(\frac{1}{\mathbb{P}(B)^{\lambda-\varphi_1(t)}}\int_B|f|^{q-\eta}d\mathbb{P}\right)^{1/q-\eta}\\ &\leq C(\theta)s^{\delta_1/(p-s)}\theta^{-\delta_1/(p-\theta)}\sup_{0<t\leq\theta}t^{\delta_1/(p-t)}\eta^{-\delta_2/(q-\eta)}\|f\|_{\mathscr{L}^{\delta_2,\varphi_2}_{q),\lambda)}},\end{aligned} \tag{65}$$

where the last inequality holds because of $\varphi_2(\eta)=\varphi_1(\phi^{-1}(\eta))=\varphi_1(t)$.

Finally, we shall show that $\sup_{0<t\leq\theta}t^{\delta_1/(p-t)}\eta^{-\delta_2/(q-\eta)}$ is bounded. Since $\lim_{x\longrightarrow 0^+}\phi(x)=0$, by l'Hospital's rule, we have

$$\lim_{x\longrightarrow 0^+}\frac{\phi(x)}{x}=\lim_{x\longrightarrow 0^+}\frac{\phi'(x)}{x'}=\lim_{x\longrightarrow 0^+}\frac{(1-\lambda)^2-\iota p^2\varphi_1'(x)}{(1-\lambda+\iota p)^2}. \tag{66}$$

Combining with the condition (iii), we have $\phi(x)\sim x$ as $x\longrightarrow 0^+$. This implies

$$\eta^{-\delta_2/(q-\eta)}\sim t^{-\delta_2/(q-\eta)},\ \text{as}\quad t\longrightarrow 0^+. \tag{67}$$

Moreover, using $\delta_1 \geq \delta_2(1 + \iota p/(1-\lambda))$, $0 < t \leq \theta < 1$, and formula (60), we obtain

$$t^{\delta_1/(p-t)}\eta^{-\delta_2/(q-\eta)} \leq Ct^{\delta_2(1+\iota p/(1-\lambda))/(p-t)}t^{-\delta_2(1/(p-t)+\iota/(1-\lambda+\varphi_1(t)))} = Ct^{\iota\delta_2[p/((1-\lambda)(p-t))-1/(1-\lambda+\varphi_1(t))]}. \quad (68)$$

Obviously, $p/[(1-\lambda)(p-t)] - 1/[1-\lambda+\varphi_1(t)] > 0$, which implies that $\sup_{0<t\leq\theta} t^{\delta_1/(p-t)}\eta^{-\delta_2/(q-\eta)} \leq C$ is bounded.

To sum up, we have

$$\|M(I^\iota f)\|_{\mathscr{L}^{\delta_1,\varphi_1}_{p),\lambda)}} \leq C(p,\delta_1,\delta_2,\varphi_1,\lambda)\|f\|_{\mathscr{L}^{\delta_2,\varphi_2}_{q),\lambda)}}, \quad (69)$$

where

$$C(p,\delta_1,\delta_2,\varphi_1,\lambda) = Cs^{\delta_1/(p-s)} \inf_{\theta\in(0,\min\{s,\epsilon\})} C(\theta)\theta^{-\delta_1/(p-\theta)}. \quad (70)$$

□

Remark 12. Recently, new results concerning the grand variable exponent Lebesgue spaces for martingales have emerged (see [33]).

Data Availability

No data is used in the manuscript.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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