Research Article

Generalized Laplace-Type Transform Method for Solving Multilayer Diffusion Problems

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Multilayer diffusion problems have found significant importance that they arise in many medical, environmental, and industrial applications of heat and mass transfer. In this article, we study the solvability of a one-dimensional nonhomogeneous multilayer diffusion problem. A new generalized Laplace-type integral transform is used, namely, the $M_{\rho,m}$-transform. First, we reduce the nonhomogeneous multilayer diffusion problem into a sequence of one-layer diffusion problems including time-varying given functions, followed by solving a general nonhomogeneous one-layer diffusion problem via the $M_{\rho,m}$-transform. Hence, by means of general interface conditions, a renewal equations’ system is determined. Finally, the $M_{\rho,m}$-transform and its analytic inverse are used to obtain an explicit solution to the renewal equations’ system. Our results are of general attractiveness and comprise a number of previous works as special cases.

1. Introduction

The multilayer diffusion problems are typical models for a variety of solute transport phenomena in layered permeable media, such as advection, dispersion, and reaction diffusions [1–10]. These problems have had their importance due to their natural prevalence in a remarkably large number of applications such as chamber-based gas flux measurements [11], contamination and decontamination in permeable media [6, 12], drug-eluting stent [13, 14], drug absorption [15, 16], moisture propagation in woven fabric composites [17], permeability of the skin [18], and wool-washing [19]. Further applications have been considered in [20, 21].

As epidemiological models, reaction-diffusion problems are widely applied to model and analyze the spread of diseases such as the global COVID-19 pandemic caused by SARS-CoV-2. These models describe the spatiotemporal prevalence of the viral pandemic and apprehend the dynamics depending on human habits and geographical features. The models estimate a qualitative harmony between the simulated prediction of the local spatiotemporal spread of a pandemic and the epidemiological collected datum (see [22, 23]). These data-driven emulations can essentially inform the respective authorities to purpose efficient pandemic-arresting measures and foresee the geographical distribution of vital medical resources. Moreover, such studies explore alternate scenarios for the repose of lockdown restrictions based on the local inhabitance densities and the qualitative dynamics of the infection. For more applications, one can refer, e.g., to [24, 25].

Although the numerical methods are usually applied to solve the diffusion problems, especially in the heterogeneous permeable media, the analytic solutions, when available, are characterized by their exactness and continuity in space and time. In the context of obtaining numerical solutions for such models, we refer to the following references [26–30]. In this work, we focus on analytic solutions of certain nonhomogeneous diffusion problems in multilayer permeable media. Here, the retardation factors are assumed to be constant, the dispersion coefficients vary across layers, but being constants within each layer, and the free terms are (arbitrary) time-varying functions.
Analytic and semianalytic solutions of multilayer diffusion problems are developed by using the integral transforms [6, 25, 31–40]. Applying Laplace transforms, to solve multilayer diffusion problems, has advantages as an applicable tool in handling different types of boundary conditions and averts solving complicated transcendental equations as demanded by eigenfunction expansion methods. Further works involving the Laplace transform have studied the permeable layered reaction diffusion problem in [41, 42]. Solutions obtained in these works are restricted to two layers as well as obtaining the inverse Laplace transform numerically. In the same context, generalized integral transform techniques, for short GITT, are well-established hybrid approaches for solving diffusion and convection-diffusion problems, in which hybrid refers to the combination of classical analytical methods with modern computational tools aimed at accurate, robust, and low-cost solutions [43–47]. In the current work, we aim to extend, generalize, and merge results in [31, 33, 38, 40, 42] to solve certain nonhomogeneous diffusion problems in one-dimensional n-layered media. We use a new generalized integral transform recently introduced in [48]. The obtained solutions are applicable to more general linear nonhomogeneous diffusion equations, finite media consisting of arbitrary many layers, continuity and dispersive flow at the contact interfaces between sequal layers, and transitory boundary conditions of the arbitrary type at the inlet and outlet. To the best knowledge of the authors, analytical solutions verifying all the above mentioned conditions have not been previously reported in the literature which strongly motivates this current work.

In the remaining part of this introductory section, in Subsection 1.1, the multilayer diffusion problem is described, and then, it is reformulated as a sequence of one-layer diffusion problems having boundary conditions including given time-depending functions. Basic properties for the \( \mathcal{M}_{p,m} \)-transform that will be needed in this work are stated in Subsection 1.2. The remaining sections are constructed as follows: in Section 2, we discuss the solvability of a general linear nonhomogeneous one-layer diffusion problem with arbitrary time-varying data, using the \( \mathcal{M}_{p,m} \)-transform. Section 3 is devoted to our main multilayer diffusion problem, where in Subsection 3.1, we solve a two-layer problem to shed light on the basic idea by considering this simple case. Further, in Subsection 3.2, we return to benefit from the results obtained in Section 2 and Subsection 3.1 to solve the main multilayer diffusion problems (2)–(8) (see Subsection 1.1 below).

### 1.1. Mathematical Modeling for Nonhomogeneous n-Layer Diffusion Systems

A one-dimensional diffusion problem in an n-layered permeable medium is set out as follows. Let

\[
\alpha = x_0 < x_1 < \cdots < x_{n-1} < x_n = \beta,
\]

be a finite partition of the interval \([\alpha, \beta]\). In each subinterval \([x_{j-1}, x_j]\), with \(j = 1, 2, \cdots, n\), the component function \(\varphi_j(x, t)\) satisfies the partial differential equation (PDE)

\[
\frac{\partial \varphi_j}{\partial t} = d_j \frac{\partial^2 \varphi_j}{\partial x^2} + \lambda(t, \tau) r_j(x, t), \quad x \in (x_{j-1}, x_j), \quad t, \tau > 0,
\]

where \(d_j \geq 0\), for all \(1 \leq j \leq n\), are the diffusion coefficients and

\[
\lambda(t, \tau) = \left( \frac{\tau^m}{\tau^m + t^m} \right)^\rho, \quad t \geq 0, \tau > 0,
\]

with \(m \in \mathbb{Z}_+ = \{1, 2, 3, \cdots\}, \rho \in \mathbb{C}, \text{Re}(\rho) > 0\). Here, the function-term \(\lambda(t, \tau) r_j(x, t)\) physically means the external source term that could be applied to the diffusion equation with \(r_j(x, t)\) depends on time and space while the other factor of the source term, i.e., \(\lambda\), depends only on time. This last factor could be, for instance, a periodic-time magnetic source.

The initial conditions (ICs) are assumed as

\[
\varphi_j(x, 0) = \eta_j(x), \quad x \in [x_{j-1}, x_j], 1 \leq j \leq n.
\]

The boundary conditions (BCs) are posited as

(i) The outer BCs (at the inlet \(x = \alpha\) and the outlet \(x = \beta\) are general Robin boundary conditions as

\[
\varphi_j(\alpha, t) + \mu_1 \frac{\partial \varphi_j}{\partial x}(\alpha, t) = \lambda(t, \tau) \zeta_1(\tau),
\]

\[
\varphi_j(\beta, t) + \mu_1 \frac{\partial \varphi_j}{\partial x}(\beta, t) = \lambda(t, \tau) \zeta_1(\tau),
\]

for all \(t \geq 0\), with \(\tau, \xi, \mu_1\), and \(l\) are constants satisfying \(|\tau|+|\xi|>0, |l| + |l|>0\).

(ii) The inner BCs (the interface conditions) are

\[
\varphi_j(x_j, t) = \Lambda_j \varphi_{j+1}(x_j, t),
\]

\[

\nu \varphi_j(x_j, t) + \mu_j \frac{\partial \varphi_j}{\partial x}(x_j, t) = \nu_{j+1} \varphi_{j+1}(x_j, t) + \mu_{j+1} \frac{\partial \varphi_{j+1}}{\partial x}(x_j, t),
\]

for all \(t \geq 0\), with \(|\nu_j| + |\mu_j|>0\) for all \(j = 1, 2, \cdots, n-1\).

For appropriate given functions \(\eta_1, \cdots, \eta_n, \zeta\), and \(\xi\), we are going to find an analytic solution of the problems (2)–(8) using the \(\mathcal{M}_{p,m}\)-generalized integral transform, introduced recently in [48]. Problems (2)–(8) can be reduced into the following sequence of one-layer diffusion problems.
(i) In the inlet layer, i.e., \( x \in [x_0, x_1] \),

\[
\frac{\partial \phi_1}{\partial t} + \frac{\partial^2 \phi_1}{\partial x^2} + \lambda(t, \tau) r_1(x, t), \quad x \in (x_0, x_1), t, \tau > 0,
\]

\[
\phi_1(x, 0) = \eta_1(x), \quad x \in [x_0, x_1],
\]

\[
n \phi_1(x_0, t) + \frac{\partial \phi_1}{\partial x}(x_0, t) = \lambda(t, \tau) \xi_1(t), \quad t \geq 0, \tau > 0,
\]

\[
\nu_1 \phi_1(x_1, t) + \mu_1 \frac{\partial \phi_1}{\partial x}(x_1, t) = \lambda(t, \tau) \xi_1(t), \quad t \geq 0, \tau > 0.
\]

(ii) In the interior layers, i.e., \( x \in [x_{j-1}, x_j] \), \( 2 \leq j \leq n - 1 \),

\[
\frac{\partial \phi_j}{\partial t} + \frac{\partial^2 \phi_j}{\partial x^2} + \lambda(t, \tau) r_j(x, t), \quad x \in (x_{j-1}, x_j), t, \tau > 0,
\]

\[
\phi_j(x, 0) = \eta_j(x), \quad x \in [x_{j-1}, x_j],
\]

\[
v_j \phi_j(x_{j-1}, t) + \mu_j \frac{\partial \phi_j}{\partial x}(x_{j-1}, t) = \lambda(t, \tau) \xi_j(t), \quad t \geq 0, \tau > 0,
\]

\[
\nu_j \phi_j(x_j, t) + \mu_j \frac{\partial \phi_j}{\partial x}(x_j, t) = \lambda(t, \tau) \xi_j(t), \quad t \geq 0, \tau > 0.
\]

(iii) In the outlet layer, i.e., \( x \in [x_{n-1}, x_n] \),

\[
\frac{\partial \phi_n}{\partial t} + \frac{\partial^2 \phi_n}{\partial x^2} + \lambda(t, \tau) r_n(x, t), \quad x \in (x_{n-1}, x_n), t, \tau > 0,
\]

\[
\phi_n(x, 0) = \eta_n(x), \quad x \in [x_{n-1}, x_n],
\]

\[
v_n \phi_n(x_{n-1}, t) + \mu_n \frac{\partial \phi_n}{\partial x}(x_{n-1}, t) = \lambda(t, \tau) \xi_n(t), \quad t \geq 0, \tau > 0,
\]

\[
\xi_n(x_n, t) + I \frac{\partial \phi_n}{\partial x}(x_n, t) = \lambda(t, \tau) \xi_n(t), \quad t \geq 0, \tau > 0.
\]

Remark 1. Each of the initial boundary value problems (9)–(11) is a case of the one-layer nonhomogeneous diffusion problem that will be discussed in Section 2 below.

Now, in view of the inner boundary conditions (7) and (8), the time-varying functions \( \xi_j \) and \( \xi_j \) for all \( 2 \leq j \leq n \) are subject to

\[
\xi_j(t) = \xi_{j-1}(t), \quad 2 \leq j \leq n,
\]

so that

\[
\xi_j(t) = \begin{cases} \xi(t), & j = 1, \\
\xi_{j-1}(t), & 2 \leq j \leq n, \\

(13)
\end{cases}
\]

While the outer boundary data \( \xi_j(t) = \xi(t) \) and \( \xi_j(t) = \xi(t) \) are given in (5) and (6), respectively, the functions \( \xi_j(2 \leq j \leq n) \) can be determined once we specify the functions \( \xi_j(1 \leq j \leq n - 1) \). Hence, we have to find \( \xi_j, 1 \leq j \leq n - 1 \). To do so, we should use the first matching condition (7).

1.2. Srivastava-Luo-Raina Generalized Integral Transform. In [48], Srivastava et al. introduced the following generalized integral transform:

\[
\mathcal{M}_{p,m}[\phi(t)](s, \tau) = \int_0^\infty e^{-s \tau} \phi(t) \left( \frac{1}{(\tau^m + \tau^m)^{\frac{1}{m}}} \right) dt,
\]

for a continuous (or piecewise continuous) function \( \phi \) on \( [0, \infty) \), where \( \rho \in \mathbb{C} ; \Re (\rho) \geq 0 ; m \in \mathbb{Z}_+, s > 0 \) is the transform variable and \( \tau > 0 \) is a parameter. The basic properties of the \( \mathcal{M}_{p,m} \)-transform are given in [48]. Next, we recall some of these properties, which are needed in the present work. Indeed, as introduced in [48] the \( \mathcal{M}_{p,m} \)-transform is closely related with the well-known integral transforms, the Laplace, natural, and Sumudu transforms. The Laplace transform is defined by

\[
L[\phi(t)](s) = \int_0^\infty e^{-s \tau} \phi(t) \, dt, \quad \Re (s) > 0.
\]

So, from (14) and (15), we have the following duality relations:

\[
L[\phi(t)](s) = \mathcal{M}_{0,m}[\phi(t)](s, 1), \quad \Re (s) > 0,
\]

\[
\mathcal{M}_{p,m}[\phi(t)](s, \tau) = L \left[ \frac{\phi(t)}{E_{m, \tau}^{(\tau^m + \tau^m)}} \right](s), \quad s, \tau > 0,
\]

\[
\mathcal{M}_{p,m}[\phi(t)](s, \tau) = \frac{1}{\tau} E_{m, \tau}^{(\tau^m + \tau^m)} \phi(t) \left( \frac{s}{\tau} \right), \quad s, \tau > 0,
\]

\[
\mathcal{M}_{p,m} \left[ \left( \frac{1}{\tau^m} + \tau^m \right)^\phi \phi(t) \right](s, \tau) = L[\phi(t)](s), \quad s, \tau > 0.
\]

Setting \( \rho = 0 \) in (14), we recover the natural transform defined as (see [49, 50])

\[
N[\phi(t)](s, \tau) = \int_0^\infty e^{-s \tau} \phi(t) \, dt, \quad s > 0, \tau > 0.
\]
Thus, we have the following $\mathbb{M}_{p,m}$-N-transform duality

$$\mathbb{N}[\varphi(t)](s, \tau) = \mathbb{M}_{0,m}[\varphi(t)](s, \tau),$$

(19)

$$\mathbb{M}_{p,m}[\varphi(t)](s, \tau) = \mathbb{N}\left[\frac{\varphi(t)}{((t^m + \tau^m)^{p})}(s, \tau)\right], \quad s > 0, \tau > 0,$$

(20)

$$\mathbb{M}_{p,m}\left[\left(\frac{t^m}{\tau^m} + \tau^m\right)^{\rho}\varphi(t)\right](s, \tau) = \mathbb{N}[\varphi(t)](s, \tau); \quad s > 0, \tau > 0.$$  

(21)

The Sumudu transform is defined by [51–53]

$$S[\varphi(t)](\tau) = \int_{0}^{\infty} e^{-\tau s} \varphi(\tau s)dt, \quad \tau > 0.$$  

(22)

Thus,

$$S[\varphi(t)](\tau) = \mathbb{M}_{0,m}[\varphi(t)](0, \tau), \quad \tau > 0,$$

(23)

$$\mathbb{M}_{p,m}[\varphi(t)](s, \tau) = \frac{1}{s} S\left[\frac{\varphi(t)}{((t^m + \tau^m)^{p})}(s, \tau)\right], \quad s > 0, \tau > 0.$$  

(24)

Based on these dualities of the $\mathbb{M}_{p,m}$-transform (14) and these well-known integral transforms, it seems to be interesting to apply the $\mathbb{M}_{p,m}$-transform (14) in solving a variety of boundary and initial-boundary value problems. In this context, we recall the following results [48]:

(i) Let $\varphi^{(n)}(t)$ be the $n$th-order $t$-derivative of the function $\varphi(t)$ and $|\varphi(t)| \leq K e^{\gamma t}$ with $K > 0, \gamma > 0$. Then,

$$\mathbb{M}_{p,m}\left[\left(\frac{t^m}{\tau^m} + \tau^m\right)^{\rho}\varphi^{(n)}(t)\right](s, \tau) = s^n \mathbb{N}[\varphi(t)](s, \tau)$$

$$- \sum_{k=0}^{n-1} s^k \varphi^{(n-k-1)}(0),$$

(25)

where $\mathbb{N}[\varphi(t)](s, \tau)$ is defined by (18). Using the duality (21) in (25), we find

$$\mathbb{N}\left[\frac{\varphi^{(n)}(t)}{((t^m + \tau^m)^{p})}(s, \tau)\right] = s^n \mathbb{N}[\varphi(t)](s, \tau)$$

$$- \sum_{k=0}^{n-1} s^k \varphi^{(n-k-1)}(0), \quad n = 0, 1, \ldots.$$  

(26)

(ii) Again, using the dualities stated before a convolution formula for the $\mathbb{M}_{p,m}$-transform (14) can be obtained as follows. Here, the convolution for the Laplace transform will be considered; that is, for the functions $\varphi$ and $\psi$, the convolution formula is given as

$$(\varphi * \psi)(t) = \int_{0}^{t} \varphi(x)(t-x)dx = \int_{0}^{t} \varphi(t-x)\psi(x)dx.$$  

(27)

If $\Phi(t, \tau) = \mathbb{M}_{p,m}[\varphi(t)](s, \tau)$ and $\Psi(t, \tau) = \mathbb{M}_{p,m}[\psi(t)](s, \tau)$, then

$$\tau \Phi(t, \tau) \Psi(t, \tau) = \tau \int_{0}^{\infty} e^{-\tau s}(\varphi(t_1)/(t_1^m + \tau^m)\rho)dt_1 \int_{0}^{\infty} (e^{-\tau \tau_2}\psi(t_2)/(t_2^m + \tau^m)\rho)dt_2 = \tau \int_{0}^{\infty} e^{-\tau (t-t_2)} \psi(t_2)dt_2,$$

where

$$\phi(t) = \frac{\varphi(t)}{(t^m + \tau^m)^{p}}; \quad \psi(t) = \frac{\psi(t)}{(t^m + \tau^m)^{p}}.$$  

(28)

Setting $t_1 + t_2 = t$ in the last equality, one gets

$$\tau \Phi(t, \tau) \Psi(t, \tau) = \tau \int_{0}^{\infty} \int_{0}^{2\pi} e^{-\tau (t-t_2)} \phi(t_2) \psi(t_2)dt_2.$$  

(29)

Here, changing of the integral order is used. Thus, using the duality of the $\mathbb{M}_{p,m}$ and N transforms (see (20)), we find

$$\tau \Phi(t, \tau) \Psi(t, \tau) = \mathbb{N}[(\varphi * \psi)](s, \tau).$$  

(30)

Remark 2. If we put $\rho = 0$ in (30), the case being interesting later in our work, then we get

$$\tau \mathbb{N}[\varphi(t)](s, \tau) \mathbb{N}[\psi(t)](s, \tau) = \mathbb{N}[(\varphi * \psi)](s, \tau).$$  

(31)

(iii) Once again, ...again, using the dualities stated before an inversion formula of the $\mathbb{M}_{p,m}$-transform (14) is given (see Theorem 4.1 of [48]) as

$$\varphi(t) = \left(\frac{t^m}{\tau^m} + \tau^m\right)^{\rho} L^{-1}\{\mathbb{M}_{p,m}[\varphi(t)](s, \tau)\left(\frac{1}{s}\right)\}$$

$$= \frac{1}{2\pi i} \left(\frac{t^m}{\tau^m} + \tau^m\right)^{\rho} \int_{c-i\infty}^{c+i\infty} e^{\tau z}\mathbb{M}_{p,m}[\varphi(t)](s, \tau)dz, \quad c, \tau > 0,$$

(32)

as long as the integral converges absolutely. In case, when $\rho = 0$ one obtains the following inversion formula of the natural transform (see Theorem 5.3 of [49])

$$\varphi(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{\tau z}\mathbb{N}[\varphi(t)](s, \tau)dz, \quad c, \tau > 0.$$  

(33)

The residue theorem (see, e.g., [54]) is usually used to calculate the contour integrals in (32) and (33).
2. One-Layer Nonhomogeneous Diffusion System

Now, we investigate the solvability for the following one-layer nonhomogeneous initial boundary value problem:

\[
\frac{\partial \varphi}{\partial t} = d \frac{\partial^2 \varphi}{\partial x^2} + \lambda(t, \tau)r(x, t), \quad x \in (\alpha, \beta), \quad t, \tau > 0, \tag{34}
\]

\[
\varphi(x, 0) = \eta(x), \quad x \in [\alpha, \beta], \tag{35}
\]

\[
\varphi(\alpha, t) + t \frac{\partial \varphi}{\partial x}(\alpha, t) = \lambda(t, \tau)\zeta(t), \quad t \geq 0, \quad \tau > 0, \tag{36}
\]

\[
\ell \varphi(\beta, t) + t \frac{\partial \varphi}{\partial x}(\beta, t) = \lambda(t, \tau)\xi(t), \quad t \geq 0, \quad \tau > 0, \tag{37}
\]

where \(d, i, t, \xi, \) and \(l\) are constants such that \(|i| + |i| > 0, |\xi| + |l| > 0, \) and \(\lambda, r, \eta, \zeta, \) and \(\xi\) are given functions with \(\lambda\) as in (3).

Applying the \(\mathcal{M}_{\rho, m}\)-transform defined by (14), to (34), yields

\[
\mathcal{M}_{\rho, m}\left[ \frac{1}{1(t, \tau)} \varphi(x, t) \right] = d \mathcal{M}_{\rho, m}\left[ \frac{1}{1(t, \tau)} \varphi_{xx}(x, t) \right] + \mathcal{M}_{\rho, m}[r(x, t)]. \tag{38}
\]

Using the duality of the \(\mathcal{M}_{\rho, m}\)-transform and the natural transform given by (21) and (25), Equation (38) can be reduced to

\[
\frac{s}{\tau d} \mathcal{N}[\varphi(x, t)](s; s, \tau) - \mathcal{N}[\varphi_{xx}(x, t)](s; s, \tau) - \frac{1}{\tau d} \varphi(x, 0) = \frac{1}{d} \mathcal{N}[\lambda(t, \tau)r(x, t)](s; s, \tau), \tag{39}
\]

where \(\mathcal{N}[\varphi(r)](s, \tau)\) is defined by (18). Setting

\[
\tilde{\varphi}(x; s, \tau) = \mathcal{N}[\varphi(x, t)](s; s, \tau), \tag{40}
\]

then, (39) can be expressed as

\[
\tilde{\varphi}_{xx}(x; s, \tau) - \frac{s}{\tau d} \tilde{\varphi}(x; s, \tau) = F(x; s, \tau), \tag{41}
\]

where

\[
F(x; s, \tau) = -\frac{1}{d} \mathcal{N}[\lambda(t, \tau)r(x, t)](s; s, \tau) - \frac{1}{\tau d} \eta(x). \tag{42}
\]

Applying the variation of the parameter method to the nonhomogeneous equation (41) gives the general solution as

\[
\tilde{\varphi}(x; s, \tau) = A \cosh \frac{s}{\tau d} x + B \sinh \frac{s}{\tau d} x + \frac{1}{s \tau d} \int_\alpha^\infty F(y; s, \tau) \sinh \frac{s}{\tau d} (x - y) dy, \tag{43}
\]

where \(A \) and \(B\) are arbitrary invariants which can depend on \(s\) and \(\tau\). Differentiating (43) with respect to \(x\), gives

\[
\tilde{\varphi}_x(x; s, \tau) = A \sqrt{\frac{s}{\tau d}} \sinh \frac{s}{\tau d} x + B \sqrt{\frac{s}{\tau d}} \cosh \frac{s}{\tau d} x + \int_\alpha^\infty F(y; s, \tau) \cosh \frac{s}{\tau d} (x - y) dy. \tag{44}
\]

Transforming the boundary conditions (36) and (37), implies

\[
i \tilde{\varphi}(\alpha; s, \tau) + i \tilde{\varphi}_x(\alpha; s, \tau) = \mathcal{M}_{\rho, m}[\zeta(t)] = \mathcal{N}[\lambda(t, \tau)\zeta(t)],
\]

\[
i \tilde{\varphi}(\beta; s, \tau) + i \tilde{\varphi}_x(\beta; s, \tau) = \mathcal{M}_{\rho, m}[\xi(t)] = \mathcal{N}[\lambda(t, \tau)\xi(t)]. \tag{45}
\]

For simplicity, we set the following vector notations:

\[
a = (t, i), \quad b = (\ell, l), \tag{46}
\]

\[
\mathcal{Q}(y; s, \tau) = \begin{pmatrix}
\cosh \frac{s}{\tau d} y & \frac{s}{\tau d} \\
\frac{s}{\tau d} & \cosh \frac{s}{\tau d} y
\end{pmatrix}, \tag{47}
\]

\[
\mathcal{G}(y; s, \tau) = \begin{pmatrix}
\sinh \frac{s}{\tau d} y & \frac{s}{\tau d} \\
\frac{s}{\tau d} & \sinh \frac{s}{\tau d} y
\end{pmatrix}. \tag{48}
\]

Obviously, we have

\[
\frac{\partial \mathcal{G}}{\partial y}(y; s, \tau) = \sqrt{\frac{s}{\tau d}} \mathcal{G}(y; s, \tau), \quad \frac{\partial \mathcal{G}}{\partial x}(y; s, \tau) = \sqrt{\frac{s}{\tau d}} \mathcal{G}(y; s, \tau). \tag{49}
\]

Substituting (43) and (44) into (45) and using the vector notation, we give the algebraic linear system

\[
\begin{pmatrix}
\langle a, \mathcal{Q}(\alpha; s, \tau) \rangle \\
\langle b, \mathcal{Q}(\beta; s, \tau) \rangle
\end{pmatrix} \begin{pmatrix}
A \\
B
\end{pmatrix} = \begin{pmatrix}
\mathcal{G}(s, \tau) \\
\mathcal{G}(s, \tau)
\end{pmatrix}, \tag{50}
\]

where \(\langle \cdot \rangle\) is the usual dot product in \(\mathbb{R}^2\), and

\[
\mathcal{G}(s, \tau) = \mathcal{N}[\lambda(t, \tau)\zeta(t)], \tag{51}
\]

\[
H^+(x; s, \tau) = H(s, \tau) - \sqrt{\frac{s}{\tau d}} \int_\alpha^\infty F(y; s, \tau) (b, \mathcal{G}(\beta - y; s, \tau)) dy, \tag{52}
\]

with \(H(s, \tau) = \mathcal{N}[\lambda(t, \tau)\zeta(t)]\) and \(\zeta(t)\) and \(\xi(t)\) are the boundary data given in (36) and (37), respectively. The
solution \((A, B)\) of system (50) is

\[
A\Delta(s) = G(s, \tau)\langle b, \mathcal{C}(\beta; s, \tau) \rangle - H^r(x; s, \tau)\langle a, \mathcal{C}(\alpha; s, \tau) \rangle,
\]

\[
B\Delta(s) = -[G(s, \tau)\langle b, \mathcal{L}(\beta; s, \tau) \rangle - H^r(x; s, \tau)\langle a, \mathcal{L}(\alpha; s, \tau) \rangle],
\]

where

\[
\Delta(s) = \langle b, \mathcal{C}(\beta; s, \tau) \rangle\langle a, \mathcal{L}(\alpha; s, \tau) \rangle - \langle a, \mathcal{C}(\alpha; s, \tau) \rangle\langle b, \mathcal{L}(\beta; s, \tau) \rangle,
\]

is the determinant of the coefficient matrix of system (50). Substituting the constants \(A\) and \(B\) into (43) gives

\[
\tilde{\varphi}(x; s, \tau) = \frac{\cosh \sqrt{(s/\tau)d}x}{\Delta(s)} \langle G(s, \tau)\langle b, \mathcal{C}(\beta; s, \tau) \rangle - H^r(x; s, \tau)\langle a, \mathcal{C}(\alpha; s, \tau) \rangle \rangle
\]

\[+ \sinh \sqrt{(s/\tau)d}x \langle -G(s, \tau)\langle b, \mathcal{L}(\beta; s, \tau) \rangle + H^r(x; s, \tau)\langle a, \mathcal{L}(\alpha; s, \tau) \rangle \rangle
\]

\[+ \sqrt{\frac{\tau d}{s}} \int_a^x \sinh \sqrt{\frac{s}{\tau d}}(x-y)F(y; s, \tau)dy,
\]

which can be rewritten as

\[
\tilde{\varphi}(x; s, \tau) = \frac{G(s, \tau)\psi(x, \beta; s, \tau, b)}{\Delta(s)} - \frac{H(s, \tau)\psi(x, \alpha; s, \tau, a)}{\Delta(s)} + \tilde{\theta}(x; s, \tau),
\]

where

\[
\psi(x, y; s, \tau, L) = \langle L, \mathcal{C}(y-x; s, \tau) \rangle \cosh \sqrt{\frac{s}{\tau d}}x
\]

\[- \langle L, \mathcal{L}(y-x; s, \tau) \rangle \sinh \sqrt{\frac{s}{\tau d}}x,
\]

\[
\tilde{\theta}(x; s, \tau) = \frac{\psi(x, \alpha; s, \tau, a)}{\Delta(s)} \sqrt{\frac{\tau d}{s}}b(\mathcal{C}(\beta-y; s, \tau))F(y; s, \tau)dy
\]

\[+ \sqrt{\frac{\tau d}{s}} \int_a^x \sinh \sqrt{\frac{s}{\tau d}}(x-y)F(y; s, \tau)dy.
\]

For further computation, we rewrite \(\tilde{\theta}(x; s, \tau)\) as

\[
\tilde{\theta}(x; s, \tau) = \sqrt{\frac{\tau d}{s}} \left( \int_a^x \psi(x, \alpha; s, \tau, a)\langle b, \mathcal{C}(\beta-y; s, \tau) \rangle \right) + \Delta(s) \sinh \sqrt{(s/\tau d)(x-y)} \left( \int_a^x \frac{\psi(x, \alpha; s, \tau, a)\langle b, \mathcal{C}(\beta-y; s, \tau) \rangle}{\Delta(s)} F(y; s, \tau)dy \right)
\]

Lemma 3. Let \(\tau, s, x, y \in \mathbb{R} \) and \(L \in \mathbb{R}^2\). Then,

\[
\psi(x, y; s, \tau, L) = \langle L, \mathcal{C}(y-x; s, \tau) \rangle,
\]

\[
\Delta(s) \sinh \sqrt{\frac{s}{\tau d}}(x-y) = -\psi(x, \alpha; s, \tau, a)\langle b, \mathcal{C}(\beta-y; s, \tau) \rangle
\]

\[+ \langle a, \mathcal{C}(\alpha-y; s, \tau) \rangle \psi(x, \beta; s, \tau, b).
\]

Consequently, for each zero \(s\) of the function \(\Delta(s) \sinh \sqrt{(s/\tau d)(x-y)}\), one has

\[
\psi(x, \alpha; s, \tau, a)\langle b, \mathcal{C}(\beta-y; s, \tau) \rangle
\]

\[= \langle a, \mathcal{C}(\alpha-y; s, \tau) \rangle \psi(x, \beta; s, \tau, b),
\]

where \(\psi\) is given by (57).

Proof. The first two conclusions of the lemma follow directly from the uniqueness theorem of the initial value problem for the second-order ordinary differential equations having constant coefficients.

For fixed \(\tau, s, y \in \mathbb{R} \) and \(L \in \mathbb{R}^2\), in view of (46) and (57), the functions \(\langle L, \mathcal{C}(y-x; s, \tau) \rangle\) and \(\psi(x, y; s, \tau, L)\) are solutions to the following initial value problem:

\[
\frac{d^2z(x)}{dx^2} - \frac{s}{\tau d}z(x) = 0,
\]

\[
z(0) = \langle L, \mathcal{C}(y; s, \tau) \rangle, \quad z'(0) = -\sqrt{\frac{s}{\tau d}}L, \mathcal{C}(y; s, \tau)\).
\]

Thus, with the uniqueness of the solution to problem (63), we conclude (60).

It is easy to see that as functions in \(x\), both sides of (61) are linear combinations of the functions \(\cosh \sqrt{(s/\tau d)x},\)
and \( \sinh \sqrt{(s/\pi d)} x \) which are linearly independent solutions to the differential equation in (63). Thus, both sides of (61) solve this differential equation.

Moreover, in view of (57), both sides of (61) satisfy

\[
z(0) = -\Delta(s) \sinh \frac{s}{\pi d} y, \quad z'(0) = \sqrt{\frac{s}{\pi d}} \cosh \frac{s}{\pi d} y. \tag{64}
\]

Hence, by the uniqueness theorem, (61) holds true.

For each \( s \), being a zero of the function \( \Delta(s) \sinh \sqrt{(s/\pi d)} (x - y) \), taking the limit in both sides of (61) as \( s \to s \) gives (62).

Applying Lemma 3, (56) and (59) respectively can be reduced to

\[
\bar{\varphi}(x; s) = \frac{G(s, \tau)(b, C(\beta - x; s, \tau))}{\Delta(s)} - \frac{H(s, \tau)(a, C(\alpha - x; s, \tau))}{\Delta(s)} + \bar{\vartheta}(x; s, \tau),
\]

\[
\bar{\vartheta}(x; s, \tau) = \sqrt{\frac{rd}{s}} \int_{s}^{rd} \frac{(a, \hat{C}(a - y; s, \tau))(b, \hat{C}(\beta - x; s, \tau))}{\Delta(s)} F(y; s, \tau) dy - \sqrt{\frac{rd}{s}} \int_{s}^{rd} \frac{(a, \hat{C}(a - y; s, \tau))(b, \hat{C}(\beta - y; s, \tau))}{\Delta(s)} F(y; s, \tau) dy.
\]

Next, in order to obtain the solution to the initial value problem (34–37), we apply the inversion formula (33) to (65) and (66). In doing so, we suppose that there are nonzero simple roots \( \{s_k\}_{k=1}^{\infty} \) of \( \Delta(s) \). That is,

\[
\Delta(s_k) = 0, \quad \Delta'(s_k) \neq 0, \quad k = 1, 2, 3, \ldots. \tag{67}
\]

\[
\Box
\]

**Lemma 4.** Suppose that (67) holds true. For each \( x, y \in \mathbb{R} \) and \( t, \tau > 0 \), we get

\[
\mathbf{N}^{-1} \left\{ \left( a, \hat{C}(x; s, \tau) \right)(b, \hat{C}(y; s, \tau)) \right\} \sqrt{(s/\pi d)\Delta(s)} = \Theta(x, y, t, \tau), \tag{68}
\]

where

\[
\Theta(x, y, t, \tau) = \Theta_0(x, y, \tau) + \sum_{k=1}^{\infty} e^{\epsilon(t, \tau) s_k} \left( a, \hat{C}(x; s, \tau) \right)(b, \hat{C}(y; s, \tau)) \sqrt{s/\pi d} \Delta'(s_k),
\]

with

\[
\text{if } \left( a, \hat{C}(x; s, \tau) \right)(b, \hat{C}(y; s, \tau)) \sqrt{s/\pi d} \Delta(s) = O(1),
\]

\[
\text{if } \left( a, \hat{C}(x; s, \tau) \right)(b, \hat{C}(y; s, \tau)) \sqrt{s/\pi d} \Delta(s) = O\left( \frac{1}{s} \right). \tag{70}
\]

Recalling (67), each \( s_k \) \( (k = 1, 2, \ldots) \) is a simple pole of \( e^{\epsilon(t, \tau) s} \hat{C}(s, \tau) \). Therefore,

\[
\text{Res} \left[ e^{\epsilon(t, \tau) s} \hat{C}(s, \tau) \right] s_k = \frac{e^{\epsilon(t, \tau) s_k} \left( a, \hat{C}(x; s_k, \tau) \right)(b, \hat{C}(y; s_k, \tau))}{\sqrt{s/\pi d} \Delta'(s_k)}, \quad k = 1, 2, \ldots. \tag{74}
\]

At \( s = 0 \), we have

\[
\text{Res} \left[ e^{\epsilon(t, \tau) s} \hat{C}(s, \tau) \right] s_0 = \lim_{s \to 0} s e^{\epsilon(t, \tau) s} \left( a, \hat{C}(x; s, \tau) \right)(b, \hat{C}(y; s, \tau)) \sqrt{s/\pi d} \Delta(s).
\]

The last integral can be usually calculated by the residue theorem [54]. Hence,

\[
\Theta(x, y, t) = \sum_{\text{poles } s_k \text{ of } \hat{C}(s, \tau)} \text{Res} \left[ e^{\epsilon(t, \tau) s} \hat{C}(s, \tau) \right] s_k. \tag{73}
\]
We see that either
\[
\frac{\langle a, \mathbf{C}(x; s, t) \rangle}{\sqrt{(s/rd)} \Delta(s)} = O(1) \quad \text{or} \quad \frac{\langle a, \mathbf{C}(x; s, t) \rangle}{\sqrt{(s/rd)} \Delta(s)} = O\left(\frac{1}{s}\right).
\]

(76)
as \(s\) tends to 0. Then, \(s = 0\) is either a removable singular point or a simple pole of \(\omega(t) \Re \Phi(s, r, t)\).

Hence, substituting (74) and (75) in (73) gives the main conclusion of the Lemma, i.e., (69) and (70). \(\square\)

In view of (68) and (42), we have
\[
\mathcal{N}(\Theta(\alpha - y, \beta - x, t)) = \frac{\langle a, \mathbf{C}(x - y, \beta) \rangle}{\sqrt{(s/rd)} \Delta(s)} \left(\frac{\lambda(y, t)}{(t^{m/r} \pm \tau^{m/r})}\right) dy.
\]

(77)

where \(\Theta\) is defined by (69) and (70), and
\[
\mathcal{N}(\Theta(\alpha - y, \beta - x, t)) = \frac{\langle a, \mathbf{C}(x - y, \beta) \rangle}{\sqrt{(s/rd)} \Delta(s)} \left(\frac{\lambda(y, t)}{(t^{m/r} \pm \tau^{m/r})}\right) dy.
\]

Hence, (66) can be rewritten as
\[
\mathcal{N}(\Theta(\alpha - y, \beta - x, t)) = \frac{\langle a, \mathbf{C}(x - y, \beta) \rangle}{\sqrt{(s/rd)} \Delta(s)} \left(\frac{\lambda(y, t)}{(t^{m/r} \pm \tau^{m/r})}\right) dy.
\]

(78)

By the convolution formula (31), the inverse natural transform of (79) is
\[
\mathcal{N}(\Theta(\alpha - y, \beta - x, t)) = \frac{\langle a, \mathbf{C}(x - y, \beta) \rangle}{\sqrt{(s/rd)} \Delta(s)} \left(\frac{\lambda(y, t)}{(t^{m/r} \pm \tau^{m/r})}\right) dy.
\]

(79)

That is,
\[
\theta(x, t, r) = -\frac{1}{r d} \int_a^b \Theta_0(\alpha - x, \beta - y) \eta(y) dy - \frac{1}{r d} \int_a^b \Theta_0(\alpha - x, \beta - y) \eta(y) dy.
\]

(80)

From Lemma 3, one has
\[
\langle a, \mathbf{C}(x - s, t) \rangle \mathbf{C}(\beta - y, s) \rangle = \langle a, \mathbf{C}(x - s, t) \rangle \mathbf{C}(\beta - y, s) \rangle.
\]

(81)

at \(s = 0\) and \(s = s_k (k = 1, 2, \ldots)\) the zeros of \(\Delta(s)\) sinh \(\sqrt{s/rd}(y - x)\). That results in
\[
\Theta_0(\alpha - y, \beta - x) = \Theta_0(\alpha - x, \beta - y),
\]

\[
\langle a, \mathbf{C}(x - s, t) \rangle \mathbf{C}(\beta - y, s) \rangle \int_x^b \langle a, \mathbf{C}(x - y, s) \rangle \theta(y) dy.
\]

(82)

The first conclusion is obvious when \(\Theta_0 = 0\) in (70). Thus, (81) can be simplified as
\[
\theta(x, t, r) = -\frac{1}{r d} \int_a^b \Theta_0(\alpha - x, \beta - y) \eta(y) dy.
\]

(83)
Next, we return to (65). Using (26) (for \( n = 1 \)) and (51), (65) can be rewritten as

\[
\tilde{\varphi}(x, s, r) = sB[\lambda(t, r)\tilde{\zeta}(t)] - sB[\lambda(t, r)\tilde{\zeta}(t)] + \tilde{\vartheta}(x, s, r) = (\tau N \frac{d}{dt}(\lambda(t, r)\tilde{\zeta}(t)) + \lambda(0, r)\tilde{\zeta}(0)) \frac{[a, \mathcal{G}(a - x + s, r)]}{s\Delta(s)} - \frac{[a, \mathcal{G}(a - x + s, r)]}{s\Delta(s)} + \tilde{\vartheta}(x, s, r),
\]

(85)

where \( \tilde{\vartheta}(x, s, r) \) is given in (66). Now, we can obtain the solution \( \varphi(x, t) \) of Problem (34)–(37) by operating the inversion formula (33) in (85). In doing so, we need the following lemma.

**Lemma 5.** Assume that (67) holds true. Then, for each \( y \in \mathbb{R}, t, r > 0 \) and \( \mathbf{L} \in \mathbb{R}^2 \), we get

\[
\mathbb{N}^{-1}\left\{ \left( \mathbf{L}, \mathcal{G}(y; s, r) \right) \right\} = \Phi(y, t; \mathbf{L}),
\]

(86)

where

\[
\Phi(y, t; \mathbf{L}) = \Phi_0(y; \mathbf{L}) + \sum_{k=1}^{\infty} \frac{s^k}{k!} \mathcal{G}(y; s_k, r),
\]

(87)

\[
\Phi_0(y; \mathbf{L}) = \left\{ \begin{array}{ll}
\lim_{t \to 0} \frac{[\mathbf{L}, \mathcal{G}(y; s, r)]}{s\Delta(s)}, & \text{if } \frac{[\mathbf{L}, \mathcal{G}(y; s, r)]}{s\Delta(s)} = O\left( \frac{1}{r} \right) \\
\lim_{s \to 0} \frac{\partial}{\partial s} \left( \frac{[\mathbf{L}, \mathcal{G}(y; s, r)]}{s\Delta(s)} \right), & \text{if } \frac{[\mathbf{L}, \mathcal{G}(y; s, r)]}{s\Delta(s)} = O\left( \frac{1}{r^2} \right) 
\end{array} \right.
\]

(88)

**Proof.** The proof is similar to Lemma 4.

From Lemma 5, we see that

\[
\mathbb{N}^{-1}\left\{ \left( \mathbf{b}, \mathcal{G}(b - x + s, r) \right) / s\Delta(s) \right\} = \Phi(b - x, t; b), \mathbb{N}^{-1}\left\{ \left( a, \mathcal{G}(a - x + s, r) \right) / s\Delta(s) \right\} = \Phi(a - x, t; a).
\]

Hence, in view of the convolution formula (31) and the inversion of natural transform (33), inverting (85) yields

\[
\varphi(x, t, r) = \int_0^1 \Phi(b - x, t - \zeta; b) \left( \tilde{\zeta}'(\zeta) + \tilde{\zeta}(0) \delta_0(\zeta) \right) d\zeta
- \int_0^1 \Phi(a - x, t - \zeta; a) \left( \tilde{\zeta}'(\zeta) + \tilde{\zeta}(0) \delta_0(\zeta) \right) d\zeta + \theta(x, t, r),
\]

(89)

where \( \tilde{\zeta} = \lambda(t, r)\tilde{\zeta}(t), \tilde{\zeta} = \lambda(t, r)\tilde{\zeta}(t), \delta_0 \) is the well-known Dirac delta function, and \( \theta(x, t, r) \) is given by (84). Then, using the basic property of the Dirac delta function, that is, \( \delta_0(\zeta) \Phi(\zeta) = \Phi(0) \), results in

\[
\varphi(x, t, r) = \int_0^1 \Phi(b - x, t - \zeta; b) \tilde{\zeta}'(\zeta) d\zeta + \tilde{\zeta}(0) \Phi(b - x, t; b)
- \int_0^1 \Phi(a - x, t - \zeta; a) \tilde{\zeta}'(\zeta) d\zeta - \tilde{\zeta}(0) \Phi(a - x, t; a) + \theta(x, t, r).
\]

(90)

Integrating by parts gives

\[
\varphi(x, t, r) = \lambda(t, r)\tilde{\zeta}(t) \Phi(b - x, 0; b)
- \int_0^t \lambda(\tau, \zeta) D_x \Phi(b - x, t - \zeta; b) \tilde{\zeta}(\zeta) d\zeta
- \lambda(t, \tau) D_x \Phi(a - x, 0; a) \tilde{\zeta}(t)
+ \int_0^t \lambda(\tau, \zeta) D_x \Phi(a - x, t - \zeta; a) \tilde{\zeta}(\zeta) d\zeta + \theta(x, t, r),
\]

(91)

where \( D_x = (\partial / \partial t) \Phi(x, t; b) \). Substituting from (87) gives

\[
\varphi(x, t, r) = \lambda(t, r)\tilde{\zeta}(t) \left( \Phi_0(b - x; b) + \sum_{k=1}^{\infty} \frac{s^k}{k!} \mathcal{G}(b - x + s, r) \right)
- \int_0^t \lambda(\tau, \zeta) \left( \sum_{k=1}^{\infty} \frac{s^k}{k!} \mathcal{G}(b - x + s, r) \right) d\zeta
- \lambda(t, r)\tilde{\zeta}(t) \Phi_0(a - x; a) + \sum_{k=1}^{\infty} \frac{s^k}{k!} \mathcal{G}(a - x + s, r)
+ \int_0^t \lambda(\tau, \zeta) \left( \sum_{k=1}^{\infty} \frac{s^k}{k!} \mathcal{G}(a - x + s, r) \right) d\zeta + \theta(x, t, r),
\]

(92)

with \( \theta(x, t, r) \) is given by (84). This result can be rewritten as

\[
\varphi(x, t, r) = \lambda(t, r)\tilde{\zeta}(t) \Phi_0(b - x; b)
+ \sum_{k=1}^{\infty} \frac{\Gamma_k \tilde{\zeta}(t)}{k!} \mathcal{G}(b - x + s, r)
- \lambda(t, r)\tilde{\zeta}(t) \Phi_0(a - x; a)
- \sum_{k=1}^{\infty} \frac{\Gamma_k \tilde{\zeta}(t)}{k!} \mathcal{G}(a - x + s, r) + \theta(x, t, r),
\]

(93)
where $\Gamma_k$ is the operator defined as

$$
\Gamma_k \phi(t) = \lambda(t, x) \phi(t) - \int_0^t \lambda(t, x) e^{k(t-s)} \phi(s) ds.
$$

The integral in (94) is the Laplacian convolution formula for $\lambda(t, x) \phi(t)$ with $e^{k(t-s)}$. As a result, (93), together with (84) and (94), expresses the solution of Problem (34)–(37).

**Remark 6.** When $\rho = 0$ and $r_j = \nu_j = 0$, for all $j = 1, \cdots, n$, Problem (34)–(37) and its solution

$$
\phi(x, t, \tau = 1) = \xi(t) \Phi(\beta - x; b) + \sum_{k=1}^\infty \frac{\tilde{T}_k \xi(t)}{s_k \Delta'(s_k)} \langle b, \Phi(\beta - x, s_k) \rangle
$$

$$
- \xi(t) \Phi(\alpha - x; a) - \sum_{k=1}^\infty \frac{\tilde{T}_k \xi(t)}{s_k \Delta'(s_k)} \langle a, \Phi(\alpha - x, s_k) \rangle + \Theta(x, t),
$$

with $\Phi(\alpha - x; a), \Phi(y, s_k) = \Phi(y - s_k, \tau = 1)$ defined as (88), (46), respectively,

$$
\tilde{T}_k \phi(t) = \phi(t) - \int_0^t e^{k(t-s)} \phi(s) ds,
$$

$$
\Theta(x, t) = -\frac{1}{d} \int_0^\beta \Theta(\alpha - x, \beta - y) \eta(y) dy
$$

$$
- \sum_{k=1}^\infty \frac{e^{k(t-s_k)} \langle b, \Phi(\beta - x, s_k) \rangle}{s_k \Delta'(s_k)} \int_a^b \langle a, \Phi(\alpha - y, s_k) \rangle \eta(y) dy,
$$

are reduced to that in Section 3 of [38].

### 2.1. Illustrative Examples.

Here, we discuss two illustrative test cases to show the accuracy and effectiveness of our technique.

**Example 1.** Heat equation with zero temperatures at finite ends.

The following initial boundary value problem with homogeneous Dirichlet boundary conditions

$$
\frac{\partial \phi}{\partial t} = d^2 \phi \frac{\partial \phi}{\partial x^2}, \quad x \in (0, L), t > 0,
$$

$$
\phi(x, 0) = f(x), \quad x \in [0, L],
$$

$$
\phi(0, t) = 0, \phi(L, t) = 0, \quad t \geq 0,
$$

is a special case of the one-layer diffusion system (34)–(37) when $a = 0, b = L, \ell = 1, i = 0, \xi = 1, \ell = 0$ and $\xi(t) = 0, \eta(x) = f(x)$. For simplicity, we will take $\tau = 1$. Thus, from (93), we have $\phi(x, t) = \theta(x, t)$. From (46), we have $a = b = (1, 0)$,

$$
\Phi(x; s) = \left(\cosh \frac{s}{d}, \sinh \frac{s}{d} \sqrt{\frac{s}{d}} \right), \quad \Phi(x; s)
$$

$$
= \left(\sinh \frac{s}{d}, \cosh \frac{s}{d} \sqrt{\frac{s}{d}} \right).
$$

Hence, (54) yields

$$
\Delta(s) = \sinh \frac{s}{d} L, \quad \Delta'(s) = \frac{L}{2d} \sqrt{\frac{s}{d}} \cosh \frac{s}{d} L,
$$

So, we have

$$
\Delta(s_k) = 0, \Delta'(s_k) \neq 0, \quad k = 1, 2, 3, \cdots,
$$

at

$$
s_k = -\frac{k^2 \pi^2 d}{L^2}, \quad k = 1, 2, 3, \cdots.
$$

Moreover,

$$
\frac{a, \Phi(x; s) \phi b, \Phi(y; s)}{\sqrt{s/d} \Delta(s)} = \frac{\sinh \sqrt{s/d} \sinh \sqrt{s/d} \sqrt{s/d} \sinh \sqrt{s/d} \Delta(s)}{s/d \sinh \sqrt{s/d} \Delta(s)} = O(1),
$$

as $s \to 0$. Therefore, (70) gives $\Theta_0(x, y) = 0$. Further,

$$
\langle a, \Phi(-y; s_k) \phi b, \Phi(L - x; s_k) \rangle = -i \sin \frac{kr}{L} y, \langle a, \Phi(L - x; s_k) \rangle
$$

$$
= (-1)^k L \sin \frac{kr}{L} x, \quad k = 1, 2, 3, \cdots,
$$

$$
\sqrt{s_k} \Delta'(s_k) = (-1)^k \frac{L}{2d}, \quad k = 1, 2, 3, \cdots.
$$

Finally, from (80), we get

$$
\psi(x, t) = \theta(x, t) = -d^2 \sum_{k=1}^{\infty} \frac{e^{i\xi_k t}}{s_k \Delta'(s_k)} \sqrt{s_k / d} \Delta'(s_k) \psi(x)
$$

$$
= \sum_{k=1}^{\infty} \frac{2L}{s_k} \frac{e^{i\xi_k t} f(y)}{s_k / d} \sin \frac{kr}{L} \Delta(s_k) \sin \frac{kr}{L} x,
$$

which recovers the solution to problem (97) obtained via the separation of variables method in [55].
Example 2. Heat flow with sources and homogeneous boundary conditions.

The following initial boundary value problem

\[
\frac{\partial \varphi}{\partial t} = d \frac{\partial^2 \varphi}{\partial x^2} + (t^m + 1)^{-\eta} e^{-t} \sin 3x, \ x \in (0, \pi), \quad t > 0, \\
\varphi(x, 0) = f(x), \ x \in [0, \pi], \\
\varphi(0, t) = 0, \ \varphi(\pi, t) = 0, \quad t \geq 0,
\]  

(104)

is a special case of the one-layer diffusion system (34)–(37) when \( \alpha = 0, \beta = \pi, d = \tau = 1, t = 1, \ell = 1, l = 0 \) and \( r(x, t) = e^{t \sin 3x}; \eta(x) = f(x), \zeta(t) = \xi(t) = 0 \). Thus, from (93), we have \( \varphi(x, t) = \Theta(t, t) \). From (46), we have \( \alpha = \beta = (1, 0) \),

\[
\mathbf{Q}(x; s) = (\cosh \sqrt{s}x, \sqrt{s} \sinh \sqrt{s}x), \ \mathbf{Q}(x; s) \\
= (\sinh \sqrt{s}x, \sqrt{s} \cosh \sqrt{s}x).
\]  

(105)

Hence, (54) yields

\[
\Delta(s) = \sinh \sqrt{s}x, \Delta'(s) = \frac{\pi}{2} \frac{1}{\sqrt{s} \cosh \sqrt{s}x}.
\]  

(106)

So, \( \Delta(s) \) has simple zeros at \( s_k = -k^2, k = 1, 2, 3, \ldots \).

Further,

\[
\langle \mathbf{a}, \mathbf{Q}(x; s) \rangle = -i \sin ky, \quad \langle \mathbf{b}, \mathbf{Q}(x; s) \rangle \\
= (-1)^{k+1} \sin kx, \quad k = 1, 2, 3, \ldots,
\]  

(107)

\[
\sqrt{s_k} \Delta'(s_k) = (-1)^k \frac{\pi}{2}, \quad k = 1, 2, 3, \ldots.
\]

On the other hand,

\[
\frac{\langle \mathbf{a}, \mathbf{Q}(x; s) \rangle \langle \mathbf{b}, \mathbf{Q}(y; s) \rangle}{\sqrt{s_2} \Delta(s)} = \frac{\sinh \sqrt{s}x \sinh \sqrt{s}y}{\sqrt{s} \sinh \sqrt{s}x} = O(1)
\]  

(108)

as \( s \to 0 \). Therefore, (70) gives \( \Theta_0(x, y) = 0 \).

Finally, from (80), we get

\[
\varphi(x, t) = \Theta(x, t) = \sum_{k=1}^{\infty} \frac{2}{\pi} \int_0^{\pi} e^{(\xi-1)z} \sin 3y \sin kx \\
\times \int_0^{\pi} e^{(\xi-1)z} \sin 3x dz dy \\
+ \sum_{k=1}^{\infty} \frac{2}{\pi} f(y) \sin ky dy \\
= \left[ \int_0^{\pi} e^{(\xi-1)z} \sin 3x dz \right] e^{k\xi t} \sin kx \\
+ \sum_{k=1}^{\infty} \frac{2}{\pi} f(y) \sin ky dy \\
= \sum_{k=1}^{\infty} a_k(t) \sin kx,
\]  

(109)

where

\[
a_k(t) = \begin{cases} 
0, & k \neq 3, \\
2 \pi \int_0^{\pi} e^{(\xi-1)z} \sin 3y \sin ky dy, & k = 3,
\end{cases}
\]  

(110)

with

\[
a_k(0) = \frac{2}{\pi} \int_0^{\pi} f(y) \sin ky dy, \quad a(t) = \frac{2}{\pi} \int_0^{\pi} e^{(\xi-1)z} dz.
\]  

(111)

When \( \rho = 0 \), formula (109) recovers the solution to problem (104) (when \( \rho = 0 \)) obtained via the eigenfunction expansion method in [55].

3. Multilayer Nonhomogeneous Diffusion System

Here, we are seeking the solution of our main problem defined in (2)–(8), which was converted into a sequence of initial boundary value problems (9)–(11). For the convenience of the reader and in order to draw the full picture in an easy way, we start with solving the bilayer diffusion problem in the following subsection; then, we move to the general case in Section 3.2.

3.1. Solution of a Two-Layer Problem. For the two-layer problem, we have

\[
\frac{\partial \varphi_1}{\partial t} = d_1 \frac{\partial^2 \varphi_1}{\partial x^2} + \lambda(t, \tau) r_1(x, t), \quad x \in (x_0, x_1), \ t, \tau > 0,
\]  

(112)

\[
\varphi_1(x, 0) = \eta_1(x), \quad x \in [x_0, x_1],
\]  

(113)

\[
\varphi_1(x_0, t) + i \frac{\partial \varphi_1}{\partial x}(x_0, t) = \lambda(t, \tau) \zeta_1(t), \quad t \geq 0, \tau > 0,
\]  

(114)
\( v_1 \varphi_1(x_1, t) + \mu_1 \frac{\partial \varphi_1}{\partial x}(x_1, t) = \lambda(t, \tau) \xi_1(t), \quad t \geq 0, \tau > 0, \)  
(115)

\[
\frac{\partial \varphi_2}{\partial t} = d_2 \frac{\partial^2 \varphi_2}{\partial x^2} + \lambda(t, \tau) \psi_2(x, t), \quad x \in (x_1, x_2), \quad t \geq 0, \tau > 0,
\]
(116)

\[
\varphi_2(x, 0) = \eta_2(x), \quad x \in [x_1, x_2],
\]
(117)

\[
v_2 \varphi_2(x_1, t) + \mu_2 \frac{\partial \varphi_2}{\partial x}(x_1, t) = \lambda(t, \tau) \xi_2(t), \quad t \geq 0, \tau > 0,
\]
(118)

\[
\xi \varphi_2(x_2, t) + \frac{\partial \varphi_2}{\partial x}(x_2, t) = \lambda(t, \tau) \xi_2(t), \quad t \geq 0, \tau > 0.
\]
(119)

Similar to what we denote in Section 2, we define the following vector notation \( \mathbf{a}_1 = (t, t), \mathbf{b}_1 = (v_1, \mu_1), \mathbf{a}_2 = (v_2, \mu_2), \mathbf{b}_2 = (\xi, \lambda) \),

\[
\mathbf{Q}_1(y; s, \tau) = \begin{pmatrix} \cosh \frac{s}{\tau d_1} & \frac{s}{\sqrt{\tau d_1}} \sinh \frac{s}{\tau d_1} \\ \frac{s}{\tau d_1} \cosh \frac{s}{\tau d_1} & \frac{s}{\sqrt{\tau d_1}} \sinh \frac{s}{\tau d_1} \end{pmatrix},
\]

\[
\mathbf{Q}_2(y; s, \tau) = \begin{pmatrix} \cosh \frac{s}{\tau d_2} & \frac{s}{\sqrt{\tau d_2}} \sinh \frac{s}{\tau d_2} \\ \frac{s}{\tau d_2} \cosh \frac{s}{\tau d_2} & \frac{s}{\sqrt{\tau d_2}} \sinh \frac{s}{\tau d_2} \end{pmatrix},
\]

\[
\mathbf{G}_1(y; s, \tau) = \begin{pmatrix} \sinh \frac{s}{\tau d_1} & \frac{s}{\sqrt{\tau d_1}} \cosh \frac{s}{\tau d_1} \\ \frac{s}{\tau d_1} \sinh \frac{s}{\tau d_1} & \frac{s}{\sqrt{\tau d_1}} \cosh \frac{s}{\tau d_1} \end{pmatrix},
\]

\[
\mathbf{G}_2(y; s, \tau) = \begin{pmatrix} \sinh \frac{s}{\tau d_2} & \frac{s}{\sqrt{\tau d_2}} \cosh \frac{s}{\tau d_2} \\ \frac{s}{\tau d_2} \sinh \frac{s}{\tau d_2} & \frac{s}{\sqrt{\tau d_2}} \cosh \frac{s}{\tau d_2} \end{pmatrix}.
\]
(120)

Also, analogues to (54), define

\[
\Delta_1(s) = \langle \mathbf{b}_1, \mathbf{G}_1(x_1, s, \tau) \rangle \mathbf{a}_1, \mathbf{Q}_1(x_0, s, \tau) \rangle
\]

\[
- \langle \mathbf{a}_1, \mathbf{G}_1(x_0, s, \tau) \rangle \langle \mathbf{b}_1, \mathbf{Q}_1(x_1, s, \tau) \rangle,
\]
(121)

\[
\Delta_2(s) = \langle \mathbf{b}_2, \mathbf{G}_2(x_2, s, \tau) \rangle \mathbf{a}_2, \mathbf{Q}_2(x_1, s, \tau) \rangle
\]

\[
- \langle \mathbf{a}_2, \mathbf{G}_2(x_1, s, \tau) \rangle \langle \mathbf{b}_2, \mathbf{Q}_2(x_2, s, \tau) \rangle.
\]

Further, similar to (67), suppose that there are nonzero simple roots \( \{s_k^{(1)}\}_{k=1}^{\infty} \) and \( \{s_k^{(2)}\}_{k=1}^{\infty} \) of the functions \( \Delta_1(s) \)

and \( \Delta_2(s) \), respectively. That is,

\[
\Delta_1(s) = 0, \quad \Delta_1(s) \neq 0, \quad \Delta_2(s) \neq 0
\]

\[
(122)
\]

Therefore, according to (84), we obtain

\[
\theta_1(x, t, \tau) = - \frac{1}{\tau d_1} \int_{x_0}^{x_1} \Theta_1^{(1)}(x_0 - x, x_1 - y) \eta_1(y) dy
\]

\[
- \frac{1}{\tau d_1} \int_{x_0}^{x_1} \Theta_1^{(1)}(x_0 - x, x_1 - y) \frac{r_1(y, \xi)}{(s/m + \tau m)^p} dy
\]

\[
= - \frac{1}{\tau d_1} \sum_{k=1}^{\infty} \left( \mathbf{b}_1, \mathbf{G}_1 \left( x_1 - x, s_k^{(1)}, \tau \right) \right) \left( \sqrt{s_k^{(1)}} / \tau d_1 \right) \Delta_1(s)
\]

\[
\left[ \int_{x_0}^{x_1} \frac{e^{i(t-r)/\tau d_1} \eta_1(y)}{(\xi - s_k^{(1)})^{1/2} \tau d_1 \Delta_1(s)} \right] \left( \sqrt{s_k^{(1)}} / \tau d_1 \right) \Delta_1(s)
\]

\[
\left[ \int_{x_0}^{x_1} \left( \mathbf{b}_1, \mathbf{G}_1 \left( x_1 - x, s_k^{(1)}, \tau \right) \right) \eta_1(y) dy \right],
\]
(123)

\[
\theta_2(x, t, \tau) = - \frac{1}{\tau d_2} \int_{x_0}^{x_1} \Theta_2^{(2)}(x_1 - x, x_2 - y) \eta_2(y) dy
\]

\[
- \frac{1}{\tau d_2} \int_{x_0}^{x_1} \Theta_2^{(2)}(x_1 - x, x_2 - y) \frac{r_2(y, \xi)}{(s/m + \tau m)^p} dy
\]

\[
= - \frac{1}{\tau d_2} \sum_{k=1}^{\infty} \left( \mathbf{b}_2, \mathbf{G}_2 \left( x_2 - x, s_k^{(2)}, \tau \right) \right) \left( \sqrt{s_k^{(2)}} / \tau d_2 \right) \Delta_2(s)
\]

\[
\left[ \int_{x_0}^{x_1} \frac{e^{i(t-r)/\tau d_2} \eta_2(y)}{(\xi - s_k^{(2)})^{1/2} \tau d_2 \Delta_2(s)} \right] \left( \sqrt{s_k^{(2)}} / \tau d_2 \right) \Delta_2(s)
\]

\[
\left[ \int_{x_0}^{x_1} \left( \mathbf{b}_2, \mathbf{G}_2 \left( x_2 - x, s_k^{(2)}, \tau \right) \right) \eta_2(y) dy \right],
\]
(124)

where \( \Theta_0^{(1)} \) and \( \Theta_0^{(2)} \) can be defined as in Lemma 4.
Also, similar to (93), with the respective forms \( \Phi_0^{(1)} \) and \( \Phi_0^{(2)} \) from Lemma 5 and the matching condition \( \xi_1(t) = \xi_2 \).
(t), we get

\[
\varphi_1(x, t, \tau) = \lambda(t, \tau)\xi_1^{(1)}(x; \tau) + \sum_{k=1}^{\infty} I_k^{(1)}(\xi_1^{(1)}; \tau) \left( b_1, \xi_1 \right)_{x, s_k^{(1)}(\tau)} + \theta_1(x, t, \tau),
\]

\[
\varphi_2(x, t, \tau) = \lambda(t, \tau)\xi_1^{(2)}(x; \tau) + \sum_{k=1}^{\infty} I_k^{(2)}(\xi_1^{(2)}; \tau) \left( b_2, \xi_1 \right)_{x, s_k^{(2)}(\tau)} + \theta_2(x, t, \tau),
\]

(125)

where the operators \( I_k^{(1)} \) and \( I_k^{(2)} \) are obtained from (94). The matching condition \( \varphi_1(x_1, t) = \Lambda_1 \varphi_2(x_1, t) \) yields

\[
\left( \Lambda_1 \lambda(t, \tau) \Phi_0^{(2)}(x_2 - x_1; b_2) + \lambda(t, \tau) \Phi_0^{(1)}(x_0 - x_1; a_1) \right) \xi_1^{(1)}(t)
+ \sum_{k=1}^{\infty} \left( a_1, \xi_1 \right)_{x, s_k^{(1)}(\tau)} I_k^{(1)}(\xi_1^{(1)}; \tau)
= \lambda(t, \tau) \xi_1^{(2)}(t) + \sum_{k=1}^{\infty} \left( a_2, \xi_1 \right)_{x, s_k^{(2)}(\tau)} I_k^{(2)}(\xi_1^{(2)}; \tau)
+ \theta_1(x_1, t, \tau),
\]

(126)

where

\[
\begin{align*}
& a_k = \left\langle a_1, \xi_1 \right\rangle_{x, s_k^{(1)}(\tau)} \left( \xi_1^{(1)}(t) \Phi_0^{(2)}(x_2 - x_1; b_2) + \Phi_0^{(1)}(x_0 - x_1; a_1) \right) \\
& b_k = \left\langle b_2, \xi_1 \right\rangle_{x, s_k^{(2)}(\tau)} \left( \xi_1^{(2)}(t) \Phi_0^{(2)}(x_2 - x_1; b_2) + \Phi_0^{(1)}(x_0 - x_1; a_1) \right) \\
& c(t) = \left( \xi_1^{(1)}(t) \Phi_0^{(2)}(x_2 - x_1; b_2) + \Phi_0^{(1)}(x_0 - x_1; a_1) \right) \xi_1^{(2)}(t) - \theta_1(x_1, t, \tau).
\end{align*}
\]

(127)

Inspired by the convolution formula (31), the natural transform of (128) is

\[
\mathcal{N} \left[ \xi_1(t) \lambda(t, \tau) \right](s, \tau) = \sum_{k=1}^{\infty} \left( a_k \mathcal{N} \left[ \xi_1(t) \lambda(t, \tau) \right](s, \tau) - \frac{s^{\xi_k^{(1)}(\tau)} - s^{\xi_k^{(2)}(\tau)}}{s - s_k^{(1)}(\tau)} \mathcal{N} \left[ \xi_1(t) \lambda(t, \tau) \right](s, \tau) \right) \\
+ \mathcal{N} \left[ \xi_1(t) \lambda(t, \tau) \right](s, \tau) - \mathcal{N} \left[ \xi_1(t) \lambda(t, \tau) \right](s, \tau) = \mathcal{N} \left[ c(t) \right](s, \tau),
\]

(130)

which can be rewritten as

\[
(1 - \mathcal{N} \left[ \Psi(t) \right](s, \tau)) \mathcal{N} \left[ \xi_1(t) \lambda(t, \tau) \right](s, \tau) = \mathcal{N} \left[ c(t) \right](s, \tau).
\]

(131)

That is,

\[
\mathcal{N} \left[ \xi_1(t) \lambda(t, \tau) \right](s, \tau) = \frac{1}{1 - \mathcal{N} \left[ \Psi(t) \right](s, \tau)} \left[ \mathcal{N} \left[ c(t) \right](s, \tau) \right]
= \left( 1 + \mathcal{N} \left[ \Psi(t) \right](s, \tau) + (\mathcal{N} \left[ \Psi(t) \right](s, \tau))^2 + \cdots \right) \left[ \mathcal{N} \left[ c(t) \right](s, \tau) \right]
= \mathcal{N} \left[ c(t) \right](s, \tau) + \mathcal{N} \left[ c(t) \right](s, \tau) \sum_{m=1}^{\infty} (\mathcal{N} \left[ \Psi(t) \right](s, \tau))^m,
\]

(132)
where

\[
\mathbb{N}[\psi(t)](s, \tau) = -\sum_{k=1}^{\infty} \left( a_k \left( 1 - \frac{\tau s_k^{(1)}}{s - s_k^{(1)}} \right) + b_k \left( 1 - \frac{\tau s_k^{(2)}}{s - s_k^{(2)}} \right) \right),
\]

for which the inverse natural transform is

\[
\psi(t) = -\tau \sum_{k=1}^{\infty} (a_k + b_k) \delta_0(t) + \tau \sum_{k=1}^{\infty} \left( a_k s_k^{(1)} e^{(s_k^{(1))/t)} + b_k s_k^{(2)} e^{(s_k^{(2))/t)} \right),
\]

where \( \delta_0 \) is the Dirac delta function. Hence, we have

\[
\xi_1(t) = \frac{1}{\lambda(t, \tau)} \left( c(t) + \sum_{m=1}^{\infty} \psi_m * c(t) \right),
\]

where for \( m \geq 2, \psi_m \) is the \( m \)-times self-convolution of \( \psi \).

Thus, one can conclude the solvability of problem (112)–(119) by the formulas (125) and (126), together with (123) and (124), with \( \xi_1 \) given in (135). Now, it is time to attack an illustrative example in the following subsection.

### 3.1.1. Illustrative Example

Here, we discuss the solvability of the following two-layer diffusion system.

**Example 3.** Temperature distribution in the two-layer slab with mixed boundary condition.

Consider the following initial boundary value system. The diffusion equations are

\[
\frac{\partial \varphi_k}{\partial t} = \frac{d^2 \varphi_k}{dx^2}, \quad x_{k-1} < x < x_k, x_0 = 0, t > 0, k = 1, 2,
\]

the initial conditions are

\[
\varphi_k(x, 0) = \eta_k(x), \quad x_{k-1} \leq x \leq x_k, k = 1, 2,
\]

the outer boundary conditions are

\[
\varphi_1(0, t) = 0, \quad \frac{\partial \varphi_2(x_2, t)}{\partial x} = 0, \quad t \geq 0,
\]

and the interface conditions at \( x = x_1 \) are

\[
\varphi_1(x_1, t) = \varphi_2(x_1, t), \quad d_1 \frac{\partial \varphi_1(x_1, t)}{\partial x} = d_2 \frac{\partial \varphi_2(x_1, t)}{\partial x}, \quad t \geq 0.
\]

Comparing this problem with the general one that is defined by (112)–(119) reveals

\[
\begin{align*}
\varphi_1(x, t) &= -\xi_1(t) \Phi_0^{(1)}(x_0 - x; a_1) \\
&\quad - \sum_{k=1}^{\infty} \frac{\Gamma_k^{(1)}(t)}{\Delta_k^{(1)}(s_k^{(1)})} \left\langle a_1, \mathfrak{C}_1 \left( x_0 - x, s_k^{(1)} \right) \right\rangle \\
&\quad + \theta_1(x, t),
\end{align*}
\]

\[
\begin{align*}
\varphi_2(x, t) &= \xi_1(t) \Phi_0^{(2)}(x_2 - x; b_2) \\
&\quad + \sum_{k=1}^{\infty} \frac{\Gamma_k^{(2)}(t)}{\Delta_k^{(2)}(s_k^{(2)})} \left\langle b_2, \mathfrak{C}_2 \left( x_2 - x, s_k^{(2)} \right) \right\rangle \\
&\quad + \theta_2(x, t),
\end{align*}
\]

with

\[
\begin{align*}
\theta_1(x, t) &= -\frac{1}{d_1} \int_{x_0}^{x_1} \Phi_0^{(1)}(x_0 - x, x_1 - y) \eta_1(y) dy \\
&\quad - \frac{1}{d_1} \sum_{k=1}^{\infty} \frac{\epsilon^{(1)}}{\sqrt{s_k^{(1)}}/d_1 \Delta_k^{(1)}(s_k^{(1)})} \left\langle a_1, \mathfrak{C}_1 \left( x_0 - y, s_k^{(1)} \right) \right\rangle \eta_1(y) dy, \\
\theta_2(x, t) &= -\frac{1}{d_2} \int_{x_1}^{x_2} \Phi_0^{(2)}(x_1 - x, x_2 - y) \eta_2(y) dy \\
&\quad - \frac{1}{d_2} \sum_{k=1}^{\infty} \frac{\epsilon^{(2)}}{\sqrt{s_k^{(2)}}/d_2 \Delta_k^{(2)}(s_k^{(2)})} \left\langle b_2, \mathfrak{C}_2 \left( x_2 - y, s_k^{(2)} \right) \right\rangle \eta_2(y) dy.
\end{align*}
\]

The operators \( \Gamma_k^{(1)} \) and \( \Gamma_k^{(2)} \) are obtained from (94).
Next, we are going to simplify these formulas. Direct computations give

\[
\begin{align*}
\Delta_1(s) &= \frac{\sqrt{s}}{d_1} \cos \left( \frac{s}{d_1} \right), \quad \Delta_2(s) = \frac{-s}{d_2} \sinh \left( \frac{\sqrt{s}}{d_2} (x_2 - x_1) \right), \\
\Delta'_1(s) &= \frac{\sqrt{s}}{d_1} \cos \left( \frac{s}{d_1} x_1 + \frac{1}{2s} \sqrt{s} \right) \cos \left( \frac{s}{d_1} x_2 - x_1 \right), \\
\Delta'_2(s) &= -\frac{1}{d_2} \sinh \left( \frac{\sqrt{s}}{d_2} (x_2 - x_1) + \frac{x_2 - x_1}{2d_2} \sqrt{s} \right) \cosh \left( \frac{s}{d_2} (x_2 - x_1) \right).
\end{align*}
\]  

(144)

It is clear that \( \Delta_1(s_k^{(1)}) = 0 \) and \( \Delta_2(s_k^{(2)}) = 0 \) when

\[
\begin{align*}
s_k^{(1)} &= \frac{(2k - 1)^2 \pi^2 d_1}{4x_1^2}, \quad s_k^{(2)} = -\frac{k^2 \pi^2 d_2}{(x_2 - x_1)^2}, \quad k = 1, 2, 3, \ldots,
\end{align*}
\]

(145)

respectively. Moreover,

\[
\begin{align*}
\frac{\langle a_1, G_1(x; s) \rangle}{\sqrt{s/d_1} \Delta_1(s)} &= \frac{\sinh \sqrt{s/d_1} x \cos \sqrt{s/d_1} y}{\sqrt{s/d_1} \cosh \sqrt{s/d_1} x} = O(1), \\
\frac{\langle a_2, G_2(x; s) \rangle}{\sqrt{s/d_2} \Delta_2(s)} &= \frac{\cosh \sqrt{s/d_2} x \cos \sqrt{s/d_2} y}{\sqrt{s/d_2} \sinh \sqrt{s/d_2} (x_2 - x_1)} = O \left( \frac{1}{s} \right).
\end{align*}
\]

(146)

as \( s \to 0 \). Therefore, (70) gives \( \Theta_0^{(1)}(x, y) = 0 \) and \( \Theta_0^{(2)}(x, y) = -(d_2/(x_2 - x_1)). \)

Further,

\[
\begin{align*}
\left\langle a_1, G_1 \left( -y; s_k^{(1)} \right) \right\rangle &= -i \sin \left( \frac{2k - 1 - \pi}{2} \right) \frac{\pi}{2x_1}, \\
\left\langle b_1, G_1 \left( x_1 - x; s_k^{(2)} \right) \right\rangle &= (-1)^{k+1} \left( \frac{2k - 1}{2} \pi \right) \frac{\sin \left( \frac{2k - 1}{2} \pi \right)}{2x_1} x_x, \\
\left\langle a_2, G_2 \left( x_1 - x; s_k^{(1)} \right) \right\rangle &= (-1)^k \left( \frac{2k - 1}{4} \right) \frac{\sqrt{s/d_1} x_1 \cos \left( \frac{s}{d_1} x_1 \right)}{x_2 - x_1} (x_1 - y), \\
\left\langle b_2, G_2 \left( x_2 - x; s_k^{(2)} \right) \right\rangle &= \left( \frac{k \pi}{x_2 - x_1} \right) \cos \left( \frac{k \pi}{x_2 - x_1} \right) (x_2 - x). \\
\end{align*}
\]

(147)

Thus,

\[
\begin{align*}
\theta_1(x, t) &= \sum_{k=1}^{\infty} \int_{0}^{\infty} \eta_1(y) \sin \left( \frac{(2k - 1)\pi y}{2x_1} \right) e^{-\left( \frac{(2k - 1)\pi x}{2x_1} \right)^2} dy \sin \left( \frac{(2k - 1)\pi x}{2x_1} \right), \\
\theta_2(x, t) &= \frac{1}{x_2 - x_1} \int_{0}^{\infty} \eta_2(y) \cos \left( \frac{k \pi y}{x_2 - x_1} (x_2 - y) \right) dy + \sum_{k=1}^{\infty} (-1)^k \\
&\cdot \frac{1}{x_2 - x_1} \int_{0}^{\infty} \eta_2(y) \cos \left( \frac{k \pi y}{x_2 - x_1} (x_2 - y) \right) dy \\
&\cdot e^{-\left( \frac{k \pi}{x_2 - x_1} \right)^2 (x_2 - x)} x_2 \cos \left( \frac{k \pi}{x_2 - x_1} (x_2 - x) \right). 
\end{align*}
\]

(148)

Similarly, we have

\[
\begin{align*}
\frac{\langle a_1, G_1(x_0 - x, s) \rangle}{sd_1 \sqrt{s/d_1} \cosh \sqrt{s/d_1} x_1} &= \sinh \sqrt{s/d_1} x_0 \cosh \sqrt{s/d_1} x_1 = O \left( \frac{1}{s} \right), \\
\frac{\langle b_2, G_2(x_2 - x, s) \rangle}{sd_2 \sqrt{s/d_2} \cosh \sqrt{s/d_2} x_2} &= \sinh \sqrt{s/d_2} x_2 \cosh \sqrt{s/d_2} x_2 = O \left( \frac{1}{s} \right). 
\end{align*}
\]

(149)

So, (88) gives

\[
\begin{align*}
\Phi_0^{(1)}(x_0 - x; a_1) &= -\frac{x}{d_1}, \Phi_0^{(2)}(x_2 - x; b_2) \\
&= \frac{1}{6d_2 (x_2 - x_1)} \left( 3v_2 - x_2 - (x_2 - x_1)^2 \right). 
\end{align*}
\]

(150)

Hence, loading the quantities in (141) and (142) gives the solution to problem (136)–(139) as

\[
\begin{align*}
\varphi_1(x, t) &= \frac{x}{d_1} \xi_1(t) + \frac{8x_1}{d_1^2} \sum_{k=1}^{\infty} \left( -1 \right)^k \frac{1}{k \pi} \xi_1(t) \sin \left( \frac{2k - 1}{2} \right) \pi x_t \theta_1(x, t), \\
\varphi_2(x, t) &= \frac{3}{6d_2 (x_2 - x_1)} \xi_1(t) - \frac{2}{d_2 \pi^2} \sum_{k=1}^{\infty} \left( -1 \right)^k \frac{1}{k \pi} \xi_1(t) \cos \left( \frac{k \pi}{x_2 - x_1} (x_2 - x) + \theta_2(x, t). 
\end{align*}
\]

(151)

Applying the matching condition \( \varphi_1(x_1, t) = \varphi_2(x_1, t) \) gives

\[
\xi_1(t) + \sum_{k=1}^{\infty} \left[ a_k \varphi_1^{(1)} \xi_1(t) + b_k \varphi_2^{(2)} \xi_1(t) \right] = c(t),
\]

(152)
where

\[ a_k = -\frac{24d_2x_1}{(2k-1)^2\pi^2(3d_2 - d_1)x_1 + d_1x_2}, \]
\[ b_k = -\frac{6d_1(x_1 - x_1)}{k^2\pi^2[3d_2 - d_1]x_1 + d_1x_2}, \]
\[ c(t) = \frac{3d_1d_2}{(3d_2 - d_1)x_1 + d_1x_2} \theta_1(x_1, t) - \theta_2(x_1, t). \]

(153)

To solve this integral equation, we use (134), to obtain

\[ \psi(t) = \delta_0(t) - \frac{6d_1d_2}{(3d_2 - d_1)x_1 + d_1x_2} \sum_{k=1}^{\infty} \left( \frac{1}{x_1} e^{-2kx_1^2/(3d_2 - d_1)x_1 + d_1x_2} + \frac{1}{x_2 - x_1} e^{-2kx_2^2/(3d_2 - d_1)x_1 + d_1x_2} \right), \]

where \( \delta_0 \) is the Dirac delta function. Here, we used

\[ \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}. \]

(154)

Hence, from (135), we have

\[ \xi(t) = c(t) + \psi(t) + (\psi * \psi)(t) + \psi * \psi * \psi(t) + \cdots = c(t) + \sum_{m=1}^{\infty} (\psi_m * \psi)(t). \]

(156)

3.2. Solution of a Multilayer Problem. Here, we investigate the solvability of the main problem (2)–(8), through solving the initial boundary value problems (9)–(11). Similar to what we have denoted in Section 2, we consider the following notations:

\[ a_j = \begin{cases} (i, i), & j = 1, \\ (v_j, \mu_j), & 2 \leq j \leq n, \end{cases} \]
\[ b_j = \begin{cases} (v_j, \mu_j), & 1 \leq j \leq n - 1, \\ (\ell, l), & j = n, \end{cases} \]

(157)

and, for all \( 1 \leq j \leq n, \)

\[ \mathcal{G}_j(y; s, t) = \left( \cosh \frac{s}{td_j} y_j, \sinh \frac{s}{td_j} y_j \right), \]
\[ \mathcal{C}_j(y; s, t) = \left( \sinh \frac{s}{td_j} y_j, \cosh \frac{s}{td_j} y_j \right). \]

Moreover, we define

\[ \Delta_j(s) = \left\langle a_j, \mathcal{G}_j(x_j, s, t) \right\rangle - \left\langle b_j, \mathcal{G}_j(x_j, s, t) \right\rangle, \]

\[ - \left\langle a_j, \mathcal{G}_j(x_j, s, t) \right\rangle + \left\langle b_j, \mathcal{G}_j(x_j, s, t) \right\rangle, \]

(159)

and let \( \{x_k^{(j)}\}_{k=1}^{\infty} \) be the sequence of zeros of the function \( \Delta_j(s) \) for all \( 1 \leq j \leq n, \)

\[ \Delta_j(x_j^{(j)}) = 0, \Delta_j(x_j^{(j)}) \neq 0 \quad (k = 1, 2, \cdots). \]

(160)

Analogue to the computations of (84) and (93), we have for the current case, for all \( j = 1, \cdots, n, \)

\[ \theta_j(x, t) = \int_{x_j}^{x_j + \tau} \left( \Theta_\phi^{(j)}(x_j, t; x_j - \gamma) \eta_j(\gamma) d\gamma \right) \]

\[ - \int_{x_j}^{x_j + \tau} \frac{\tau_j(\gamma, \tau)}{(\tau(\tau + \tau) + \tau)^2} d\gamma \]

\[ - \int_{x_j}^{x_j + \tau} \frac{\tau_j(\gamma, \tau)}{\sqrt{s_j^{(j)}/2}} d\gamma \]

\[ \int_{x_j}^{x_j + \tau} \frac{\tau_j(\gamma, \tau)}{\sqrt{s_j^{(j)}/2}} d\gamma \]

\[ \int_{x_j}^{x_j + \tau} \left( a_j, \mathcal{G}_j(x_j, t; x_j - \gamma) \right) \eta_j(\gamma) d\gamma \]

(161)

where \( \Theta_\phi^{(j)} \) can be defined in a similar way as in Lemma 4, and

\[ \varphi_j(x, t; \tau) = \lambda(\tau, \tau) \xi_j(t) \theta_j(x, t; b_j) \]

\[ - \int^t_0 \lambda(\tau, \tau) \theta_j(x, t; b_j) \xi_j(t) d\tau \]

\[ - \frac{\lambda(\tau, \tau) \theta_j(t; b_j) \xi_j(t)}{\lambda(\tau, \tau) \theta_j(t; b_j) \xi_j(t) + \lambda(\tau, \tau) \theta_j(t; b_j) \xi_j(t) + \lambda(\tau, \tau) \theta_j(t; b_j) \xi_j(t)} \]

\[ + \theta_j(x, t, \tau), \]

(162)

with the respective forms \( \Phi_j \) defined by (87) in Lemma 5. This last equation (162) can be rewritten as

\[ \varphi_j(x, t, \tau) = \tilde{T}_j(\tau) \xi_j(t) - \tilde{T}_j(\tau) \xi_j(t) + \theta_j(x, t, \tau), \]

(163)

in which \( \tilde{T}_j = \lambda(\tau, \tau) \xi_j(t), \xi_j = \lambda(\tau, \tau) \xi_j(t) \) and the linear
operator \( T_j \) is defined by

\[
T_j \varphi (y, t; L) = \Phi_j (y, 0; L) \varphi (t) - \int_0^t D_j \Phi_j (y, t - \zeta; L) \varphi (\zeta) d\zeta,
\]

(164)

for all \( j = 1, \cdots, n, L \in \mathbb{R}^2 \).

The matching conditions \( \varphi_j (x_j, t, \tau) = \Lambda_j \varphi_{j+1} (x_j, t, \tau), \) \( j = 1, \cdots, n - 1 \), lead to

\[
T_j \hat{\xi}_1 (0, t; b_j) - T_j \hat{\xi}_j (x_j - x_j, t; a_j) + \theta_j (x_j, t, \tau) = \Lambda_j \hat{\xi}_{j+1} (x_{j+1} - x_j, t; b_{j+1}) + \Lambda_j \hat{\xi}_{j+1} (0, t; a_{j+1}) + \theta_{j+1} (x_j, t, \tau),
\]

(165)

Using the matching conditions (12), we have \( \hat{\xi}_{j+1} (t) = \hat{\xi}_j (t) \) for all \( 1 \leq j \leq n - 1 \). Thus, for \( j = 1, \)

\[
-T_1 \hat{\xi}_1 (x_1 - x_1, t; a_1) = \Lambda_1 T_1 \hat{\xi}_1 (x_1 - x_1, t; b_2) + \Lambda_1 T_1 \hat{\xi}_2 (0, t; a_2) = \Lambda_1 \theta_1 (x_1, t, \tau) - \theta_1 (x_1, t, \tau) - T_1 \hat{\xi}_1 (0, t; b_1).
\]

(166)

For \( 2 \leq j \leq n - 2, \)

\[
T_j \hat{\xi}_{j+1} (0, t; b_j) - T_j \hat{\xi}_j (x_j - x_j, t; a_j) = \Lambda_j T_j \hat{\xi}_{j+1} (x_{j+1} - x_j, t; b_{j+1}) + \Lambda_j T_j \hat{\xi}_{j+1} (0, t; a_{j+1}) = \Lambda_j \theta_{j+1} (x_j, t, \tau) - \theta_j (x_j, t, \tau).
\]

(167)

For \( j = n - 1, \)

\[
T_{n-1} \hat{\xi}_{n-2} (0, t; b_{n-1}) - T_{n-1} \hat{\xi}_{n-1} (x_{n-2} - x_{n-1}, t; a_{n-1}) = \Lambda_{n-1} T_{n-1} \hat{\xi}_{n-1} (x_{n-1} - x_{n-1}, t; b_n) + \Lambda_{n-1} \theta_{n-1} (x_{n-1}, t, \tau) - \theta_{n-1} (x_{n-1}, t, \tau) - T_{n-1} \hat{\xi}_{n-1} (0, t; a_n).
\]

(168)

System (166), (167), and (168), of \( (n - 1) \) integral equations of the unknowns \( \hat{\xi}_j; 1 \leq j \leq n - 1 \), can be adjusted as a matrix equation

\[
\mathcal{A} (0) \mathbf{h} (t) + \left( \mathcal{A}' \ast \mathbf{h} \right) (t) = \mathbf{b} (t), \tag{169}
\]

with \( \mathcal{A}(t) \) is a tridiagonal matrix of order \( n - 1 \) whose entries are as follows:

\[
\begin{align*}
\mathcal{A}_j (x_j - x_j, t; a_j) - \Lambda_j \mathcal{A}_j (x_j - x_j, t; b_{j+1}), & \quad 1 \leq j \leq n - 1 \text{ (main diagonal),} \\
\Lambda_{j+1} \mathcal{A}_j (0, t; a_j), & \quad 1 \leq j \leq n - 2 \text{ (super diagonal)} \\
\Lambda_j \mathcal{A}_j (0, t; b_j). & \quad 2 \leq j \leq n - 1 \text{ (subdiagonal),}
\end{align*}
\]

(170)

and the vectors \( \mathbf{h} (t) \) and \( \mathbf{b} (t) \) are defined as

\[
\mathbf{h} (t) = \left( \begin{array}{c}
\hat{\xi}_1 (t) \\
\vdots \\
\hat{\xi}_{n-1} (t)
\end{array} \right),
\]

\[
\mathbf{b} (t) = \left( \begin{array}{c}
\Lambda_1 \theta_1 (x_1, t, \tau) - \theta_1 (x_1, t, \tau) - T_1 \hat{\xi}_1 (0, t; b_1) \\
\Lambda_2 \theta_2 (x_2, t, \tau) - \theta_2 (x_2, t, \tau) \\
\vdots \\
\Lambda_{n-1} \theta_{n-1} (x_{n-1}, t, \tau) - \theta_{n-1} (x_{n-1}, t, \tau) - T_{n-1} \hat{\xi}_{n-1} (0, t; a_n)
\end{array} \right).
\]

(171)

In fact, we can rewrite (169) as

\[
\mathbf{h} (t) = \tilde{\mathcal{C}} (t) + (\mathcal{R} \ast \mathbf{h}) (t), \tag{172}
\]

with \( \tilde{\mathcal{C}} (t) = \mathcal{A} (0)^{-1} \mathbf{b} (t) \) and \( \mathcal{R} (t) = -\mathcal{A} (0)^{-1} \mathcal{A}' (t) \). In view of the convolution formula (31), the natural transform of (172) reads

\[
\mathcal{N} [\mathbf{h} (t)] (s, \tau) = \mathcal{N} [\tilde{\mathcal{C}} (t)] (s, \tau) + \tau \mathcal{N} [\mathcal{R} (t)] (s, \tau) \mathcal{N} [\mathbf{h} (t)] (s, \tau), \tag{173}
\]

which is equivalent to

\[
\mathcal{N} [\mathbf{h} (t)] (s, \tau) = (I - \tau \mathcal{N} [\mathcal{R} (t)] (s, \tau))^{-1} \mathcal{N} [\mathcal{C} (t)] (s, \tau)
\]

\[
= (I + \tau \mathcal{N} [\mathcal{R} (t)] (s, \tau) + (\tau \mathcal{N} [\mathcal{R} (t)] (s, \tau))^2 + \cdots) \mathcal{N} [\mathcal{C} (t)] (s, \tau), \tag{174}
\]

where \( I \) is the \( (n - 1) \times (n - 1) \) identity matrix. Once again, throughout the convolution sense (31), the natural transform inversion of (174) is

\[
\mathbf{h} (t) = \tilde{\mathcal{C}} (t) + (\mathcal{R} \ast \tilde{\mathcal{C}}) (t) + (\mathcal{R} \ast \mathcal{R} \ast \tilde{\mathcal{C}}) (t) + \cdots = \mathcal{C} (t) + \sum_{m=1}^{\infty} (\mathcal{R} \ast \mathcal{C}) (t), \tag{175}
\]

where \( \mathcal{R} m \) is the \( m \)-times self-convolution of \( \mathcal{R} \). Finally, the solution of the nonhomogeneous multilayer diffusion systems (9)–(11) and hence that of the main problem (2)–(8)
is concluded as
\[
\varphi_j(x, t, \tau) = \lambda(t, \tau) b_j(x_t) \frac{d^{(j)}}{dx^{(j-1)}} (x_j - x; b_j) \\
- \lambda(t, \tau) \xi_j(t) \frac{d^{(j)}}{dx^{(j-1)}} (x_{j-1} - x; a_j) \\
+ \sum_{k=1}^{\infty} \lambda(t, \tau) \xi_j(t) - \int_0^\tau \xi_j(t) \frac{d^{(j)}}{dx^{(j-1)}} (x_j - x; b_j) \\
\quad + \lambda(t, \tau) \xi_j(t) \frac{d^{(j)}}{dx^{(j-1)}} (x_{j-1} - x; a_j) \\
\quad + \sum_{k=1}^{\infty} \lambda(t, \tau) \xi_j(t) - \int_0^\tau \xi_j(t) \frac{d^{(j)}}{dx^{(j-1)}} (x_j - x; b_j) \\
\quad + \lambda(t, \tau) \xi_j(t) \frac{d^{(j)}}{dx^{(j-1)}} (x_{j-1} - x; a_j) \\
\quad + \theta_j(x, t, \tau),
\]
(176)
with the respective forms \( \Phi^{(j)}_b \) and \( \theta_j \) defined as in (88) and (161), respectively, for all \( j = 1, \cdots, n \).

4. Conclusion

Throughout the current contribution, a one-dimensional \( n \)-layer nonhomogeneous diffusion problem with time-varying data and general interface conditions has been concluded by means of a generalized integral transform. Although most of the previous works have been focused on solving the problems of the homogeneous diffusion equation, the nonhomogeneous diffusion equation problem arises in many physical applications. We have obtained the exact solutions for one- and multilayer nonhomogeneous diffusion problems. The former case has been solved by a new generalized integral transform; the latter one (\( n \)-layer problem) has been recast in a sequence of one-layer problems. The obtained results generalize and extend those in [11, 33, 38, 40, 42]. Our results motivate to deal with other types of diffusion problems, for example, reaction diffusion problems, advection-reaction diffusion problems, and nonautonomous reaction diffusion problems.

On the other hand, more general partial differential equations (PDEs) and systems can be considered, for example, system of coupled PDEs, nonlinear diffusion PDEs, and nonautonomous reaction diffusion PDEs. Those kinds of PDEs appear widely as epidemiological models to study and analyze the spread of diseases and pandemics [22–25].

Data Availability

No data were used to support the study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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