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Research Article

Kannan Nonexpansive Mappings on Nakano Sequence Space of Soft Reals with Some Applications

Awad A. Bakery[1,2] **and Mustafa M. Mohammed**[1,3]

[1]*University of Jeddah, College of Science and Arts at Khulis, Department of Mathematics, Jeddah, Saudi Arabia*
[2]*Department of Mathematics, Faculty of Science, Ain Shams University, Abbassia, Egypt*
[3]*Department of Statistics, Faculty of Science, Sudan University of Science & Technology, Khartoum, Sudan*

Correspondence should be addressed to Mustafa M. Mohammed; mustasta@gmail.com

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We developed the operators ideal in this article by extending s-soft reals and a particular space of sequences with soft real numbers. The criteria necessary for the Nakano sequence space of soft real numbers given with the definite function to be prequasi Banach and closed are investigated. This space's (R) and normal structural features are illustrated. Fixed points have been introduced for Kannan contraction and nonexpansive mapping. Finally, we investigate whether the Kannan contraction mapping has a fixed point in the prequasi operator ideal with which it is linked. By examining some real-world instances and their applications, it is demonstrated that there exist solutions to nonlinear difference equations.

1. Introduction

The study of variable exponent Lebesgue spaces received additional impetus from the mathematical explanation of non-Newtonian fluids' hydrodynamics (see [1, 2]). Electrorheological fluids have various applications in various fields, including military science, civil engineering, and orthopedics. Since the publication of the Banach fixed point theorem [3], there have been numerous developments in the field of mathematics. While contractions have fixed point actions, Kannan [4] illustrated a noncontinuous mapping. In Reference [5], a single attempt was made to explain Kannan operators in modular vector spaces, and this was the only one that worked. Mitrovic$'$ et al. [6] defined a cone $b_v(s)$-metric space over Banach algebra as a generalization of metric spaces, rectangular metric spaces, b-metric spaces, rectangular b-metric spaces, v-generalized metric spaces, cone b-metric spaces over Banach algebra, and rectangular cone b-metric spaces over Banach algebra. They provided fixed point results for Banach and Kannan in cone $b_v(s)$-metric spaces over Banach algebra. Debnath et al. [7] showed the existence and uniqueness of common fixed points for pairs of self-maps of the Kannan, Reich, and Chatterjea types in a complete metric space. Younis et al. [8] used concepts from graph theory and fixed point theory to provide a fixed point result for Kannan-type mappings in the context of freshly published graphical b-metric spaces. They provided suitable examples of graphs that corroborated the existing theory. They demonstrated the anticipated results by applying them to several nonlinear issues encountered in engineering and research. Younis and Singh [9] discovered adequate conditions for the existence of solutions to certain classes of Hammerstein integral equations and fractional differential equations. They extended the concept of Kannan mappings in terms of F-contraction in the context of b-metric-like spaces and provided a series of novel and nontrivial instances, as well as computer simulations, to demonstrate the established results, therefore introducing the concept in a novel way. On the other hand, several unresolved issues are offered to enthusiastic readers. More information on Kannan's fixed point theorems can be found here (see [10–15]). The mathematics underpinnings of fuzzy set theory, which were pioneered by Zadeh [16] in 1965 and have made significant progress, are well understood in fuzzy theory. The fuzzy theory has the potential to be applied to various real-world problems. The possibility theory, for example, has been developed by several researchers, including Dubois and Prade [17] and Nahmias [18]. The contribution of probability theory, fuzzy set theory, and rough sets to the study of uncertainty is critical. Yet, these theories have some limitations as well as advantages. The theory of soft sets, developed by Molodtsov [19], was introduced as a new mathematical strategy for dealing with uncertainties to overcome these characteristics. Soft sets have been widely used in various disciplines and technologies. In particular, Maji et al. [20, 21] studied several operations on soft sets and applied their findings to decision-making problems in the literature. Several writers, including Chen [22], Pei and Miao [23], Zou and Xiao [24], and Kong et al. [25], have discovered significant characteristics of soft sets. Soft semirings, soft ideals, and idealistic soft semirings were all investigated by Feng et al. [26]. Das and Samanta developed the ideas of a soft real number and a soft real set in [27] and discussed the characteristics of each concept. These principles served as the foundation for their investigation into the concept of "soft metrics" in "[28]." (See [29, 30] for a more in-depth examination.) Based on the idea of soft elements of soft metric spaces, Abbas et al. [31] developed the concept of soft contraction mapping, which they named "soft contraction mapping." They focused on fixed points of soft contraction maps and obtained, among other things, a soft Banach contraction principle as a result of their efforts. In their paper, Abbas et al. [32] demonstrated that every complete soft metric induces an equivalent complete usual metric. They obtained in a direct way soft metric versions of various significant fixed point theorems for metric spaces, such as the Banach contraction principle, Kannan and Meir-Keeler fixed point theorems, and Caristi theorem, Kirk's, among other things. In [33], Chen and Lin presented an extension of the Meir and Keeler fixed point theorem to soft metric spaces, which was previously published. Many researchers working on sequence spaces and summability theory were involved in introducing fuzzy sequence spaces and studying their many characteristics. When it comes to fuzzy numbers, Nuray and Savas [34] defined and explored the Nakano sequences of fuzzy numbers, $\ell^F(\tau)$ equipped with a definite function. The following theories use operators' ideals: fixed point theory, Banach space geometry, normal series theory, approximation theory, and ideal transformations. For additional evidence, see [35–37]. According to Faried and Bakery [38], prequasi operator ideals are broader than quasioperator ideals. This study is aimed at introducing a certain space of soft real number sequences, abbreviated (csss), under a pre-quasi-quasi function (csss). The structure of the ideal operators has been described using this space and s-numbers. The conditions essential to generate prequasi Banach and closed (csss) $(\ell^S(\tau))_h$ supplied with the definite function h are investigated. This space's (R) and normal structure properties are illustrated. Fixed points have been introduced for Kannan contraction and nonexpansive mapping. Finally, we investigate whether the Kannan contraction mapping has a fixed point in the prequasi operator ideal with which it is linked. A few real-world examples and applications demonstrate the existence of solutions to nonlinear difference equations.

2. Definitions and Preliminaries

Assume that $\mathfrak{R}$ is the set of real numbers and $\mathcal{N}$ is the set of nonnegative integers. We denote the collection of all nonempty bounded subsets of $\mathfrak{R}$ by $\mathfrak{B}(\mathfrak{R})$ and E is the set of parameters.

Definition 1 (see [27]). A soft real set denoted by $(\tilde{f}, A)$, or simply by $\tilde{f}$, is a mapping $\tilde{f} : A \longrightarrow \mathfrak{B}(\mathfrak{R})$. If $\tilde{f}$ is a single-valued mapping on $A \subset E$ taking values in $\mathfrak{R}$, then $\tilde{f}$ is called a soft element of $\mathfrak{R}$ or a soft real number. If $\tilde{f}$ is a single-valued mapping on $A \subset E$ taking values in the set $\mathfrak{R}^+$ of nonnegative real numbers, then $\tilde{f}$ is called a nonnegative soft real number. We shall denote the set of nonnegative soft real numbers (corresponding to A) by $\mathfrak{R}(A)^*$. A constant soft real number $\tilde{c}$ is a soft real number such that for each $a \in A$, we have $\tilde{c}(a) = c$, where c is some real number.

Definition 2 (see [39]). For two soft real numbers $\tilde{f}$, $\tilde{g}$, we say that

(a) $\tilde{f} \tilde{\leq} \tilde{g}$ if $\tilde{f}(a) \tilde{\leq} \tilde{g}(a)$, for all $a \in A$

(b) $\tilde{f} \tilde{\geq} \tilde{g}$ if $\tilde{f}(a) \tilde{\geq} \tilde{g}(a)$, for all $a \in A$

(c) $\tilde{f} \tilde{<} \tilde{g}$ if $\tilde{f}(a) \tilde{<} \tilde{g}(a)$, for all $a \in A$

(d) $\tilde{f} \tilde{>} \tilde{g}$ if $\tilde{f}(a) \tilde{>} \tilde{g}(a)$, for all $a \in A$

Note that the relation $\tilde{\leq}$ is a partial order on $\mathfrak{R}(A)$. The additive identity and multiplicative identity in $\mathfrak{R}(A)$ are denoted by $\tilde{0}$ and $\tilde{1}$, respectively.

The arithmetic operations on $\mathfrak{R}(A)$ are defined as follows:

$$\begin{aligned} \left(\tilde{f} \oplus \tilde{g}\right)(\lambda) &= \left\{\tilde{f}(\lambda) + \tilde{g}(\lambda) : \lambda \in A\right\}, \\ \left(\tilde{f} ! \tilde{g}\right)(\lambda) &= \left\{\tilde{f}(\lambda) - \tilde{g}(\lambda) : \lambda \in A\right\}, \\ \left(\tilde{f} \otimes \tilde{g}\right)(\lambda) &= \left\{\tilde{f}(\lambda)\tilde{g}(\lambda) : \lambda \in A\right\}, \\ \left(\frac{\tilde{f}}{\tilde{g}}\right)(\lambda) &= \left\{\frac{\tilde{f}(\lambda)}{\tilde{g}(\lambda)} : \lambda \in A \text{ and } 0 \notin \tilde{g}(\lambda)\right\}. \end{aligned} \tag{1}$$

The absolute value $|\tilde{f}|$ of $\tilde{f} \in \mathfrak{R}(A)$ is defined by

$$\left|\tilde{f}\right|(\lambda) = \left\{\left|\tilde{f}(\lambda)\right| : \lambda \in A\right\}. \tag{2}$$

Let $d : \mathfrak{R}(A) \times \mathfrak{R}(A) \longrightarrow \mathfrak{R}(A)^*$, where $d(\tilde{f}, \tilde{g}) = |\tilde{f} - \tilde{g}|$ for all $\tilde{f}, \tilde{g} \in \mathfrak{R}(A)$. Assume $m_d : \mathfrak{R}(A) \times \mathfrak{R}(A) \longrightarrow \mathfrak{R}^+$ is defined by $m_d(\tilde{f}, \tilde{g}) = \max_{\lambda \in A} d(\tilde{f}, \tilde{g})(\lambda)$.

Note that

(1) $(\mathfrak{R}(A), m_d)$ is a complete metric space

(2) $m_d(\tilde{f}+\tilde{k}, \tilde{g}+\tilde{k}) = m_d(\tilde{f}, \tilde{g})$ for all $\tilde{f}, \tilde{g}, \tilde{k} \in \mathfrak{R}(A)$

$$m_d\left(\tilde{f}+\tilde{k}, \tilde{g}+\tilde{l}\right) \le m_d\left(\tilde{f}, \tilde{g}\right) + m_d\left(\tilde{k}, \tilde{l}\right). \quad (3)$$

(3) $m_d(\xi\tilde{f}, \xi\tilde{g}) = |\xi| m_d(f, g)$, for all $\xi \in \mathfrak{R}$

Definition 3. A sequence $\tilde{f} = (\widetilde{f_j})$ of soft real numbers is said to be

(a) bounded if the set $\{\widetilde{f_j} : j \in \mathcal{N}\}$ of soft real numbers is bounded; i.e., if a sequence $(\widetilde{f_j})$ is bounded, then there are two soft real numbers $\tilde{g}, \tilde{l}$ such that $\tilde{g} \tilde{\le} \widetilde{f_j} \tilde{\le} \tilde{l}$

(b) convergent to a soft real number $\widetilde{f_0}$ if, for every $\varepsilon > 0$, there exists $n_0 \in \mathcal{N}$ such that $m_d(\widetilde{f_j}, \widetilde{f_0}) < \varepsilon$, for all $j \ge j_0$

By ℓ_∞ and ℓ_r, we indicate the spaces of bounded and r-absolutely summable sequences of reals. Assume $\omega(S)$ is the classes of all sequence spaces of soft reals. If $\tau = (\tau_a) \in \mathfrak{R}^{+\mathcal{N}}$, where $\mathfrak{R}^{+\mathcal{N}}$ is the space of positive real sequences, we introduce Nakano sequences of soft reals such as [34] and marked it by $\ell^S(\tau) = \{\tilde{v} = (\widetilde{v_a}) \in \omega(S): h(\mu\tilde{v}) < \infty, \text{for some } \mu > 0\}$, where $h(\tilde{v}) = \sum_{a=0}^{\infty} [m_d(\widetilde{v_a}, \tilde{0})]^{\tau_a}$. The space $(\ell^S(\tau), \|.\|)$, where $\|\tilde{v}\| = \inf\{\kappa > 0 : h(\tilde{v}/\kappa) \le 1\}$ and $\tau_a \ge 1$, for all $a \in \mathcal{N}$, is a Banach space. Suppose $(\tau_a) \in \ell_\infty$, one has

$$\begin{aligned} \ell^S(\tau) &= \{\tilde{v} = (\widetilde{v_a}) \in \omega(S): h(\mu\tilde{v}) < \infty, \text{for some } \mu > 0\} \\ &= \left\{ \tilde{v} = (\widetilde{v_a}) \in \omega(S): \inf_a |\mu|^{\tau_a} \sum_{a=0}^{\infty} \left[m_d\left(\widetilde{v_a}, \tilde{0}\right)\right]^{\tau_a} \right. \\ &\le \left. \sum_{a=0}^{\infty} \left[m_d\left(\mu\widetilde{v_a}, \tilde{0}\right)\right]^{\tau_a} < \infty, \text{for some } \mu > 0 \right\} \\ &= \left\{ \tilde{v} = (\widetilde{v_a}) \in \omega(S): \sum_{a=0}^{\infty} \left[m_d\left(\widetilde{v_a}, \tilde{0}\right)\right]^{\tau_a} < \infty \right\} \\ &= \{\tilde{v} = (\widetilde{v_a}) \in \omega(S): h(\mu\tilde{v}) < \infty, \text{for any } \mu > 0\}. \end{aligned} \quad (4)$$

Lemma 4 (see [40]). *If $\tau_a > 0$ and $v_a, t_a \in \mathfrak{R}$, for all $a \in \mathcal{N}$, one gets $|v_a + t_a|^{\tau_a} \le 2^{K-1}(|v_a|^{\tau_a} + |t_a|^{\tau_a})$, where $K = \max\{1, \sup_a \tau_a\}$.*

3. Some Properties of $\ell^S(\tau)$

We have investigated in this section the certain space of sequences of soft real numbers under definite function to form prequasi (csss). We present sufficient conditions of $\ell^S(\tau)$ under definite function h to construct prequasi Banach and closed (csss). The Fatou property of different prequasi norms h on $\ell^S(\tau)$ has been explained. We have explored the uniform convexity (UUC2), the property (R), and this space's h-normal structure property.

Definition 5. The linear space U is called a certain space of sequences of soft reals (csss), when

(1) $\{\tilde{b}_q\}_{q\in\mathcal{N}} \subseteq \mathbf{U}$, where $\tilde{b}_q = \{\tilde{0}, \tilde{0}, \cdots, \tilde{1}, \tilde{0}, \tilde{0}, \cdots\}$, for $\tilde{1}$ marks at the q^{th} place

(2) $\mathbf{U}$ is solid, i.e., if $\tilde{Y} = (\widetilde{Y_q}) \in \omega(S)$, $\tilde{Z} = (\widetilde{Z_q}) \in \mathbf{U}$, and $|\widetilde{Y_q}| \tilde{\le} |\widetilde{Z_q}|$, for all $q \in \mathcal{N}$, one has $\tilde{Y} \in \mathbf{U}$

(3) $(\widetilde{Y_{[q/2]}})_{q=0}^{\infty} \in \mathbf{U}$, where $[q/2]$ indicates the integral part of $q/2$, assume $(\widetilde{Y_q})_{q=0}^{\infty} \in \mathbf{U}$

Definition 6. A subclass U_h of U is said to be a premodular (csss), if one has $h \in [0,\infty)^U$ holds the following conditions:

(i) Suppose $\tilde{Y} \in \mathbf{U}$, $\tilde{Y} = \tilde{\vartheta} \Leftrightarrow h(\tilde{Y}) = 0$ with $h(\tilde{Y}) \ge 0$, where $\tilde{\vartheta} = (\tilde{0}, \tilde{0}, \tilde{0},)$

(ii) We have $Q \ge 1$, the inequality $h(\alpha\tilde{Y}) \le Q|\alpha| h(\tilde{Y})$ holds, for all $\tilde{Y} \in \mathbf{U}$ and $\alpha \in \mathfrak{R}$

(iii) One has $P \ge 1$, the inequality $h(\tilde{Y} + \tilde{Z}) \le P(h(\tilde{Y}) + h(\tilde{Z}))$ satisfies, for all $\tilde{Y}, \tilde{Z} \in \mathbf{U}$

(iv) When $|\widetilde{Y_q}| \tilde{\le} |\widetilde{Z_q}|$, for all $q \in \mathcal{N}$, we have $h((\widetilde{Y_q})) \le h((\widetilde{Z_q}))$

(v) The inequality $h((\widetilde{Y_q})) \le h((\widetilde{Y_{[q/2]}})) \le P_0 h((\widetilde{Y_q}))$ verifies, for some $P_0 \ge 1$

(vi) Assume $\mathbb{E}$ is the space of finite sequences of soft real numbers, one has the closure of $\mathbb{E} = \mathbf{U}_h$

(vii) We have $\sigma > 0$ with $h(\tilde{\alpha}, \tilde{0}, \tilde{0}, \tilde{0}, \cdots) \ge \sigma|\alpha| h(\tilde{1}, \tilde{0}, \tilde{0}, \tilde{0}, \cdots)$, where $\tilde{\alpha}(a) = \alpha$, for every $a \in A$

Definition 7. If U is a (csss). The function $h \in [0,\infty)^U$ is said to be a prequasi norm on U, if it satisfies the following settings:

(i) Suppose $\tilde{Y} \in \mathbf{U}$, $\tilde{Y} = \tilde{\vartheta} \Leftrightarrow h(\tilde{Y}) = 0$ with $h(\tilde{Y}) \ge 0$, where $\tilde{\vartheta} = (\tilde{0}, \tilde{0}, \tilde{0},)$

(ii) One has $Q \ge 1$, the inequality $h(\alpha\tilde{Y}) \le Q|\alpha| h(\tilde{Y})$ verifies, for all $\tilde{Y} \in \mathbf{U}$ and $\alpha \in \mathfrak{R}$

(iii) We have $P \ge 1$, the inequality $h(\tilde{Y} + \tilde{Z}) \le P(h(\tilde{Y}) + h(\tilde{Z}))$ satisfies, for all $\tilde{Y}, \tilde{Z} \in \mathbf{U}$

Evidently, by the last two definitions, one has the following two theorems.

Theorem 8. *Assume U is a premodular (csss), then it is prequasi normed (csss).*

Theorem 9. *U is a prequasi normed (csss), when it is quasi-normed (csss).*

Definition 10.

(a) The function h on $\ell^S(\tau)$ is called h-convex, when

$$h\left(\alpha\tilde{Y}+(1-\alpha)\tilde{Z}\right)\leq\alpha h\left(\tilde{Y}\right)+(1-\alpha)h\left(\tilde{Z}\right), \quad (5)$$

for all $\alpha\in[0,1]$ and $\tilde{Y},\tilde{Z}\in\ell^S(\tau)$

(b) $\{\widetilde{Y_q}\}_{q\in\mathcal{N}}\subseteq(\ell^S(\tau))_h$ is h-convergent to $\tilde{Y}\in(\ell^S(\tau))_h$, if and only if, $\lim_{q\longrightarrow\infty}h(\widetilde{Y_q}-\tilde{Y})=0$. If the h-limit exists, then it is unique

(c) $\{\widetilde{Y_q}\}_{q\in\mathcal{N}}\subseteq(\ell^S(\tau))_h$ is h-Cauchy, if $\lim_{q,r\longrightarrow\infty}h(\widetilde{Y_q}-\widetilde{Y_r})=0$

(d) $\Gamma\subset(\ell^S(\tau))_h$ is h-closed, if for every h-converges $\{\widetilde{Y_q}\}_{a\in\mathcal{N}}\subset\Gamma$ to $\tilde{Y}$, one has $\tilde{Y}\in\Gamma$

(e) $\Gamma\subset(\ell^S(\tau))_h$ is h-bounded, assume $\delta_h(\Gamma)=\sup\{h(\tilde{Y}-\tilde{Z}):\tilde{Y},\tilde{Z}\in\Gamma\}<\infty$

(f) The h-ball of radius $\varepsilon\geq0$ and center $\tilde{Y}$, for all $\tilde{Y}\in(\ell^S(\tau))_h$, is denoted by

$$\mathbf{B}_h\left(\tilde{Y},\varepsilon\right)=\left\{\tilde{Z}\in\left(\ell^S(\tau)\right)_h:h\left(\tilde{Y}-\tilde{Z}\right)\leq\varepsilon\right\}. \quad (6)$$

(g) A prequasi norm h on $\ell^S(\tau)$ verifies the Fatou property, if for all sequence $\{\widetilde{Z^{(q)}}\}\subseteq(\ell^S(\tau))_h$ with $\lim_{q\longrightarrow\infty}h(\widetilde{Z^{(q)}}-\tilde{Z})=0$ and every $\tilde{Y}\in(\ell^S(\tau))_h$, we have $h(\tilde{Y}-\tilde{Z})\leq\sup_r\inf_{q\geq r}h(\tilde{Y}-\widetilde{Z^{(q)}})$

Recall that the Fatou property gives the h-closedness of the h-balls. We will indicate the space of all increasing sequences of reals by **I**.

Theorem 11. $(\ell^S(\tau))_h$, *where* $h(Y)=[\sum_{q=0}^{\infty}[m_d(\widetilde{Y_q},\tilde{0})]^{\tau_q}]^{1/K}$, *for every* $\tilde{Y}\in\ell^S(\tau)$, *is a premodular (csss), if* $(\tau_q)_{q\in\mathcal{N}}\in\ell_\infty\cap I$ *with* $\tau_0>0$.

Proof. (i) Clearly, $h(\tilde{Y})\geq0$ and $h(\tilde{Y})=0\Leftrightarrow\tilde{Y}=\tilde{\vartheta}$.

(1-i) Assume $\tilde{Y},\tilde{Z}\in\ell^S(\tau)$. Then,

$$\begin{aligned}h\left(\tilde{Y}+\tilde{Z}\right)&=\left[\sum_{q=0}^{\infty}\left[m_d\left(\widetilde{Y_q}+\widetilde{Z_q},\tilde{0}\right)\right]^{\tau_q}\right]^{1/K}\\&\leq\left[\sum_{q=0}^{\infty}\left[m_d\left(\widetilde{Y_q},\tilde{0}\right)\right]^{\tau_q}\right]^{1/K}+\left[\sum_{q=0}^{\infty}\left[m_d\left(\widetilde{Z_q},\tilde{0}\right)\right]^{\tau_q}\right]^{1/K}\\&=h\left(\tilde{Y}\right)+h\left(\tilde{Z}\right)<\infty.\end{aligned} \quad (7)$$

Hence, $\tilde{Y}+\tilde{Z}\in\ell^S(\tau)$.

(ii) We have $P\geq1$ with $h(\tilde{Y}+\tilde{Z})\leq P(h(\tilde{Y})+h(\tilde{Z}))$, for every $\tilde{Y},\tilde{Z}\in\ell^S(\tau)$.

(1-ii) Suppose $\alpha\in\mathfrak{R}$ and $\tilde{Y}\in\ell^S(\tau)$, one has

$$\begin{aligned}h\left(\alpha\tilde{Y}\right)&=\left[\sum_{q=0}^{\infty}\left[m_d\left(\widetilde{\alpha Y_q},\tilde{0}\right)\right]^{\tau_q}\right]^{1/K}\\&\leq\sup_q|\alpha|^{\tau_q/K}\left[\sum_{q=0}^{\infty}\left[m_d\left(\widetilde{Y_q},\tilde{0}\right)\right]^{\tau_q}\right]^{1/K}\\&\leq Q|\alpha|h\left(\tilde{Y}\right)<\infty.\end{aligned} \quad (8)$$

Since $\alpha\tilde{Y}\in\ell^S(\tau)$. By parts (1-i) and (1-ii), we have $\ell^S(\tau)$ is linear. And $\tilde{b}_p\in\ell^S(\tau)$, for every $p\in\mathcal{N}$, as $h(\tilde{b}_p)=[\sum_{q=0}^{\infty}[m_d(\tilde{b}_p,\tilde{0})]^{\tau_q}]^{1/K}=1$.

(iii) One has $Q=\max\{1,\sup_q|\alpha|^{(\tau_q/K)-1}\}\geq1$ with $h(\alpha\tilde{Y})\leq Q|\alpha|h(\tilde{Y})$, for every $\tilde{Y}\in\ell^S(\tau)$ and $\alpha\in\mathfrak{R}$.

(2) If $|\widetilde{Y_q}|\leq|\widetilde{Z_q}|$, for every $q\in\mathcal{N}$ and $\tilde{Z}\in\ell^S(\tau)$. Then

$$\begin{aligned}h\left(\tilde{Y}\right)&=\left[\sum_{q=0}^{\infty}\left[m_d\left(\widetilde{Y_q},\tilde{0}\right)\right]^{\tau_q}\right]^{1/K}\\&\leq\left[\sum_{q=0}^{\infty}\left[m_d\left(\widetilde{Z_q},\tilde{0}\right)\right]^{\tau_q}\right]^{1/K}=h\left(\tilde{Z}\right)<\infty,\end{aligned} \quad (9)$$

then $\tilde{Y}\in\ell^S(\tau)$.

(iv) Evidently, from (24).

(3) Assume $(Y_q)\in\ell^S(\tau)$, one has

$$\begin{aligned}h\left(\left(\widetilde{Y_{[q/2]}}\right)\right)&=\left[\sum_{q=0}^{\infty}\left[m_d\left(\widetilde{Y_{[q/2]}},\tilde{0}\right)\right]^{\tau_q}\right]^{1/K}\\&=\left[\sum_{q=0}^{\infty}\left[m_d\left(\widetilde{Y_q},\tilde{0}\right)\right]^{\tau_{2q}}+\sum_{q=0}^{\infty}\left[m_d\left(\widetilde{Y_q},\tilde{0}\right)\right]^{\tau_{2q+1}}\right]^{1/K}\\&\leq2^{1/K}\left[\sum_{q=0}^{\infty}\left[m_d\left(\widetilde{Y_q},\tilde{0}\right)\right]^{\tau_q}\right]^{1/K}=2^{1/K}h\left(\left(\widetilde{Y_q}\right)\right),\end{aligned} \quad (10)$$

so $(\widetilde{Y_{[q/2]}})\in\ell^S(\tau)$. (v) From (25), there are $P_0=2^{1/K}\geq1$.

(vi) Clearly the closure of $\mathbb{E} = \ell^S(\tau)$.

(vii) One gets $0 < \sigma \le |\alpha|^{(\tau_0/K)-1}$, for $\alpha \ne 0$ or $\sigma > 0$, for $\alpha = 0$ with

$$\left(\tilde{\alpha}, \tilde{0}, \tilde{0}, \tilde{0}, \cdots\right) \ge \sigma|\alpha| h\left(\tilde{1}, \tilde{0}, \tilde{0}, \tilde{0}, \cdots\right). \tag{11}$$

□

Theorem 12. *Assume $(\tau_q)_{q\in\mathcal{N}} \in \ell_\infty \cap I$ with $\tau_0 > 0$, one has $(\ell^S(\tau))_h$ which is a prequasi Banach (csss), where $h(\tilde{Y}) = [\sum_{q=0}^{\infty}[m_d(\widetilde{Y_q}, \tilde{0})]^{\tau_q}]^{1/K}$, for all $\tilde{Y} \in \ell^S(\tau)$.*

Proof. From Theorems 11 and 8, the space $(\ell^S(\tau))_h$ is a prequasi normed (csss). If $\widetilde{Y^l} = (\widetilde{Y_q^l})_{q=0}^{\infty}$ is a Cauchy sequence in $(\ell^S(\tau))_h$, then for all $\varepsilon \in (0,1)$, we have $l_0 \in \mathcal{N}$ such that for every $l, m \ge l_0$, we obtain

$$h\left(\widetilde{Y^l} - \widetilde{Y^m}\right) = \left[\sum_{q=0}^{\infty}\left[m_d\left(\widetilde{Y_q^l} - \widetilde{Y_q^m}, \tilde{0}\right)\right]^{\tau_q}\right]^{1/K} < \varepsilon. \tag{12}$$

Therefore, $m_d(\widetilde{Y_q^l} - \widetilde{Y_q^m}, \tilde{0}) < \varepsilon$. Since $(\mathfrak{R}(A), m_d)$ is a complete metric space, so $(\widetilde{Y_q^m})$ is a Cauchy sequence in $\mathfrak{R}(A)$, for constant $q \in \mathcal{N}$. Then, $\lim_{m\longrightarrow\infty}\widetilde{Y_q^m} = \widetilde{Y_q^0}$, for fixed $q \in \mathcal{N}$. So $h(\widetilde{Y^l} - \widetilde{Y^0}) < \varepsilon$, for all $l \ge l_0$. As $h(\widetilde{Y^0}) = h(\widetilde{Y^0} - \widetilde{Y^l} + \widetilde{Y^l}) \le h(\widetilde{Y^l} - \widetilde{Y^0}) + h(\widetilde{Y^l}) < \infty$. Then, $\widetilde{Y^0} \in \ell^S(\tau)$. □

Theorem 13. *If $(\tau_q)_{q\in\mathcal{N}} \in \ell_\infty \cap I$ with $\tau_0 > 0$, we have $(\ell^S(\tau))_h$ a prequasi closed (csss), where $h(\tilde{Y}) = [\sum_{q=0}^{\infty}[m_d(\widetilde{Y_q}, \tilde{0})]^{\tau_q}]^{1/K}$, for all $\tilde{Y} \in \ell^S(\tau)$.*

Proof. By Theorems 11 and 8, the space $(\ell^S(\tau))_h$ is a prequasi normed (csss). When $\widetilde{Y^l} = (\widetilde{Y_q^l})_{q=0}^{\infty} \in (\ell^S(\tau))_h$ and $\lim_{l\longrightarrow\infty} h(\widetilde{Y^l} - \widetilde{Y^0}) = 0$, one has for every $\varepsilon \in (0,1)$, there is $l_0 \in \mathcal{N}$ such that for every $l \ge l_0$, one gets

$$\varepsilon > h\left(\widetilde{Y^l} - \widetilde{Y^0}\right) = \left[\sum_{q=0}^{\infty}\left[m_d\left(\widetilde{Y_q^l} - \widetilde{Y_q^0}, \tilde{0}\right)\right]^{\tau_q}\right]^{1/K}. \tag{13}$$

Therefore, $m_d(\widetilde{Y_q^l} - \widetilde{Y_q^0}, \tilde{0}) < \varepsilon$. Since $(\mathfrak{R}(A), m_d)$ is a complete metric space, so $(\widetilde{Y_q^l})$ is a convergent sequence in $\mathfrak{R}(A)$, for constant $q \in \mathcal{N}$. Then, $\lim_{l\longrightarrow\infty}\widetilde{Y_q^l} = \widetilde{Y_q^0}$, for fixed $q \in \mathcal{N}$. As $h(\widetilde{Y^0}) = h(\widetilde{Y^0} - \widetilde{Y^l} + \widetilde{Y^l}) \le h(\widetilde{Y^l} - \widetilde{Y^0}) + h(\widetilde{Y^l}) < \infty$. We have $\widetilde{Y^0} \in \ell^S(\tau)$. □

Theorem 14. *The function $h(\tilde{Y}) = [\sum_{q=0}^{\infty}[m_d(\widetilde{Y_q}, \tilde{0})]^{\tau_q}]^{1/K}$ verifies the Fatou property, when $(\tau_q)_{q\in\mathcal{N}} \in \ell_\infty \cap I$ so that $\tau_0 > 0$, for every $\tilde{Y} \in \ell^S(\tau)$.*

Proof. Assume $\{\widetilde{Z^r}\} \subseteq (\ell^S(\tau))_h$ with $\lim_{r\longrightarrow\infty} h(\widetilde{Z^r} - \tilde{Z}) = 0$. As $(\ell^S(\tau))_h$ is a prequasi closed space, we have $\tilde{Z} \in (\ell^S(\tau))_h$. For every $\tilde{Y} \in (\ell^S(\tau))_h$, then

$$\begin{aligned} h\left(\tilde{Y} - \tilde{Z}\right) &= \left[\sum_{q=0}^{\infty}\left[m_d\left(\widetilde{Y_q} - \widetilde{Z_q}, \tilde{0}\right)\right]^{\tau_q}\right]^{1/K} \\ &\le \left[\sum_{q=0}^{\infty}\left[m_d\left(\widetilde{Y_q} - \widetilde{Z_q^r}, \tilde{0}\right)\right]^{\tau_q}\right]^{1/K} \\ &\quad + \left[\sum_{q=0}^{\infty}\left[m_d\left(\widetilde{Z_q^r} - \widetilde{Z_q}, \tilde{0}\right)\right]^{\tau_q}\right]^{1/K} \\ &\le \sup_m \inf_{r\ge m} h\left(\tilde{Y} - \widetilde{Z^r}\right). \end{aligned} \tag{14}$$

□

Theorem 15. *The function $h(\tilde{Y}) = \sum_{q=0}^{\infty}[m_d(\widetilde{Y_q}, \tilde{0})]^{\tau_q}$ does not satisfy the Fatou property, for every $\tilde{Y} \in \ell^S(\tau)$, if $(\tau_q) \in \ell_\infty$ and $\tau_q > 1$, for every $q \in \mathcal{N}$.*

Proof. Assume $\{\widetilde{Z^r}\} \subseteq (\ell^S(\tau))_h$ with $\lim_{r\longrightarrow\infty} h(\widetilde{Z^r} - \tilde{Z}) = 0$. As $(\ell^S(\tau))_h$ is a prequasi closed space, we have $\tilde{Z} \in (\ell^S(\tau))_h$. For all $\tilde{Z} \in (\ell^S(\tau))_h$, one can see

$$\begin{aligned} h\left(\tilde{Y} - \tilde{Z}\right) &= \sum_{q=0}^{\infty}\left[m_d\left(\widetilde{Y_q} - \widetilde{Z_q}, \tilde{0}\right)\right]^{\tau_q} \\ &\le 2^{\sup_q \tau_q - 1}\left(\sum_{q=0}^{\infty}\left[m_d\left(\widetilde{Y_q} - \widetilde{Z_q^r}, \tilde{0}\right)\right]^{\tau_q} \right. \\ &\quad \left. + \sum_{q=0}^{\infty}\left[m_d\left(\widetilde{Z_q^r} - \widetilde{Z_q}, \tilde{0}\right)\right]^{\tau_q}\right) \\ &\le 2^{\sup_q \tau_q - 1} \sup_m \inf_{r\ge m} h\left(\tilde{Y} - \widetilde{Z^r}\right). \end{aligned} \tag{15}$$

□

Example 16. For $(\tau_q) \in [1,\infty)^{\mathcal{N}}$, the function $h(\tilde{Y}) = \inf\{\alpha > 0 : \sum_{q\in\mathcal{N}}[m_d(\widetilde{Y_q}/\alpha, \tilde{0})]^{\tau_q} \le 1\}$ is a norm on $\ell^S(\tau)$.

Example 17. The function $h(\tilde{Y}) = \sqrt[3]{\sum_{q\in\mathcal{N}}[m_d(\widetilde{Y_q}, \tilde{0})]^{(3q+2)/(q+1)}}$ is a prequasi norm (not a norm) on $\ell^S(((3q+2)/(q+1))_{q=0}^{\infty})$.

Example 18. The function $h(\tilde{Y}) = \sum_{q\in\mathcal{N}}[m_d(\widetilde{Y_q}, \tilde{0})]^{(3q+2)/(q+1)}$ is a prequasi norm (not a quasinorm) on $\ell^S(((3q+2)/(q+1))_{q=0}^{\infty})$.

Example 19. The function $h(\tilde{Y}) = \sqrt[d]{\sum_{q\in\mathcal{N}}[m_d(\widetilde{Y_q}, \tilde{0})]^d}$ is a prequasi norm, quasi norm, and not a norm on ℓ_d^S, for $0 < d < 1$.

Definition 20.

(1) [41] If $p > 0$ and $q > 0$. Mark

$$\mathbb{K}_2(p,q) = \left\{ \left(\tilde{Y}, \tilde{Z}\right) : \tilde{Y}, \tilde{Z} \in \mathbf{U}_h, h(\tilde{Y}) \le p, h\left(\tilde{Z}\right) \le p, h\left(\frac{\tilde{Y} - \tilde{Z}}{2}\right) \ge pq \right\}. \tag{16}$$

For $\mathbb{K}_2(p,q) \neq \varnothing$, let

$$\mathrm{K}_2(p,q) = \inf \left\{ 1 - \frac{1}{p} h\left(\frac{\tilde{Y} + \tilde{Z}}{2}\right) : \left(\tilde{Y}, \tilde{Z}\right) \in \mathbb{K}_2(p,q) \right\}. \tag{17}$$

Suppose $\mathbb{K}_2(p,q) = \varnothing$, we take $\mathrm{K}_2(p,q) = 1$.

(2) [41] The function h holds (UUC2) when for all $r \ge 0$ and $q > 0$, one has $\beta_2(r,q)$ such that

$$\mathrm{K}_2(p,q) > \beta_2(r,q) > 0, \text{for } p > r. \tag{18}$$

(3) [42] The function h is strictly convex, (SC), when for every $\tilde{Y}, \tilde{Z} \in \mathbf{U}_h$ with $h(\tilde{Y}) = h(\tilde{Z})$ and $h((\tilde{Y} + \tilde{Z})/2) = (h(\tilde{Y}) + h(\tilde{Z}))/2$, one gets $\tilde{Y} = \tilde{Z}$

Lemma 21.

(i) [43] If $t \ge 2$ and for every $f, g \in \mathfrak{R}$, one has

$$\left|\frac{f+g}{2}\right|^t + \left|\frac{f-g}{2}\right|^t \le \frac{1}{2}\left(|f|^t + |g|^t\right). \tag{19}$$

(ii) [44] Assume $1 < t \le 2$ and for all $f, g \in \mathfrak{R}$ with $|f| + |g| \neq 0$, one obtains

$$\left|\frac{f+g}{2}\right|^t + \frac{t(t-1)}{2}\left|\frac{f-g}{|f|+|g|}\right|^{2-t}\left|\frac{f-g}{2}\right|^t \le \frac{1}{2}\left(|f|^t + |g|^t\right). \tag{20}$$

In the next part of this section, we will use the function h as $h(\tilde{g}) = [\sum_{p=0}^{\infty}(m_d(\widetilde{g_p}, \tilde{0}))^{\tau_p}]^{1/K}$, for all $\tilde{g} \in \ell^S(\tau)$.

Theorem 22. *If $(\tau_q)_{q\in\mathcal{N}} \in \ell_\infty \cap I$ so that $\tau_0 > 1$, one has h is (UUC2).*

Proof. Suppose $b > 0$ and $a > r \ge 0$. If $\tilde{f}, \tilde{g} \in \ell^S(\tau)_h$ with

$$h\left(\tilde{f}\right) \le a, h(\tilde{g}) \le a \text{ and } h\left(\frac{\tilde{f} - \tilde{g}}{2}\right) \ge ab. \tag{21}$$

By using the definition of h, one can see

$$\begin{aligned} ab \le h\left(\frac{\tilde{f} - \tilde{g}}{2}\right) &= \left[\sum_{m=0}^{\infty}\left(m_d\left(\frac{\widetilde{f_m} - \widetilde{g_m}}{2}, \tilde{0}\right)\right)^{\tau_m}\right]^{1/K} \\ &\le 2^{-\tau_0/K}\left(\left[\sum_{m=0}^{\infty}\left(m_d\left(\widetilde{f_m}, \tilde{0}\right)\right)^{\tau_m}\right]^{1/K} + \left[\sum_{m=0}^{\infty}\left(m_d(\widetilde{g_m}, \tilde{0})\right)^{\tau_m}\right]^{1/K}\right) \\ &= 2^{-\tau_0/K}\left(h\left(\tilde{f}\right) + h(\tilde{g})\right) \le 2a, \end{aligned} \tag{22}$$

then $b \le 2$. Assume $Q = \{x \in \mathcal{N} : 1 < \tau_x < 2\}$ and $P = \{x \in \mathcal{N} : \tau_x \ge 2\} = \mathcal{N} \setminus Q$. For all $\tilde{w} \in \ell^S(\tau)_h$, one has $h^K(\tilde{w}) = h_P^K(\tilde{w}) + h_Q^K(\tilde{w})$. Therefore, $h_P((\tilde{f} - \tilde{g})/2) \ge ab/2$ or $h_Q((\tilde{f} - \tilde{g})/2) \ge ab/2$. Let first $h_P((\tilde{f} - \tilde{g})/2) \ge ab/2$. In view of Lemma 21, part (i), one gets

$$h_P^K\left(\frac{\tilde{f} + \tilde{g}}{2}\right) + h_P^K\left(\frac{\tilde{f} - \tilde{g}}{2}\right) \le \frac{h_P^K\left(\tilde{f}\right) + h_P^K(\tilde{g})}{2}, \tag{23}$$

then

$$h_P^K\left(\frac{\tilde{f} + \tilde{g}}{2}\right) \le \frac{h_P^K\left(\tilde{f}\right) + h_P^K(\tilde{g})}{2} - \left(\frac{ab}{2}\right)^K. \tag{24}$$

Since

$$h_Q^K\left(\frac{\tilde{f} + \tilde{g}}{2}\right) \le \frac{h_Q^K\left(\tilde{f}\right) + h_Q^K(\tilde{g})}{2}, \tag{25}$$

by summing inequalities 2 and 3, and from inequality 1, one can see

$$h^K\left(\frac{\tilde{f} + \tilde{g}}{2}\right) \le \frac{h^K\left(\tilde{f}\right) + h^K(\tilde{g})}{2} - \left(\frac{ab}{2}\right)^K \le a^K\left(1 - \left(\frac{b}{2}\right)^K\right). \tag{26}$$

This implies

$$h\left(\frac{\tilde{f}+\tilde{g}}{2}\right) \leq a\left(1-\left(\frac{b}{2}\right)^{K}\right)^{1/K}. \tag{27}$$

After, assume $h_Q((\tilde{f}-\tilde{g})/2) \geq ab/2$. Put $B=(b/4)^K$,

$$\begin{aligned} Q_1 = \Big\{ m \in Q : m_d\left(\widetilde{f_m}-\widetilde{g_m}, \tilde{0}\right) \\ \leq B\left(m_d\left(\widetilde{f_m}, \tilde{0}\right)+m_d\left(\widetilde{g_m}, \tilde{0}\right)\right)\Big\} \text{ and } Q_2 = Q \setminus Q_1. \end{aligned} \tag{28}$$

Since $B \leq 1$ and the power function is convex. Hence,

$$\begin{aligned} h_{Q_1}^K\left(\frac{\tilde{f}-\tilde{g}}{2}\right) &\leq \sum_{m\in Q_1} B^{\tau_m}\left(m_d\left(\frac{\widetilde{f_m}+\widetilde{g_m}}{2}, \tilde{0}\right)\right)^{\tau_m} \\ &\leq \left(\frac{B}{2}\right)^{\tau_0}\left(h_{Q_1}^K\left(\tilde{f}\right)+h_{Q_1}^K(\tilde{g})\right) \\ &\leq \frac{B}{2}\left(h_Q^K\left(\tilde{f}\right)+h_Q^K(\tilde{g})\right) \\ &\leq \frac{B}{2}\left(h^K\left(\tilde{f}\right)+h^K(\tilde{g})\right) \leq Ba^K. \end{aligned} \tag{29}$$

As $h_Q((\tilde{f}-\tilde{g})/2) \geq ab/2$, one has

$$\begin{aligned} h_{Q_2}^K\left(\frac{\tilde{f}-\tilde{g}}{2}\right) &= h_Q^K\left(\frac{\tilde{f}-\tilde{g}}{2}\right) - h_{Q_1}^K\left(\frac{\tilde{f}-\tilde{g}}{2}\right) \\ &\geq a^K\left(\left(\frac{b}{2}\right)^K-\left(\frac{b}{4}\right)^K\right). \end{aligned} \tag{30}$$

For all $m \in Q_2$, one obtains

$$\begin{aligned} &\tau_0 - 1 < \tau_0(\tau_0-1) \leq \leq \tau_{m-1}(\tau_{m-1}-1) \leq \tau_m(\tau_m - 1), \\ &B < B^{2-\tau_m} < \left|\frac{m_d\left(\widetilde{f_m}-\widetilde{g_m}, \tilde{0}\right)}{m_d\left(\widetilde{f_m}, \tilde{0}\right)+m_d\left(\widetilde{g_m}, \tilde{0}\right)}\right|^{2-\tau_m}. \end{aligned} \tag{31}$$

In view of Lemma 21, part (ii), one gets

$$\begin{aligned} &\left(m_d\left(\frac{\widetilde{f_m}+\widetilde{g_m}}{2}, \tilde{0}\right)\right)^{\tau_m} + \frac{(\tau_0-1)B}{2}\left(m_d\left(\frac{\widetilde{f_m}-\widetilde{g_m}}{2}, \tilde{0}\right)\right)^{\tau_m} \\ &\leq \frac{1}{2}\left(\left(m_d\left(\widetilde{f_m}, \tilde{0}\right)\right)^{\tau_m} + \left(m_d\left(\widetilde{g_m}, \tilde{0}\right)\right)^{\tau_m}\right). \end{aligned} \tag{32}$$

So

$$h_{Q_2}^K\left(\frac{\tilde{f}+\tilde{g}}{2}\right) + \frac{(\tau_0-1)B}{2}h_{Q_2}^K\left(\frac{\tilde{f}-\tilde{g}}{2}\right) \leq \frac{h_{Q_2}^K\left(\tilde{f}\right)+h_{Q_2}^K(\tilde{g})}{2}, \tag{33}$$

then

$$\begin{aligned} h_{Q_2}^K\left(\frac{\tilde{f}+\tilde{g}}{2}\right) \leq &\frac{h_{Q_2}^K\left(\tilde{f}\right)+h_{Q_2}^K(\tilde{g})}{2} \\ &- \frac{(\tau_0-1)B}{2}a^K\left(\left(\frac{b}{2}\right)^K-\left(\frac{b}{4}\right)^K\right). \end{aligned} \tag{34}$$

As

$$h_{Q_1}^K\left(\frac{\tilde{f}+\tilde{g}}{2}\right) \leq \frac{h_{Q_1}^K\left(\tilde{f}\right)+h_{Q_1}^K(\tilde{g})}{2}, \tag{35}$$

by summing inequalities 5 and 6, we have

$$\begin{aligned} h_Q^K\left(\frac{\tilde{f}+\tilde{g}}{2}\right) \leq &\frac{h_Q^K\left(\tilde{f}\right)+h_Q^K(\tilde{g})}{2} \\ &- \frac{(\tau_0-1)B}{2}a^K\left(\left(\frac{b}{2}\right)^K-\left(\frac{b}{4}\right)^K\right) \\ \leq &\frac{h_Q^K\left(\tilde{f}\right)+h_Q^K(\tilde{g})}{2} - \frac{(\tau_0-1)}{2}\left(\frac{b}{4}\right)^{2K}a^K\left(2^K-1\right) \\ \leq &\frac{h_Q^K\left(\tilde{f}\right)+h_Q^K(\tilde{g})}{2} - \frac{(\tau_0-1)}{2^K-1}\left(\frac{b}{4}\right)^{2K}a^K. \end{aligned} \tag{36}$$

As

$$h_P^K\left(\frac{\tilde{f}+\tilde{g}}{2}\right) \leq \frac{h_P^K\left(\tilde{f}\right)+h_P^K(\tilde{g})}{2}, \tag{37}$$

by summing inequalities 7 and 8, and from inequality 1, then

$$\begin{aligned} h^K\left(\frac{\tilde{f}+\tilde{g}}{2}\right) &\leq \frac{h^K\left(\tilde{f}\right)+h^K(\tilde{g})}{2} - \frac{(\tau_0-1)}{2^K-1}\left(\frac{b}{4}\right)^{2K}a^K \\ &\leq a^K\left[1-\frac{(\tau_0-1)}{2^K-1}\left(\frac{b}{4}\right)^{2K}\right]. \end{aligned} \tag{38}$$

So

$$h\left(\frac{\tilde{f}+\tilde{g}}{2}\right) \leq a\left[1-\frac{(\tau_0-1)}{2^K-1}\left(\frac{b}{4}\right)^{2K}\right]^{1/K}. \tag{39}$$

Evidently,

$$1 < \tau_0 \leq K < 2^K \Rightarrow 0 < \frac{\tau_0-1}{2^K-1} < 1. \tag{40}$$

From inequalities 4 and 9, and Definition 20, when we take

$$\beta_2(r,b)=\min\left(1-\left(1-\left(\frac{b}{2}\right)^K\right)^{1/K},1-\left[1-\frac{(\tau_0-1)}{2^K-1}\left(\frac{b}{4}\right)^{2K}\right]^{1/K}\right). \tag{41}$$

Therefore, we have $K_2(a,b)>\beta_2(r,b)>0$, so h is (UUC2). □

Definition 23. The space U_h verifies the property (R), if and only if, for every decreasing sequence $\{\Gamma_j\}_{j\in\mathcal{N}}$ of h-closed and h-convex nonempty subsets of U_h so that $\sup_{j\in\mathcal{N}}\mathfrak{K}_h(\tilde{Y},\Gamma_j)<\infty$, for some $\tilde{Y}\in U_h$, then $\bigcap_{j\in\mathcal{N}}\Gamma_j\neq\varnothing$.

By denoting Γ a nonempty h-closed and h-convex subset of $(\ell^S(\tau))_h$.

Theorem 24. *Suppose $(\tau_q)_{q\in\mathcal{N}}\in\ell_\infty\cap I$ so that $\tau_0>1$, we have*

(i) if $\tilde{Y}\in(\ell^S(\tau))_h$ such that

$$\mathfrak{K}_h(\tilde{Y},\Gamma)=\inf\left\{h(\tilde{Y}-\tilde{Z}):\tilde{Z}\in\Gamma\right\}<\infty. \tag{42}$$

One has a unique $\tilde{\alpha}\in\Gamma$ with $\mathfrak{K}_h(\tilde{Y},\Gamma)=h(\tilde{Y}-\tilde{\alpha})$.

(ii) $(\ell^S(\tau))_h$ satisfies the property (R).

Proof. To prove (i), if $\tilde{Y}\notin\Gamma$ as Γ is h-closed, we have $C:=\mathfrak{K}_h(\tilde{Y},\Gamma)>0$. Then, for every $r\in\mathcal{N}$, we have $\widetilde{Z_r}\in\Gamma$ so that $h(\tilde{Y}-\widetilde{Z_r})<C(1+(1/r))$. Assume $\{\widetilde{Z_r}/2\}$ is not h-Cauchy. There is a subsequence $\{\widetilde{Z_{g(r)}}/2\}$ and $l_0>0$ so that $h((\widetilde{Z_{g(r)}-Z_{g(j)}})/2)\geq l_0$, for all $r>j\geq 0$. Also, we obtain $K_2(C(1+(1/r)),l_0/2C)>\alpha:=\beta_2(C(1+(1/r)),l_0/2C)>0$, for every $r\in\mathcal{N}$. As

$$\begin{aligned}\max\left(h\left(\tilde{Y}-\widetilde{Z_{g(r)}}\right),h\left(\tilde{Y}-\widetilde{Z_{g(j)}}\right)\right)&\leq C\left(1+\frac{1}{g(j)}\right),\\ h\left(\frac{\widetilde{Z_{g(r)}-Z_{g(j)}}}{2}\right)&\geq l_0\geq C\left(1+\frac{1}{g(j)}\right)\frac{l_0}{2C},\end{aligned} \tag{43}$$

for all $r>j\geq 0$, one has

$$h\left(\tilde{Y}-\frac{\widetilde{Z_{g(r)}+Z_{g(j)}}}{2}\right)\leq C\left(1+\frac{1}{g(j)}\right)(1-\alpha). \tag{44}$$

So

$$C=\mathfrak{K}_h(\tilde{Y},\Gamma)\leq C\left(1+\frac{1}{g(j)}\right)(1-\alpha), \tag{45}$$

for every $j\in\mathcal{N}$. By choosing $j\longrightarrow\infty$, we have

$$0<C\leq C\left(1+\frac{1}{g(j)}\right)(1-\alpha)<C. \tag{46}$$

This is a contradiction. Hence, $\{\widetilde{Z_r}/2\}$ is h-Cauchy. Since $(\ell^S(\tau))_h$ is h-complete, one has $\{\widetilde{Z_r}/2\}h$-converges to some $\tilde{Z}$. For every $j\in\mathcal{N}$, we have $\{(\widetilde{Z_r+Z_j})/2\}h$-converges to $\tilde{Z}+(\widetilde{Z_j}/2)$. As Γ is h-closed and h-convex, we have $\tilde{Z}+(\widetilde{Z_j}/2)\in\Gamma$. As $\tilde{Z}+(\widetilde{Z_j}/2)h$-converges to $2\tilde{Z}$, one gets $2\tilde{Z}\in\Gamma$. Suppose $\tilde{\lambda}=2\tilde{z}$ and from Theorem 14, as h verifies the Fatou property, we get

$$\begin{aligned}\mathfrak{K}_h(\tilde{Y},\Gamma)&\leq h(\tilde{Y}-\tilde{\lambda})\leq\sup_i\inf_{j\geq i}h\left(\tilde{Y}-\left(\tilde{Z}+\frac{\widetilde{Z_j}}{2}\right)\right)\\&\leq\sup_i\inf_{j\geq i}\sup_i\inf_{r\geq i}h\left(\tilde{Y}-\frac{\widetilde{Z_r+Z_j}}{2}\right)\\&\leq\frac{1}{2}\sup_i\inf_{r\geq i}\sup_i\inf_{r\geq i}\left[h\left(\tilde{Y}-\widetilde{Z_r}\right)+h\left(\tilde{Y}-\widetilde{Z_j}\right)\right]\\&=\mathfrak{K}_h(\tilde{Y},\Gamma).\end{aligned} \tag{47}$$

So $h(\tilde{Y}-\tilde{\lambda})=\mathfrak{K}_h(\tilde{Y},\Gamma)$. As h is (UUC2), then it is (SC), which explains the uniqueness of $\tilde{\lambda}$. To prove (ii), if $\tilde{Y}\notin\Gamma_{r_0}$, for some $r_0\in\mathcal{N}$. As $(\mathfrak{K}_h(\tilde{Y},\Gamma_r))_{r\in\mathcal{N}}\in\ell_\infty$ is increasing, take $\lim_{r\longrightarrow\infty}\mathfrak{K}_h(\tilde{Y},\Gamma_r)=C$. If $C>0$, otherwise, $\tilde{Y}\in\Gamma_r$, for every $r\in\mathcal{N}$. From (i), one has one point $\widetilde{Z_r}\in\Gamma_r$ so that $\mathfrak{K}_h(\tilde{Y},\Gamma_r)=h(\tilde{Y}-\widetilde{Z_r})$, for all $r\in\mathcal{N}$. A similar proof will show that $\{\widetilde{Z_r}/2\}h$-converges to some $\tilde{Z}\in(\ell^S(\tau))_h$. Since $\{\Gamma_r\}$ are h-convex, decreasing, and h-closed, we have $2\tilde{Z}\in\cap_{r\in\mathcal{N}}\Gamma_r$. □

Definition 25. U_h verifies the h-normal structure property, if and only if, for every nonempty h-bounded, h-convex, and h-closed subset Γ of U_h not decreased to one point, then $\tilde{Y}\in\Gamma$ so that

$$\sup_{\tilde{Z}\in\Gamma}h(\tilde{Y}-\tilde{Z})<\delta_h(\Gamma):=\sup\left\{h(\tilde{Y}-\tilde{Z}):\tilde{Y},\tilde{Z}\in\Gamma\right\}<\infty. \tag{48}$$

Theorem 26. *Suppose $(\tau_q)_{q\in\mathcal{N}}\in\ell_\infty\cap I$ so that $\tau_0>1$, then $(\ell^S(\tau))_h$ satisfies the h-normal structure property.*

Proof. Theorem 22 implies that h is (UUC2). Suppose Γ is a h-bounded, h-convex, and h-closed subset of $(\ell^S(\tau))_h$ not decreased to one point. Then, $\delta_h(\Gamma)>0$. Put $C=\delta_h(\Gamma)$. If

$\tilde{Y}, \tilde{Z} \in \Gamma$ with $\tilde{Y} \neq \tilde{Z}$, then $h((\tilde{Y} - \tilde{Z})/2) = l > 0$. For all $\tilde{\alpha} \in \Gamma$, we have $h(\tilde{Y} - \tilde{\alpha}) \leq C$ and $h(\tilde{Z} - \tilde{\alpha}) \leq C$. Since Γ is h-convex, we have $(\tilde{Y} + \tilde{Z})/2 \in \Gamma$. Since

$$\begin{aligned} h\left(\frac{\tilde{Y} + \tilde{Z}}{2} - \tilde{\alpha}\right) &= h\left(\frac{(\tilde{Y} - \tilde{\alpha}) + (\tilde{Z} - \tilde{\alpha})}{2}\right) \\ &\leq C\left(1 - \mathrm{K}_2\left(C, \frac{l}{C}\right)\right), \end{aligned} \tag{49}$$

for every $\tilde{\alpha} \in \Gamma$. We get

$$\sup_{\tilde{\alpha} \in \Gamma} h\left(\frac{\tilde{Y} + \tilde{Z}}{2} - \tilde{\alpha}\right) \leq C\left(1 - \mathrm{K}_2\left(C, \frac{l}{C}\right)\right) < C = \delta_h(\Gamma). \tag{50}$$

□

4. Kannan Contraction Mapping on $\ell^S(\tau)$

In this section, we have constructed $(\ell^S(\tau))_h$ with distinct h so that one has a unique fixed point of Kannan contraction mapping.

Definition 27. A mapping $V : U_h \longrightarrow U_h$ is called a Kannan h-contraction, when we have $\alpha \in [0, 1/2)$ so that $h(V\tilde{Y} - V\tilde{Z}) \leq \alpha(h(V\tilde{Y} - \tilde{Y}) + h(V\tilde{Z} - \tilde{Z}))$, for every $\tilde{Y}, \tilde{Z} \in U_h$. The mapping V is said to be Kannan h-nonexpansive, if $\alpha = 1/2$.

A vector $\tilde{Y} \in \mathbf{U}_h$ is said to be a fixed point of V, if $V(\tilde{Y}) = \tilde{Y}$.

Theorem 28. *Suppose $(\tau_q)_{q \in \mathcal{N}} \in \ell_\infty \cap I$ so that $\tau_0 > 0$ and $V : (\ell^S(\tau))_h \longrightarrow (\ell^S(\tau))_h$ is Kannan h-contraction mapping, where $h(\tilde{Y}) = [\sum_{q=0}^{\infty} [m_d(\widetilde{Y_q}, \tilde{0})]^{\tau_q}]^{1/K}$, for every $\tilde{Y} \in \ell^S(\tau)$, then V has a unique fixed point.*

Proof. Let $\tilde{Y} \in \ell^S(\tau)$, we have $V^p\tilde{Y} \in \ell^S(\tau)$. Since V is a Kannan h-contraction mapping, then

$$\begin{aligned} &h\left(V^{l+1}\tilde{Y} - V^l\tilde{Y}\right) \leq \alpha\left(h\left(V^{l+1}\tilde{Y} - V^l\tilde{Y}\right) + h\left(V^l\tilde{Y} - V^{l-1}\tilde{Y}\right)\right) \Longrightarrow \\ &h\left(V^{l+1}\tilde{Y} - V^l\tilde{Y}\right) \leq \frac{\alpha}{1-\alpha} h\left(V^l\tilde{Y} - V^{l-1}\tilde{Y}\right) \\ &\leq \left(\frac{\alpha}{1-\alpha}\right)^2 h\left(V^{l-1}\tilde{Y} - V^{l-2}\tilde{Y}\right) \\ &\leq \cdots \leq \left(\frac{\alpha}{1-\alpha}\right)^l h\left(V\tilde{Y} - \tilde{Y}\right). \end{aligned} \tag{51}$$

Hence, for every $l, m \in \mathcal{N}$ so that $m > l$, then

$$\begin{aligned} h\left(V^l\tilde{Y} - V^m\tilde{Y}\right) &\leq \alpha\left(h\left(V^l\tilde{Y} - V^{l-1}\tilde{Y}\right) + h\left(V^m\tilde{Y} - V^{m-1}\tilde{Y}\right)\right) \\ &\leq \alpha\left(\left(\frac{\alpha}{1-\alpha}\right)^{l-1} + \left(\frac{\alpha}{1-\alpha}\right)^{m-1}\right) h\left(V\tilde{Y} - \tilde{Y}\right). \end{aligned} \tag{52}$$

Therefore, $\{V^l\tilde{Y}\}$ is a Cauchy sequence in $(\ell^S(\tau))_h$. Since the space $(\ell^S(\tau))_h$ is prequasi Banach space, we have $\tilde{Z} \in (\ell^S(\tau))_h$ so that $\lim_{l \longrightarrow \infty} V^l\tilde{Y} = \tilde{Z}$. To show that $V\tilde{Z} = \tilde{Z}$, as h holds the Fatou property, we get

$$\begin{aligned} h\left(V\tilde{Z} - \tilde{Z}\right) &\leq \sup_i \inf_{l \geq i} h\left(V^{l+1}\tilde{Y} - V^l\tilde{Y}\right) \\ &\leq \sup_i \inf_{l \geq i} \left(\frac{\alpha}{1-\alpha}\right)^l h\left(V\tilde{Y} - \tilde{Y}\right) = 0, \end{aligned} \tag{53}$$

so $V\tilde{Z} = \tilde{Z}$. Hence, $\tilde{Z}$ is a fixed point of V. To prove the uniqueness, assume $\tilde{Y}, \tilde{Z} \in (\ell^S(\tau))_h$ are two not equal fixed points of V. Then,

$$\begin{aligned} h\left(\tilde{Y} - \tilde{Z}\right) &\leq h\left(V\tilde{Y} - V\tilde{Z}\right) \\ &\leq \alpha\left(h\left(V\tilde{Y} - \tilde{Y}\right) + h\left(V\tilde{Z} - \tilde{Z}\right)\right) = 0. \end{aligned} \tag{54}$$

Hence, $\tilde{Y} = \tilde{Z}$. □

Corollary 29. *Assume $(\tau_q)_{q \in \mathcal{N}} \in \ell_\infty \cap I$ so that $\tau_0 > 1$, and $V : (\ell^S(\tau))_h \longrightarrow (\ell^S(\tau))_h$ is Kannan h-contraction mapping, where $h(\tilde{Y}) = [\sum_{q=0}^{\infty} [m_d(\widetilde{Y_q}, \tilde{0})]^{\tau_q}]^{1/K}$, for every $\tilde{Y} \in \ell^S(\tau)$, then V has unique fixed point $\tilde{Z}$ with $h(V^l\tilde{Y} - \tilde{Z}) \leq \alpha (\alpha/(1-\alpha))^{l-1} h(V\tilde{Y} - \tilde{Y})$.*

Proof. By Theorem 28, we have a unique fixed point $\tilde{Z}$ of V. Then,

$$\begin{aligned} h\left(V^l\tilde{Y} - \tilde{Z}\right) &= h\left(V^l\tilde{Y} - V\tilde{Z}\right) \\ &\leq \alpha\left(h\left(V^l\tilde{Y} - V^{l-1}\tilde{Y}\right) + h\left(V\tilde{Z} - \tilde{Z}\right)\right) \\ &= \alpha\left(\frac{\alpha}{1-\alpha}\right)^{l-1} h\left(V\tilde{Y} - \tilde{Y}\right). \end{aligned} \tag{55}$$

□

Example 30. If $V : (\ell^S(((2q+3)/(q+2))_{q=0}^{\infty}))_h \longrightarrow (\ell^S(((2q+3)/(q+2))_{q=0}^{\infty}))_h$, where $h(\tilde{g}) = \sqrt{\sum_{q=0}^{\infty} (m_d(\tilde{g}_q, \tilde{0}))^{(2q+3)/(q+2)}}$, for all $\tilde{g} \in \ell^S(((2q+3)/(q+2))_{q=0}^{\infty})$ and

$$V(\tilde{g}) = \begin{cases} \dfrac{\tilde{g}}{4}, & h(\tilde{g}) \in [0, 1), \\ \dfrac{\tilde{g}}{5}, & h(\tilde{g}) \in [1, \infty). \end{cases} \tag{56}$$

Since for all $\widetilde{g_1}, \widetilde{g_2} \in (\ell^S(((2q+3)/(q+2))_{q=0}^{\infty}))_h$ so that $h(\widetilde{g_1}), h(\widetilde{g_2}) \in [0,1)$, we have

$$h(V\widetilde{g_1} - V\widetilde{g_2}) = h\left(\frac{\widetilde{g_1}}{4} - \frac{\widetilde{g_2}}{4}\right) \le \frac{1}{\sqrt[4]{27}}\left(h\left(\frac{3\widetilde{g_1}}{4}\right) + h\left(\frac{3\widetilde{g_2}}{4}\right)\right) = \frac{1}{\sqrt[4]{27}}(h(V\widetilde{g_1} - \widetilde{g_1}) + h(V\widetilde{g_2} - \widetilde{g_2})). \quad (57)$$

For every $\widetilde{g_1}, \widetilde{g_2} \in (\ell^S(((2q+3)/(q+2))_{q=0}^{\infty}))_h$ with $h(\widetilde{g_1}), h(\widetilde{g_2}) \in [1,\infty)$, we get

$$h(V\widetilde{g_1} - V\widetilde{g_2}) = h\left(\frac{\widetilde{g_1}}{5} - \frac{\widetilde{g_2}}{5}\right) \le \frac{1}{\sqrt[4]{64}}\left(h\left(\frac{4\widetilde{g_1}}{5}\right) + h\left(\frac{4\widetilde{g_2}}{5}\right)\right) = \frac{1}{\sqrt[4]{64}}(h(V\widetilde{g_1} - \widetilde{g_1}) + h(V\widetilde{g_2} - \widetilde{g_2})). \quad (58)$$

For each $\widetilde{g_1}, \widetilde{g_2} \in (\ell^S(((2q+3)/(q+2))_{q=0}^{\infty}))_h$ with $h(\widetilde{g_1}) \in [0,1)$ and $h(\widetilde{g_2}) \in [1,\infty)$, one has

$$\begin{aligned} h(V\widetilde{g_1} - V\widetilde{g_2}) &= h\left(\frac{\widetilde{g_1}}{4} - \frac{\widetilde{g_2}}{5}\right) \\ &\le \frac{1}{\sqrt[4]{27}} h\left(\frac{3\widetilde{g_1}}{4}\right) + \frac{1}{\sqrt[4]{64}} h\left(\frac{4\widetilde{g_2}}{5}\right) \\ &\le \frac{1}{\sqrt[4]{27}}\left(h\left(\frac{3\widetilde{g_1}}{4}\right) + h\left(\frac{4\widetilde{g_2}}{5}\right)\right) \\ &= \frac{1}{\sqrt[4]{27}}(h(V\widetilde{g_1} - \widetilde{g_1}) + h(V\widetilde{g_2} - \widetilde{g_2})). \end{aligned} \quad (59)$$

Therefore, V is Kannan h-contraction. Since h holds the Fatou property, by Theorem 28, we have V that holds unique fixed point $\tilde{\vartheta} \in (\ell^S(((2q+3)/(q+2))_{q=0}^{\infty}))_h$.

Definition 31. If U_h is a prequasi normed (csss), $V : U_h \longrightarrow U_h$ and $\tilde{Z} \in U_h$. The mapping V is said to be h-sequentially continuous at $\tilde{Z}$, if and only if, assume $\lim_{q\longrightarrow\infty} h(\widetilde{Y_q} - \tilde{Z}) = 0$, one has $\lim_{q\longrightarrow\infty} h(V\widetilde{Y_q} - V\tilde{Z}) = 0$.

Example 32. If $V : (\ell^S(((q+1)/(2q+4))_{q=0}^{\infty}))_h \longrightarrow (\ell^S(((q+1)/(2q+4))_{q=0}^{\infty}))_h$, where $h(\tilde{Z}) = \sum_{q=0}^{\infty}(m_d(\widetilde{Z_q}, \tilde{0}))^{(q+1)/(2q+4)}$, for all $\tilde{Z} \in \ell^S(((q+1)/(2q+4))_{q=0}^{\infty})$ and

$$V(\tilde{Z}) = \begin{cases} \frac{1}{18}(\tilde{b}_0 + \tilde{Z}), & \widetilde{Z_0}(a) \in \left[0, \frac{1}{17}\right), \\ \frac{1}{17}\tilde{b}_0, & \widetilde{Z_0}(a) = \frac{1}{17}, \\ \frac{1}{18}\tilde{b}_0, & \widetilde{Z_0}(a) \in \left(\frac{1}{17}, 1\right]. \end{cases} \quad (60)$$

V is obviously both h-sequentially continuous and discontinuous at $1/17\tilde{b}_0 \in (\ell^S(((q+1)/(2q+4))_{q=0}^{\infty}))_h$.

Example 33. Suppose V is defined as in Example 30. If $\{\widetilde{Z^{(n)}}\} \subseteq (\ell^S(((2q+3)/(q+2))_{q=0}^{\infty}))_h$ with $\lim_{n\longrightarrow\infty} h(\widetilde{Z^{(n)}} - \widetilde{Z^{(0)}}) = 0$, where $\widetilde{Z^{(0)}} \in (\ell^S(((2q+3)/(q+2))_{q=0}^{\infty}))_h$ so that $h(\widetilde{Z^{(0)}}) = 1$.

Since the prequasi norm h is continuous, one obtains

$$\lim_{n\longrightarrow\infty} h\left(V\widetilde{Z^{(n)}} - V\widetilde{Z^{(0)}}\right) = \lim_{n\longrightarrow\infty} h\left(\frac{\widetilde{Z^{(n)}}}{4} - \frac{\widetilde{Z^{(0)}}}{5}\right) = h\left(\frac{\widetilde{Z^{(0)}}}{20}\right) > 0. \quad (61)$$

Hence, V is not h-sequentially continuous at $\widetilde{Z^{(0)}}$.

Theorem 34. *Assume $(\tau_q)_{q\in\mathcal{N}} \in \ell_\infty \cap I$ so that $\tau_0 > 1$, and $V : (\ell^S(\tau))_h \longrightarrow (\ell^S(\tau))_h$, where $h(\tilde{Y}) = \sum_{q=0}^{\infty}[m_d(\widetilde{Y_q}, \tilde{0})]^{\tau_q}$, for every $\tilde{Y} \in \ell^S(\tau)$. If*

(1) V is Kannan h-contraction mapping

(2) V is h-sequentially continuous at $\tilde{Z} \in (\ell^S(\tau))_h$

(3) One has $\tilde{Y} \in (\ell^S(\tau))_h$ so that $\{V^l\tilde{Y}\}$ has $\{V^{l_j}\tilde{Y}\}$ converging to $\tilde{Z}$

Then, $\tilde{Z} \in (\ell^S(\tau))_h$ is the only fixed point of V.

Proof. Suppose $\tilde{Z}$ is not a fixed point of V, we have $V\tilde{Z} \ne \tilde{Z}$. By using conditions (24) and (25), one has

$$\lim_{l_j\longrightarrow\infty} h\left(V^{l_j}\tilde{Y} - \tilde{Z}\right) = 0, \quad \lim_{l_j\longrightarrow\infty} h\left(V^{l_j+1}\tilde{Y} - V\tilde{Z}\right) = 0. \quad (62)$$

Since V is Kannan h-contraction, then

$$\begin{aligned} 0 &< h\left(V\tilde{Z} - \tilde{Z}\right) \\ &= h\left(\left(V\tilde{Z} - V^{l_j+1}\tilde{Y}\right) + \left(V^{l_j}\tilde{Y} - \tilde{Z}\right) + \left(V^{l_j+1}\tilde{Y} - V^{l_j}\tilde{Y}\right)\right) \\ &\le 2^{2^{\sup_i \tau_i - 2}} h\left(V^{l_j+1}\tilde{Y} - V\tilde{Z}\right) + 2^{2^{\sup_i \tau_i - 2}} h\left(V^{l_j}\tilde{Y} - \tilde{Z}\right) \\ &\quad + 2^{\sup_i \tau_i - 1} \alpha\left(\frac{\alpha}{1-\alpha}\right)^{l_j - 1} h\left(V\tilde{Y} - \tilde{Y}\right). \end{aligned} \quad (63)$$

Since $l_j \longrightarrow \infty$, this gives a contradiction. So $\tilde{Z}$ is a fixed point of V. To prove the uniqueness, assume $\tilde{Z}, \tilde{Y} \in (\ell^S(\tau))_h$ is two not equal fixed points of V. We have

$$\begin{aligned} h\left(\tilde{Z}-\tilde{Y}\right) &\leq h\left(V\tilde{Z}-V\tilde{Y}\right) \\ &\leq \alpha\left(h\left(V\tilde{Z}-\tilde{Z}\right)+h\left(V\tilde{Y}-\tilde{Y}\right)\right)=0. \end{aligned} \tag{64}$$

Therefore, $\tilde{Z}=\tilde{Y}$. □

Example 35. If V is defined as in Example 30. Suppose $h(\tilde{Y})=\sum_{q\in\mathcal{N}}(m_d(\widetilde{Y_q},\tilde{0}))^{(2q+3)/(q+2)}$, for every $\tilde{Y}\in\ell^S(((2q+3)/(q+2))_{q=0}^{\infty})$. As for every $\widetilde{Y_1},\widetilde{Y_2}\in(\ell^S(((2a+3)/(q+2))_{q=0}^{\infty}))_h$ so that $h(\widetilde{Y_1}),h(\widetilde{Y_2})\in[0,1)$, we have

$$\begin{aligned} h\left(V\widetilde{Y_1}-V\widetilde{Y_2}\right) &= h\left(\frac{\widetilde{Y_1}}{4}-\frac{\widetilde{Y_2}}{4}\right) \\ &\leq \frac{2}{\sqrt{27}}\left(h\left(\frac{3\widetilde{Y_1}}{4}\right)+h\left(\frac{3\widetilde{Y_2}}{4}\right)\right) \\ &= \frac{2}{\sqrt{27}}\left(h\left(V\widetilde{Y_1}-\widetilde{Y_1}\right)+h\left(V\widetilde{Y_2}-\widetilde{Y_2}\right)\right). \end{aligned} \tag{65}$$

For every $\widetilde{Y_1},\widetilde{Y_2}\in(\ell^S(((2q+3)/(q+2))_{q=0}^{\infty}))_h$ such that $h(\widetilde{Y_1}),h(\widetilde{Y_2})\in[1,\infty)$, then

$$\begin{aligned} h\left(V\widetilde{Y_1}-V\widetilde{Y_2}\right) &= h\left(\frac{\widetilde{Y_1}}{5}-\frac{\widetilde{Y_2}}{5}\right) \\ &\leq \frac{1}{4}\left(h\left(\frac{4\widetilde{Y_1}}{5}\right)+h\left(\frac{4\widetilde{Y_2}}{5}\right)\right) \\ &= \frac{1}{4}\left(h\left(V\widetilde{Y_1}-\widetilde{Y_1}\right)+h\left(V\widetilde{Y_2}-\widetilde{Y_2}\right)\right). \end{aligned} \tag{66}$$

For every $\widetilde{Y_1},\widetilde{Y_2}\in(\ell^S(((2q+3)/(q+2))_{q=0}^{\infty}))_h$ with $h(\widetilde{Y_1})\in[0,1)$ and $h(\widetilde{Y_2})\in[1,\infty)$, we have

$$\begin{aligned} h\left(V\widetilde{Y_1}-V\widetilde{Y_2}\right) &= h\left(\frac{\widetilde{Y_1}}{4}-\frac{\widetilde{Y_2}}{5}\right) \\ &\leq \frac{2}{\sqrt{27}}h\left(\frac{3\widetilde{Y_1}}{4}\right)+\frac{1}{4}h\left(\frac{4\widetilde{Y_2}}{5}\right) \\ &\leq \frac{2}{\sqrt{27}}\left(h\left(\frac{3\widetilde{Y_1}}{4}\right)+h\left(\frac{4\widetilde{Y_2}}{5}\right)\right) \\ &= \frac{2}{\sqrt{27}}\left(h\left(V\widetilde{Y_1}-\widetilde{Y_1}\right)+h\left(V\widetilde{Y_2}-\widetilde{Y_2}\right)\right). \end{aligned} \tag{67}$$

Then, V is Kannan h-contraction and

$$V^l\left(\tilde{Y}\right)=\begin{cases} \dfrac{\tilde{Y}}{4^l}, & h\left(\tilde{Y}\right)\in[0,1), \\ \dfrac{\tilde{Y}}{5^l}, & h\left(\tilde{Y}\right)\in[1,\infty). \end{cases} \tag{68}$$

Clearly, V is h-sequentially continuous at $\tilde{\vartheta}\in(\ell^S(((2a+3)/(q+2))_{q=0}^{\infty}))_h$ and $\{V^l\tilde{Y}\}$ verifies $\{V^{l_j}\tilde{Y}\}$ converges to $\tilde{\vartheta}$. From Theorem 34, the element $\tilde{\vartheta}\in(\ell^S(((2a+3)/(q+2))_{q=0}^{\infty}))_h$ is the only fixed point of V.

5. Kannan Nonexpansive Mapping on $(\ell^S(\tau))_h$

The enough setups of $(\ell^S(\tau))_h$, where $h(\tilde{g})=[\sum_{p=0}^{\infty}(m_d(\tilde{g}_p,\tilde{0}))^{\tau_p}]^{1/K}$, for all $\tilde{g}\in\ell^S(\tau)$, so that the Kannan nonexpansive mapping on it has a fixed point are presented.

By letting Γ a nonempty h-bounded, h-convex, and h-closed subset of $(\ell^S(\tau))_h$.

Lemma 36. *Suppose $(\ell^S(\tau))_h$ verifies the (R) property and the h-quasinormal property. If $V:\Gamma\longrightarrow\Gamma$ is a Kannan h-nonexpansive mapping, for $t>0$, put $G_t=\{\tilde{Y}\in\Gamma: h(\tilde{Y}-V(\tilde{Y}))\leq t\}\neq\varnothing$. Let*

$$\Gamma_t=\bigcap\{\mathbf{B}_h(r,j): V(G_t)\subset\mathbf{B}_h(r,j)\}\cap\Gamma. \tag{69}$$

Hence, $\Gamma_t\neq\varnothing$, h-convex, h-closed subset of Γ and $V(\Gamma_t)\subset\Gamma_t\subset G_t$ and $\delta_h(\Gamma_t)\leq t$.

Proof. As $V(G_t)\subset\Gamma_t$, one has $\Gamma_t\neq\varnothing$. Since the h-balls are h-convex and h-closed, one gets Γ_t is a h-closed and h-convex subset of Γ. To prove that $\Gamma_t\subset G_t$, let $\tilde{Y}\in\Gamma_t$. If $h(\tilde{Y}-V(\tilde{Y}))=0$, we have $\tilde{Y}\in G_t$. Otherwise, when $h(\tilde{Y}-V(\tilde{Y}))>0$, let

$$r=\sup\left\{h\left(V\left(\tilde{Z}\right)-V\left(\tilde{Y}\right)\right): \tilde{Z}\in G_t\right\}. \tag{70}$$

From the definition of r, we have $V(G_t)\subset\mathbf{B}_h(V(\tilde{Y}),r)$. Hence, $\Gamma_t\subset\mathbf{B}_h(V(\tilde{Y}),r)$, so $h(\tilde{Y}-V(\tilde{Y}))\leq r$. By taking $l>0$, we have $\tilde{Z}\in G_t$ so that $r-l\leq h(V(\tilde{Z})-V(\tilde{Y}))$. Then,

$$\begin{aligned} h(\tilde{Y}-V(\tilde{Y}))-l\leq r-l &\leq h\left(V\left(\tilde{Z}\right)-V\left(\tilde{Y}\right)\right) \\ &\leq \frac{1}{2}\left(h(\tilde{Y}-V(\tilde{Y}))+h\left(\tilde{Z}-V\left(\tilde{Z}\right)\right)\right) \\ &\leq \frac{1}{2}\left(h(\tilde{Y}-V(\tilde{Y}))+t\right). \end{aligned} \tag{71}$$

Since l is an arbitrary positive, we have $h(\tilde{Y} - V(\tilde{Y})) \leq t$, so $\tilde{Y} \in G_t$. As $V(G_t) \subset \Gamma_t$, we have $V(\Gamma_t) \subset V(G_t) \subset \Gamma_t$, then Γ_t is V-invariant. To prove that $\delta_h(\Gamma_t) \leq t$. As

$$h\left(V\left(\tilde{Y}\right) - V\left(\tilde{Z}\right)\right) \leq \frac{1}{2}\left(h\left(\tilde{Y} - V\left(\tilde{Y}\right)\right) + h\left(\tilde{Z} - V\left(\tilde{Z}\right)\right)\right), \tag{72}$$

for every $\tilde{Y}, \tilde{Z} \in G_t$. If $\tilde{Y} \in G_t$. We get $V(G_t) \subset \mathbf{B}_h(V(\tilde{Y}), t)$. The definition of Γ_t implies $\Gamma_t \subset \mathbf{B}_h(V(\tilde{Y}), t)$. Hence, $V(\tilde{Y}) \in \bigcap_{t \in \Gamma_t} \mathbf{B}_h(\tilde{Z}, t)$. Then, $h(\tilde{Z} - \tilde{Y}) \leq t$, for all $\tilde{Z}, \tilde{Y} \in \Gamma_t$, this implies $\delta_h(\Gamma_t) \leq t$. □

Theorem 37. *Assume $(\ell^S(\tau))_h$ verifies the h-quasinormal property and the (R) property. If $V : \Gamma \longrightarrow \Gamma$ is a Kannan h-nonexpansive mapping, so V has a fixed point.*

Proof. Put $t_0 = \inf \{h(\tilde{Y} - V(\tilde{Y})) \colon \tilde{Y} \in \Gamma\}$ and $t_r = t_0 + (1/r)$, for all $r \geq 1$. By the definition of t_0, we have $G_{t_r} = \{\tilde{Y} \in \Gamma : h(\tilde{Y} - V(\tilde{Y})) \leq t_r\} \neq \varnothing$, for all $r \geq 1$. If Γ_{t_r} is defined as in Lemma 36, it is obvious that $\{\Gamma_{t_r}\}$ is a decreasing sequence of nonempty h-bounded, h-closed, and h-convex subsets of Γ. The property (R) holds that $\Gamma_\infty = \bigcap_{r \geq 1} \Gamma_{t_r} \neq \varnothing$. Put $\tilde{Y} \in \Gamma_\infty$, then $h(\tilde{Y} - V(\tilde{Y})) \leq t_r$, for every $r \geq 1$. If $r \longrightarrow \infty$, one has $h(\tilde{Y} - V(\tilde{Y})) \leq t_0$, then $h(\tilde{Y} - V(\tilde{Y})) = t_0$. Hence, $G_{t_0} \neq \varnothing$. Therefore, $t_0 = 0$. Otherwise, $t_0 > 0$ then V fails to have a fixed point. Put Γ_{t_0} as defined in Lemma 36. Since V fails to have a fixed point and Γ_{t_0} is V-invariant, so Γ_{t_0} has more than one point, then $\delta_h(\Gamma_{t_0}) > 0$. By the h-quasinormal property, we have $\tilde{Y} \in \Gamma_{t_0}$ so that

$$h\left(\tilde{Y} - \tilde{Z}\right) < \delta_h\left(\Gamma_{t_0}\right) \leq t_0, \tag{73}$$

for every $\tilde{Z} \in \Gamma_{t_0}$. In view of Lemma 36, one has $\Gamma_{t_0} \subset G_{t_0}$. By definition of Γ_{t_0}, then $V(\tilde{Y}) \in G_{t_0} \subset \Gamma_{t_0}$. We have

$$h\left(\tilde{Y} - V\left(\tilde{Y}\right)\right) < \delta_h\left(\Gamma_{t_0}\right) \leq t_0, \tag{74}$$

which contradicts the definition of t_0. So $t_0 = 0$ which gives that any point in G_{t_0} is a fixed point of V. □

In view of Theorems 24, 26, and 28, we have the following.

Corollary 38. *If $(\tau_q)_{q \in \mathcal{N}} \in \ell_\infty \cap I$ so that $\tau_0 > 1$ and $V : \Gamma \longrightarrow \Gamma$ is a Kannan h-nonexpansive mapping. One has V that holds a fixed point.*

Example 39. Suppose $V : \Gamma \longrightarrow \Gamma$ so that

$$V\left(\tilde{Y}\right) = \begin{cases} \dfrac{\tilde{Y}}{4}, & h\left(\tilde{Y}\right) \in [0, 1), \\ \dfrac{\tilde{Y}}{5}, & h\left(\tilde{Y}\right) \in [1, \infty), \end{cases} \tag{75}$$

where $\Gamma = \{\tilde{Y} \in (\ell^S(((2q+3)/(q+2))_{q=0}^{\infty}))_h : \widetilde{Y_0} = \widetilde{Y_1} = \tilde{0}\}$ and $h(\tilde{Y}) = \sqrt{\sum_{q \in \mathcal{N}} (m_d(\widetilde{Y_q}, \tilde{0}))^{(2q+3)/(q+2)}}$, for all $\tilde{Y} \in (\ell^S(((2q+3)/(q+2))_{q=0}^{\infty}))_h$. From Example 35, V is Kannan h-contraction. Therefore, it is Kannan h-nonexpansive. From Corollary 38, then V has a fixed point $\tilde{\vartheta}$ in Γ.

6. Kannan Contraction and Structure of Operators Ideal

The structure of the operators ideal by $(\ell^S(\tau))_h$ under definite function h, where $h(\tilde{g}) = [\sum_{p=0}^{\infty} (m_d(\tilde{g}_p, \tilde{0}))^{\tau_p}]^{1/K}$, for all $\tilde{g} \in \ell^S(\tau)$, and s-soft reals has been offered. Finally, we study the idea of Kannan contraction mapping in its linked pre-quasi operator ideal. Also, the existence of a fixed point of Kannan contraction mapping has been offered. We mark the space of all bounded, finite rank linear operators from a Banach space Δ into a Banach space Λ by $\mathscr{L}(\Delta, \Lambda)$, and $\mathfrak{F}(\Delta, \Lambda)$ and if $\Delta = \Lambda$, we indicate $\mathscr{L}(\Delta)$ and $\mathfrak{F}(\Delta)$.

Definition 40 (see [45]). An s -number function is $s : \mathscr{L}(\Delta, \Lambda) \longrightarrow \mathfrak{R}^{+\mathcal{N}}$ which gives all $V \in \mathscr{L}(\Delta, \Lambda)$ a $(s_d(V))_{d=0}^{\infty}$ holds the next conditions:

(a) $\|V\| = s_0(V) \geq s_1(V) \geq s_2(V) \geq \cdots \geq 0$, for every $V \in \mathscr{L}(\Delta, \Lambda)$

(b) $s_{l+d-1}(V_1 + V_2) \leq s_l(V_1) + s_d(V_2)$, for every $V_1, V_2 \in \mathscr{L}(\Delta, \Lambda)$ and $l, d \in \mathcal{N}$

(c) $s_d(VYW) \leq \|V\| s_d(Y) \|W\|$, for all $W \in \mathscr{L}(\Delta_0, \Delta)$, $Y \in \mathscr{L}(\Delta, \Lambda)$ and $V \in \mathscr{L}(\Lambda, \Lambda_0)$, where Δ_0 and Λ_0 are arbitrary Banach spaces

(d) Suppose $V \in \mathscr{L}(\Delta, \Lambda)$ and $\gamma \in \mathfrak{R}$, one has $s_d(\gamma V) = |\gamma| s_d(V)$

(e) If rank $(V) \leq d$, then $s_d(V) = 0$, for all $V \in \mathscr{L}(\Delta, \Lambda)$

(f) $s_{l \geq a}(I_a) = 0$ or $s_{l < a}(I_a) = 1$, where I_a marks the unit map on the a-dimensional Hilbert space ℓ_2^a

Definition 41 (see [37]). Suppose $\mathscr{L}$ is the class of all bounded linear operators between any arbitrary Banach spaces. A subclass $\mathscr{U}$ of $\mathscr{L}$ is called an operator ideal, when every $\mathscr{U}(\Delta, \Lambda) = \mathscr{U} \cap \mathscr{L}(\Delta, \Lambda)$ holds the next setups:

(i) $I_\Gamma \in \mathscr{U}$, where Γ marks Banach space of one dimension

(ii) The space $\mathscr{U}(\Delta, \Lambda)$ is linear over $\mathfrak{R}$

(iii) If $W \in \mathscr{L}(\Delta_0, \Delta)$, $X \in \mathscr{U}(\Delta, \Lambda)$ and $Y \in \mathscr{L}(\Lambda, \Lambda_0)$, one has $YXW \in \mathscr{U}(\Delta_0, \Lambda_0)$

Notations 42.

$$\begin{aligned}\tilde{\mathfrak{F}}_{\mathbf{U}} &:= \{\tilde{\mathfrak{F}}_{\mathbf{U}}(\Delta, \Lambda)\}, \text{where } \tilde{\mathfrak{F}}_{\mathbf{U}}(\Delta, \Lambda) \\ &:= \left\{V \in \mathscr{L}(\Delta, \Lambda): \left(\left(\widetilde{s_d(V)}\right)_{d=0}^{\infty} \in \mathbf{U}\right\}.\end{aligned} \tag{76}$$

$\widetilde{s_d(V)}(x) = s_d(V)$, for every $x \in A$.

Theorem 43. *If U is a (csss), one has $\tilde{\mathfrak{F}}_U$ an operator ideal.*

Proof.

(i) Suppose $V \in \mathfrak{F}(\Delta, \Lambda)$ and rank $(V) = n$, for every $n \in \mathscr{N}$, since $\tilde{b}_i \in \mathbf{U}$, for every $i \in \mathscr{N}$, and $\mathbf{U}$ is a linear space, then $(\widetilde{s_i(V)})_{i=0}^{\infty} = (\widetilde{s_0(V)}, \widetilde{s_1(V)}, \cdots, \widetilde{s_{n-1}(V)}, \tilde{0}, \tilde{0}, \tilde{0}, \cdots) = \sum_{i=0}^{n-1} \widetilde{s_i(V)}\tilde{b}_i \in \mathbf{U}$; for that $V \in \tilde{\mathfrak{F}}_{\mathbf{U}}(\Delta, \Lambda)$ then $\mathfrak{F}(\Delta, \Lambda) \subseteq \tilde{\mathfrak{F}}_{\mathbf{U}}(\Delta, \Lambda)$

(ii) If $V_1, V_2 \in \tilde{\mathfrak{F}}_{\mathbf{U}}(\Delta, \Lambda)$ and $\beta_1, \beta_2 \in \mathfrak{R}$ so by Definition 5 condition (25) one has $(\widetilde{s_{[i/2]}(V_1)})_{i=0}^{\infty} \in \mathbf{U}$ and $(\widetilde{s_{[i/2]}(V_1)})_{i=0}^{\infty} \in \mathbf{U}$, as $i \geq 2[i/2]$, by the definition of s-numbers and $s_i(V)$ is decreasing, we have $\widetilde{s_i(\beta_1 V_1 + \beta_2 V_2)} \leq \widetilde{s_{2[i/2]}(\beta_1 V_1 + \beta_2 V_2)} \leq \widetilde{s_{[i/2]}(\beta_1 V_1)} + \widetilde{s_{[i/2]}(\beta_2 V_2)} = |\beta_1|\widetilde{s_{[i/2]}(V_1)} + |\beta_2|\widetilde{s_{[i/2]}(V_2)}$ for all $i \in \mathscr{N}$. By Definition 5 part (2) and $\mathbf{U}$ is a linear space, we get $(\widetilde{s_i(\beta_1 V_1 + \beta_2 V_2)})_{i=0}^{\infty} \in \mathbf{U}$; hence, $\beta_1 V_1 + \beta_2 V_2 \in \tilde{\mathfrak{F}}_{\mathbf{U}}(\Delta, \Lambda)$

(iii) Assume $P \in \mathscr{L}(\Delta_0, \Delta)$, $T \in \tilde{\mathfrak{F}}_{\mathbf{U}}(\Delta, \Lambda)$, and $R \in \mathscr{L}(\Lambda, \Lambda_0)$, then $(\widetilde{s_i(T)})_{i=0}^{\infty} \in \mathbf{U}$ and since $\widetilde{s_i(RTP)} \leq \|R\|\widetilde{s_i(T)}\|P\|$, from Definition 5 parts (1) and (2), then $(\widetilde{s_i(RTP)})_{i=0}^{\infty} \in \mathbf{U}$, then $RTP \in \tilde{\mathfrak{F}}_{\mathbf{U}}(\Delta_0, \Lambda_0)$

□

In view of Theorems 11 and 43, we have the following theorem.

Theorem 44. *If $(\tau_q)_{q\in\mathscr{N}} \in \ell_\infty \cap I$ so that $\tau_0 > 0$, then $\tilde{\mathfrak{F}}_{(\ell^S(\tau))_h}$ is an operator ideal.*

Definition 45 [38]. A function $H \in [0,\infty)^{\mathscr{U}}$ is said to be a prequasi norm on the ideal $\mathscr{U}$, when the next setups are verified.

(1) If $V \in \mathscr{U}(\Delta, \Lambda)$, $H(V) \geq 0$, and $H(V) = 0$, if and only if, $V = 0$

(2) One has $Q \geq 1$ so as to $H(\alpha V) \leq D|\alpha|H(V)$, for all $V \in \mathscr{U}(\Delta, \Lambda)$ and $\alpha \in \mathfrak{R}$

(3) One has $P \geq 1$ with $H(V_1 + V_2) \leq P[H(V_1) + H(V_2)]$, for all $V_1, V_2 \in \mathscr{U}(\Delta, \Lambda)$

(4) One has $\sigma \geq 1$ for to if $V \in \mathscr{L}(\Delta_0, \Delta)$, $X \in \mathscr{U}(\Delta, \Lambda)$, and $Y \in \mathscr{L}(\Lambda, \Lambda_0)$, one has $H(YXV) \leq \sigma\|Y\| H(X)\|V\|$

Theorem 46 (see [38]). *H is a prequasi norm on the ideal $\mathscr{U}$, whenever H is a quasinorm on the ideal $\mathscr{U}$.*

Theorem 47. *Suppose $(\tau_q)_{q\in\mathscr{N}} \in \ell_\infty \cap I$ so that $\tau_0 > 0$; hence, the function H is a prequasi norm on $\tilde{\mathfrak{F}}_{(\ell^S(\tau))_h}$, with $H(Z) = h(\widetilde{s_q(Z)})_{q=0}^{\infty}$, for every $Z \in \tilde{\mathfrak{F}}_{(\ell^S(\tau))_h}(\Delta, \Lambda)$.*

Proof.

(1) If $X \in \tilde{\mathfrak{F}}_{(\ell^S(\tau))_h}(\Delta, \Lambda)$, $H(X) = h(\widetilde{s_q(X)})_{q=0}^{\infty} \geq 0$ and $H(X) = h(\widetilde{s_q(X)})_{q=0}^{\infty} = 0$, if and only if, $\widetilde{s_q(X)} = \tilde{0}$, for all $q \in \mathscr{N}$, if and only if, $X = 0$

(2) One has $Q \geq 1$ with $H(\alpha X) = h(\widetilde{s_q(\alpha X)})_{q=0}^{\infty} \leq Q|\alpha| H(X)$, for every $X \in \tilde{\mathfrak{F}}_{(\ell^S(\tau))_h}(\Delta, \Lambda)$ and $\alpha \in \mathfrak{R}$

(3) There are $PP_0 \geq 1$ with for $X_1, X_2 \in \tilde{\mathfrak{F}}_{(\ell^S(\tau))_h}(\Delta, \Lambda)$, we have

$$\begin{aligned}H(X_1 + X_2) &= h\left(\widetilde{s_q(X_1 + X_2)}\right)_{q=0}^{\infty} \\ &\leq P\left(h\left(\widetilde{s_{[q/2]}(X_1)}\right)_{q=0}^{\infty} + h\left(\widetilde{s_{[q/2]}(X_2)}\right)_{q=0}^{\infty}\right) \\ &\leq PP_0\left(h\left(\widetilde{s_q(X_1)}\right)_{q=0}^{\infty} + h\left(\widetilde{s_q(X_2)}\right)_{q=0}^{\infty}\right).\end{aligned} \tag{77}$$

(4) There are $\rho \geq 1$, assume $X \in \mathscr{L}(\Delta_0, \Delta)$, $Y \in \tilde{\mathfrak{F}}_{(\ell^S(\tau))_h}(\Delta, \Lambda)$ and $Z \in \mathscr{L}(\Lambda, \Lambda_0)$, one has $H(ZYX) = h(\widetilde{s_q(ZYX)})_{q=0}^{\infty} \leq h(\|X\|\|Z\|\widetilde{s_q(Y)})_{q=0}^{\infty} \leq \rho\|X\|H(Y)\|Z\|$

□

In the next theorems, we will use the notation $(\tilde{\mathfrak{F}}_{(\ell^S(\tau))_h}, H)$, where $H(V) = h((\widetilde{s_q(V)})_{q=0}^{\infty})$, for every $V \in \tilde{\mathfrak{F}}_{(\ell^S(\tau))_h}$.

Theorem 48. *If $(\tau_q)_{q\in\mathscr{N}} \in \ell_\infty \cap I$ so that $\tau_0 > 0$, then $(\tilde{\mathfrak{F}}_{(\ell^S(\tau))_h}, H)$ is a prequasi Banach operator ideal.*

Proof. Assume $(V_a)_{a\in\mathscr{N}}$ is a Cauchy sequence in $\tilde{\mathfrak{F}}_{(\ell^S(\tau))_h}(\Delta, \Lambda)$. Since $\mathscr{L}(\Delta, \Lambda) \supseteq S_{(\ell^S(\tau))_h}(\Delta, \Lambda)$, we have

$$\begin{aligned}H(V_r - V_a) &= h\left(\left(\widetilde{s_q(V_r - V_a)}\right)_{q=0}^{\infty}\right) \\ &\geq h\left(\widetilde{s_0(V_r - V_a)}, \tilde{0}, \tilde{0}, \tilde{0}, \cdots\right) \geq \|V_r - V_a\|^{\tau_0/K}.\end{aligned} \tag{78}$$

Then, $(V_a)_{a\in\mathcal{N}}$ is a Cauchy sequence in $\mathscr{L}(\Delta,\Lambda)$. Since $\mathscr{L}(\Delta,\Lambda)$ is a Banach space, one has $V\in\mathscr{L}(\Delta,\Lambda)$ with $\lim_{a\longrightarrow\infty}\|V_a-V\|=0$ and as $(s_q\widetilde{(V_a)})_{q=0}^{\infty}\in(\ell^S(\tau))_h$, for every $a\in\mathcal{N}$ and $(\ell^S(\tau))_h$ is a premodular (csss). Then,

$$\begin{aligned}H(V)&=h\left(\left(\widetilde{s_q(V)}\right)_{q=0}^{\infty}\right)\\&\le h\left(\left(s_{[q/2]}\widetilde{(V-V_a)}\right)_{q=0}^{\infty}\right)+h\left(\left(\widetilde{s_{[q/2]}}(V_a)_{q=0}^{\infty}\right)\right)\\&\le h\left(\left(\|V_a-V\|\tilde{1}\right)_{q=0}^{\infty}\right)+(2)^{1/K}h\left(\left(s_q\widetilde{(V_a)}\right)_{q=0}^{\infty}\right)<\varepsilon,\end{aligned}\tag{79}$$

one gets $(\widetilde{s_q(V)})_{q=0}^{\infty}\in(\ell^S(\tau))_h$, then $V\in\tilde{\mathfrak{F}}_{(\ell^S(\tau))_h}(\Delta,\Lambda)$. □

Theorem 49. *Assume $(\tau_q)_{q\in\mathcal{N}}\in\ell_\infty\cap I$ so that $\tau_0>0$, then $(\tilde{\mathfrak{F}}_{(\ell^S(\tau))_h},H)$ is a prequasi closed operator ideal.*

Proof. If $V_a\in\tilde{\mathfrak{F}}_{(\ell^S(\tau))_h}(\Delta,\Lambda)$, for every $a\in\mathcal{N}$ and $\lim_{a\longrightarrow\infty}H(V_a-V)=0$. Since $\mathscr{L}(\Delta,\Lambda)\supseteq S_{(\ell^S(\tau))_h}(\Delta,\Lambda)$, we have

$$\begin{aligned}H(V_a-V)&=h\left(\left(s_q\widetilde{(V_a-V)}\right)_{q=0}^{\infty}\right)\\&\ge h\left(s_0\widetilde{(V_a-V)},\tilde{0},\tilde{0},\tilde{0},\cdots\right)\ge\|V_a-V\|^{\tau_0/K}.\end{aligned}\tag{80}$$

Hence, $(V_a)_{a\in\mathcal{N}}$ is convergent in $\mathscr{L}(\Delta,\Lambda)$; i.e., $\lim_{a\longrightarrow\infty}\|V_a-V\|=0$ and as $(s_q\widetilde{(V_a)})_{q=0}^{\infty}\in(\ell^S(\tau))_h$, for every $q\in\mathcal{N}$ and $(\ell^S(\tau))_h$ is a premodular (csss). Then,

$$\begin{aligned}H(V)&=h\left(\left(\widetilde{s_q(V)}\right)_{q=0}^{\infty}\right)\\&\le h\left(\left(s_{[q/2]}\widetilde{(V-V_a)}\right)_{q=0}^{\infty}\right)+h\left(\left(\widetilde{s_{[q/2]}}(V_a)_{q=0}^{\infty}\right)\right)\\&\le h\left(\left(\|V_a-V\|\tilde{1}\right)_{q=0}^{\infty}\right)+(2)^{1/K}h\left(\left(s_q\widetilde{(V_a)}\right)_{q=0}^{\infty}\right)<\varepsilon.\end{aligned}\tag{81}$$

We obtain $(\widetilde{s_q(V)})_{q=0}^{\infty}\in(\ell^S(\tau))_h$; hence, $V\in\tilde{\mathfrak{F}}_{(\ell^S(\tau))_h}(\Delta,\Lambda)$. □

Definition 50. A prequasi norm H on the ideal $\tilde{\mathfrak{F}}_{U_h}$ holds the Fatou property if for all $\{T_q\}_{q\in\mathcal{N}}\subseteq\tilde{\mathfrak{F}}_{U_h}(\Delta,\Lambda)$ with $\lim_{q\longrightarrow\infty}H(T_q-T)=0$ and $M\in\tilde{\mathfrak{F}}_{U_h}(\Delta,\Lambda)$, then

$$H(M-T)\le\sup_q\inf_{j\ge q}H\left(M-T_j\right).\tag{82}$$

Theorem 51. *If $(\tau_q)_{q\in\mathcal{N}}\in\ell_\infty\cap I$ so that $\tau_0>0$, then $(\tilde{\mathfrak{F}}_{(\ell^S(\tau))_h},H)$ does not satisfy the Fatou property.*

Proof. Let $\{T_q\}_{q\in\mathcal{N}}\subseteq\tilde{\mathfrak{F}}_{(\ell^S(\tau))_h}(\Delta,\Lambda)$ so that $\lim_{q\longrightarrow\infty}H(T_q-T)=0$. As $\tilde{\mathfrak{F}}_{(\ell^S(\tau))_h}$ is a prequasi closed ideal, one has $T\in\tilde{\mathfrak{F}}_{(\ell^S(\tau))_h}(\Delta,\Lambda)$. So for all $M\in\tilde{\mathfrak{F}}_{(\ell^S(\tau))_h}(\Delta,\Lambda)$, then

$$\begin{aligned}H(M-T)&=\left[\sum_{q=0}^{\infty}\left(m_d\left(s_q\widetilde{(M-T)},\tilde{0}\right)\right)^{\tau_q}\right]^{1/K}\\&\le\left[\sum_{q=0}^{\infty}\left(m_d\left(s_{[q/2]}\widetilde{(M-T_i)},\tilde{0}\right)\right)^{\tau_q}\right]^{1/K}\\&\quad+\left[\sum_{q=0}^{\infty}\left(m_d\left(s_{[q/2]}\widetilde{(T_i-T)},\tilde{0}\right)\right)^{\tau_q}\right]^{1/K}\\&\le(2)^{1/K}\sup_r\inf_{j\ge r}\left[\sum_{q=0}^{\infty}\left(m_d\left(s_q\widetilde{(M-T_j)},\tilde{0}\right)\right)^{\tau_q}\right]^{1/K}.\end{aligned}\tag{83}$$

□

Definition 52. An operator $V:\tilde{\mathfrak{F}}_{U_h}(\Delta,\Lambda)\longrightarrow\tilde{\mathfrak{F}}_{U_h}(\Delta,\Lambda)$ is called a Kannan H-contraction, if there is $\alpha\in[0,1/2)$ so that $H(VT-VM)\le\alpha(H(VT-T)+H(VM-M))$, for every $T,M\in\tilde{\mathfrak{F}}_{U_h}(\Delta,\Lambda)$.

Definition 53. An operator $V:\tilde{\mathfrak{F}}_{U_h}(\Delta,\Lambda)\longrightarrow\tilde{\mathfrak{F}}_{U_h}(\Delta,\Lambda)$ is called H-sequentially continuous at M, where $M\in\tilde{\mathfrak{F}}_{U_h}(\Delta,\Lambda)$, if and only if, $\lim_{r\longrightarrow\infty}H(T_r-M)=0\Rightarrow\lim_{r\longrightarrow\infty}H(VT_r-VM)=0$.

Example 54. Assume $V:\tilde{\mathfrak{F}}_{(\ell^S(((2q+3)/(q+2))_{q=0}^{\infty}))_h}(\Delta,\Lambda)\longrightarrow\tilde{\mathfrak{F}}_{(\ell^S(((2q+3)/(q+2))_{q=0}^{\infty}))_h}(\Delta,\Lambda)$, where $H(T)=\sqrt{\sum_{q=0}^{\infty}(m_d(\widetilde{s_q(T)},\tilde{0}))^{(2q+3)/(q+2)}}$, for all $T\in\tilde{\mathfrak{F}}_{(\ell^S(((2q+3)/(q+2))_{q=0}^{\infty}))_h}(\Delta,\Lambda)$ and

$$V(T)=\begin{cases}\dfrac{T}{6}, & H(T)\in[0,1),\\[2mm]\dfrac{T}{7}, & H(T)\in[1,\infty).\end{cases}\tag{84}$$

Clearly, V is H-sequentially continuous at the zero operator $\Theta\in\tilde{\mathfrak{F}}_{(\ell^S(((2q+3)/(q+2))_{q=0}^{\infty}))_h}$. Suppose $\{T^{(j)}\}\subseteq\tilde{\mathfrak{F}}_{(\ell^S(((2q+3)/(q+2))_{q=0}^{\infty}))_h}$ so that $\lim_{j\longrightarrow\infty}H(T^{(j)}-T^{(0)})=0$, where $T^{(0)}\in\tilde{\mathfrak{F}}_{(\ell^S(((2q+3)/(q+2))_{q=0}^{\infty}))_h}$ with $H(T^{(0)})=1$. As the prequasi norm H is continuous, we have

$$\begin{aligned}\lim_{j\longrightarrow\infty}H\left(VT^{(j)}-VT^{(0)}\right)&=\lim_{j\longrightarrow\infty}H\left(\frac{T^{(0)}}{6}-\frac{T^{(0)}}{7}\right)\\&=H\left(\frac{T^{(0)}}{42}\right)>0.\end{aligned}\tag{85}$$

Hence, V is not H-sequentially continuous at $T^{(0)}$.

Theorem 55. *If* $(\tau_q)_{q\in\mathcal{N}} \in \ell_\infty \cap I$ *so that* $\tau_0 > 0$ *and* $V : \tilde{\Xi}_{(\ell^S(\tau))_h}(\Delta, \Lambda) \longrightarrow \tilde{\Xi}_{(\ell^S(\tau))_h}(\Delta, \Lambda)$. *Suppose*

(i) *V is Kannan H-contraction mapping*

(ii) *V is H-sequentially continuous at a vector* $M \in \tilde{\Xi}_{(\ell^S(\tau))_h}(\Delta, \Lambda)$

(iii) *we have* $G \in \tilde{\Xi}_{(\ell^S(\tau))_h}(\Delta, \Lambda)$ *so that the sequence of iterates* $\{V^r G\}$ *has a* $\{V^{r_i} G\}$ *converging to M*

Then, $M \in \tilde{\Xi}_{(\ell^S(\tau))_h}(\Delta, \Lambda)$ is the unique fixed point of V.

Proof. Assume M is not a fixed point of V, one has $VM \neq M$. By using conditions (ii) and (iii), one has

$$\lim_{r_i\longrightarrow\infty} H(V^{r_i}G - M) = 0 \text{ and } \lim_{r_i\longrightarrow\infty} H\left(V^{r_i+1}G - VM\right) = 0. \tag{86}$$

As V is Kannan H-contraction, we get

$$\begin{aligned} 0 &< H(VM - M) \\ &= H\left(\left(VM - V^{r_i+1}G\right) + \left(V^{r_i}G - M\right) + \left(V^{r_i+1}G - V^{r_i}G\right)\right) \\ &\le (2)^{1/K} H\left(V^{r_i+1}G - VM\right) + (2)^{2/K} H\left(V^{r_i}G - M\right) \\ &\quad + (2)^{2/K}\alpha\left(\frac{\alpha}{1-\alpha}\right)^{r_i-1} H(VG - G). \end{aligned} \tag{87}$$

Since $r_i \longrightarrow \infty$, we have a contradiction. Therefore, M is a fixed point of V. To prove the uniqueness of the fixed point M, assume there are two not equal fixed points M, $J \in \tilde{\Xi}_{(\ell^S(\tau))_h}(\Delta, \Lambda)$ of V. We get $H(M - J) \le H(VM - VJ) \le \alpha(H(VM - M) + H(VJ - J)) = 0$. So, $M = J$. □

Example 56. According to Example 54, as for every $T_1, T_2 \in \tilde{\Xi}_{(\ell^S(((2q+3)/(q+2))_{q=0}^\infty))_h}$ with $H(T_1), H(T_2) \in [0, 1)$, then

$$\begin{aligned} H(VT_1 - VT_2) &= H\left(\frac{T_1}{6} - \frac{T_2}{6}\right) \\ &\le \frac{\sqrt{2}}{\sqrt[4]{125}}\left(H\left(\frac{5T_1}{6}\right) + H\left(\frac{5T_2}{6}\right)\right) \\ &= \frac{\sqrt{2}}{\sqrt[4]{125}}(H(VT_1 - T_1) + H(VT_2 - T_2)). \end{aligned} \tag{88}$$

For every $T_1, T_2 \in \tilde{\Xi}_{(\ell^S(((2q+3)/(q+2))_{q=0}^\infty))_h}$ with $H(T_1)$, $H(T_2) \in [1,\infty)$, then

$$\begin{aligned} H(VT_1 - VT_2) &= H\left(\frac{T_1}{7} - \frac{T_2}{7}\right) \\ &\le \frac{\sqrt{2}}{\sqrt[4]{216}}\left(H\left(\frac{6T_1}{7}\right) + H\left(\frac{6T_2}{7}\right)\right) \\ &= \frac{\sqrt{2}}{\sqrt[4]{216}}(H(VT_1 - T_1) + H(VT_2 - T_2)). \end{aligned} \tag{89}$$

For each $T_1, T_2 \in \tilde{\Xi}_{(\ell^S(((2q+3)/(q+2))_{q=0}^\infty))_h}$ with $H(T_1) \in [0, 1)$ and $H(T_2) \in [1,\infty)$, one gets

$$\begin{aligned} H(VT_1 - VT_2) &= H\left(\frac{T_1}{6} - \frac{T_2}{7}\right) \\ &\le \frac{\sqrt{2}}{\sqrt[4]{125}} H\left(\frac{5T_1}{6}\right) + \frac{\sqrt{2}}{\sqrt[4]{216}} H\left(\frac{6T_2}{7}\right) \\ &\le \frac{\sqrt{2}}{\sqrt[4]{125}}(H(VT_1 - T_1) + H(VT_2 - T_2)). \end{aligned} \tag{90}$$

Therefore, V is Kannan H-contraction and

$$V^r(T) = \begin{cases} \dfrac{T}{6^r}, & H(T) \in [0, 1), \\ \dfrac{T}{7^r}, & H(T) \in [1,\infty). \end{cases} \tag{91}$$

Clearly, V is H-sequentially continuous at $\Theta \in \tilde{\Xi}_{(\ell^S(((2q+3)/(q+2))_{q=0}^\infty))_h}$ and $\{V^r T\}$ has a subsequence $\{V^{r_i} T\}$ that converges to Θ. According to Theorem 55, Θ is the only fixed point of G.

7. Applications

In this section, some successful applications to the existence of solutions of nonlinear difference equations of soft functions are introduced.

Theorem 57. *Assume the summable equations*

$$Y_q = R_q + \sum_{r=0}^{\infty} D(q, r) m(r, Y_r), \tag{92}$$

which are considered by many authors [46–48], and let $V : (\ell^S(\tau))_h \longrightarrow (\ell^S(\tau))_h$, *where* $(\tau_q)_{q\in\mathcal{N}} \in \ell_\infty \cap \mathbf{I}$ *with* $\tau_0 > 1$ *and* $h(\tilde{Y}) = \left[\sum_{j=0}^{\infty}\left(m_d(\widetilde{Y_j}, \tilde{0})\right)^{\tau_j}\right]^{1/K}$, *for all* $\tilde{Y} \in \ell^S(\tau)$, *given by*

$$V\left(\widetilde{Y_q}\right)_{q\in\mathcal{N}} = \left(\widetilde{R_q} + \sum_{r=0}^{\infty} D(q, r) v\left(r, \widetilde{Y_r}\right)\right)_{q\in\mathcal{N}}. \tag{93}$$

The summable equation (92) has a unique solution in $(\ell^S(\tau))_h$, *when* $D:\mathcal{N}^2 \longrightarrow \mathfrak{R}$,$v:\mathcal{N}\times\mathfrak{R}(A)\longrightarrow\mathfrak{R}(A)$, $\tilde{R}:\mathcal{N}\longrightarrow\mathfrak{R}(A)$,$\tilde{Z}:\mathcal{N}\longrightarrow\mathfrak{R}(A)$, *and for all* $q\in\mathcal{N}$, *suppose*

$$\begin{aligned}&\left|\sum_{r\in\mathcal{N}} D(q,r)\left(v\left(r,\widetilde{Y_r}\right)-v\left(r,\widetilde{Z_r}\right)\right)\right|\\&\quad\tilde{\leq}\frac{1}{2^K}\left[\left|\widetilde{R_q}-\widetilde{Y_q}+\sum_{r=0}^{\infty}D(q,r)v\left(r,\widetilde{Y_r}\right)\right|\right.\\&\quad\left.+\left|\widetilde{R_q}-\widetilde{Z_q}+\sum_{r=0}^{\infty}D(q,r)v\left(r,\widetilde{Z_r}\right)\right|\right].\end{aligned}\tag{94}$$

Proof. We have

$$\begin{aligned}&h\left(V\tilde{Y}-V\tilde{Z}\right)\\&=\left[\sum_{q\in\mathcal{N}}\left(m_d\left(V\widetilde{Y_q}-V\widetilde{Z_q},\tilde{0}\right)\right)^{\tau_q}\right]^{1/K}\\&=\left[\sum_{q\in\mathcal{N}}\left(m_d\left(\sum_{r\in\mathcal{N}}D(q,r)\left[v\left(r,\widetilde{Y_r}\right)-v\left(r,\widetilde{Z_r}\right)\right],\tilde{0}\right)\right)^{\tau_q}\right]^{1/K}\\&\leq\frac{1}{2}\left[\sum_{q\in\mathcal{N}}\left(m_d\left(\widetilde{R_q}-\widetilde{Y_q}+\sum_{r=0}^{\infty}D(q,r)v\left(r,\widetilde{Y_r}\right),\tilde{0}\right)\right)^{\tau_q}\right]^{1/K}\\&\quad+\frac{1}{2}\left[\sum_{q\in\mathcal{N}}\left(m_d\left(\widetilde{R_q}-\widetilde{Z_q}+\sum_{r=0}^{\infty}D(q,r)v\left(r,\widetilde{Z_r}\right),\tilde{0}\right)\right)^{\tau_q}\right]^{1/K}\\&=\frac{1}{2}\left(h\left(V\tilde{Y}-\tilde{Y}\right)+h\left(V\tilde{Z}-\tilde{Z}\right)\right).\end{aligned}\tag{95}$$

In view of Theorem 37, there is a unique solution of equation (92) in $(\ell^S(\tau))_h$. □

Example 58. If $(\ell^S(((2q+3)/(q+2))_{q=0}^{\infty}))_h$, where $h(\tilde{Y})=\sqrt{\sum_{q\in\mathcal{N}}(m_d(Y_q,\tilde{0}))^{(2q+3)/(q+2)}}$, for every $\tilde{Y}\in\ell^S(((2q+3)/(q+2))_{q=0}^{\infty})$. Assume the summable equations

$$\widetilde{Y_q}=\widetilde{R_q}+\sum_{r=0}^{\infty}(-1)^{q+r}\left(\frac{\widetilde{Y_q}}{q^2+r^2+1}\right)^t,\tag{96}$$

so that $q\geq 2$ and $t>0$. Let $\Gamma=\{\tilde{Y}\in(\ell^S(((2q+3)/(q+2))_{q=0}^{\infty}))_h:\widetilde{Y_0}=\widetilde{Y_1}=\tilde{0}\}$. Clearly, Γ is a nonempty, h-convex, h-closed, and h-bounded subset of $(\ell^S(((2q+3)/(q+2))_{q=0}^{\infty}))_h$. Suppose $V:\Gamma\longrightarrow\Gamma$ is defined as

$$V\left(\widetilde{Y_q}\right)_{q\geq 2}=\left(\widetilde{R_q}+\sum_{r=0}^{\infty}(-1)^{q+r}\left(\frac{\widetilde{Y_q}}{q^2+r^2+1}\right)^t\right)_{q\geq 2}.\tag{97}$$

Evidently,

$$\begin{aligned}&\left|\sum_{r=0}^{\infty}(-1)^q\left(\frac{\widetilde{Y_q}}{q^2+r^2+1}\right)^t((-1)^r-(-1)^r)\right|\\&\quad\tilde{\leq}\frac{1}{4}\left[\left|\widetilde{R_q}-\widetilde{Y_q}+\sum_{r=0}^{\infty}(-1)^{q+r}\left(\frac{\widetilde{Y_q}}{q^2+r^2+1}\right)^t\right|\right.\\&\quad\left.+\left|\widetilde{R_q}-\widetilde{Z_q}+\sum_{r=0}^{\infty}(-1)^{q+r}\left(\frac{\widetilde{Z_q}}{q^2+r^2+1}\right)^t\right|\right].\end{aligned}\tag{98}$$

According to Theorem 57 and Corollary 38, the summable equations (96) have a solution in Γ.

Example 59. If $(\ell^S(((2q+3)/(q+2))_{q=0}^{\infty}))_h$, where $h(\tilde{Y})=\sqrt{\sum_{q\in\mathcal{N}}(m_d(Y_q,\tilde{0}))^{(2q+3)/(q+2)}}$, for every $\tilde{Y}\in\ell^S(((2q+3)/(q+2))_{q=0}^{\infty})$, let the nonlinear difference equations

$$\widetilde{Y_q}=\widetilde{R_q}+\sum_{l=0}^{\infty}(-1)^{q+l}\frac{\widetilde{Y_{q-2}^r}}{\widetilde{Y_{q-1}^q}+\widetilde{l^2}+\tilde{1}},\tag{99}$$

so that $r,q>0$, $\widetilde{Y_{-2}}(x),\widetilde{Y_{-1}}(x)>0$, for every $x\in A$, and suppose $V:\ell^S(((2q+3)/(q+2))_{q=0}^{\infty})\longrightarrow\ell^S(((2q+3)/(q+2))_{q=0}^{\infty})$, explained by

$$V\left(Y_q\right)_{q=0}^{\infty}=\left(\widetilde{R_q}+\sum_{l=0}^{\infty}(-1)^{q+l}\frac{\widetilde{Y_{q-2}^r}}{\widetilde{Y_{q-1}^q}+\widetilde{l^2}+\tilde{1}}\right)_{q=0}^{\infty}.\tag{100}$$

It is clear that

$$\begin{aligned}&\left|\sum_{l=0}^{\infty}(-1)^q\frac{\widetilde{Y_{q-2}^r}}{\widetilde{Y_{q-1}^q}+\widetilde{l^2}+\tilde{1}}\left((-1)^l-(-1)^l\right)\right|\\&\quad\leq\frac{1}{4}\left[\left|\widetilde{R_q}-\widetilde{Y_q}+\sum_{l=0}^{\infty}(-1)^{q+l}\frac{\widetilde{Y_{q-2}^r}}{\widetilde{Y_{q-1}^q}+\widetilde{l^2}+\tilde{1}}\right|\right.\\&\quad\left.+\left|\widetilde{R_q}-\widetilde{Z_q}+\sum_{l=0}^{\infty}(-1)^{q+l}\frac{\widetilde{Z_{q-2}^r}}{\widetilde{Z_{q-1}^q}+\widetilde{l^2}+\tilde{1}}\right|\right].\end{aligned}\tag{101}$$

In view of Theorem 57, the nonlinear difference equations (99) have a unique solution in $\ell^S(((2q+3)/(q+2))_{q=0}^{\infty})$.

8. Conclusion

The site we discussed was a "pre-quasinormed" place rather than a "quasinormed" location. In the prequasi Banach space, the concept of a fixed point of the Kannan prequasi norm contraction mapping is introduced (csss). Both (R) and the pre-quasinormal structure are supported. The occurrence of a fixed point in the Kannan nonexpansive mapping was studied in this study. A fixed point of Kannan contraction mapping in the prequasi Banach operator ideal formed by Nakano (csss) and the s-soft real numbers has also been investigated for a fixed point of Kannan contraction mapping. Finally, we have demonstrated how the results can be applied to a problem by presenting a few examples of how this has happened. Under a wide range of flexible conditions, the presence of a sequence can be established using the Nakano sequence space. Specifically, when it comes to the variable exponent in the previously described space, our key conclusions have helped to strengthen several well-established ideas.

Data Availability

No data were used to support this study.

Ethical Approval

This article does not contain any studies with human participants or animals performed by any of the authors.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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